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The Transmission Problem to Thermoelastic Plate of Hyperbolic Type*

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Abstract

In this paper we consider the thermoelastic plate equation with localized thermal dissipation of memory type, proposed by Gurtin and Pipkin [11]. We will show that the solution of the corresponding model decays exponentially as time goes to infinity, provided the relaxation function decays exponentially. The main difference with others thermoelastic system is that the whole system is of hyperbolic type, and the *dissipation* is weaker (indefinite) than such given by the Fourier Law for the heat flux.

Keywords – Exponential stability, transmission problem, thermoelasticity.

1 Introduction

In the classical linear theory of thermoelasticity Fourier's law is used to describe the heat conduction of the body. Therefore, the corresponding thermoelastic equations consist of an hyperbolic equation for the displacemente field coupled with a parabolic equation for the heat equation. This theory has two shortcomings: First, it is unable to take into account the memory effect which may prevail in some materials, particularly at low temperatures. Second, the corresponding parabolic part of the system predicts an unrealistic result in the sense that the thermal disturbance at one point of the body is instantly felt everywhere in the body. Although, at first sight, this outcome of the theory seems to contradict the physical intuition, it can be justified by resorting to the fact that molecular motion, which plays a crucial part in transport phenomena, is very rapid except at extremely low temperatures. Hence a finite velocity of propagation for

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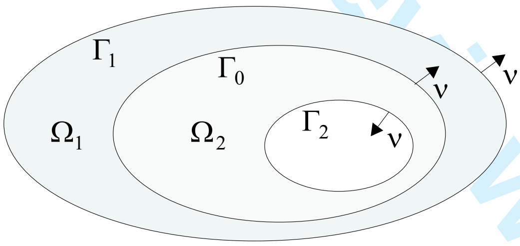
thermal perturbations is usually nononservable unless experiments are performed in some neighbourhood of absolute zero as in the case of liquid helium. In fact, thermal waves, commonly known as second sound, are detected in some metals cooled approximately down to 20^oK. For example Brorson et al [2] observed electron temperature transport velocities of $8.4 \times 10^5 m/s$ in thin gold films upon sudden heating with ultrafast femtosecond laser irradiation. Other result in this directions can be found in [25, 9, 8, 3, 7, 4] among others. Very limited documented experimental results appear for these situations in the literature. For a short survey the reader is referred to the works of Ackerman and Guyer [1], Taylor et al. [27], and Jackson and Walker [14].

To take into account the memory effect at low temperatures, Gurtin and Pipkin [11] introduce a new constitutive law for the heat flux. This constitutive law depends on the heat memory and as a first consequence the parabolicity of the system is removed. Therefore the thermoelastic system is fully hyperbolic. So we have finite speed of propagation (see [18]).

In this paper we study the transmission problem for a partial thermoelastic plate. That is, we consider a plate composed by two components, a thermoelastic part and an elastic part insensible to changes of temperature. This in particular means that the thermal constants are discontinuous on the plate, positive over the thermoelastic region and zero over the elastic part.

More precisely, let us denote by Ω an open bounded set of $\mathbb{R}^2$ with smooth boundary $\partial\Omega = \Gamma_1 \cup \Gamma_2$. We assume that over the region Ω_1 the plate is sensitive to the change of temperature, while in the complementary part $\Omega_2 = \Omega\backslash\Omega_1$, the plate is indifferent to changes of temperature. Let us denote by Γ_0 the interphace, that is a curve between Ω_1 and Ω_2, a typical example of Ω is given by the next picture,



Let us denote by u and v the transverse oscillation over Ω_1 and Ω_2, respectively and by θ the difference of temperature, then the transmission problem for the thermoelastic plate equation is written as

$$\rho_1 u_{tt} - \gamma_1 \triangle u_{tt} + \beta_1 \triangle^2 u + \mu \triangle \theta = 0 \quad \text{in} \quad \Omega_1 \times \mathbb{R}^+ \tag{1.1}$$

$$\rho_0 \theta_t - \beta_0 (k * \triangle \theta) - \mu \triangle u_t = 0 \quad \text{in} \quad \Omega_1 \times \mathbb{R}^+ \tag{1.2}$$

$$\rho_2 v_{tt} - \gamma_2 \triangle v_{tt} + \beta_2 \triangle^2 v = 0 \quad \text{in} \quad \Omega_2 \times \mathbb{R}^+ \tag{1.3}$$

with the following boundary conditions

$$u = \frac{\partial u}{\partial \nu} = 0 \quad \text{on} \quad \Gamma_1 \times \mathbb{R}^+, \qquad v = \frac{\partial v}{\partial \nu} = 0 \quad \text{on} \quad \Gamma_2 \times \mathbb{R}^+ \tag{1.4}$$

$$\theta = 0 \quad \text{on} \quad \Gamma_0 \times \mathbb{R}^+ \quad \text{and} \quad \theta = 0 \quad \text{on} \quad \Gamma_1 \times \mathbb{R}^+ \tag{1.5}$$

and the transmission conditions over Γ_0

$$u = v, \quad \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} \tag{1.6}$$

$$\beta_1 \triangle u = \beta_2 \triangle v \tag{1.7}$$

$$\gamma_1 \frac{\partial u_{tt}}{\partial \nu} - \beta_1 \frac{\partial \triangle u}{\partial \nu} - \mu \frac{\partial \theta}{\partial \nu} = \gamma_2 \frac{\partial v_{tt}}{\partial \nu} - \beta_2 \frac{\partial \triangle v}{\partial \nu}. \tag{1.8}$$

