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Yihong Song<sup>a</sup> & Christopher T. H. Baker<sup>c</sup>

<sup>a</sup> Suzhou University, Department of Mathematics, Suzhou, Jiangsu, 215006, P.R. China

<sup>b</sup> University of Chester, Department of Mathematics, Chester, CH1 4BJ, UK

<sup>c</sup> School of Mathematics, University of Manchester, Manchester, M13 9PL, UK

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# Admissibility for discrete Volterra equations

YIHONG SONG<sup>†\*</sup> and CHRISTOPHER T. H. BAKER<sup>‡¶</sup>

<sup>†</sup>Department of Mathematics, Suzhou University, Suzhou, Jiangsu 215006, P.R. China

<sup>‡</sup>Department of Mathematics, University of Chester, Chester CH1 4BJ, UK

<sup>¶</sup>School of Mathematics, University of Manchester, Manchester M13 9PL, UK

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In this paper we develop the theory of admissibility for linear discrete Volterra operators and obtain several necessary and sufficient conditions for admissibility in various sequence spaces. Using the results obtained, we study the existence of solutions (such as bounded, exponential or convergent solutions), of linear or nonlinear discrete Volterra summation equations.

**Keywords:** Discrete Volterra operator; Admissibility; Discrete Volterra summation equations; Convergent solutions

## 1. Introduction

In this paper we study, using concepts of admissibility, the discrete Volterra “summation” equations

$$x(n) = h(n) + \sum_{j=0}^n B(n, j)x(j), \quad n \in \mathbb{Z}^+ \quad (1.1)$$

(where we denote by  $\mathbb{Z}^+$  the set of all nonnegative integers,  $\{0, 1, \dots\}$ ) and the associated nonlinear equation

$$x(n) = h(n) + \sum_{j=0}^n B(n, j)f(j, x(j)), \quad n \in \mathbb{Z}^+ \quad (1.2)$$

(the  $n$ th equation is implicit in  $x(n)$ ).

### 1.1 The framework

We denote by  $\mathbb{R}^d$  the  $d$ -dimensional real space, by  $\mathbb{C}^d$  the  $d$ -dimensional complex space and  $\mathbb{E}^d$  denotes either  $\mathbb{R}^d$  or  $\mathbb{C}^d$  (Euclidean space);  $x = \{x(n)\}_{n \in \mathbb{Z}^+}$  is a sequence, with  $x(n) \in \mathbb{E}^d$ , i.e. to be determined. The kernel of equation (1.1),  $B(n, j)$ , is a  $d \times d$  matrix for each  $j, n \in \mathbb{Z}^+$  with  $j \leq n$  (the matrices  $B(n, j)$  with  $j > n$  do not enter the summation equations

\*Corresponding author. Email: yihongsong@hotmail.com

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¶Email: cthbaker@na-net.ornl.gov. Visiting professor, University of Chester; Emeritus professor, University of Manchester.

and may be set to zero). Additionally,  $h = \{h(n)\}_{n \in \mathbb{Z}^+}$  a given sequence in  $\mathbb{E}^d$ , for  $u \in \mathbb{E}^d$ ,  $f(j, u) \in \mathbb{E}^d$ . We suppose (see Remark 1.1 for a generalization) that  $f : \mathbb{Z}^+ \times \mathbb{E}^d \rightarrow \mathbb{E}^d$ .

*Remark 1.1* For the nonlinear equation (1.2), we may generalize the framework indicated by supposing that for some integer  $k > 0$ ,  $h(n) \in \mathbb{E}^k$  (for  $n \in \mathbb{Z}^+$ ), we have  $f : \mathbb{Z}^+ \times \mathbb{E}^k \rightarrow \mathbb{E}^d$  and  $B(n, j)$ , is a  $k \times d$  matrix for each  $j, n \in \mathbb{Z}^+$  with  $j \leq n$ .  $\square$

## 1.2 Background material

To provide some background on Volterra equations, we first note the difference between equations (1.1) and (1.2) and the explicit equations:

$$x(n) = h(n) + \sum_{j=0}^{n-1} B(n, j)x(j), \quad x(n) = h(n) + \sum_{j=0}^{n-1} B(n, j)f(j, x(j)), \quad (1.3)$$

special cases in which  $B(n, n)$  is absent from the summation. Equation (1.3) are sometimes classed as “Volterra difference equations” in the literature (and this term has also been found in use for equations (1.1) and (1.2)). We term equations (1.1) and (1.2), which includes equation (1.3), Volterra summation equations to indicate that they may be regarded as appropriate discrete analogues of classical Volterra integral equations

$$x(t) = h(t) + \int_0^t B(t, s)x(s) ds \quad (t \geq 0), \quad (1.4a)$$

$$x(t) = h(t) + \int_0^t B(t, s)f(s, x(s)) ds \quad (t \geq 0), \quad (1.4b)$$

where for illustration, we suppose all the functions involved are continuous.

