to each positive number ϵ, two numbers δ_1 and δ_2 exist such that

$$|u - u_0| < \frac{\epsilon}{2} \qquad \text{whenever } 0 < (x - x_0)^2 + (y - y_0)^2 < \delta_1^2$$

and

$$|v - v_0| < \frac{\epsilon}{2} \qquad \text{whenever } 0 < (x - x_0)^2 + (y - y_0)^2 < \delta_2^2.$$

Let δ denote the smaller of the two numbers δ_1 and δ_2. Then, since

$$|u - u_0 + i(v - v_0)| \leqq |u - u_0| + |v - v_0|,$$

condition (3) follows. Thus statement (1) is a consequence of statements (2), and the proof of the theorem is complete.

Theorem 2. *Let f and F be functions whose limits exist at z_0:*

$$(5) \qquad \lim_{z \to z_0} f(z) = w_0, \qquad \lim_{z \to z_0} F(z) = W_0.$$

Then

$$(6) \qquad \lim_{z \to z_0} [f(z) + F(z)] = w_0 + W_0,$$

$$(7) \qquad \lim_{z \to z_0} [f(z)F(z)] = w_0 W_0,$$

and, if $W_0 \neq 0$,

$$(8) \qquad \lim_{z \to z_0} \frac{f(z)}{F(z)} = \frac{w_0}{W_0}.$$

This fundamental theorem can be established directly from the definition (Sec. 12) of the limit of a function of a complex variable. But with the aid of Theorem 1 it follows almost immediately from theorems on limits of real-valued functions of two real variables.

Consider the proof of property (7), for example. We write

$$f(z) = u(x,y) + iv(x,y), \qquad F(z) = U(x,y) + iV(x,y),$$
$$z_0 = x_0 + iy_0, \qquad w_0 = u_0 + iv_0, \qquad W_0 = U_0 + iV_0.$$

Then, according to our hypotheses (5) and Theorem 1, the limits, as (x,y) approaches (x_0,y_0), of u, v, U, and V exist and have the values u_0, v_0, U_0, and V_0, respectively. The real and imaginary components of the function

$$f(z)F(z) = uU - vV + i(uV + vU)$$

therefore have the limits $(u_0U_0 - v_0V_0)$ and $(u_0V_0 + v_0U_0)$, according to theorems on limits of sums and products of functions.

Therefore $f(z)F(z)$ has the limit

$$u_0U_0 - v_0V_0 + i(u_0V_0 + v_0U_0),$$

which is equal to w_0W_0, and statement (7) follows.

Corresponding proofs of conclusions (6) and (8) can be written.

In order to prove (8) directly from the definition of the limit of a function of a complex variable, a useful auxiliary result, concerning a function F whose limit W_0 is not zero, should be noted. It is convenient to define $F(z_0)$ to be W_0. Then there is a neighborhood of z_0 such that $|F(z)|$ is *bounded away from zero;* that is, $|F(z)|$ exceeds some positive constant, for all z in that neighborhood. This can be seen by writing δ_0 for a value of δ that corresponds to the value $\frac{1}{2}|W_0|$ for ϵ. Then

$$(9) \qquad |F(z) - W_0| < \tfrac{1}{2}|W_0| \qquad \text{whenever } |z - z_0| < \delta_0.$$

It is left to the exercises to show that, as a consequence,

$$(10) \qquad |F(z)| > \tfrac{1}{2}|W_0| \qquad \text{whenever } |z - z_0| < \delta_0.$$

In particular, $F(z) \neq 0$ for any value of z in that neighborhood of z_0.

From the definition of the limit we can see that

$$\lim_{z \to z_0} z = z_0,$$

since we can write $\delta = \epsilon$ when $f(z) = z$. It follows from statement (7) on the limit of a product, and by induction, that

$$(11) \qquad \lim_{z \to z_0} z^n = z_0^n \qquad (n = 1, 2, \ldots),$$

where z_0 is any complex number. Also, the limit of a constant is that constant. In view of Theorem 2, then, the limit of a polynomial

$$P(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n$$

is the value of that polynomial at z_0, for every number z_0:

$$(12) \qquad \lim_{z \to z_0} P(z) = P(z_0).$$

EXERCISES

1. Describe the domain of definition of the function

$$g(z) = \frac{y}{x} + \frac{1}{1 - y} i.$$

For all z in the domain of definition of the function f_5 described in Sec. 10, show that $g(z) = f_5(z)$.

