of the domain

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}, \tag{1}$$

and therefore

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial x\, \partial y}, \qquad \frac{\partial^2 u}{\partial y^2} = -\frac{\partial^2 v}{\partial y\, \partial x}, \tag{2}$$

provided these second derivatives exist. We shall show in Chap. 5 (Sec. 52) that, when f is analytic, the partial derivatives of u and v of all orders exist and are continuous functions of x and y. Granting this for the present, it follows that the two cross derivatives in equations (2) are equal and therefore that

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \tag{3}$$

throughout the domain.

Equation (3) is *Laplace's* partial differential equation in two independent variables x and y. Any function that has continuous partial derivatives of the second order and that satisfies Laplace's equation is called a *harmonic function.*

The function v, as well as u, is harmonic when the function $f = u + iv$ is an analytic function. This can be shown by differentiating the first of equations (1) with respect to y, the second with respect to x, and subtracting to get the equation

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0. \tag{4}$$

If the function $f = u + iv$ is analytic, then u and v are called *conjugate harmonic functions.* This is a different use of the word *conjugate* from that employed in defining $\bar{z}$.

Given one of two conjugate harmonic functions, the Cauchy-Riemann equations (1) can be used to find the other. We shall now illustrate one method of obtaining the conjugate harmonic of a given harmonic function.

The function

$$u = y^3 - 3x^2y$$

is readily seen, by direct substitution into Laplace's equation, to be a harmonic function. In order to find its harmonic con-

jugate v, we note that

$$\frac{\partial u}{\partial x} = -6xy,$$

from which, by using one of the Cauchy-Riemann equations, we may conclude that

$$\frac{\partial v}{\partial y} = -6xy.$$

Integrating this equation with respect to y with x held fixed, we find that

$$v = -3xy^2 + \phi(x),$$

where $\phi(x)$ is at present an arbitrary function of x. But since $\partial v/\partial x = -\partial u/\partial y$, it follows that

$$-3y^2 + \phi'(x) = -3y^2 + 3x^2;$$

therefore $\phi'(x) = 3x^2$ and $\phi(x) = x^3 + c$, where c is an arbitrary constant. Hence the harmonic conjugate of the function $u = y^3 - 3x^2y$ is

$$v = -3xy^2 + x^3 + c.$$

The corresponding function $f = u + iv$ is

$$f(z) = y^3 - 3x^2y + i(x^3 - 3xy^2) + ic. \tag{5}$$

It is easily verified that

$$f(z) = i(z^3 + c).$$

This form is suggested by noting that when $y = 0$, equation (5) becomes

$$f(x) = i(x^3 + c).$$

Later on (Sec. 78) we shall show that, corresponding to each harmonic function u, a conjugate harmonic function v exists. We shall use a line integral to write an explicit formula for v in terms of u.

EXERCISES

1. Prove that each of these functions is entire:

(*a*) $f(z) = 3x + y + i(3y - x)$;
(*b*) $f(z) = \sin x \cosh y + i \cos x \sinh y$;
(*c*) $f(z) = e^{-y}(\cos x + i \sin x)$;
(*d*) $f(z) = (z^2 - 2)e^{-x}(\cos y - i \sin y)$.

2. Show why each of these functions is nowhere analytic:

(*a*) $f(z) = xy + iy$; (*b*) $f(z) = e^y(\cos x + i \sin x)$.

3. Determine the singular points of each of these functions and state why the function is analytic everywhere except at those points:

(*a*) $\dfrac{2z+1}{z(z^2+1)}$; (*b*) $\dfrac{z^3+i}{z^2-3z+2}$; (*c*) $(z+2)^{-1}(z^2+2z+2)^{-1}$.

Ans. (*a*) $z = 0, \pm i$; (*c*) $z = -2, -1 \pm i$.

4. If $z = r(\cos\theta + i \sin\theta)$, prove that the function

$$F(z) = \log r + i\theta \qquad \left(r > 0, -\frac{\pi}{2} < \theta < \frac{\pi}{2}\right)$$

is analytic in the domain of definition indicated and that $F'(z) = 1/z$ there. Then show why the composite function $F(2z + i - 2)$ is an analytic function of z in the domain $x > 1$.

