Strong Solutions of Navier-Stokes Equations of Quantum Incompressible Fluids

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Abstract

In this paper, we deal with the motion of the incompressible quantum fluids of Navier-Stokes type in a d-dimensional torus $\mathbb{T}^d$, with $d \leq 3$. In applications, such kind of fluids are used to describe superfluids, quantum semiconductors and quantum trajectories of Bohmian mechanics. By using the concept of effective velocity the equations of the motion are deduced and we obtain results on the existence and uniqueness of solutions of the problem (by using semi-spectral nonlinear Galerkin method and weak compactness results); besides that, results on continuity of velocity-solution and density-solution are shown in adequate spaces.

1. Introduction

In his work [15], Louis de Broglie proposed that the wave-particle duality would be a general property of microscopic objects. Broglie suggested that microscopic particles, besides of the material behaviour, do behave like particles (with definite position and momentum at any moment); he also showed that the particles have intrinsic characteristics of wave phenomena. This would imply a new type of wave in coexistence with the material point, the wave would act as a kind of pilot wave guiding the particle.

David Bohm [4], [5] rediscovered the de Broglie's hypothesis and developed a new physical theory. In his quantum mechanics formulation, Bohm represents the probability distribution n of a single particle as a classical fluid (in the sense of a set of particles) that moves on both classic effect of an external field and a quantum field which is known as potential Bohm and described by the equation

$$V_{qu} = -\frac{\varepsilon^2}{4m}\left[\frac{\Delta n}{n} - \frac{1}{2}\left(\frac{\nabla n}{n}\right)^2\right] = -\frac{\varepsilon^2}{2m}\frac{\Delta\sqrt{n}}{\sqrt{n}},$$

with the equations of motion,

$$\begin{aligned} & n_t + div\,(n\nabla S/m) = 0, \\ & S_t + \frac{(\nabla S)^2}{2m} + V_{qu} + V = 0. \end{aligned}$$

Based on the work of Bohm, Harvey ([22]) introduced the fluid quantum theory by using the following system of equations:

$$\begin{aligned} & \rho_t + div(n\mathbf{v}) = 0, \\ & m\mathbf{v}_t + m(\mathbf{v}\cdot\nabla)\mathbf{v} = -\nabla(V_{qu} + V), \end{aligned}$$

where $\mathbf{v} = \nabla S/m$ (where m is the mass of the particle, and S is the phase function of the equation of the particle wave associated).

The theory of quantum fluid was initially devised to describe trajectories of particles in quantum mechanics of Bohm [23], [20], currently also being used to describe superfluids [16] and quantum semiconductor [7]. In [1], it was shown a connection between Quantum Euler model

$$\begin{aligned} & n_t + div(n\mathbf{w}) = \nu\Delta n \\ & (n\mathbf{w})_t + div(n\mathbf{w}\otimes\mathbf{w}) + \nabla p(n) - 2\varepsilon_0^2 n\nabla\left(\frac{\Delta\sqrt{n}}{\sqrt{n}}\right) - n\mathbf{f} = \nu\Delta(n\mathbf{w}) \end{aligned}$$

(introduced by [17]) and the Quantum Navier-Stokes model

$$\begin{aligned} & n_t + div(n\mathbf{u}) = 0 \\ & (n\mathbf{u})_t + div(n\mathbf{u}\otimes\mathbf{u}) + \nabla p(n) - 2\varepsilon^2 n\nabla\left(\frac{\Delta\sqrt{n}}{\sqrt{n}}\right) - n\mathbf{f} = 2\nu div(nD(\mathbf{u})) \end{aligned}$$

(this last one introduced by [24]). By using the effective velocity $\mathbf{w} = \mathbf{u} + \nu\nabla log\ n$ (which was firstly used in [6]), Jüngel [1] proved the existence of global weak solution for the Navier-Stokes quantum baratropic compressible fluid, with constant viscosity and smaller than the Plank constant, considering the region as the d-dimentional torus $\mathbb{T}^d$, with $d \leq 3$. Subsequently, the results obtained in [1] were extended to the case of the viscosity being equal to Planck cosntant and the viscosity greater than Plank constant (in [12] and [9], respectively). In [11], Brenner model suggests the following modified Navier-Stokes model

$$n_t + div(n\mathbf{w}) = 0, \qquad (n\mathbf{u})_t + div(n\mathbf{u}\otimes\mathbf{w}) + \nabla p = divS,$$

which interprets $\mathbf{u}$ and $\mathbf{w}$ as the volume velocity and mass velocity respectively and it holds the following relationship $\mathbf{u} = \mathbf{w} + \nu\nabla\log n$ (for $\nu > 0$ constant). In [2], Jüngel suggests the following modified Quantum Navier-Stokes problem

$$n_t + div(n\mathbf{u}) = \nu\Delta n, \tag{1}$$

$$(n\mathbf{u})_t + div(n\mathbf{u}\otimes\mathbf{u}) + \nabla p = n\mathbf{f} + \nu\Delta(n\mathbf{u}) + 2\varepsilon^2 n\nabla\left(\frac{\Delta\sqrt{n}}{\sqrt{n}}\right). \tag{2}$$

Now, in total analogy to the classical physics, we add the hypothesis $div\ u = 0$ and consider the following identities:

$$div(n\mathbf{u} \otimes \mathbf{u}) = (\mathbf{u} \cdot \nabla n)\mathbf{u} + (n\mathbf{u} \cdot \nabla)\mathbf{u}$$
$$\nu\Delta(n\mathbf{u}) = \nu(\nabla n \cdot \nabla)\mathbf{u} + \nu n\Delta\mathbf{u} + \nu(\nabla n \cdot \nabla)\mathbf{u}$$
$$div(n\mathbf{u}) = \mathbf{u} \cdot \nabla n$$

in order to obtain the Navier-Stokes problem for incompressible Quantum fluids:

$$\begin{aligned}
&n_t + \mathbf{u}.\nabla n = \nu\Delta n \quad x \in \mathbb{T}^d \ \ (d \leq 3), \quad t > 0,\\
&(n\mathbf{u})_t + (n\mathbf{u}.\nabla)\mathbf{u} + (\mathbf{u}.\nabla n)\mathbf{u} + \nabla p = \nu n\Delta\mathbf{u} + \nu(\nabla n.\nabla)\mathbf{u} + \nu(\mathbf{u}.\nabla)\nabla n + n\mathbf{f} + 2\varepsilon^2 n\nabla\left(\frac{\Delta\sqrt{n}}{\sqrt{n}}\right)\\
&n(.,0) = n_0, \quad \mathbf{u}(.,0) = \mathbf{u}_0 \quad em\ \mathbb{T}^d \ \ (d \leq 3)\\
&div\ \mathbf{u} = 0 \quad x \in \mathbb{T}^d \ \ (d \leq 3),
\end{aligned}$$

where $\mathbf{u}$ is the vector of volume velocity and n is the density, $\mathbb{T}^d$ is the d-dimentional torus, with $(d \leq 3)$. The function p is the fluid pressure and $\mathbf{f}$ describes the external forces from, for instance, an electric field. The physical parameters $\nu, \varepsilon > 0$ are the viscosity coefficient and Plank constant respectively, the term $-\Delta\sqrt{n}/\sqrt{n}$ represents the quantum potential, introduced by David Bohm in [4], [5], the quantum potential intended to induce quantum behavior to the fluid; hence, when $-\Delta\sqrt{n}/\sqrt{n} = 0$ we have the classical fluid behavior.

By using the following identities

$$\begin{aligned}
&n_t = \nu\Delta n - \mathbf{u} \cdot \nabla n,\\
&2\varepsilon^2 n\nabla\left(\frac{\Delta\sqrt{n}}{\sqrt{n}}\right) = \varepsilon^2\nabla\Delta n - \varepsilon^2\frac{1}{n}\Delta n\nabla n - \varepsilon^2\frac{1}{n}(\nabla n \cdot \nabla)\nabla n + \varepsilon^2\frac{1}{n^2}(\nabla n \cdot \nabla n)\nabla n
\end{aligned}$$

we otain a striking fact: the equation obtained is similar to the equations of motion of viscous incompressible fluids with diffusion phenomena ([19], [10]) and, therefore, we have the Navier-Stokes problem for incompressible quantum fluids:

$$\left.\begin{aligned}
&n_t + \mathbf{u}.\nabla n = \nu\Delta n, \quad x \in \mathbb{T}^d \ \ (d \leq 3), \quad t > 0,\\
&n\mathbf{u}_t + (n\mathbf{u}.\nabla)u - \nu n\Delta\mathbf{u} - \nu(\mathbf{u}.\nabla)\nabla n - \nu(\nabla n.\nabla)\mathbf{u} + \varepsilon^2\frac{1}{n}(\nabla n.\nabla)\nabla n - \varepsilon^2\frac{1}{n^2}(\nabla n.\nabla n)\nabla n\\
&+\varepsilon^2\frac{1}{n}\Delta n\nabla n + \nabla p = n\mathbf{f} - \nu\mathbf{u}\Delta n + \varepsilon^2\nabla\Delta n \quad x \in \mathbb{T}^d \ \ (d \leq 3), \quad t > 0,\\
&n(.,0) = n_0, \quad \mathbf{u}(.,0) = \mathbf{u}_0 \quad em\ \mathbb{T}^d \ \ (d \leq 3)\\
&div\ \mathbf{u} = 0 \quad x \in \mathbb{T}^d \ \ (d \leq 3).
\end{aligned}\right\} \tag{3}$$

This intimate relationship between quantum fluids and classical fluids confirms the theories of David Bohm in the sense that we have uniqueness of particle's trajectory (which solely depends on the initial datum).