Finally, we prescrive the initial conditions

$$u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \quad \theta(x,0) = \theta_0(x) \quad \text{in} \quad \Omega_1 \tag{1.9}$$

$$v(x,0) = v_0(x), \quad v_t(x,0) = v_1(x) \quad \text{in} \quad \Omega_2. \tag{1.10}$$

The constants $\rho_0, \rho_1, \rho_2, \beta_0, \beta_1, \beta_2, \gamma_1, \gamma_2, \mu$ are all positive. We denote by $k \in C^1(0,\infty)$ the relaxation function and by $*$ the convolution product given by

$$k * \varphi(t) = \int_0^t k(t-\tau)\varphi(\tau)\, d\tau.$$

Problem (1.1)–(1.3) is know as Volterra's integral differential equation see [5]. To be precise in our formulation, let us first introduce the following definition.

Definition 1.1 *Let $k \in L^1(\mathbb{R}_+)$, We say that k is a positive definite function when*

$$\int_0^t \varphi k * \varphi\, ds \geq 0, \qquad \forall t \geq 0, \quad \forall \varphi \in C^0(\mathbb{R}^+).$$

We say that k is strongly positive definite function when there exists $\delta > 0$ such that $k(t) - \delta e^{-t}$ is positive definite.

One important characterization of positive definite function is given in the following theorem, which is proved in [23].

Theorem 1.1 *Let $k \in L^1(\mathbb{R}_+)$, then k is a positive definite function if and only if*

$$\text{Re}\, \widehat{k}(i\xi) \geq 0 \qquad \textit{where} \qquad \widehat{k}(\lambda) = \int_0^\infty k(t) e^{-\lambda t}\, dt.$$

The hypotheses we use to show the existence result and the exponential decay are the following

H1 $k \in C^2$ is a strongly positive definite function, satisfying $k(0) \geq 0$

H2 k decays to zero exponentially.

Remark 1.1 *Hypotheses* **H2** *we will use only to show the exponential decay. It is not necessary to show the existence of solution.*

One important remark is that the dissipation produced for the Gurtin and Pipkin's law is of indefinite type. That is, the derivative of the energy function can change its sign. In fact, the total energy associated with system (1.1)-(1.10) is given by

$$\begin{aligned} E(t) &= \frac{1}{2}\int_{\Omega_1} \rho_1|u_t|^2 + \gamma_1|\nabla u_t|^2 + \beta_1|\triangle u|^2 + \rho_0|\theta|^2 dx + \\ &+\frac{1}{2}\int_{\Omega_2} \rho_2|v_t|^2 + \gamma_2|\nabla v_t|^2 + \beta_2|\triangle v|^2 dx. \end{aligned} \tag{1.11}$$

Using the equations we can verify that

$$\frac{d}{dt}E(t) = -\beta_0\int_{\Omega_1}(k*\nabla\theta)\cdot\nabla\theta dx. \tag{1.12}$$

Note that the right hand side of the above equation, does not have a definite sign. This is because positive definite function as k, does not make that the right hand side of equation (1.12) is negative. In fact, let us consider the functions

$$k(t) = e^{-t}\cos t, \quad y(t) = e^{-2t}.$$

We have that k satisfies

$$k \in L^1(0,\infty) \quad \text{and} \quad \text{Re}\,\hat{k}(i\xi) \geq \frac{1}{2(1+\xi^2)}, \quad \forall\xi\in\mathbb{R},$$

then k is strongly definite positive (see Remark 0.1.1). On the other hand we have that

$$\mathcal{Y}(t) \equiv (k*y)(t)\cdot y(t) = \frac{e^{-4t}}{2}[e^t(\cos t + \sin t) - 1],$$

change of sign. To see this take $t = \frac{\pi}{2} + 2m\pi$ to get $\mathcal{Y}(t) > 0$, while for $t = -\frac{\pi}{2} + 2m\pi$ we have $\mathcal{Y}(t) < 0$. Therefore $\mathcal{Y}(t)$ change of sign. But $\int_0^T \mathcal{Y}(t)dt > 0$ for any $T > 0$.

Integrating (1.12) over $[0,t[$, we get

$$E(t) = E(0) - \beta_0\int_0^t\int_{\Omega_1}(k*\nabla\theta)\cdot\nabla\theta dt \tag{1.13}$$

and using the fact that k is positive definite we conclude that $E(t)$ is bounded by $E(0)$. Therefore, system (1.1)-(1.10) is not of dissipative type. Note also that the thermal effect, which produces the *indefinite dissipation* is active only in one equation. So, we may ask whether the energy associated to system decays or not to zero uniformly as time goes to infinity.

