$z = 2 + iy \quad (0 \leqq y \leqq 1)$, and dz is replaced by $i\,dy$. Therefore

$$I_2 = \int_0^2 x^2\,dx + \int_0^1 (2+iy)^2 i\,dy$$
$$= \tfrac{8}{3} + i\left[\int_0^1 (4-y^2)\,dy + 4i\int_0^1 y\,dy\right] = \tfrac{2}{3} + \tfrac{11}{3}i.$$

Incidentally, the equations of contour OAB here can be written in the form $x = \phi(t)$ $y = \psi(t)$ $\quad (0 \leqq t \leqq 3)$, where

$$\phi(t) = \begin{cases} t & (0 \leqq t \leqq 2) \\ 2 & (2 \leqq t \leqq 3), \end{cases}$$
$$\psi(t) = \begin{cases} 0 & (0 \leqq t \leqq 2) \\ t-2 & (2 \leqq t \leqq 3). \end{cases}$$

We note that $I_2 = I_1$. Thus the integral of z^2 over the closed contour $OABO$ has the value $I_2 - I_1 = 0$, and we shall soon see that this is a consequence of the fact that the integrand z^2 is analytic interior to and on the contour.

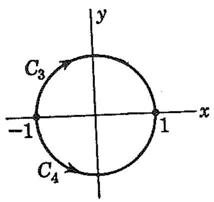

FIG. 31

As a third example, let the integrand be the function

$$f(z) = \bar{z},$$

which is everywhere continuous. If the path C_3 is the upper half of the circle $|z| = 1$ from $z = -1$ to $z = 1$ (Fig. 31), its parametric equations can be written

$$x = \cos\theta, \qquad y = \sin\theta, \qquad \text{or} \qquad z = e^{i\theta} \qquad (0 \leqq \theta \leqq \pi).$$

Then, by replacing dz by $(-\sin\theta + i\cos\theta)\,d\theta$, we find that

$$I_3 = \int_{C_3} \bar{z}\,dz = \int_\pi^0 (\cos\theta - i\sin\theta)(-\sin\theta + i\cos\theta)\,d\theta$$
$$= \int_\pi^0 e^{-i\theta}\,ie^{i\theta}\,d\theta = i\int_\pi^0 d\theta = -\pi i.$$

The integral I_4 between the same two points along the lower semicircle C_4 (Fig. 31), represented by the equations

$$x = \cos\theta, \qquad y = \sin\theta \qquad \text{or} \qquad z = e^{i\theta} \qquad (\pi \leqq \theta \leqq 2\pi),$$

is found in like manner:

$$I_4 = \int_{C_4} \bar{z}\,dz = i\int_\pi^{2\pi} d\theta = \pi i.$$

Note that $I_4 \neq I_3$ and that the integral I_C of $\bar{z}$ around the entire

circle C in the counterclockwise direction does not vanish, since

$$I_C = \int_C \bar{z}\,dz = I_4 - I_3 = 2\pi i.$$

When z is on the unit circle C,

$$\frac{1}{z} = \frac{\bar{z}}{|z|^2} = \bar{z};$$

thus the integrands of the integrals I_3, I_4, and I_C can be replaced by $1/z$. In particular,

$$I_C = \int_C \frac{dz}{z} = 2\pi i.$$

As a final example, let C_5 denote the line segment from the point $z = i$ to $z = 1$. Without evaluating the integral

$$I_5 = \int_{C_5} \frac{dz}{z^4},$$

let us determine an upper bound for its absolute value. The integrand is continuous on C_5, since its only discontinuity is at the origin.

If the parameter t is chosen as the coordinate x, the parametric equations of C_5 reduce to $y = 1 - x \quad (0 \leqq x \leqq 1)$. When z is on C_5,

$$|z^4| = (x^2 + y^2)^2 = [x^2 + (1-x)^2]^2 = (2x^2 - 2x + 1)^2,$$

and, to find a lower bound for this quantity, we note that

$$|z^4| = [2(x - \tfrac{1}{2})^2 + \tfrac{1}{2}]^2 \geqq \tfrac{1}{4}$$

since $(x - \frac{1}{2})^2 \geqq 0$. Consequently, for all z on C_5,

$$\left|\frac{1}{z^4}\right| \leqq 4,$$

a fact that is also evident from a figure. Thus we may write $M = 4$ in the inequality (12), Sec. 44. Since the length of C_5 is $\sqrt{2}$, it follows that

$$|I_5| \leqq 4\sqrt{2}.$$

EXERCISES

For each function f and path C given in Exercises 1 to 6, find the value of

$$\int_C f(z)\,dz$$

after observing that f is at least sectionally continuous on C and that C is a contour.