Remark 1.1 We can see from the above deduction that the choice of region of the motion of the fluid (in this case, the torus $\mathbb{T}^d$) is made in a such way that the boundary of the region is empty. The main reason for that is to avoid the classical question in quantum mechanics of defining precisely the notion of "boundary of a region".

In this article, we are assuming the existence of constants $m, M > 0$, such that

$$0 < m \le n_0 \le M \quad \text{em} \;\; \mathbb{T}^d. \tag{4}$$

To our knowledge, there are no results on the existence and uniqueness of (multidimensional) strong solution; the results we obtain in this article confirm that David Bohm's theory is mathematically self-consistent. The paper is organized as follow: in section 2 we give the framework for this kind of problems and present the variational formulation and our main teorem. Section 3 is devoted to *a priori* estimates for the aprrroximated solutions (and for the exact solution). The procedure of passing to the limit is treated in section 4 e the question of uniqueness of solution is proved in section 5.

2. Functional spaces and semi-Galerkin Formulation

In this article, we denote by $(.,.)$ the inner product space L^2. Now we introduce the functional spaces of the usual Navier-Stokes equations:

$$\mathbf{H} = \Big\{ \mathbf{u} : \mathbf{u} \in \mathbf{L}^2(\mathbb{T}^d),\ div\ \mathbf{u} = 0 \Big\}$$
$$\mathbf{V} = \Big\{ \mathbf{u} : \mathbf{u} \in \mathbf{H}^1(\mathbb{T}^d),\ div\ \mathbf{u} = 0 \Big\}$$

The usual norms $|\mathbf{u}|_{H^1}$ e $|\nabla \mathbf{u}|_{L^2} = |\nabla \mathbf{u}|$ are equivalent in $\mathbf{V}$, and $|\mathbf{u}|_{H^2}$ and $|A\mathbf{u}|$ are equivalent in $H^2(\mathbb{T}^d) \cap \mathbf{V}$ (see [18], [13]); we also have that usual norms $|n|_{H^2}$ and $|\Delta n|$ are equivalent in $H^2(\mathbb{T}^d)$ and $|n|_{H^3}$ and $|\nabla \Delta n|$ are equivalent in $H^3(\mathbb{T}^d)$ (see [3]).

The Stokes operator $A : D(A) \to \mathbf{H}$ is defined by $A = P(-\Delta)$, with domain $D(A) = H^2(\mathbb{T}^d) \cap \mathbf{V}$ where $P : L^2(\mathbb{T}^d) \to \mathbf{H}$ is the operator of orthogonal projection. We denote by V_k the finite dimensional space spanned by the first k eingenfunctions of the Stokes operator, or, $V_k = [\varphi_1, ..., \varphi_k]$ and P_k the orthogonal projection of $L^2(\mathbb{T}^d)$ over V_k.

In an entirely analogous way to the case of the Navier-Stokes equations (see [25]), it is posible to show that the above formulation is equivalent to the following weak form:

$$\left.\begin{array}{l}
(n\mathbf{u}_t, v) + ((n\mathbf{u} \cdot \nabla)\mathbf{u}, v) + \nu(nA\mathbf{u}, v) - \nu((\mathbf{u} \cdot \nabla)\nabla n, v) - \nu((\nabla n \cdot \nabla)\mathbf{u}, v) \\
+\varepsilon^2(\dfrac{1}{n}(\nabla n \cdot \nabla)\nabla n, v) - \varepsilon^2(\dfrac{1}{n^2}(\nabla n \cdot \nabla n)\nabla n, v) + \varepsilon^2(\dfrac{1}{n}\Delta n \nabla n, v) \\
= (n\mathbf{f}, v) - \nu(\mathbf{u}\Delta n, v) + \varepsilon^2(\nabla \Delta n, v) \;\; \forall\, v \in V \\
\dfrac{\partial n}{\partial t} + \mathbf{u} \cdot \nabla n - \lambda \Delta n = 0 \qquad \text{for} \;\; 0 < t < T \\
\mathbf{u}(0) = \mathbf{u}_0,\ n(0) = n_0,
\end{array}\right\} \tag{5}$$

We define $(\mathbf{u}^k, n^k) \in C^1([0, T^k]; H^2(\mathbb{T}^d) \cap V) \times C^2(\mathbb{T}^d \times [0, T^k])$, for each $k \in I\!N$, as the spectral semi-Galerkin approximation of the solution $(\mathbf{u}, n)$ by:

$$\left.\begin{aligned}
&(n^k\mathbf{u}_t^k,v)+((n^k\mathbf{u}^k\cdot\nabla)\mathbf{u}^k,v)+\nu(n^kA\mathbf{u}^k,v)-\nu((\mathbf{u}^k\cdot\nabla)\nabla n^k,v)-\nu((\nabla n^k\cdot\nabla)\mathbf{u}^k,v)\\
&+\varepsilon^2(\frac{1}{n^k}(\nabla n^k\cdot\nabla)\nabla n^k,v)-\varepsilon^2(\frac{1}{(n^k)^2}(\nabla n^k\cdot\nabla n^k)\nabla n^k,v)+\varepsilon^2(\frac{1}{n^k}\Delta n^k\nabla n^k,v)\\
&=(n^k\mathbf{f},v)-\nu(\mathbf{u}^k\Delta n^k,v)+\varepsilon^2(\nabla\Delta n^k,v)\ \ \forall\, v\in V_k\\
&n_t^k+\mathbf{u}^k\cdot\nabla n^k-\nu\Delta n^k=0\qquad\forall\ \ (x,t)\in\mathbb{T}^d\times(0,T^k)\\
&\mathbf{u}^k(x,0)=P_k\mathbf{u}_0,\ \ n^k(0,x)=n_0(x),\quad\forall\ \ x\in\mathbb{T}^d.
\end{aligned}\right\}\tag{6}$$

Remark 2.1 Here, by *"semi-Galerkin spectral approximations"* we mean finite-dimensional approximations for the velocity $\mathbf{u}$ and infinite-dimensional approximations for the density n. For sake of simplicity, we have chosen $P_k\mathbf{u}_0$ as the initial condition for the velocity in the approximated problem (6); of course, we could choose another initial condition $v_k(0)$ which converges strong (in the required norms) to $\mathbf{u}_0$. The results that we have obtained for the spectral basis (of Stokes operator) remain valid for any other orthonormal basis of $L^2(\mathbb{T}^d)$ and $H_0^1(\mathbb{T}^d)$; the main reason for having chosen such a basis is due to the fact that it is possible to obtain better error estimates in the approximation process (which will be shown in a next paper).

We remember that, for all $k\in I\!N$ the above system (of ODEs) admits an unique solution $(\mathbf{u}^k,n^k)$ defined on $[0,T^k]$, with $0<T^k\le T$ (as *Carathéodory* theorem, see for example, [8]). However, the estimates that we obtain (which will be independent of the level of approximation k) allow us to take $T^k=T^*$, for all $k\ge 1$.

Our main objective in this paper is to show that the approximations $(\mathbf{u}^k,n^k)$ converge in a suitable way for the solution $(\mathbf{u},n)$ of problem (5), as $k\to\infty$. We present our main result:

Teorema 2.1. *Let $\mathbf{u}_0\in D(A)$, $n_0\in H^3(\mathbb{T}^d)$, $0<\alpha\le n_0\le\beta$, $\mathbf{f}\in L^2(0,T;H^1(\mathbb{T}^d))$ and $\mathbf{f}_t\in L^2(0,T;L^2(\mathbb{T}^d))$. Then, for any $T>0$, there exist $T^*\in\,]0,T]$, $n\in L^\infty(0,T^*;H^3(\mathbb{T}^d))\cap L^2(0,T^*;H^4(\mathbb{T}^d))\cap C([0,T^*];H^2(\mathbb{T}^d))$ and $\mathbf{u}\in L^\infty(0,T^*;D(A))\cap L^2(0,T^*;W^{1,\infty}(\mathbb{T}^d))\cap C([0,T^*];\mathbf{V})$ such that $(\mathbf{u},n)$ is the unique solution of the problem (3) in $[0,T^*]\times\mathbb{T}^d$.*

Furthermore, the approximations $\mathbf{u}^k,n^k$ satisfy the following estimates:

$$\left.\begin{aligned}
&\alpha\le n^k\le\beta;\qquad|\nabla\mathbf{u}^k(t)|\le F_1(t);\qquad|A\mathbf{u}^k(t)|\le F_2(t);\\
&|\mathbf{u}_t^k(t)|^2+\int_0^t|\nabla\mathbf{u}_t^k(s)|^2ds\le F_3(t);\\
&\int_0^t|\nabla\mathbf{u}^k(s)|_\infty^2ds\le F_4(t);\qquad\int_0^t|\nabla n^k(t)|_\infty^2\le F_5(t);\\
&|\nabla\Delta n^k(t)|^2+\int_0^t|n^k(s)|_{H^4}^2ds\le F_6(t),\\
&|\nabla n_t^k(t)|^2+\int_0^t|n_t^k(s)|_{H^2}^2ds\le F_7(t),
\end{aligned}\right\}\tag{7}$$

Analogous estimates are verified by the solution $(\mathbf{u}, n)$.