Concerning the literature relative to our problem, we have the work of Fabrizio-Lazzari-Rivera[6], that proves the exponential stability for a thermoelastic plate, when the temperature is active over the whole plate configurated over Ω, $(\Omega_2 = \emptyset)$. Here we follow similar ideas but we have to deal also with estimates over the interphase of the plate which in general produces several problems. In [10], it is considered the one dimensional thermoelastic system with Gurtin and Pipkin's law for the temperature, working over the whole domain, it is proved that the corresponding solution decays exponentially to zero, provided that the relaxation function is strongly positive and also decays exponentially to zero. Finally, in [20] the authors consider the transmission problem for the thermoelastic plate equation with the Fourier law. It is proved that the corresponding solution decays exponentially to zero as time goes to infinity, no matter how small is the thermoelastic part (the dissipative part) of the plate.

The main result of this paper is to prove that the solutions of the partial thermoelastic plate system decays exponentially to zero, no matter the size of the thermoelastic component of the plate Ω_1, which produces the *thermal dissipation* over the plate. The main difference concerning the result in [10] is because of the interphase conditions. For plates the interphase conditions are more complicated, so it is necessary to take care to estimate terms over the interphase of the body configurated over Ω. Concerning the work [20], the system is of hyperbolic-parabolic type because of the Fourier law. This fact, produce a strong dissipation that gives an important help to arrive to the exponential decay. In our case the system is not dissipative and does not have a parabolic part. Therefore the regularizing properties dessapear and we have to look for others techniques to achieve the estimates necessary to show the exponential stability.

The method we use to prove the main result is based on observabilities inequatilies for transmission problems. We also introduce new multipliers which combining with Volterra's method to solve integral equations produce crux estimates to achieve the exponential estability.

The rest of the paper is organized as follows. In section 0.2 we introduce some notations and establish some results which will be useful to show the existence of solutions as well as the exponential decay. In section 0.3 we establish the existence and regularity of solutions. Finally, in section 0.4 we show the exponential decay.

2 Notations and Preliminaries

Let us introduce, the following functional space

$$\mathbb{H}^1 = \{(\varphi,\psi) \in H^1(\Omega_1) \times H^1(\Omega_2); \quad \text{satisfying} \quad (2.14)\}$$

$$\begin{cases} \varphi = 0 & \text{on} \quad \Gamma_1 \\ \psi = 0 & \text{on} \quad \Gamma_2 \\ \varphi = \psi & \text{on} \quad \Gamma_0 \end{cases} \tag{2.14}$$

$$\mathbb{H}^2 = \{(\varphi,\psi) \in [H^2(\Omega_1)\times H^2(\Omega_2)]\cap \mathbb{H}^1; \quad \text{satisfying} \quad (2.15)\}$$

$$\begin{cases} \dfrac{\partial \varphi}{\partial \nu} = 0 \quad \text{on} \quad \Gamma_1 \\ \dfrac{\partial \psi}{\partial \nu} = 0 \quad \text{on} \quad \Gamma_2 \\ \dfrac{\partial \varphi}{\partial \nu} = \dfrac{\partial \psi}{\partial \nu} \quad \text{on} \quad \Gamma_0 \end{cases} \tag{2.15}$$

$$\mathbb{V}^2 = H^2(\Omega_1)\times H^2(\Omega_2)\cap \mathbb{H}^1,$$

$$\mathbb{V}^3 = H^3(\Omega_1)\times H^3(\Omega_2)\cap \mathbb{H}^2,$$

$$\mathbb{V}^4 = H^4(\Omega_1)\times H^4(\Omega_2)\cap \mathbb{H}^2.$$

Lemma 2.1 *The space $\mathbb{H}^1$ with the inner product*

$$\langle(\varphi^1,\psi^1),(\varphi^2,\psi^2)\rangle_{\mathbb{H}^1} = \int_{\Omega_1}(\rho_1\varphi^1\varphi^2 + \gamma_1\nabla\varphi^1\cdot\nabla\varphi^2)dx + \int_{\Omega_2}(\rho_2\psi^1\psi^2 + \gamma_2\nabla\psi^1\cdot\nabla\psi^2)dx$$

is a Hilbert space.

Lemma 2.2 *The space $\mathbb{H}^2$ with the inner product*

$$\langle(\varphi^1,\psi^1),(\varphi^2,\psi^2)\rangle_{\mathbb{H}^2} = \beta_1\int_{\Omega_1}\triangle\varphi^1\triangle\varphi^2 dx + \beta_2\int_{\Omega_2}\triangle\psi^1\triangle\psi^2 dx$$

is a Hilbert space.