5. If $u + iv$ is analytic, state why $-v + iu$ is also analytic. Consequently, show that, if u and v are conjugate harmonic functions, then u and $-v$ are also conjugate harmonic functions. In the example given in Sec. 20, therefore, $3xy^2 - x^3 + C$ is another harmonic conjugate to the function $y^3 - 3x^2y$.

6. Show that u is harmonic in some domain and find a harmonic conjugate v, when

(*a*) $u = 2x(1 - y)$; (*b*) $u = 2x - x^3 + 3xy^2$;
(*c*) $u = \sinh x \sin y$; (*d*) $u = y(x^2 + y^2)^{-1}$.

Ans. (*a*) $v = x^2 - y^2 + 2y$; (*c*) $v = -\cosh x \cos y$.

7. Let u and v be conjugate harmonic functions. Their contour curves or level lines are the families of curves $u = c_1$ and $v = c_2$. Prove that these families of curves are orthogonal. More precisely, show that at any point (x_0,y_0) that is common to a curve $u = c_1$ and a curve $v = c_2$, the tangents (or normals) to the two curves are perpendicular, provided $\partial u/\partial x$ and $\partial u/\partial y$ do not both vanish at the point, that is, provided $f'(z_0) \neq 0$ where $f = u + iv$.

8. Show that when

$$f(z) = u + iv = z^2,$$

the families of curves $u = c_1$ and $v = c_2$ are those shown in Fig. 14. Note the orthogonality of these curves as proved in Exercise 7. The curves $u = 0$ and $v = 0$ intersect at the origin and are not orthogonal to each other. Why is this fact in agreement with the result of Exercise 7?

9. Sketch the families of curves $u = c_1$ and $v = c_2$ when $f(z) = 1/z$ and note the orthogonality proved in Exercise 7.

10. Sketch the families of curves $u = c_1$ and $v = c_2$ when

$$f(z) = \frac{z-1}{z+1},$$

and note how the results of Exercise 7 are illustrated here.

11. Solve Exercise 9 using polar coordinates.

12. Let a function f be analytic in a domain D that does not include the point $z = 0$. If $f(z) = u(r,\theta) + iv(r,\theta)$, use the Cauchy-Riemann

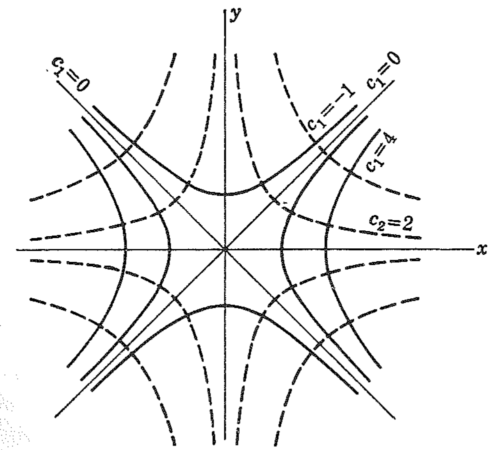

FIG. 14

conditions in polar coordinates to show that, in D, both u and v satisfy Laplace's equation in polar coordinates,

$$r^2\frac{\partial^2 u}{\partial r^2} + r\frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial \theta^2} = 0 \qquad (\theta \text{ in radians}),$$

given that all partial derivatives of u and v up to the second order are continuous.

13. In the domain $r > 0$, $0 < \theta < 2\pi$, show that the function $u = \log r$ is harmonic (Exercise 12) and find its harmonic conjugate.

Ans. $v = \theta + c$.

14. If in some domain a function $f = u + iv$ and its complex conjugate $\bar{f} = u - iv$ are both analytic, prove that f is a constant.

15. If f is analytic in some domain, prove that its absolute value $|f|$ cannot be a constant there unless f is a constant.

16. Point out why the final statement in Sec. 19, that the composition $f(g)$ of two entire functions is an entire function, is true without qualification. Also, state why a *linear combination* $bf + cg$, where b and c are complex constants, of entire functions f and g is again entire.