The functions on the right hand side of the above estimates depend on the argument t, T^* *and the initial data of the problem. In the interval in question such functions are continuous in time.*

The proof of the main theorem will be done in many stages in the following sections.

3. *A Priori* Estimates

Consider the second equation (6); by the maximum principle for the solution of parabolic equations, we have that for all $k \in \mathbb{N}$, it is valid $\alpha \le n^k \le \beta$.

Next, we mention an essential result on differential inequalities which we will use later on to ensure the existence of an interval $[0, T^*]$, where all the approximated solutions of the initial problem are defined. (see [14])

Lema 3.1. *Let* $g \in W^{1,1}(0,T)$ *e* $h \in L^1(0,T)$ *satisfying*

$$\frac{dg}{dt} \le F(g) + h \quad \text{in } [0,T], \quad g(0) \le g_0$$

where $F : \mathbb{R} \to \mathbb{R}$ *is bounded function in bounded sets. Then for any* $\varepsilon > 0$, *there exists* $T_\varepsilon > 0$ *which is independent of* g *such that*

$$g(t) \le g_0 + \varepsilon \quad \forall t \le T_\varepsilon.$$

Lema 3.2. *In the conditions of Theorem 2.1, the solution* $(\mathbf{u}^k, n^k)$ *of the approximated problem (6) satisfies:*

$$\mathbf{u}^k \in L^\infty(0,T^*;\mathbf{H}) \cap L^2(0,T^*;\mathbf{V}), \tag{8}$$

$$\mathbf{u}^k \in L^\infty(0,T^*;\mathbf{V}) \cap L^2(0,T^*;D(A)), \tag{9}$$

$$\mathbf{u}^k_t \in L^2(0,T^*;\mathbf{H}), \tag{10}$$

$$n^k \in L^\infty(0,T^*;H^2(\mathbb{T}^d)) \cap L^2(0,T^*;H^3(\mathbb{T}^d)), \tag{11}$$

uniformly in k.

Proof.:

Making $v = \mathbf{u}^k$ the first equation (6) we:

$$\frac{1}{2}\frac{d}{dt}|(n^k)^{\frac{1}{2}}\mathbf{u}^k|^2 + \nu\alpha|\nabla \mathbf{u}^k|^2 \le \frac{1}{2}(n^k_t\mathbf{u}^k, \mathbf{u}^k) - (n^k\mathbf{u}^k \cdot \nabla \mathbf{u}^k, \mathbf{u}^k) + \nu((\nabla n^k \cdot \nabla)\mathbf{u}^k, \mathbf{u}^k)$$
$$+\nu((\mathbf{u}^k \cdot \nabla)\nabla n^k, \mathbf{u}^k) - \varepsilon^2(\frac{1}{n^k}(\nabla n^k \cdot \nabla)\nabla n^k, \mathbf{u}^k) + \varepsilon^2(\frac{1}{(n^k)^2}(\nabla n^k \cdot \nabla n^k)\nabla n^k, \mathbf{u}^k)$$
$$-\varepsilon^2(\frac{1}{n^k}\Delta n^k \nabla n^k, \mathbf{u}^k) + (n^k\mathbf{f}, \mathbf{u}^k) - \nu(\mathbf{u}^k\Delta n^k, \mathbf{u}^k) + \varepsilon^2(\nabla\Delta n^k, \mathbf{u}^k)$$

The next step is to estimate the terms on the right hand side of the above equality. We note that, as is usually done to obtain a priori estimates, we will proceed by using Hölder's inequality, Young's interpolation inequality (for classical Sobolev immersions), the Gronwall lemma, etc.. Thus, we obtain the following differential inequality:

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}|(n^k)^{\frac{1}{2}}\mathbf{u}^k|^2 + \frac{\nu\alpha}{2}|\nabla\mathbf{u}^k|^2 \leq C|\nabla\mathbf{u}^k|^4 + C|\nabla\mathbf{u}^k|^6 + 3\delta|A\mathbf{u}^k|^2 \\ +C|(n^k)^{\frac{1}{2}}\mathbf{u}^k|^2 + C|\Delta n^k|^2 + C|\Delta n^k|^4 + C|\Delta n^k|^6 + C|\Delta n^k|^8 + C|\mathbf{f}|^2.\end{aligned} \tag{12}$$

Now, making $v = \mathbf{u}_t^k$ in the first equation (6), we obtain, after an integration by parts in $\mathbb{T}^d$:

$$\frac{\nu\alpha}{2}\frac{d}{dt}|\nabla\mathbf{u}^k|^2 + |(n^k)^{1/2}\mathbf{u}_t^k|^2 \leq |(n^k\mathbf{f},\mathbf{u}_t^k)| + |((n^k\mathbf{u}^k\cdot\nabla)\mathbf{u}^k,\mathbf{u}_t^k)| + \nu|((\nabla n^k\cdot\nabla)\mathbf{u}^k,\mathbf{u}_t^k)|$$

$$+\nu|((\mathbf{u}^k\cdot\nabla)\nabla n^k,\mathbf{u}_t^k)| + \varepsilon^2|(\frac{1}{n^k}(\nabla n^k\cdot\nabla)\nabla n^k,\mathbf{u}_t^k)| + \varepsilon^2|(\frac{1}{(n^k)^2}(\nabla n^k\cdot\nabla n^k)\nabla n^k,\mathbf{u}_t^k)|$$

$$+\varepsilon^2|(\frac{1}{n^k}\Delta n^k\nabla n^k,\mathbf{u}_t^k)| + \nu|(\mathbf{u}^k\Delta n^k,\mathbf{u}_t^k)| + \varepsilon^2|(\nabla\Delta n^k,\mathbf{u}_t^k)|.$$

Estimating the terms of the right of the above expression, we obtain the following differential inequality:

$$\begin{aligned}\frac{\nu\alpha}{2}\frac{d}{dt}|\nabla\mathbf{u}^k|^2 + \frac{\alpha}{2}|\mathbf{u}_t^k|^2 &\leq C|\mathbf{f}|^2 + C|\nabla\mathbf{u}^k|^4 + C|\nabla\mathbf{u}^k|^6 + C|\Delta n^k|^6 \\ &+ 2\delta|A\mathbf{u}^k|^2 + C|\Delta n^k|^8 + 3\gamma|\nabla\Delta n^k|^2.\end{aligned} \tag{13}$$

where γ e δ are arbitrary positive constants (to be chosen later).

Consider now the second equation (6); applying the operator Δ and taking the $L^2(\mathbb{T}^d)$ inner product with Δn^k we obtain:

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}|\Delta n^k|^2 + (\nu - 2\gamma)|\nabla\Delta n^k|^2 &\leq \delta|A\mathbf{u}^k|^2 + C|\nabla\mathbf{u}^k|^4 \\ &+ C|\nabla\mathbf{u}^k|^6 + C|\Delta n^k|^8.\end{aligned} \tag{14}$$

Summing up the inequalities (12), (13) and (14), it follows that (after choosing $\gamma = \frac{\lambda}{10}$):

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\left\{|(n^k)^{\frac{1}{2}}\mathbf{u}^k|^2 + \nu\alpha|\nabla\mathbf{u}^k|^2 + |\Delta n^k|^2\right\} + \frac{\nu\alpha}{2}|\nabla\mathbf{u}^k|^2 + \frac{\alpha}{2}|\mathbf{u}_t^k|^2 \\ +\frac{\nu}{2}|\nabla\Delta n^k|^2 \leq 6\delta|A\mathbf{u}^k|^2 + C|\Delta n^k|^2 + C|\Delta n^k|^4 + C|\Delta n^k|^6 + C|\Delta n^k|^8 \\ +C|\nabla\mathbf{u}^k|^4 + C|\nabla\mathbf{u}^k|^6 + C|(n^k)^{\frac{1}{2}}\mathbf{u}^k|^2 + C|\mathbf{f}|^2.\end{aligned} \tag{15}$$