We finish this section establishing the following Lemma whose proof can be found in [26]

Lemma 2.3 *Let us suppose that $k \in L^1(\mathbb{R}^+)$ is a strongly positive definite kernel satisfying $k' \in \quad L^1(\mathbb{R}^+)$; then we have*

$$\int_0^t |k * y(\tau)|^2\, d\tau \le \beta_0 K \int_0^t k * y(\tau)\; y(\tau)\; d\tau$$

for any $y \in L^1_{\text{loc}}(\mathbb{R}^+)$, where $K = |k|_1^2 + 4|k'|_1^2$ and $\beta_0 > 0$ such that the function $k(t) - \beta_0^{-1}e^{-t}$ is a positive definite kernel.

3 Existence and uniqueness of solutions

Here we establish the existence and regularity of weak solutions of (1.1)-(1.10). To this end we assume that the hypotheses **H1** holds. Our starting point is to define weak solution to (1.1)–(1.3)

Definition 3.1 *We say that (u, v, θ) is a weak solution of the system (1.1)-(1.10) when*

$$(u, v) \in L^\infty(0, T; \mathbb{H}^2) \qquad (u_t, v_t) \in L^\infty(0, T; \mathbb{H}^1)$$

$$\theta \in L^\infty(0, T; L^2(\Omega_1)), \quad (k * \theta) \in L^2(0, T; H^1(\Omega_1))$$

and satisfies the following identities

$$\begin{aligned}
-\rho_1 \int_{\Omega_1} u_1(x)\varphi(x,0)dx - \gamma_1 \int_{\Omega_1} \nabla u_1 \cdot \nabla\varphi(x,0)dx + \rho_1 \int_{\Omega_1} u_0(x)\varphi_t(x,0)dx \quad & - \\
-\gamma_1 \int_{\Omega_1} \nabla u_0 \cdot \nabla\varphi_t(x,0)dx - \rho_2 \int_{\Omega_2} v_1(x)\psi(x,0)dx - \gamma_2 \int_{\Omega_2} \nabla v_1 \cdot \nabla\psi(x,0)dx \quad & + \\
+\rho_2 \int_{\Omega_2} v_0(x)\psi_t(x,0)dx - \gamma_2 \int_{\Omega_2} \nabla v_0 \cdot \nabla\psi_t(x,0)dx + \int_0^T \int_{\Omega_1} [\rho_1 u\varphi_{tt} + \gamma_1 \nabla u \cdot \nabla\varphi_{tt}]dxdt \quad & + \\
+\beta_1 \int_0^T \int_{\Omega_1} \triangle u \triangle\varphi dxdt + \mu \int_0^T \int_{\Omega_1} \theta \triangle\varphi dxdt + \int_0^T \int_{\Omega_2} [\rho_2 v\psi_{tt} + \gamma_2 \nabla u \cdot \nabla\psi_{tt}]dxdt \quad & + \\
+\beta_2 \int_0^T \int_{\Omega_2} \triangle v \triangle\psi dxdt \quad & = \quad 0
\end{aligned}$$

$$\begin{aligned}
\rho_0 \int_{\Omega_1} \theta_0(x)\phi(x,0)dx - \rho_0 \int_0^T \int_{\Omega_1} \theta\phi_t dxdt + \beta_0 \int_0^T \int_{\Omega_1} (k * \nabla\theta) \cdot \nabla\phi dxdt \quad & + \\
+\mu \int_0^T \int_{\Omega_1} \nabla u_t \cdot \nabla\phi dxdt \quad & = \quad 0
\end{aligned}$$

$\forall(\varphi, \psi) \in \mathcal{C}^2([0, T]; \mathbb{H}^2)$ *such that*

$$\varphi(\cdot, T) = \varphi_t(\cdot, T) = \nabla\varphi(\cdot, T) = \nabla\varphi_t(\cdot, T) = 0$$

$$\psi(\cdot, T) = \psi_t(\cdot, T) = \nabla\psi(\cdot, T) = \nabla\psi_t(\cdot, T) = 0$$

$\forall\phi \in \mathcal{C}^1([0, T]; H_0^1(\Omega_1))$

Using Galerkin method, and standard estimates we can show the following theorem.

Theorem 3.1 (Existence of solutions) *Let us suppose that hypotheses* **H1** *holds, if* $(u^0, v^0) \in \mathbb{H}^2$, $\quad (u^1, v^1) \in \mathbb{H}^1$ *and* $\theta^0 \in L^2(\Omega_1)$. *Then, there exists a unique weak solution for (1.1)-(1.10). On the other hand, if* $(u^0, v^0) \in \mathbb{V}^4$, $\quad (u^1, v^1) \in \mathbb{V}^3$ *and* $\theta^0 \in H^2(\Omega_1) \cap H_0^1(\Omega_1)$ *with:*

$$\beta_1 \triangle u^0 = \beta_2 \triangle v^0, \qquad \beta_1 \frac{\partial \triangle u^0}{\partial \nu} = \beta_2 \frac{\partial \triangle v^0}{\partial \nu}$$

then, there exists a unique solution for (1.1)-(1.10), satisfying:

$$(u, v) \in C([0, \infty[; \mathbb{V}^4) \cap C^1([0, \infty[; \mathbb{V}^3) \cap C^2([0, \infty[; \mathbb{H}^2)$$

$$\theta \in C([0, \infty[; H_0^1(\Omega_1) \cap H^2(\Omega_1)) \cap C^1([0, \infty[; H_0^1(\Omega_1))$$

$$(k * \theta) \in L^2(0, T; H^3(\Omega_1))$$

For a more general method to prove existente to transmission problems, we refer to [13, 15].