1. $f(z) = y - x - 3x^2 i$; C is the straight-line segment from $z = 0$ to $z = 1 + i$. *Ans.* $1 - i$.

2. $f(z) = y - x - 3x^2 i$; C consists of two straight-line segments, one from $z = 0$ to $z = i$ and the other from $z = i$ to $z = 1 + i$. *Ans.* $\frac{1}{2}(1 - i)$.

3. $f(z) = (z + 2)/z$ and C is

(*a*) the semicircle $z = 2e^{i\theta}$ where θ varies from 0 to π;
(*b*) the semicircle $z = 2e^{i\theta}$ where θ varies from 0 to $-\pi$;
(*c*) the circle $z = 2e^{i\theta}$ where θ varies from $-\pi$ to π.

Ans. (*a*) $-4 + 2\pi i$; (*b*) $-4 - 2\pi i$; (*c*) $4\pi i$.

4. $f(z) = z - 1$, and C is this arc from $z = 0$ to $z = 2$:

(*a*) the semicircle $z - 1 = e^{i\theta}$ $\quad (0 \leqq \theta \leqq \pi)$;
(*b*) the segment of the x axis. *Ans.* (*a*) 0; (*b*) 0.

5. C is the arc from $z = -1 - i$ to $z = 1 + i$ of the curve $y = x^3$ and

$$f(z) = \begin{cases} 4y & \text{when } y > 0, \\ 1 & \text{when } y < 0. \end{cases}$$

Ans. $2 + 3i$.

6. $f(z) = e^z$, and C is this path from $z = \pi i$ to $z = 1$:

(*a*) the straight-line segment;
(*b*) the broken-line segment along the coordinate axes.

Ans. (*a*) $1 + e$; (*b*) $1 + e$.

7. If C is the boundary of the square with vertices at the points $z = 0$, $z = 1$, $z = 1 + i$, and $z = i$, show that

$$\int_C (3z + 1)\, dz = 0.$$

8. If C is the boundary of the square in Exercise 7, evaluate

$$\int_C \pi \exp(\pi \bar{z})\, dz.$$

Ans. $4(e^\pi - 1)$.

9. Evaluate the integral I_3, Sec. 45, using these equations of C_3:

$$x = t, \qquad y = \sqrt{1 - t^2} \qquad (-1 \leqq t \leqq 1).$$

10. Let C be the arc of the circle $|z| = 2$ that lies in the first quadrant. Without finding the actual value of the integral, show that

$$\left| \int_C \frac{dz}{z^2 + 1} \right| \leqq \frac{\pi}{3}.$$

11. When C denotes the boundary of the triangle with vertices at the points $z = 0$, $z = -4$, and $z = 3i$, show that

$$\left| \int_C (e^z - \bar{z})\, dz \right| \leqq 60.$$

12. If C is a circle $|z| = R$, where $R > 1$, show that

$$\left| \int_C \frac{\operatorname{Log} z}{z^2}\, dz \right| < 2\pi \frac{\pi + \operatorname{Log} R}{R}$$

and hence that the value of the integral tends to zero as $R \to \infty$.

13. By writing the integral in terms of real integrals, prove that

$$\int_\alpha^\beta dz = \beta - \alpha$$

whenever the path of integration from point $z = \alpha$ to point $z = \beta$ is (*a*) a smooth Jordan arc; (*b*) a contour.

14. Prove that

$$2 \int_\alpha^\beta z\, dz = \beta^2 - \alpha^2$$

whenever the path of integration is (*a*) a smooth Jordan arc; (*b*) a contour. (*c*) As a consequence, show that the integral of z around any closed contour vanishes.

15. When C_0 is a circle

$$z - z_0 = r_0 e^{i\theta} \qquad (0 \leqq \theta \leqq 2\pi,\ r_0 > 0).$$

described counterclockwise, show that

$$\int_{C_0} f(z)\, dz = i r_0 \int_0^{2\pi} f(z_0 + r_0 e^{i\theta}) e^{i\theta}\, d\theta$$

if f is continuous on C_0.