Furthermore, making $v = A\mathbf{u}^k$ in the first equation of (6) we get:

$$\nu\alpha|A\mathbf{u}^k|^2 \leq |(n^k\mathbf{f},A\mathbf{u}^k)| + |(n^k\mathbf{u}_t^k,A\mathbf{u}^k)| + |((n^k\mathbf{u}^k\cdot\nabla)\mathbf{u}^k,A\mathbf{u}^k)| + \nu|((\nabla n^k\cdot\nabla)\mathbf{u}^k,A\mathbf{u}^k)|$$

$$+\nu|((\mathbf{u}^k\cdot\nabla)\nabla n^k,A\mathbf{u}^k)| + \varepsilon^2|(\frac{1}{n^k}(\nabla n^k\cdot\nabla)\nabla n^k,A\mathbf{u}^k)| + \varepsilon^2|(\frac{1}{(n^k)^2}(\nabla n^k\cdot\nabla n^k)\nabla n^k,A\mathbf{u}^k)|$$

$$+\varepsilon^2|(\frac{1}{n^k}\Delta n^k\nabla n^k,A\mathbf{u}^k)| + \nu|(\mathbf{u}^k\Delta n^k,A\mathbf{u}^{\mathbf{k}})| + \varepsilon^2|(\nabla\Delta n^k,A\mathbf{u}^{\mathbf{k}})|.$$

Then, we estimate the terms on the right hand side in the above expression and obtain the following inequality:

$$\begin{aligned}\frac{\nu\alpha}{4}|A\mathbf{u}^k|^2 \leq \frac{\beta^2}{2\nu\alpha}|\mathbf{u}_t^k|^2 + C|\nabla\mathbf{u}^k|^4 + C|\nabla\mathbf{u}^k|^6 \\ +C|\Delta n^k|^6 + C|\Delta n|^8 + 4\eta|\nabla\Delta n|^2 + C|\mathbf{f}|^2.\end{aligned}$$

By multiplying the above inequality $\dfrac{\alpha\mu}{2\beta^2}$ and adding it to (15) we obtain:

$$\frac{1}{2}\frac{d}{dt}(|(n^k)^{\frac{1}{2}}\mathbf{u}^k|^2+\nu\alpha|\nabla\mathbf{u}^k|^2+|\Delta n^k|^2)+\frac{\nu\alpha}{2}|\nabla\mathbf{u}^k|^2+\frac{\alpha}{4}|\mathbf{u}_t^k|^2$$

$$+(\frac{\nu}{2}-4\eta\frac{\alpha^2\nu}{2\beta^2})|\nabla\Delta n^k|^2+(\frac{\alpha^3\nu^2}{8\beta^2}-6\delta)|A\mathbf{u}^k|^2\leq C|\nabla\mathbf{u}^k|^4+C|\nabla\mathbf{u}^k|^6$$

$$+C|\Delta n^k|^2+C|\Delta n^k|^4+C|\Delta n^k|^6+C|\Delta n^k|^8+C|(n^k)^{\frac{1}{2}}\mathbf{u}^k|^2+C|\mathbf{f}|^2.$$

Finally, we choose $\eta=\dfrac{\beta^2}{8\alpha^2}$ and $\delta=\dfrac{\alpha\nu^2}{96\beta^2}$; thus we obtain:

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}(|(n^k)^{\frac{1}{2}}\mathbf{u}^k|^2+\nu\alpha|\nabla\mathbf{u}^k|^2+|\Delta n^k|^2)+\frac{\nu\alpha}{2}|\nabla\mathbf{u}^k|^2+\frac{\alpha}{4}|\mathbf{u}_t^k|^2+\frac{\alpha^3\nu^2}{16\beta^2}|A\mathbf{u}^k|^2\\&+\frac{\nu}{4}|\nabla\Delta n^k|^2\leq C\{|\mathbf{f}|^2+|\nabla\mathbf{u}^k|^4+|\nabla\mathbf{u}^k|^6+|(n^k)^{\frac{1}{2}}\mathbf{u}^k|^2\\&+|\Delta n^k|^2+|\Delta n^k|^4+|\Delta n^k|^6+|\Delta n^k|^8\}.\end{aligned}\tag{16}$$

Defining

$$G(t)=|(n^k(t))^{\frac{1}{2}}\mathbf{u}^k(t)|^2+\nu\alpha|\nabla\mathbf{u}^k(t)|^2+|\Delta n^k(t)|^2,$$

we can see from the above inequality that:

$$\frac{d}{dt}G(t)\quad\leq\quad C|\mathbf{f}|^2+CG(t)+CG^2(t)+CG^3(t)+CG^4(t)$$

Thus, we get the following system:

$$\begin{cases}G'(t)\leq C|\mathbf{f}|^2+CG(t)+CG^2(t)+CG^3(t)+CG^4(t)\\ G(0)=|n_\circ^{\frac{1}{2}}u_\circ|+\nu\alpha|\nabla\mathbf{u}_\circ|^2+|\Delta n_\circ|^2.\end{cases}$$

Using the Lemma 3.1 for differential inequalities, we have:

$$G(t)\leq\varphi(t),$$

for all t in the maximal interval of existence of φ, where

$$\begin{cases}\varphi'(t)&=&C|\mathbf{f}|^2+C\varphi+C\varphi^2+C\varphi^3+C\varphi^4\\ \varphi(0)&=&G(0).\end{cases}\tag{17}$$

Therefore there exists T^*, with $0<T^*\leq T$ such that for some constant $M>0$, we:

$$G(t)\leq M,\qquad\forall\, t\ \in\ [0,T^*].$$

Coming back to the expression (16) and integrating it from 0 to t, with $t\in[0,T^*]$, we obtain:

$$|(n^k)^{\frac{1}{2}}(t)\mathbf{u}^k(t)|^2+\nu\alpha|\nabla\mathbf{u}^k(t)|^2+|\Delta n^k(t)|^2+\nu\alpha\int_0^t|\nabla\mathbf{u}^k(s)|^2+\frac{\alpha}{2}\int_0^t|\mathbf{u}_t^k(s)|^2ds$$

$$
\begin{aligned}
&+\frac{\alpha^3\nu^2}{16\beta^2}\int_0^t |A\mathbf{u}^k(s)|^2 ds + \frac{\nu}{2}\int_0^t |\nabla\Delta n^k(s)|^2 ds \\
&\leq |n_\circ^{\frac{1}{2}}\mathbf{u}_\circ|^2 + \nu\alpha|\nabla\mathbf{u}_\circ|^2 + |\Delta n_\circ|^2 + C\int_0^t |\mathbf{f}(s)|^2 ds \\
&+C\int_0^t \{|(n^k(s))^{\frac{1}{2}}\mathbf{u}^k(s)|^2 + |\nabla\mathbf{u}^k(s)|^4 + |\nabla\mathbf{u}^k(s)|^6 \\
&+|\Delta n^k|^2 + |\Delta n^k(s)|^4 + |\Delta n^k|^6 + |\Delta n^k(s)|^8\} ds \\
&\leq |n_\circ^{\frac{1}{2}}\mathbf{u}_\circ| + \mu|\nabla\mathbf{u}_\circ|^2 + |\Delta n_\circ|^2 + C|\mathbf{f}|^2_{L^2(0,T^*;L^2(\mathbb{T}^d))} + CT^*. \quad \blacksquare
\end{aligned}
$$

Lema 3.3. *In the conditions of Theorem 2.1, the density-solution n^k of the approximated problem (6) satisfies:*

$$n_t^k \in L^\infty(0,T^*;L^2(\mathbb{T}^d)) \cap L^2(0,T^*;H^1(\mathbb{T}^d)) \tag{18}$$

uniformly in k.

Proof. Again using the second equation (6) we see that:

$$|n_t^k| \leq |\mathbf{u}^k \cdot \nabla n^k| + \lambda|\Delta n^k|.$$

Thus,

$$|n_t^k|_{L^\infty(0,T^*,L^2(\mathbb{T}^d))} \leq C|n^k|_{L^\infty(0,T^*;H^2(\mathbb{T}^d))} \cdot (1 + |\mathbf{u}^k|_{L^\infty(0,T^*;V)}) \leq C$$

This tells us that

$$n_t^k \in L^\infty(0,T^*;L^2(\mathbb{T}^d)),$$

uniformly k.

Now, applying the operator ∇ in the second equation (6) we obtain:

$$\nabla n_t^k = \lambda\nabla\Delta n^k - \nabla(\mathbf{u}^k \cdot \nabla n^k).$$

Taking the L^2-norm in the above equation and estimating the terms right and integrating from 0 to t we get:

$$\int_0^t |\nabla n_t^k(s)|^2 ds \leq C\int_0^t |n^k(s)|^2_{H^3} ds + C\int_0^t |A\mathbf{u}^k(s)|^2 ds \leq C.$$

and therefore, we can conclude that

$$n_t^k \in L^2(0,T^*;H^1(\mathbb{T}^d))$$

uniformly k. $\blacksquare$

Lema 3.4. *In the conditions of Theorem 2.1, the solution $(\mathbf{u}^k, n^k)$ of the approximated problem (6) satisfies:*

$$\mathbf{u}_t^k \in L^\infty(0,T^*;\mathbf{H}) \cap L^2(0,T^*;\mathbf{V}) \tag{19}$$

$$n_t^k \in L^\infty(0,T^*;H^1(\mathbb{T}^d)) \cap L^2(0,T^*;H^2(\mathbb{T}^d)) \tag{20}$$

$$\mathbf{u}^k \in L^\infty(0,T^*;D(A)) \tag{21}$$

$$n^k \in L^\infty(0,T^*;H^3(\mathbb{T}^d)) \cap L^2(0,T^*;H^4(\mathbb{T}^d)), \tag{22}$$

uniformly k.