Remark 3.1 *To show the existence of a weak solution we only need that the strongly positive function k be in C^0, and $k(0) > 0$. Instead, to get the regularity result we need hypotheses* **H1***.*

4 Exponential decay

In this section we will assume that the domain Ω has the following geometric property: There exists $x_0 \in \mathbb{R}^2$ such that the function $m(x) = x - x_0$ satisfies:

$$\begin{array}{lll} m \cdot \nu \geq \delta_0 \quad > 0 & \text{in} & \Gamma_0 \\ m \cdot \nu \leq 0 & \text{in} & \Gamma_2 \end{array}$$

for $\delta_0 > 0$.

To show that the solution decays exponentially to zero as time goes to infinity, we introduce the following functions

$$U(x,t) = u(x,t)e^{\eta t}, \quad \Theta(x,t) = \theta(x,t)e^{\eta t}, \quad \text{and} \quad V(x,t) = v(x,t)e^{\eta t}.$$

Therefore, to prove the exponential decay, we only have to prove that the above functions are uniformly bounded with respect to the time for η small enough. To do this, let us differentiate with respect to the time to get

$$\begin{aligned} U_t &= \eta U + e^{\eta t}u_t, \quad U_{tt} = 2\eta U_t + e^{\eta t}u_{tt} - \eta^2 U \\ V_t &= \eta V + e^{\eta t}v_t, \quad V_{tt} = 2\eta V_t + e^{\eta t}v_{tt} - \eta^2 V \\ \Theta_t &= \eta\Theta + e^{\eta t}\theta_t, \quad \tilde{k} * \triangle\Theta = e^{\eta t}(k * \triangle\theta). \end{aligned}$$

From the above identities and using equations (1.1)-(1.10), we see that (U, Θ, V) satisfies the following system

$$\rho_1 U_{tt} - \gamma_1 \triangle U_{tt} + \beta_1 \triangle^2 U + \mu \triangle\Theta = \mathcal{P}, \quad \text{in} \quad \Omega_1 \times \mathbb{R}^+ \tag{4.16}$$

$$\rho_0 \Theta_t - \beta_0(\tilde{k} * \triangle\Theta) - \mu \triangle U_t = \mathcal{Q}, \quad \text{in} \quad \Omega_1 \times \mathbb{R}^+ \tag{4.17}$$

$$\rho_2 V_{tt} - \gamma_2 \triangle V_{tt} + \beta_2 \triangle^2 V = \mathcal{R}, \quad \text{in} \quad \Omega_2 \times \mathbb{R}^+ \tag{4.18}$$

with the following boundary conditions:

$$U = \frac{\partial U}{\partial \nu} = 0 \quad \text{on} \quad \Gamma_1 \times \mathbb{R}^+, \qquad V = \frac{\partial V}{\partial \nu} = 0 \quad \text{on} \quad \Gamma_2 \times \mathbb{R}^+ \tag{4.19}$$

$$\Theta = 0 \quad \text{on} \quad \Gamma_0 \times \mathbb{R}^+ \quad \text{and} \quad \Theta = 0 \quad \text{on} \quad \Gamma_1 \times \mathbb{R}^+ \tag{4.20}$$

where $\mathcal{P}$, $\mathcal{Q}$ and $\mathcal{R}$ are given by:

$$\begin{aligned}\mathcal{P} &= 2\eta\rho_1 U_t - \rho_1\eta^2 U + \gamma_1\eta^2\triangle U - 2\gamma_1\eta\triangle U_t,\\ \mathcal{Q} &= \rho_0\eta\Theta - \mu\eta\triangle U,\\ \mathcal{R} &= 2\eta\rho_2 V_t - \rho_2\eta^2 V + \gamma_2\eta^2\triangle V - 2\gamma_2\eta\triangle V_t.\end{aligned}$$