16. As particular cases of Exercise 15, show that

$$\int_{C_0} \frac{dz}{z - z_0} = 2\pi i, \qquad \int_{C_0} \frac{dz}{(z - z_0)^n} = 0 \qquad (n = 2, 3, \ldots).$$

46. The Cauchy-Goursat Theorem. According to Green's theorem on real line integrals, if two functions $P(x,y)$ and $Q(x,y)$, together with their partial derivatives of the first order, are continuous throughout a closed region R consisting of the interior of a closed contour C together with the boundary C itself, then

$$\int_C (P\, dx + Q\, dy) = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx\, dy,$$

whose values coincide with those of the function (5) in the lower half plane. The analytic function

$$\frac{2}{3} z^{\frac{3}{2}} = \frac{2}{3} r^{\frac{3}{2}} \exp \frac{3i\theta}{2} \qquad \left(r > 0, \frac{\pi}{2} < \theta < \frac{5\pi}{2}\right)$$

is an indefinite integral of $f_2(z)$; thus, along the lower paths,

$$\int_{-1}^{1} z^{\frac{1}{2}}\, dz = \tfrac{2}{3} r^{\frac{3}{2}} e^{3i\theta/2}\Big]_{-1}^{1} = \tfrac{2}{3}(e^{3\pi i} - e^{3\pi i/2}) = \tfrac{2}{3}(-1 + i).$$

The integral of the function (5) in the positive sense around a closed contour consisting of a path of the second group combined with one of the first therefore has the value

$$\tfrac{2}{3}(-1 + i) - \tfrac{2}{3}(1 + i) = -\tfrac{4}{3}.$$

EXERCISES

1. Determine the domain of analyticity of the function f and apply the Cauchy-Goursat theorem to show that

$$\int_C f(z)\, dz = 0$$

when the closed contour C is the circle $|z| = 1$ and when

(*a*) $f(z) = \dfrac{z^2}{z - 3}$; (*b*) $f(z) = ze^{-z}$; (*c*) $f(z) = \dfrac{1}{z^2 + 2z + 2}$;
(*d*) $f(z) = \operatorname{sech} z$; (*e*) $f(z) = \tan z$; (*f*) $f(z) = \operatorname{Log}(z + 2)$.

2. If B is the oriented boundary of the region between the circle $|z| = 4$ and the square with sides along the lines $x = \pm 1$, $y = \pm 1$, where B is described so that the region lies on the left of B, state why

$$\int_B f(z)\, dz = 0$$

when

(*a*) $f(z) = \dfrac{1}{3z^2 + 1}$; (*b*) $f(z) = \dfrac{z + 2}{\sin(z/2)}$; (*c*) $f(z) = \dfrac{z}{1 - e^z}$.

3. Let C_1 be a closed contour in the domain interior to a closed contour C_2, where both C_1 and C_2 are oriented in the positive (counterclockwise) direction. If a function f is analytic in the closed region between C_1 and C_2, state why

$$\int_{C_1} f(z)\, dz = \int_{C_2} f(z)\, dz.$$

4. Use the results of Exercise 3 above and Exercise 16, Sec. 45, to show that

$$\int_C \frac{dz}{z - 2 - i} = 2\pi i, \qquad \int_C \frac{dz}{(z - 2 - i)^n} = 0 \qquad (n = 2, 3, \ldots),$$

when C is the boundary of the rectangle $0 \leqq x \leqq 3$, $0 \leqq y \leqq 2$, described in the positive sense.

5. Use the indefinite integral to show that, for every contour C extending from a point α to a point β,

$$\int_C z^n\, dz = \frac{1}{n + 1} (\beta^{n+1} - \alpha^{n+1}) \qquad (n = 0, 1, 2, \ldots).$$

6. Evaluate each of these integrals where the path is an arbitrary contour between the points represented by the limits:

(*a*) $\displaystyle\int_i^{i/2} e^{\pi z}\, dz$; (*b*) $\displaystyle\int_0^{\pi+2i} \cos\frac{z}{2}\, dz$; (*c*) $\displaystyle\int_1^3 (z - 2)^3\, dz$.

Ans. (*a*) $(1 + i)/\pi$; (*b*) $e + 1/e$; (*c*) 0.