Proof. Using the first equation (6) and remembering that we are working with the basic spectral we see that

$$\begin{aligned}\nu n^k A\mathbf{u}^k = P_k\{-n^k\mathbf{u}_t^k - (n^k\mathbf{u}^k\cdot\nabla)\mathbf{u}^k + \nu[(\mathbf{u}^k\cdot\nabla)\nabla n^k + (\nabla n^k\cdot\nabla)\mathbf{u}^k - \mathbf{u}^k\Delta n^k] \\ +n^k\mathbf{f} - \varepsilon^2[\frac{1}{n^k}(\nabla n^k\cdot\nabla)\nabla n^k - \frac{1}{(n^k)^2}(\nabla n^k\cdot\nabla n^k)\nabla n^k + \frac{1}{n^k}\Delta n^k\nabla n^k + \nabla\Delta n^k]\}.\end{aligned} \tag{23}$$

Taking the L^2-norm in the above equation and estimating the terms on the right hand side, we obtain the following inequality:

$$|A\mathbf{u}^k|^2 \le C + C|\mathbf{f}|^2 + C|\mathbf{u}_t^k|^2 + C|\nabla\Delta n^k|^2 \le C + C|\mathbf{u}_t^k|^2 + C|\nabla\Delta n^k|^2. \tag{24}$$

Since $\mathbf{f}\in L^2(0,T^*;H^1(\mathbb{T}^d))$ and $\mathbf{f}_t\in L^2(0,T^*;L^2(\mathbb{T}^d))$, we can conclude, from Aubin-Lion's lemma, that $\mathbf{f}\in C([0,T^*];L^2(\mathbb{T}^d))$.

Calculating the derivative with respect to t of the second equation of the system (6) we have:

$$n_{tt}^k - \nu\Delta n_t^k = -\mathbf{u}_t^k\cdot\nabla n^k - \mathbf{u}^k\cdot\nabla n_t^k,$$

Applying the operator ∇ in the above equation, we are left with the expression:

$$\nabla n_{tt}^k - \nu\nabla\Delta n_t^k = -\nabla\mathbf{u}_t^k\cdot\nabla n^k - \mathbf{u}_t^k\cdot\nabla^2 n^k - \nabla\mathbf{u}^k\cdot\nabla n_t^k - \mathbf{u}^k\cdot\nabla^2 n_t^k.$$

Taking the inner product of $L^2(\mathbb{T}^d)$ the terms of the above equation with the term ∇n_t^k obtain, after integration by parts, the following differential inequality:

$$\frac{1}{2}\frac{d}{dt}|\nabla n_t^k|^2 + \nu|\Delta n_t^k|^2 \le |(\nabla\mathbf{u}_t^k\cdot\nabla n^k, \nabla n_t^k)|$$

$$+|(\mathbf{u}_t^k\cdot\nabla^2 n^k, \nabla n_t^k)| + |(\nabla\mathbf{u}^k\cdot\nabla n_t^k, \nabla n_t^k)| + |(\mathbf{u}^k\cdot\nabla^2 n_t^k, \nabla n_t^k)|.$$

Estimating the terms the right hand side of the above equation, we obtain the following differential inequality:

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}|\nabla n_t^k|^2 + \nu|\Delta n_t^k|^2 \le 2\gamma|\nabla\mathbf{u}_t^k|^2 + 2\delta|\Delta n_t^k|^2 \\ +C(|A\mathbf{u}^k|^2 + |\nabla\Delta n^k|^2)|\nabla n_t^k|^2.\end{aligned} \tag{25}$$

Now, calculating the derivative of equation (23) concerning to t and taking the $\mathbf{u}_t^k$ inner product (and remembering that $0<\alpha\le n^k\le\beta$), we obtain the following differential inequality:

$$
\begin{aligned}
\frac{1}{2}\frac{d}{dt}|(n^k)^{\frac{1}{2}}\mathbf{u}_t^k|^2+\nu\alpha|\nabla\mathbf{u}_t^k|^2 \;\; \leq \;\; & -\frac{1}{2}(n_t^k\mathbf{u}_t^k,\mathbf{u}_t^k)-(n_t^k(\mathbf{u}^k\cdot\nabla)\mathbf{u}^k,\mathbf{u}_t^k)\\
&-(n^k(\mathbf{u}_t^k\cdot\nabla)\mathbf{u}^k,\mathbf{u}_t^k)-(n^k(\mathbf{u}^k\cdot\nabla)\mathbf{u}_t^k,\mathbf{u}_t^k)+(n_t^k\mathbf{f},\mathbf{u}_t^k)+(n^k\mathbf{f}_t,\mathbf{u}_t^k)\\
&+\nu\{((\nabla n_t^k\cdot\nabla)\mathbf{u}^k,\mathbf{u}_t^k)+((\nabla n^k\cdot\nabla)\mathbf{u}_t^k,\mathbf{u}_t^k)+((\mathbf{u}_t^k\cdot\nabla)\nabla n^k,\mathbf{u}_t^k)\}\\
&-\nu\{(n_t^kA\mathbf{u}^{\mathbf{k}},u_t^k)+(\mathbf{u}_t^k\Delta n^k,\mathbf{u}_t^k)+(\mathbf{u}^k\Delta n_t^k,\mathbf{u}_t^k)\}\\
&+\nu((\mathbf{u}^k\cdot\nabla)\nabla n_t^k,\mathbf{u}_t^k)+\varepsilon^2(\frac{1}{(n^k)^2}n_t^k(\nabla n^k\cdot\nabla)\nabla n^k,\mathbf{u}_t^k)\\
&-\varepsilon^2\{(\frac{1}{n^k}(\nabla n_t^k\cdot\nabla)\nabla n^k,\mathbf{u}_t^k)-(\frac{1}{n^k}(\nabla n^k\cdot\nabla)\nabla n_t^k,\mathbf{u}_t^k)+(\nabla\Delta n_t^k,\mathbf{u}_t^k)\}\\
&-\varepsilon^2\{(\frac{2n_t^k}{(n^k)^3}(\nabla n^k\cdot\nabla n^k)\nabla n^k,\mathbf{u}_t^k)+(\frac{1}{(n^k)^2}(\nabla n_t^k\cdot\nabla n^k)\nabla n^k,\mathbf{u}_t^k)\}\\
&+\varepsilon^2\{(\frac{1}{(n^k)^2}(\nabla n^k\cdot\nabla n_t^k)\nabla n^k,\mathbf{u}_t^k)+(\frac{1}{(n^k)^2}(\nabla n^k\cdot\nabla n^k)\nabla n_t^k,\mathbf{u}_t^k)\}\\
&+\varepsilon^2\{(\frac{n_t^k}{(n^k)^2}\Delta n^k\nabla n^k,\mathbf{u}_t^k)-(\frac{1}{n^k}\Delta n_t^k\nabla n^k,\mathbf{u}_t^k)-(\frac{1}{n^k}\Delta n^k\nabla n_t^k,\mathbf{u}_t^k)\}.
\end{aligned}
$$

Estimating the terms in the right hand side of the above inequality, it allows us to reach the following differential inequality (making use of the fact that $\nabla\mathbf{u}^k$ and Δn^k are uniformly bounded):

$$
\frac{1}{2}\frac{d}{dt}|(n^k)^{\frac{1}{2}}\mathbf{u}_t^k|^2+\nu\alpha|\nabla\mathbf{u}_t^k|^2\leq C(|\mathbf{f}|^2_{H^1}|n_t^k|^2+|\mathbf{f}_t|^2)
$$

$$
+C(|n_t^k|^2_{H^1}+|A\mathbf{u}^k|^2+|\nabla\Delta n^k|^2+1)|\mathbf{u}_t^k|^2
$$

$$
+C(|A\mathbf{u}^k|^2+|\nabla\Delta n^k|^2)|\nabla n_t^k|^2+12\gamma|\nabla\mathbf{u}_t^k|^2+7\delta|\Delta n_t^k|^2.
$$

Adding the above inequality to that one given by (25), and choosing γ e δ conveniently, we obtain:

$$
\frac{1}{2}\frac{d}{dt}(|(n^k)^{\frac{1}{2}}\mathbf{u}_t^k|^2+|\nabla n_t^k|^2)+\frac{\nu\alpha}{2}|\nabla\mathbf{u}_t^k|^2+\frac{\lambda}{2}|\Delta n_t^k|^2\leq C(|\mathbf{f}|^2_{H^1}|n_t^k|^2+|\mathbf{f}_t|^2)
$$

$$
+C(|n_t^k|^2_{H^1}+|A\mathbf{u}^k|^2+|\nabla\Delta n^k|^2+1)(|(n^k)^{\frac{1}{2}}\mathbf{u}_t^k|^2+|\nabla n_t^k|^2).
$$

$$
=C\varphi(t)+C\psi(t)(|(n^k)^{\frac{1}{2}}\mathbf{u}_t^k|^2+|\nabla n_t^k|^2),
$$

with $\varphi,\ \psi\in L^1(0,T^*)$.