Finally, the couple (U, V) must verify the following transmission conditions:

$$U = V, \qquad \frac{\partial U}{\partial\nu} = \frac{\partial V}{\partial\nu} \tag{4.21}$$

$$\beta_1\triangle U = \beta_2\triangle V \tag{4.22}$$

$$\gamma_1\Big\{\frac{\partial U_{tt}}{\partial\nu} - 2\eta\frac{\partial U_t}{\partial\nu} + \eta^2\frac{\partial U}{\partial\nu}\Big\} - \beta_1\frac{\partial\triangle U}{\partial\nu} - \mu\frac{\partial\Theta}{\partial\nu} = \gamma_2\Big\{\frac{\partial V_{tt}}{\partial\nu} - 2\eta\frac{\partial V_t}{\partial\nu} + \eta^2\frac{\partial V}{\partial\nu}\Big\} - \beta_2\frac{\partial\triangle V}{\partial\nu} \tag{4.23}$$

in $\Gamma_0 \times \mathbb{R}^+$; and initial conditions:

$$U(x,0) = u_0(x), \quad U_t(x,0) = u_1(x) + \eta u_0(x) \quad \text{in} \quad \Omega_1 \tag{4.24}$$

$$\Theta(x,0) = \theta_0(x) \quad \text{in} \quad \Omega_1 \tag{4.25}$$

$$V(x,0) = v_0(x), \quad V_t(x,0) = v_1(x) + \eta v_0(x) \quad \text{in} \quad \Omega_2 \tag{4.26}$$

Let us introduce the energy function $\mathcal{E}(t)$ associated with the above equations,

$$\begin{aligned}\mathcal{E}(t) &= \frac{1}{2}\int_{\Omega_1}\rho_1|U_t|^2 + \gamma_1|\nabla U_t|^2 + \beta_1|\triangle U|^2 + \rho_0|\Theta|^2\,dx + \frac{\eta^2}{2}\int_{\Omega_1}\rho_1|U|^2 + \gamma_1|\nabla U|^2dx\\ &\quad + \frac{\rho_2}{2}\int_{\Omega_2}|V_t|^2 + \gamma_2|\nabla V_t|^2 + \beta_2|\triangle V|^2\,dx + \frac{\eta^2}{2}\int_{\Omega_2}\rho_2|V|^2 + \gamma_2|\nabla V|^2dx.\end{aligned}$$

To show the exponential decay of $E(t)$ it is enough to show that $\mathcal{E}(t)$ is uniformly bounded for any $t > 0$. To this end, we start with the following Lemma.

Lemma 4.1 *Let us suppose that the initial data satisfy* $(u_0, v_0) \in \mathbb{V}^4$, $(u_1, v_1) \in \mathbb{V}^3$ *e* $\theta_0 \in H^2(\Omega_1) \cap H_0^1(\Omega_1)$ *with*

$$\beta_1\triangle u^0 = \beta_2\triangle v^0, \qquad \beta_1\frac{\partial\triangle u^0}{\partial\nu} = \beta_2\frac{\partial\triangle v^0}{\partial\nu}$$

Then, there exist positive constants c *and* β_0 *satisfying*

$$\frac{d}{dt}\mathcal{E}(t) \le -\beta_0\int_{\Omega_1}(\tilde{k} * \nabla\Theta)\cdot\nabla\Theta dx + c\eta\mathcal{E}(t).$$

Proof: Multiplying equation (4.16) by U_t , (4.17) by Θ and (4.18) by V_t and summing up the product result our conclusion follows.

Theorem 4.1 *Under the same hypotheses as in Lemma 0.4.1, if in addition we assume that*

$$\rho_1 \geq \rho_2, \quad \gamma_1 \geq \gamma_2 \quad e \quad \beta_1 \leq \beta_2, \tag{4.27}$$

then the solution of system (1.1)-(1.10) decays exponentially to zero as time goes to infinity. That is, there exist positive constants $C > 0$, $\lambda > 0$, *such that*

$$E(t) \leq CE(0)e^{-\lambda t}.$$

Proof: Let us denote by $\mathcal{K}U = m \cdot \nabla U - \frac{1}{2}U$, multiplying equation (4.16) by $\mathcal{K}U$, equation (4.18) by $\mathcal{K}V$, applying hypotheses (4.27) and using integration by parts we get

$$\begin{aligned} \frac{d}{dt}I(t) \leq\ & \frac{\beta_1}{2}\int_{\Gamma_1}(m\cdot\nu)|\triangle U|^2dx + \frac{\rho_1\eta^2}{2}\int_{\Gamma_0}(m\cdot\nu)|U|^2dx + \frac{\gamma_1\eta^2}{2}\int_{\Gamma_0}(m\cdot\nu)|\nabla U|^2dx \\ & -\frac{3\rho_1}{2}\int_{\Omega_1}|U_t|^2dx - \frac{\gamma_1}{2}\int_{\Omega_1}|\nabla U_t|^2dx - \frac{\beta_1}{2}\int_{\Omega_1}|\triangle U|^2dx - \frac{3\rho_2}{2}\int_{\Omega_2}|V_t|^2dx \\ & -\frac{\gamma_2}{2}\int_{\Omega_2}|\nabla V_t|^2dx - \frac{\beta_2}{2}\int_{\Omega_2}|\triangle V|^2dx + \mu\int_{\Omega_1}\nabla\Theta\cdot\nabla(m\cdot\nabla U)dx \\ & + \frac{\mu}{2}\int_{\Omega_1}\Theta\triangle U dx + \eta c_0\mathcal{E}(t), \end{aligned}$$

where

$$I(t) = \rho_1\int_{\Omega_1}U_t\mathcal{K}U + \gamma_1\int_{\Omega_1}\nabla U_t\cdot\nabla\mathcal{K}U dx - \frac{\eta\rho_1}{2}\int_{\Omega_1}|U|^2dx - \frac{\eta\gamma_1}{2}\int_{\Omega_1}|\nabla U|^2dx + \rho_2\int_{\Omega_2}V_t\mathcal{K}V$$

$$+\gamma_2\int_{\Omega_2}\nabla V_t\cdot\nabla\mathcal{K}V dx - \frac{\eta\rho_2}{2}\int_{\Omega_2}|V|^2dx - \frac{\eta\gamma_2}{2}\int_{\Omega_2}|\nabla V|^2dx.$$