7. If $z_1 \neq 0$ and $z_2 \neq 0$ and $z_1 \neq z_2$, show why

$$\int_{z_1}^{z_2} \frac{dz}{z^2} = \frac{1}{z_1} - \frac{1}{z_2}$$

whenever the contour of integration is interior to a simply connected domain which does not contain the origin. Show how it follows that, for every closed contour C for which the origin is either an interior point or an exterior point,

$$\int_C \frac{dz}{z^2} = 0.$$

8. Let z_0, z_1, and z_2 denote three distinct points of a simply connected domain D. Given that a function f and its derivative f' are both analytic throughout D except at z_0, generalize the result in Exercise 7 to show that, for each contour in D drawn from z_1 to z_2 but not passing through z_0,

$$\int_{z_1}^{z_2} f'(z)\, dz = f(z_2) - f(z_1); \qquad \text{thus} \qquad \int_C f'(z)\, dz = 0$$

when the closed contour C in D does not pass through z_0. Give examples of such functions and domains.

9. Use an indefinite integral to find the value of the integral

$$\int_{-2i}^{2i} \frac{dz}{z}$$

over every contour from $z = -2i$ to $z = 2i$ lying in the right half plane. Note that the principal branch $\operatorname{Log} z$ is an indefinite integral of $1/z$ that is analytic in the half plane $x \geqq 0$ except at the origin.

The proof of this theorem by purely algebraic methods is difficult, but it follows easily from Liouville's theorem. For let us suppose that $P(z)$ is not zero for any value of z. Then the function

$$f(z) = \frac{1}{P(z)}$$

is everywhere analytic. Also $|f(z)|$ approaches zero as $|z|$ tends to infinity, so that $|f(z)|$ is bounded for all z. Consequently $f(z)$ is a constant. We have therefore arrived at a contradiction, for $P(z)$ is not a constant when $m = 1, 2, \ldots$, and $a_m \neq 0$. Hence $P(z)$ is zero for at least one value of z.

In elementary algebra courses the fundamental theorem is usually stated without proof; then as a consequence it is shown that an algebraic equation of degree m has not more than m roots.

EXERCISES

1. If C is the circle $|z| = 3$ described in the positive sense and if

$$g(z_0) = \int_C \frac{2z^2 - z - 2}{z - z_0}\,dz \qquad (|z_0| \neq 3),$$

show that $g(2) = 8\pi i$. What is the value of $g(z_0)$ when $|z_0| > 3$?

2. If C is a closed contour described in the positive sense and

$$g(z_0) = \int_C \frac{z^3 + 2z}{(z - z_0)^3}\,dz,$$

show why $g(z_0) = 6\pi i z_0$ when z_0 is inside C, and $g(z_0) = 0$ when z_0 is outside C.

3. Let C denote the boundary of the square whose sides lie along the lines $x = \pm 2$ and $y = \pm 2$, where C is described in the positive sense. Give the value of each of these integrals:

$$(a) \int_C \frac{e^{-z}\,dz}{z - \pi i/2}; \quad (b) \int_C \frac{\cos z}{z(z^2 + 8)}\,dz; \quad (c) \int_C \frac{z\,dz}{2z + 1};$$

$$(d) \int_C \frac{\tan (z/2)}{(z - x_0)^2}\,dz \quad (|x_0| < 2); \quad (e) \int_C \frac{\cosh z}{z^4}\,dz.$$

Ans. (a) 2π; (b) $\pi i/4$; (c) $-\pi i/2$; (d) $i\pi \sec^2 (x_0/2)$; (e) 0.

4. Give the value of the integral of $g(z)$ around the closed contour $|z - i| = 2$ in the positive sense when

$$(a)\ g(z) = \frac{1}{z^2 + 4}; \quad (b)\ g(z) = \frac{1}{(z^2 + 4)^2}.$$

Ans. (a) $\pi/2$; (b) $\pi/16$.