*Remark 1.2* The (so-called<sup>†</sup>—we here follow [7, p. 26]) resolvent kernel  $R(t, s)$ , associated with  $B(t, s)$ , permits the solution (1.4a) to be written  $x(t) = h(t) + \int_0^t R(t, s)h(s) ds$ . The nonlinear equation (1.4b) is often called a Hammerstein equation, and the operator  $F$  with  $(Fx)(s) = f(s, x(s))$  is called the Niemytzki operator.

Discrete equations of the form (1.1) and (1.2) may arise from certain discretization procedures [1, 2, 11] for the numerical solution of the integral equations (1.4a) and (1.4b) and from modelling systems [10]. Certain integral equations reduce to ordinary or delay-differential equations. Likewise, if  $B(n, j) = 0$  when  $n - j > k$  then our summation equations reduce to finite-term recurrence relations, and if  $\{B(n, j)\}$  satisfy an appropriate recurrence relation then the summation equations can be reduced to related difference equations.  $\square$

The literature (see [9, 11, 14, 16, 22], and references therein) on various types of discrete Volterra equations, is quite extensive. For Volterra summation equations of the type (1.1) and (1.2) there are—to the best of our knowledge—relatively few papers (e.g. [3, 4, 12, 15, 18–21, 23]). Much of the general qualitative theory for Volterra summation equations (1.1) and (1.2) remains to be developed; for further reading see section 5.

<sup>†</sup>The term “resolvent” has more than one use in analysis; see [13, p. 194].

Let us place this paper in context. Admissibility theory for ordinary differential equations was originated by Bohl early in the 20th century, and consolidated by Perron and Bellman and then by Massera and Schäffer (cf. [17]). Admissibility theory for Volterra integral equations (1.4) originated with Corduneanu [5] (cf. [6,7]) and was taken up by Cushing [8]. Our aim in this paper is to develop admissibility theory for the corresponding discrete Volterra equations (1.1) and (1.2). Admissibility theory provides a framework for investigation of solvability of equations and stability of solutions, and our program is as follows: In section 2, we present some Banach sequence spaces and define admissibility for discrete Volterra operators. In section 3, we present and prove results on admissibility. Finally, we discuss some properties of solutions of linear discrete Volterra equations, and bounded, exponential and convergent solutions of nonlinear discrete Volterra equations, in section 4.

## 2. Preliminaries

### 2.1 Some Banach sequence spaces

We need to consider some spaces of sequences, or functions defined on  $\mathbb{Z}^+$ . We denote by  $\ell(\mathbb{E}^d)$  the linear topological space

$$\ell(\mathbb{E}^d) = \{x : x = \{x(n)\}_{n=0}^\infty, x(n) \in \mathbb{E}^d\}, \quad (2.1)$$

(which with the obvious definition of addition and scalar multiplication, is a linear space of sequences of elements of  $\mathbb{E}^d$ ), the topology being that of convergence on any finite subset of  $\mathbb{Z}^+$ . This means that  $\{x^m = \{x^m(n)\}_{n \in \mathbb{Z}^+}\}_{m \in \mathbb{Z}^+}$  converges to  $x = \{x(n)\}_{n \in \mathbb{Z}^+}$  if and only if  $\lim_{m \rightarrow \infty} x^m(n) = x(n)$  for each  $n \in \mathbb{Z}^+$  (point-wise).

For any  $x = \{x(n)\}_{n=0}^\infty \in \ell(\mathbb{E}^d)$  and any integer  $m \geq 0$ , write

$${}_m\|x\| = \max \{|x(n)| : 0 \leq n \leq m\}, \quad (2.2)$$

where  $|\cdot|$  stands for the Euclidean norm in  $\mathbb{E}^d$ . It is clear that the mapping  $x \rightarrow {}_m\|x\|$  is a semi-norm on  $\ell(\mathbb{E}^d)$  and the topology of convergence on any finite subset of  $\mathbb{Z}^+$  is that generated by the family of semi-norms  ${}_m\|x\| : m = 0, 1, \dots$ . A distance can be defined on  $\ell(\mathbb{E}^d)$  by the metric

$$\rho(x, y) : \sum_{m=0}^{\infty} \frac{1}{2^m} \frac{{}_m\|x - y\|}{1 + {}_m\|x - y\|}.$$

The topology induced by  $\rho$  is the same as the topology of convergence on any finite subset of  $\mathbb{Z}^+$ .

A sequence  $g = \{g(n)\}_{n \in \mathbb{Z}^+} \in \ell(\mathbb{E})$  is non-vanishing if  $g(n) \neq 0$  for all  $n \in \mathbb{Z}^+$ . If  $g \in \ell(\mathbb{E})$  is non-vanishing, we will denote by  $\ell_g(\mathbb{E}^d)$  the linear space of all sequences of  $\ell(\mathbb{E}^d)$  such that  $\sup_{n \in \mathbb{Z}^+} |x(n)|/|g(n)| < \infty$ . If we define

$$\|x\|_{\ell_g} = \sup_{n \in \mathbb{Z}^+} \frac{|x(n)|}{|g(n)|} \quad (2.3)$$

for all  $x \in \ell_g(\mathbb{E}^d)$ , then  $x \rightarrow \|x\|_{\ell_g}$  is clearly a norm; likewise,  $\ell_g(\mathbb{E}^d)$  is the linear space with the norm  $\|x\|_{\ell_g}$  for a given non-vanishing sequence  $g = \{g(n)\}_{n \in \mathbb{Z}^+} \in \ell(\mathbb{E})$ . Obviously, the following lemma is true.