2. Let b, c, and z_0 denote complex constants. Use the definition of the limit (Sec. 12) to prove that

(a) $\lim_{z \to z_0} c = c$; (b) $\lim_{z \to z_0} (bz + c) = bz_0 + c$;

(c) $\lim_{z \to z_0} (z^2 + c) = z_0^2 + c$; (d) $\lim_{z \to z_0} \mathcal{R}(z) = \mathcal{R}(z_0)$;

(e) $\lim_{z \to z_0} \bar{z} = \bar{z}_0$; (f) $\lim_{z \to 1-i} [x + i(2x + y)] = 1 + i$.

3. Prove statement (6) in Theorem 2, (a) by using Theorem 1 and properties of limits of real-valued functions; (b) directly from the definition (Sec. 12) of the limit of a function.

4. Prove that condition (10) follows from condition (9).

5. If n is a positive integer and P and Q are polynomials and if $Q(z_0) \neq 0$, use Theorem 2 and established limits to find

(a) $\lim_{z \to z_0} \frac{1}{z^n}$ $(z_0 \neq 0)$; (b) $\lim_{z \to i} \frac{iz^3 - 1}{z + i}$; (c) $\lim_{z \to z_0} \frac{P(z)}{Q(z)}$.

Ans. (a) $1/z_0^n$; (b) 0; (c) $P(z_0)/Q(z_0)$.

14. Continuity. A function f is *continuous* at a point z_0 if and only if all three of the following conditions are satisfied:

(1) $f(z_0)$ exists,

(2) $\lim_{z \to z_0} f(z)$ exists,

(3) $\lim_{z \to z_0} f(z) = f(z_0)$.

Those conditions imply that $f(z)$ is defined throughout some neighborhood of the point z_0. A natural modification of this definition is needed in order to define the continuity of a function at a point on the boundary of a region in which the function is defined. Suppose that $f(z)$ is defined throughout a region extending up to and including a curve C, but not extending across C. Then f is continuous at a point z_0 on C if and only if the conditions (2) and (3) are satisfied, where, in this case, the limit is the limit from the interior of the region; that is, the neighborhood $|z - z_0| < \delta$ used in defining the limit is to be replaced by that part of the neighborhood which lies in the given region.

As a result of the theorems on limits, if any two functions are continuous, their sum and their product are also continuous,

and their quotient is continuous except for those values of z for which the denominator vanishes.

Every polynomial in z is continuous at each point, according to formula (12), Sec. 13. The quotient of two polynomials is continuous for each value of z for which the denominator is different from zero.

From Theorem 1, Sec. 13, it follows that

(4) $f = u + iv$ *is continuous if and only if* u *and* v *are continuous.*

Properties of continuous functions of z may thus be deduced from properties of continuous functions u and v of x and y. For instance, when f is a continuous function of z at every point in a closed region R, then u and v are continuous in R, and therefore bounded in R; consequently f is *bounded* in R; that is, for some constant M,

$$|f(z)| < M \qquad \text{for all } z \text{ in } R.$$

According to statement (4), $xy^2 + i(2x - y)$ is a continuous function of z everywhere, because polynomials xy^2 and $2x - y$ in x and y are everywhere continuous functions of the two variables x and y. Likewise, $e^x + i \sin (xy)$ is continuous for all z because of the continuity of the exponential and sine functions and the function xy.

Condition (3) can be written as follows. Corresponding to each positive number ϵ, a number δ exists such that

(5) $$|f(z) - f(z_0)| < \epsilon \qquad \text{whenever } |z - z_0| < \delta.$$

The number δ that corresponds to a given ϵ may depend on z_0. However, if f is continuous at every point of a *closed* region R, then f is *uniformly continuous* there; that is, for each given ϵ, a number δ, independent of z_0, exists such that condition (5) is satisfied simultaneously for every point z_0 in R. This follows from statement (4) and the corresponding property for real-valued functions u and v.

Let D be a domain of definition of a function f. For all z in some neighborhood N of a point z_0 let the range of a function g be included in D. Then $f[g(z)]$ is defined when z is in N. If g is continuous at z_0 and f is continuous at the point $g(z_0)$, then the *composite function* of z, $f(g)$, is continuous at z_0. In brief, a