5. If f is analytic within and on an oriented closed contour C and if z_0 is not on C, show why it follows that

$$\int_C \frac{f'(z)\,dz}{z - z_0} = \int_C \frac{f(z)\,dz}{(z - z_0)^2}.$$

6. Let f denote a function that is *continuous on a closed contour* C. Follow the procedure used in Sec. 52 to prove that the function

$$g(s) = \frac{1}{2\pi i} \int_C \frac{f(z)\,dz}{z - s}$$

is *analytic* for all s interior to C and in fact that, for each such s,

$$g'(s) = \frac{1}{2\pi i} \int_C \frac{f(z)\,dz}{(z - s)^2}.$$

7. If C is the unit circle $z = \exp (i\theta)$ described from $\theta = -\pi$ to $\theta = \pi$ and k is any real constant, first show that

$$\int_C \frac{e^{kz}}{z}\,dz = 2\pi i;$$

then write the integral in terms of θ to derive the formula

$$\int_0^{\pi} e^{k \cos \theta} \cos (k \sin \theta)\,d\theta = \pi.$$

8. Let f be analytic in a bounded domain D and continuous in the closure $\bar{D}$ and assume that $f(z) \neq 0$ anywhere in $\bar{D}$. If N is the minimum value of $|f(z)|$ in $\bar{D}$, consider the function $1/f$ to prove that

$$|f(z)| > N \qquad \text{for every point } z \text{ in } D,$$

unless f is a constant. This is a *minimum modulus theorem.*

9. Give an example to show that $|f(z)|$ may assume its minimum value at an interior point of a domain in which f is analytic, if that minimum value is zero.

10. Illustrate the maximum modulus theorem and Exercise 8 when $f(z) = (z + 1)^2$ and D is the interior of the triangle with vertices at $z = 0$, $z = 2$, and $z = i$, by finding the points in $\bar{D}$ where $|f(z)|$ has its greatest and least values. *Ans.* $z = 2$, $z = 0$.

11. Let f be analytic in a bounded domain D and continuous in $\bar{D}$ and write $f = u + iv$. Prove that the harmonic function $u(x,y)$ assumes its *minimum* value on the boundary of D, never at an interior point, unless u is a constant.

12. Illustrate Exercise 11 and the maximum principle for harmonic functions (Sec. 54) by writing $u = e^x \cos y$ and taking D as the interior of the rectangle $0 < x < 1$, $0 < y < \pi$, and finding those points in $\bar{D}$ where $u(x,y)$ has its least and greatest values.

Ans. $z = 1 + \pi i$, $z = 1$.

13. Let f be an entire function and u its real component. If the harmonic function u has an upper bound u_0, $u(x,y) < u_0$ for all points in the xy plane, prove that $u(x,y)$ is a constant.

14. Complete the derivation of formula (3), Sec. 52.

15. Carry out the induction to establish formula (4), Sec. 52.

CHAPTER 6

POWER SERIES

56. Taylor's Series. We begin with one of the most important results of this chapter.

Theorem. *Let $f(z)$ be analytic at all points within a circle C_0 with center at z_0 and radius r_0. Then at each point z inside C_0*

$$(1)\quad f(z) = f(z_0) + f'(z_0)(z - z_0) + \frac{f''(z_0)}{2!}(z - z_0)^2 + \cdots + \frac{f^{(n)}(z_0)}{n!}(z - z_0)^n + \cdots ;$$

that is, the infinite series here converges to $f(z)$.

This is the expansion of the function f by Taylor's series about the point z_0. As a special case, when z, z_0, and $f(z)$ are real, it includes the expansion of a real-valued function by Taylor's series, introduced in elementary calculus.

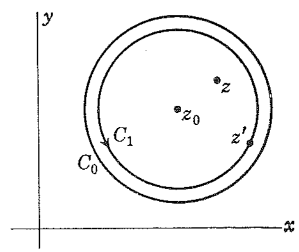

FIG. 39

To prove the theorem, let z be any fixed point inside the circle C_0 and write $|z - z_0| = r$; thus $r < r_0$. Let z' denote any point on a circle $|z' - z_0| = r_1$, denoted by C_1, where $r < r_1 < r_0$. As illustrated in Fig. 39, then, z is inside C_1, and f is analytic within and on C_1. According to the Cauchy integral formula, it follows that

$$(2)\quad f(z) = \frac{1}{2\pi i}\int_{C_1} \frac{f(z')\,dz'}{z' - z}.$$

Now

$$\frac{1}{z' - z} = \frac{1}{(z' - z_0) - (z - z_0)} = \frac{1}{z' - z_0}\,\frac{1}{1 - \dfrac{z - z_0}{z' - z_0}}.$$