Multiplying the above inequality by 2, integrating it from 0 to t, using the Generalized Gronwall's lemma (see [21]) and the fact $0<\alpha\leq n^k\leq\beta$ we obtain:

$$
\alpha|\mathbf{u}_t^k(t)|^2+|\nabla n_t^k(t)|^2+\nu\alpha\int_0^t|\nabla\mathbf{u}_t^k(s)|^2ds+\nu\int_0^t|\Delta n_t^k(s)|^2ds
$$

$$
\leq C\left(|n_\circ^{\frac{1}{2}}\mathbf{u}_t^k(0)|^2+|\nabla n_t^k(0)|^2+\int_0^t\varphi(s)ds\right)\cdot\ \exp\,(C\int_0^t\psi(s)ds)<+\infty.
$$

Now, we need to show that $|\mathbf{u}_t^k(0)|$ and $|\nabla n_t^k(0)|$ are uniformly bounded (in k). In fact, using the first equation (6) with the multiplier $v = \mathbf{u}_t^k$ we:

$$|(n^k)^{\frac{1}{2}}\mathbf{u}_t^k|^2 = (-(n^k\mathbf{u}^k\cdot\nabla)\mathbf{u}^k - \nu n^k A\mathbf{u}^k + \nu(\mathbf{u}^k\cdot\nabla)\nabla n^k + \nu((\nabla n^k\cdot\nabla)\mathbf{u}^k$$

$$-\varepsilon^2\frac{1}{n^k}(\nabla n^k\cdot\nabla)\nabla n^k + \varepsilon^2\frac{1}{(n^k)^2}(\nabla n^k\cdot\nabla n^k)\nabla n^k - \varepsilon^2(\frac{1}{n^k}\Delta n^k\nabla n^k) + (n^k\mathbf{f}, \mathbf{u}_t^k)$$

$$-\nu\mathbf{u}^k\Delta n^k + \varepsilon^2\nabla\Delta n^k = (\Phi^k, \mathbf{u}_t^k).$$

Using the fact $|(n^k)^{\frac{1}{2}}\mathbf{u}_t^k|^2 \geq \alpha|\mathbf{u}_t^k|^2$ we

$$\alpha|\mathbf{u}_t^k|^2 \leq |(\Phi^k, \mathbf{u}_t^k)| \Longrightarrow |\mathbf{u}_t^k|^2 \leq \frac{1}{\alpha}|(\Phi^k, \mathbf{u}_t^k)| \leq C|\Phi^k|^2 + \frac{1}{2}|\mathbf{u}_t^k|^2 \Longrightarrow |\mathbf{u}_t^k|^2 \leq C|\Phi^k|^2.$$

In particular, for $t = 0$ we:

$$|\mathbf{u}_t^k(0)|^2 \leq C|\Phi^k(0)|^2 \leq C$$

and therefore, $|\mathbf{u}_t^k(0)|$ is uniformly bounded (in k). Furthermore,

$$|\nabla n_t^k(0)| \leq |\nabla\mathbf{u}^k(0)\cdot\nabla n^k(0)| + |\mathbf{u}^k(0)\cdot\nabla^2 n^k(0)| + \nu|\nabla\Delta n^k(0)|$$

$$\leq C(|A\mathbf{u}_\circ||\Delta n_\circ| + |\nabla\Delta n_\circ|) \leq C,$$

and so, $|\nabla n_t^k(0)|$ is also uniformly bounded (in k). We conclude that

$$\begin{aligned}&\mathbf{u}_t^k \in L^\infty(0,T^*;\mathbf{H}) \cap L^2(0,T^*;\mathbf{V})\\ &n_t^k \in L^\infty(0,T^*;H^1(\mathbb{T}^d)) \cap L^2(0,T^*;H^2(\mathbb{T}^d))\end{aligned}$$

uniformly in k. As the second problem equation (6) is typically a problem parabolic, the fact that $n_t^k \in L^\infty(0,T^*;H^1(\mathbb{T}^d))$ implies:

$$n^k \in L^\infty(0,T^*;H^3(\mathbb{T}^d))$$

uniformly in k and due to (24) we can conclude that:

$$\mathbf{u}^k \in L^\infty(0,T^*;D(A)),$$

uniformly in k. Furthermore,

$$\nu|\Delta^2 n^k| = |\Delta n_t^k + \Delta\mathbf{u}^k\cdot\nabla n^k + \mathbf{u}\cdot\Delta\nabla n^k|$$

$$\leq C(|\Delta n_t^k| + |\Delta\mathbf{u}^k||\nabla n^k|_\infty + |\mathbf{u}^k|_\infty|\nabla\Delta n^k|)$$

$$\leq C(|\Delta n_t^k| + |A\mathbf{u}^k||n^k|_{H^3}),$$

and therefore

$$\nu\int_0^t |\Delta^2 n^k|^2 ds \leq C\int_0^t (|\Delta n_t^k|^2 + |A\mathbf{u}^k|^2|n^k|_{H^3}^2)ds \leq C.$$

We conclude that

$$n^k \in L^2(0,T^*;H^4(\mathbb{T}^d)) \quad \blacksquare$$

Lema 3.5. *In the conditions of Theorem 2.1, the velocity-solution* $\mathbf{u}^k$ *of the approximated problem (6) satisfies:*

$$\mathbf{u}^k \in L^2(0, T^*; W^{1,\infty}(\mathbb{T}^d)) \tag{26}$$

uniformly in k.

Proof. We can see from the first equation (6) that:

$$\begin{aligned} \nu n^k A\mathbf{u}^k &= -P_k[n^k \mathbf{u}_t^k + (n^k \mathbf{u}^k \cdot \nabla)\mathbf{u}^k - n^k \mathbf{f} - \nu(\mathbf{u}^k \cdot \nabla)\nabla n^k + \mathbf{u}^k \Delta n^k \\ &\quad -\nu(\nabla n^k \cdot \nabla)\mathbf{u}^k)] - \varepsilon^2 P_k[\frac{1}{n^k}(\nabla n^k \cdot \nabla)\nabla n^k - \nabla \Delta n^k \\ &\quad -\frac{1}{(n^k)^2}(\nabla n^k \cdot \nabla n^k)\nabla n^k + \frac{1}{n^k}\Delta n^k \nabla n^k]. \end{aligned} \tag{27}$$

Estimating the above equation in terms of L^6-norm and integrating it from 0 to t we get:

$$\int_0^t |\mathbf{u}^k(s)|^2_{W^{2,6}} ds \le C.$$

Then, we use the Sobolev immersion of $W^{2,6}(\mathbb{T}^d) \subset W^{1,\infty}(\mathbb{T}^d)$ for:

$$\int_0^t |\nabla \mathbf{u}^k(s)|^2_\infty ds \le C.$$

Therefore, we have

$$\mathbf{u}^k \in L^2(0, T^*; W^{1,\infty}(\mathbb{T}^d)),$$

uniformly in k. ■

4. Passing to the limit

All the previous uniform bounds involve the following convergences (passing to subsequences, if necessary):

(1) $\mathbf{u}^k \rightharpoonup \mathbf{u}$ weakly - $*$ in $L^\infty(0, T^*; \mathbf{V})$ and $L^\infty(0, T^*; D(A))$; ;
(2) $\mathbf{u}^k \rightharpoonup \mathbf{u}$ weakly in $L^2(0, T^*; D(A))$; and $L^2(0, T^*; W^{1,\infty}(\mathbb{T}^d))$;
(3) $\mathbf{u}_t^k \rightharpoonup \mathbf{u}_t$ weakly - $*$ in $L^\infty(0, T^*; \mathbf{H})$; and weak in $L^2(0, T^*; \mathbf{V})$;
(4) $n^k \rightharpoonup n$ weakly - $*$ in $L^\infty(0, T^*; H^3(\mathbb{T}^d))$; and weakly in $L^2(0, T^*; H^4(\mathbb{T}^d))$;
(5) $\mathbf{u}^k \longrightarrow \mathbf{u}$ strongly in $L^2(0, T^*; \mathbf{V})$;
(6) $n^k \longrightarrow n$ strongly in $L^q((0, T^*) \times \mathbb{T}^d)$, $\forall q \ge 1$ and strongly in $L^2(0, T^*; H^2(\mathbb{T}^d))$;
(7) $n_t^k \rightharpoonup n_t$ weakly - $*$ in $L^\infty(0, T^*; H^1(\mathbb{T}^d))$, weakly in $L^2(0, T^*; H^2(\mathbb{T}^d)), L^2(0, T^*; H^1(\mathbb{T}^d))$ and $L^2((0, T^*) \times \mathbb{T}^d))$.

We observed that by using *Aubin-Lions*'s lemma we obtain the convergences (5) (for $\mathbf{u}^k$) and (6) (for n^k in $L^q((0,T)\times\mathbb{T}^d)$).