The main problem in the above inequality is with the term $\mu\int_{\Omega_1}\nabla\Theta\cdot\nabla(m\cdot\nabla U)dx$, which is not possible to estimate in terms of the first order energy $\mathcal{E}(t)$. For this reason we introduce the functional,

$$\mathcal{M}(t) = \rho_0\int_{\Omega_1}\Theta m_i\frac{\partial}{\partial x_i}(\tilde{k}*\Theta)dx - \mu\int_{\Omega_1}\triangle U m_i\frac{\partial}{\partial x_i}(\tilde{k}*\Theta)dx.$$

That is to say, let us multiply equation (4.17) by $m_i\frac{\partial}{\partial x_i}(\tilde{k}*\Theta)$. Therefore for $\eta > 0$ small enough, we have that there exist positive constants c_1, c_2, c_3, such that:

$$\begin{aligned} \frac{d}{dt}\mathcal{M}(t) \leq\ & -\frac{\beta_0\delta_0}{2}\int_{\Gamma_0}|\tilde{k}*\frac{\partial\Theta}{\partial\nu}|^2dx + \frac{\beta_0}{2}\int_{\Gamma_1}(m\cdot\nu)|\tilde{k}*\frac{\partial\Theta}{\partial\nu}|^2dx - c_1\int_{\Omega_1}|\Theta|^2dx \\ & -\mu\tilde{k}(0)\int_{\Omega_1}\nabla\Theta\cdot\nabla(m\cdot\nabla U)dx + c_2\int_{\Omega_1}|\tilde{k}*\nabla\Theta|^2 + \eta c_3\mathcal{E}(t), \end{aligned}$$

where we have used that

$$\int_{\Omega_1}\nabla\Theta\cdot\nabla(m\nabla U)\,dx = \int_{\Omega_1}(m\cdot\nabla\Theta)\triangle U\,dx.$$

To simplifly notations, let us introduce the functional $G(t) = I(t) + \frac{1}{\tilde{k}(0)}\mathcal{M}(t)$. For $\eta > 0$ small enough there exist positive constants c_0, c_1, c_2 such that

$$\frac{d}{dt}G(t) \leq \frac{\beta_1 c_0}{2}\underbrace{\int_{\Gamma_1}|\triangle U|^2dx}_{:=J_1} + \frac{\beta_0 c_0}{2\tilde{k}(0)}\underbrace{\int_{\Gamma_1}\left|\tilde{k}*\frac{\partial\Theta}{\partial\nu}\right|^2dx}_{:=J_2} - \frac{\beta_0\delta_0}{2\tilde{k}(0)}\int_{\Gamma_0}\left|\tilde{k}*\frac{\partial\Theta}{\partial\nu}\right|^2dx$$
$$- c_1\mathcal{E}(t) + c_2\int_{\Omega_1}|\tilde{k}*\nabla\Theta|^2dx + \frac{\mu^2}{4\beta_1}\int_{\Omega_1}|\Theta|^2dx.$$

From the above inequality, we see that our next problem is to estimate the boundary terms J_1 and J_2. To do this, we apply an observability technique which consist in to use some multipliers at the boundary, such as $h = (h_1, h_2) \in [C^2(\Omega)]^n$ defined by

$$h(x) = \begin{cases} -\nu(x) & \text{se} \quad x \in \Gamma_1 \\ 0 & \text{se} \quad x \in B_\delta(\Omega_2). \end{cases}$$

where

$$B_\delta(\Omega_2) = \{x \in \Omega, \ \ \mathrm{dist}(x, \Omega_2) \leq \delta\}.$$

Let us introduce the functional

$$H(t) = \rho_1\int_{\Omega_1}U_t(h\cdot\nabla U)dx + \gamma_1\int_{\Omega_1}\nabla U_t\cdot\nabla(h\cdot\nabla U)dx$$
$$+\frac{1}{\tilde{k}(0)}\Big\{\rho_0\int_{\Omega_1}\Theta h_i\frac{\partial}{\partial x_i}(\tilde{k}*\Theta)dx - \mu\int_{\Omega_1}\triangle U h_i\frac{\partial}{\partial x_i}(\tilde{k}*\Theta)dx\Big\}.$$