Now, passing to the limit is a standard procedure (see [13]) and it left to the reader. For that, we choose

$$v=\phi^m=\sum_{i=1}^{m} c_{im}(t)\varphi_i(x), \tag{28}$$

where $\varphi_i(x)$ is i-th eigenfunction of the Stokes operator; then, we consider $k>m$ and pass to the limit (as $k\to\infty$) in the equation:

$$\begin{aligned}\int_0^{T^*}&(n^k\mathbf{u}_t^k+(n^k\mathbf{u}^k\cdot\nabla)\mathbf{u}^k+\nu n^k\Delta\mathbf{u}^k-\nu(\nabla n^k\cdot\nabla)\mathbf{u}^k-\nu(\mathbf{u}^k\cdot\nabla)\nabla n^k+\frac{\varepsilon^2}{n^k}(\nabla n^k\cdot\nabla)\nabla n^k\\&-\frac{\varepsilon^2}{(n^k)^2}(\nabla n^k\cdot\nabla n^k)\nabla n^k+\frac{\varepsilon^2}{n^k}\Delta n^k\nabla n^k-n^k\mathbf{f}-\nu\mathbf{u}^k\Delta n^k+\varepsilon^2\nabla\Delta n^k,\phi^m)dt=0,\end{aligned} \tag{29}$$

Thus, we obtain:

$$\begin{aligned}\int_0^{T^*}&(n\mathbf{u}_t+(n\mathbf{u}\cdot\nabla)\mathbf{u}+\nu n\Delta\mathbf{u}-\nu(\nabla n\cdot\nabla)\mathbf{u}-\nu(\mathbf{u}\cdot\nabla)\nabla n+\frac{\varepsilon^2}{n}(\nabla n\cdot\nabla)\nabla n\\&-\frac{\varepsilon^2}{n^2}(\nabla n\cdot\nabla n)\nabla n+\frac{\varepsilon^2}{n}\Delta n\nabla n-n\mathbf{f}-\nu\mathbf{u}\Delta n+\varepsilon^2\nabla\Delta n,\phi^m)dt=0,\end{aligned} \tag{30}$$

for all ϕ^m given by (28).

Moreover, it is easy to show that;

$$\begin{aligned}\Big|&n\mathbf{u}_t+(n\mathbf{u}\cdot\nabla)\mathbf{u}+\nu n\Delta\mathbf{u}-\nu(\nabla n\cdot\nabla)\mathbf{u}-\nu(\mathbf{u}\cdot\nabla)\nabla n+\frac{\varepsilon^2}{n}(\nabla n\cdot\nabla)\nabla n\\&-\frac{\varepsilon^2}{n^2}(\nabla n\cdot\nabla n)\nabla n+\frac{\varepsilon^2}{n}\Delta n\nabla n-nf-\nu\mathbf{u}\Delta n+\varepsilon^2\nabla\Delta n\Big|_{L^2(0,T^*;L^2)}\leq C.\end{aligned}$$

But this means that

$$\begin{aligned}L\mathbf{u}=n\mathbf{u}_t&+(n\mathbf{u}\cdot\nabla)\mathbf{u}-\nu n\Delta\mathbf{u}-\nu(\nabla n\cdot\nabla)\mathbf{u}-\nu(\mathbf{u}\cdot\nabla)\nabla n+\frac{\varepsilon^2}{n}(\nabla n\cdot\nabla)\nabla n\\-\frac{\varepsilon^2}{n^2}(\nabla n\cdot\nabla n)\nabla n&+\frac{\varepsilon^2}{n}\Delta n\nabla n-nf-\nu\mathbf{u}\Delta n+\varepsilon^2\nabla\Delta n\quad\in\quad L^2(0,T^*;L^2(\mathbb{T}^d)).\end{aligned}$$

Due to the fact the functions ϕ^m are dense in $L^2(0,T^*;\mathbf{H})$ we have that (30) is also valid for all $\phi\in L^2(0,T^*;\mathbf{H})$, and so, $L\mathbf{u}\in L^2(0,T^*;\mathbf{H})^{\perp}$. Therefore, by *De Rham*'s Lemma, there exist some function $p\in L^2(0,T^*;H^1(\mathbb{T}^d))$ such that

$$\begin{aligned}n\mathbf{u}_t+(n\mathbf{u}\cdot\nabla)\mathbf{u}-\nu n\Delta\mathbf{u}-\nu(\nabla n\cdot\nabla)\mathbf{u}-\nu(\mathbf{u}\cdot\nabla)\nabla n+\frac{\varepsilon^2}{n}(\nabla n\cdot\nabla)\nabla n\\-\frac{\varepsilon^2}{n^2}(\nabla n\cdot\nabla n)\nabla n+\frac{\varepsilon^2}{n}\Delta n\nabla n-\nu\mathbf{u}\Delta n+\varepsilon^2\nabla\Delta n-n\mathbf{f}=\nabla p.\end{aligned} \tag{31}$$

Using *Du Bois Raymond*'s lemma, it can be shown that

$$n_t + \mathbf{u} \cdot \nabla n - \lambda \Delta n = 0$$

q.t.p. in Q_{T^*}.

Furthermore, we have:

$$|n_t| \leq |\mathbf{u} \cdot \nabla n| + \lambda |\Delta n| \leq C |\nabla \mathbf{u}| |\nabla^2 n| + \lambda |\Delta n|,$$

implying

$$|n_t|_{L^\infty(0,T^*;L^2(\mathbb{T}^d))} \leq C.$$

Therefore,

$$n_t + \mathbf{u} \cdot \nabla n - \lambda \Delta n = 0 \text{ in } L^\infty(0, T^*; L^2(D(A))).$$

Remark 4.1 We can see from equations (21), (19) and (22), (20) that $\mathbf{u} \in L^\infty((0,T^*); D(A))$, $\mathbf{u}_t \in L^2(0,T^*;\mathbf{V})$, $n \in L^\infty(0,T^*;H^3(\mathbb{T}^d))$ and $n_t \in L^2(0,T^*;H^2(\mathbb{T}^d))$; since $D(A) \overset{c}{\hookrightarrow} \mathbf{V}$ and $H^3(\mathbb{T}^d) \overset{c}{\hookrightarrow} H^2(\mathbb{T}^d)$, the Aubin-Lion's lemma claims that there exist $\widetilde{\mathbf{u}} \in C([0,T^*];\mathbf{V})$ and $\widetilde{n} \in C([0,T^*];H^2(\mathbb{T}^d))$ such that $\widetilde{\mathbf{u}}(t) = \mathbf{u}(t)$ and $\widetilde{n} = n(t)$, a.e. in $[0,T^*]$; this means that we can consider that the solution $(\mathbf{u}, n)$ assumes the initial datum continuously.

5. Uniqueness of Solution

Let $(\mathbf{u}, n)$ e $(\mathbf{u}^1, n^1)$ two solutions of the initial problem (5); define $z = n - n^1$ e $\mathbf{w} = \mathbf{u} - \mathbf{u}^1$.

Using the second equation (5), we get:

$$(z_t + \mathbf{u} \cdot \nabla z + \mathbf{w} \cdot \nabla n^1 - \nu \Delta z, \psi) = 0,$$

for all $\psi \in L^2(Q_{T^*})$.

In particular, making $\psi = z$ we obtain:

$$\frac{1}{2}\frac{d}{dt}|z|^2 = \nu(\Delta z, z) - (\mathbf{w} \cdot \nabla n^1, z)$$

since $(\mathbf{u} \cdot \nabla z, z) = 0$. The two terms on the right hand side in the above expression can be estimated as:

$$|(\Delta z, z)| \leq C|z|^2 + \delta |\Delta z|^2;$$

$$|(\mathbf{w} \cdot \nabla n^1, z)| \leq |\mathbf{w}||\nabla n^1|_\infty |z| \leq C|\mathbf{w}|^2 + C|z|^2.$$

So,

$$\frac{1}{2}\frac{d}{dt}|z|^2 \leq C|\mathbf{w}|^2 + C|z|^2 + \delta|\Delta z|^2 \tag{32}$$

Moreover, making $\psi = -\Delta z$ we obtain:

$$-(z_t,\Delta z)+\nu(\Delta z,\Delta z)-(\mathbf{u}\cdot\nabla z,\Delta z)-(\mathbf{w}\cdot\nabla n^1,\Delta z)=0.$$

And this implies

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}|\nabla z|^2+\nu|\Delta z|^2&=(\mathbf{u}\cdot\nabla z,\Delta z)+(\mathbf{w}\cdot\nabla n^1,\Delta z)\\&\leq|\mathbf{u}|_\infty|\nabla z||\Delta z|+|\mathbf{w}||\nabla n^1|_\infty|\Delta z|\\&\leq 2\mu|\Delta z|^2+C|A\mathbf{u}|^2|\nabla z|^2+C|n^1|^2_{_{H^3}}|\mathbf{w}|^2\end{aligned}$$

After choosing $\mu=\dfrac{\nu}{4}$ and taking into account the bounds for $|A\mathbf{u}|$ and $|n^1|_{_{H^3}}$, the above inequality takes the following form:

$$\frac{1}{2}\frac{d}{dt}|\nabla z|^2+\frac{\nu}{2}|\Delta z|^2\leq C|z|^2_{_{H^1}}+C|\mathbf{w}|^2. \tag{33}$$

Next, we consider the first equation (5) applied to $(\mathbf{u},n)$ and $(\mathbf{u}^1,n^1)$.