Multiplying equation (4.16) by $(h\cdot\nabla U)$ and equation (4.17) by $h_i\frac{\partial}{\partial x_i}(\tilde{k}*\Theta)$. Summing up the product results, we get for $\eta > 0$ small enough that there exist positive constants C_i with $i = 1, 2, ..., 5$ such that:

$$\frac{d}{dt}H(t) \leq -\frac{\beta_1}{2}\int_{\Gamma_1}|\triangle U|^2dx - \frac{\beta_0}{2\tilde{k}(0)}\int_{\Gamma_1}|\tilde{k}*\frac{\partial\Theta}{\partial\nu}|^2dx - C_1\int_{\Omega_1}|U_t|^2dx + \eta C_2\int_{\Omega_1}|\nabla U_t|^2dx$$
$$-C_3\int_{\Omega_1}|\triangle U|^2dx - C_4\int_{\Omega_1}|\Theta|^2dx + C_5\int_{\Omega_1}|\tilde{k}*\nabla\Theta|^2dx.$$

To simplify once more, let us denote by $F(t)$ the function $F(t) = G(t) + \epsilon_0 H(t)$. So we have that for $\epsilon_0 > 0$ large enough and for $\eta > 0$ small, there exists a positive constant, we denote by c_i with $i = 1, 2, ...5$, such that

$$\frac{d}{dt}F(t) \leq -c_1\int_{\Gamma_1}|\triangle U|^2dx - c_2\int_{\Gamma_1}|\tilde{k}*\frac{\partial\Theta}{\partial\nu}|^2dx - c_3\int_{\Gamma_0}|\tilde{k}*\frac{\partial\Theta}{\partial\nu}|^2dx$$
$$-c_4E(t) + c_5\int_{\Omega_1}|\tilde{k}*\nabla\Theta|^2dx. \tag{4.28}$$

Let us define the Lyapunov functional $\mathcal{L}$, given by

$$\mathcal{L}(t) = N\mathcal{E}(t) + F(t),$$

where N denotes a large positive constant to be fixed later. Combining Lemma 0.4.1 and inequality (4.28) and using Lemma 0.2.3 we conclude that:

$$\begin{aligned}\frac{d}{dt}\mathcal{L}(t) &\leq -\beta_0 N\int_{\Omega_1}(\tilde{k}*\nabla\Theta)\cdot\nabla\Theta dx + c_5\int_{\Omega_1}|\tilde{k}*\nabla\Theta|^2dx - (c_4-\eta cN)E(t,U,\Theta,V) - \\ &- c_1\int_{\Gamma_1}|\triangle U|^2dx - c_2\int_{\Gamma_1}|\tilde{k}*\frac{\partial\Theta}{\partial\nu}|^2dx - c_3\int_{\Gamma_0}|\tilde{k}*\frac{\partial\Theta}{\partial\nu}|^2dx.\end{aligned}$$

Integrating over $[0,t[$ we get

$$\begin{aligned}\mathcal{L}(t) &\leq \mathcal{L}(0) - \beta_0 N\int_0^t\int_{\Omega_1}(\tilde{k}*\nabla\Theta)\cdot\nabla\Theta dxd\tau + c_5\int_0^t\int_{\Omega_1}|\tilde{k}*\nabla\Theta|^2dxd\tau - \\ &-(c_4-\eta cN)\int_0^t\mathcal{E}(\tau)d\tau - c_1\int_0^t\int_{\Gamma_1}|\triangle U|^2dxd\tau - \\ &-c_2\int_0^t\int_{\Gamma_1}|\tilde{k}*\frac{\partial\Theta}{\partial\nu}|^2dxd\tau - c_3\int_0^t\int_{\Gamma_0}|\tilde{k}*\frac{\partial\Theta}{\partial\nu}|^2dxd\tau. \qquad (4.29)\end{aligned}$$

Using Young's inequality we can to prove that for N large enough, there exist positive constants C_1, C_2 such that

$$e^{2\eta t}E(t) \leq 2\mathcal{E}(t) \leq C_1\mathcal{L}(t) + C_2\int_{\Omega_1}|\tilde{k}*\nabla\Theta|^2dx. \qquad (4.30)$$

Combining (4.29) and (4.30) we conclude that for N large enough and η small enough we have:

$$E(t) \leq C_1\mathcal{L}(0)e^{-2\eta t} \leq CE(0)e^{-2\eta t}.$$

Which implies the exponential stability.

Remark 4.1 *As a final remark, since the problem is linear, Theorem 0.4.1 can be extended by using standard density method, to weak solution. That is when the initial data satisfies*

$$(u^0, v^0) \in \mathbb{H}^2, \quad (u^1, v^1) \in \mathbb{H}^1, \quad \theta^0 \in L^2(\Omega_1).$$

In this case we get from (4.29) the regularity result to weak solutions

$$\frac{\partial(k*\theta)}{\partial\nu} \in L^2(0,T;L^2(\Gamma_0\cup\Gamma_1)), \quad \triangle u \in L^2(0,T;L^2(\Gamma_1))$$

which can not be obtained as a consequence of the regularity result of the weak solution.

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