Making the difference between them and taking the L^2-inner product with $v=\mathbf{w}$ we obtain:

$$(n\mathbf{w}_t,\mathbf{w})+\nu(n^1A\mathbf{w},\mathbf{w})=-(z\mathbf{u}^1_t,\mathbf{w})-((z\mathbf{u}\cdot\nabla)\mathbf{u},\mathbf{w})-\nu(zA\mathbf{u},\mathbf{w})$$

$$-((n^1\mathbf{w}\cdot\nabla)\mathbf{u},\mathbf{w})-((n^1\mathbf{u}^1\cdot\nabla)\mathbf{w},\mathbf{w})+\nu((\mathbf{u}\cdot\nabla)\nabla z,\mathbf{w})+\nu((\mathbf{w}\cdot\nabla)\nabla n^1,\mathbf{w})$$

$$+\nu((\nabla z\cdot\nabla)\mathbf{u},\mathbf{w})+\nu((\nabla n^1\cdot\nabla)\mathbf{w},\mathbf{w})+(z\mathbf{f},\mathbf{w})-\nu(\mathbf{w}\Delta n,\mathbf{w})-\nu(\mathbf{u}^1\Delta z,\mathbf{w})$$

$$-\varepsilon^2\{(-\frac{z}{nn^1}(\nabla n\cdot\nabla)\nabla n+\frac{1}{n^1}(\nabla z\cdot\nabla)\nabla n-\nabla\Delta z+\frac{1}{n^1}(\nabla n^1\cdot\nabla)\nabla z$$

$$+\frac{z(n^1+n)}{(nn^1)^2}(\nabla n\cdot\nabla n)\nabla n-\frac{1}{(n^1)^2}(\nabla z\cdot\nabla n)\nabla n-\frac{1}{(n^1)^2}(\nabla n^1\cdot\nabla z)\nabla n$$

$$-\frac{1}{(n^1)^2}(\nabla n^1\cdot\nabla n^1)\nabla z-\frac{z}{nn^1}\Delta n\nabla n+\frac{1}{n^1}\Delta z\nabla n+\frac{1}{n^1}\Delta n^1\nabla z,\mathbf{w})\}.$$

We note that the first term on the right hand side in the above equation can be rewriten as

$$(n\mathbf{w}_t,\mathbf{w})\ =\ \frac{1}{2}\frac{d}{dt}(n\mathbf{w},\mathbf{w})+\frac{1}{2}((\mathbf{u}\cdot\nabla n)\mathbf{w},\mathbf{w})-\frac{\nu}{2}(\Delta n\mathbf{w},\mathbf{w});$$

and then, such equation becomes:

$$\frac{1}{2}\frac{d}{dt}(n\mathbf{w},\mathbf{w})+\nu\alpha|\nabla\mathbf{w}|^2\leq-\frac{1}{2}((\mathbf{u}\cdot\nabla n)\mathbf{w},\mathbf{w})+\frac{\nu}{2}(\Delta n\mathbf{w},\mathbf{w})-\nu(zA\mathbf{u},\mathbf{w})$$

$$+(z\mathbf{f},\mathbf{w})-(z\mathbf{u}^1_t,\mathbf{w})-((z\mathbf{u}\cdot\nabla)\mathbf{u},\mathbf{w})-((n^1\mathbf{w}\cdot\nabla)\mathbf{u},\mathbf{w})-((n^1\mathbf{u}^1\cdot\nabla)\mathbf{w},\mathbf{w})-\nu(\mathbf{w}\Delta n,\mathbf{w})$$

$$+\nu\{((\mathbf{u}\cdot\nabla)\nabla z,\mathbf{w})+((\mathbf{w}\cdot\nabla)\nabla n^1,\mathbf{w})+((\nabla z\cdot\nabla)\mathbf{u},\mathbf{w})+((\nabla n^1\cdot\nabla)\mathbf{w},\mathbf{w})\}-(\mathbf{u}^1\Delta z,\mathbf{w})$$

$$-\varepsilon^2\{(-\frac{z}{nn^1}(\nabla n\cdot\nabla)\nabla n+\frac{1}{n^1}(\nabla z\cdot\nabla)\nabla n-\nabla\Delta z+\frac{1}{n^1}(\nabla n^1\cdot\nabla)\nabla z$$

$$+\frac{z(n^1+n)}{(nn^1)^2}(\nabla n\cdot\nabla n)\nabla n-\frac{1}{(n^1)^2}(\nabla z\cdot\nabla n)\nabla n-\frac{1}{(n^1)^2}(\nabla n^1\cdot\nabla z)\nabla n$$

$$-\frac{1}{(n^1)^2}(\nabla n^1\cdot\nabla n^1)\nabla z-\frac{z}{nn^1}\Delta n\nabla n+\frac{1}{n^1}\Delta z\nabla n+\frac{1}{n^1}\Delta n^1\nabla z,\mathbf{w})\}.$$

The terms on the right hand side in the above equation are estimated in a standard way, by using Hölder's inequality, Sobolev's inequality and Young's inequality in a convenient way. Then, we get the following inequality:

$$\frac{1}{2}\frac{d}{dt}(n\mathbf{w},\mathbf{w})+(\nu\alpha-\mu)|\nabla\mathbf{w}|^2\le C_{\mu,\delta}\{|A\mathbf{u}||n|_{_{H^3}}+|\nabla n|^2+|\mathbf{u}_t^1|^2$$

$$+|A\mathbf{u}|^2|\nabla\mathbf{u}|^2+|A\mathbf{u}|^2+|A\mathbf{u}^1|^2+|n^1|^2_{_{H^3}}\}|\mathbf{w}|^2$$

$$+C_\mu\{|\mathbf{f}|^2+|\Delta n|^4+|\Delta n|^6+|n|^2_{_{H^3}}+|n^1|^2_{_{H^3}}$$

$$+|\Delta n^1|^4+|\Delta n^1|^2|\Delta n|^2\}|z|^2_{_{H^1}}+\delta|\Delta z|^2,$$

and using the estimates obtained in Section 1, we obtain:

$$\frac{1}{2}\frac{d}{dt}(n\mathbf{w},\mathbf{w})+(\nu\alpha-\mu)|\nabla\mathbf{w}|^2\le C|\mathbf{w}|^2+C|z|^2_{_{H^1}}+\delta|\Delta z|^2 \tag{34}$$

Summing up the inequalities (32), (33) and (34) we get:

$$\frac{1}{2}\frac{d}{dt}\{(n\mathbf{w},\mathbf{w})+|z|^2+|z|^2_{_{H^1}}\}+(\nu\alpha-\mu)|\nabla\mathbf{w}|^2+(\frac{\nu}{2}-5\delta)|\Delta z|^2$$

$$\le C|\mathbf{w}|^2+C|z|^2+C|z|^2_{_{H^1}}.$$

By choosing μ and δ conveniently, it follows that:

$$\frac{1}{2}\frac{d}{dt}\{(n\mathbf{w},\mathbf{w})+|z|^2_{_{H^1}}\}+\frac{\nu\alpha}{2}|\nabla\mathbf{w}|^2+\frac{\nu}{4}|\Delta z|^2$$
$$\le C|\mathbf{w}|^2+C|z|^2_{_{H^1}}+C|z|^2,$$

Since $|\mathbf{w}|^2\le\frac{1}{\alpha}(n\mathbf{w},\mathbf{w})$, we have:

$$\frac{d}{dt}\{(n\mathbf{w},\mathbf{w})+|z|^2+|z|^2_{_{H^1}}\}\le C((n\mathbf{w},\mathbf{w})+|z|^2+|z|^2_{_{H^1}}),$$

and integrating the 0 the t, with $t\in[0,T^*]$:

$$(n\mathbf{w},\mathbf{w})+|z|^2+|z|^2_{_{H^1}}\le(n_\circ\mathbf{w}(0),\mathbf{w}(0))+|z(0)|^2+|z(0)|^2_{_{H^1}}$$

$$+C\int_0^t((n(s)\mathbf{w}(s),\mathbf{w}(s))+|z(s)|^2+|z(s)|^2_{_{H^1}})ds.$$

Now, by using Gronwall's lemma we obtain:

$$\begin{aligned}(n\mathbf{w},\mathbf{w})+|z|^2+|z|^2_{_{H^1}} &\le [(n_\circ\mathbf{w}(0),\mathbf{w}(0))+|z(0)|^2+|z(0)|^2_{_{H^1}}]\cdot e^{C\int_0^{T^*}ds}\\ &\le C[(n_\circ\mathbf{w}(0),\mathbf{w}(0))+|z(0)|^2+|z(0)|^2_{_{H^1}}].\end{aligned}$$

Since $\mathbf{w}(0) = 0$, $|z(0)|^2_{H^1} = 0$ we conclude that $(n\mathbf{w}, \mathbf{w}) = 0$ e $|z|^2 = 0$ and because $n > 0$ we get $\mathbf{w} = 0$ and $z = 0$ a.e..

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