LEMMA 2.1  $\ell_g(\mathbb{E}^d)$  is a Banach space under the norm  $\|x\|_{\ell_g}$ .

Remark 2.2 The topology of  $\ell_g(\mathbb{E}^d)$  is stronger than that of  $\ell(\mathbb{E}^d)$ . Indeed, suppose that  $x^m \rightarrow x$  in  $\ell_g(\mathbb{E}^d)$  as  $m \rightarrow \infty$ . Then, given  $\epsilon > 0$ , there exists  $M(\epsilon) > 0$  such that  $|x^m(n) - x(n)| < \epsilon|g(n)|$ ,  $n \in \mathbb{Z}^+$ , whenever  $m \geq M(\epsilon)$ . Since  $|g(n)|$  is finite, it follows that  $\lim x^m(n) = x(n)$  as  $m \rightarrow \infty$ . That is we have convergence in  $\ell(\mathbb{E}^d)$ .  $\square$

For  $\{g(n)\}_{n \in \mathbb{Z}^+}$  with  $|g(n)| = 1$  for  $n \in \mathbb{Z}^+$ , the space  $\ell_g(\mathbb{E}^d)$  becomes the well-known space  $\ell^\infty(\mathbb{E}^d)$  with norm  $\|\cdot\|_\infty$  given by

$$\|x\|_\infty = \sup_{n \geq 0} |x(n)| \quad (2.4)$$

It is obvious that if  $|g(n)| \geq \gamma_* > 0$  for all  $n \in \mathbb{Z}^+$  then  $\ell_g(\mathbb{E}^d)$  is isomorphic to  $\ell^\infty(\mathbb{E}^d)$ . (The isomorphic mapping from  $\ell^\infty(\mathbb{E}^d)$  into  $\ell_g(\mathbb{E}^d)$  is  $x = \{x(n)\}_{n \in \mathbb{Z}^+} \leftrightarrow xg = \{x(n)g(n)\}_{n \in \mathbb{Z}^+}$ .) Nevertheless, the opportunities for the choice of  $\{g(n)\}_{n \in \mathbb{Z}^+}$  provide a large variety of spaces consisting of sequences with a required behaviour. Several subspaces of the space  $\ell^\infty(\mathbb{E}^d)$  are needed in the sequel. By  $\ell_c^\infty(\mathbb{E}^d)$  we denote the space of convergent sequences, namely,

$$\ell_c^\infty(\mathbb{E}^d) = \{x | x \in \ell^\infty(\mathbb{E}^d) \text{ with } \lim_{n \rightarrow \infty} x(n) = x(\infty) < \infty\} \quad (2.5)$$

with the norm of  $\ell^\infty(\mathbb{E}^d)$  and

$$\ell_0^\infty(\mathbb{E}^d) = \{x | x \in \ell_c^\infty(\mathbb{E}^d) \text{ with } \lim_{n \rightarrow \infty} x(n) = 0\} \quad (2.6)$$

with the norm of  $\ell^\infty(\mathbb{E}^d)$ . We note that the topology of  $\ell_c^\infty(\mathbb{E}^d)$  is stronger than that of  $\ell(\mathbb{E}^d)$ . If there is no confusion, we sometimes write  $\ell$ ,  $\ell_g$ ,  $\ell^\infty$ ,  $\ell_c$  and  $\ell_0$  for  $\ell(\mathbb{E}^d)$ ,  $\ell_g(\mathbb{E}^d)$ ,  $\ell^\infty(\mathbb{E}^d)$ ,  $\ell_c(\mathbb{E}^d)$  and  $\ell_0(\mathbb{E}^d)$ , respectively.

DEFINITION 2.3 A matrix-valued function  $C(n)$  on  $\mathbb{Z}^+$  is said to be exponential, or to decay exponentially if it satisfies  $|C(n)| \leq Mv^n$  for some  $M > 0$  and  $v \in (0, 1)$ .

If  $g(n) = v^n$  for  $v \in (0, 1)$ , then any element of  $\ell_g(\mathbb{E}^d)$  is exponential. We say that a summation or difference equation has an exponential (or convergent) solution if the solution is in  $\ell_g(\mathbb{E}^d)$  with  $g(n) = v^n$  for  $v \in (0, 1)$  (or in  $\ell_c(\mathbb{E}^d)$ ).

## 2.2 The definition of admissibility for discrete Volterra operators

We now turn to equations (1.1) and (1.2) namely,

$$x(n) = h(n) + \sum_{j=0}^n B(n, j)x(j), \quad n \in \mathbb{Z}^+, \quad (2.7)$$

and

$$x(n) = h(n) + \sum_{j=0}^n B(n, j)f(j, x(j)), \quad n \in \mathbb{Z}^+. \quad (2.8)$$

The behaviour of the solutions of equations (2.7) and (2.8) is, in part, determined by the associated linear discrete Volterra operator  $B$  on  $\ell(\mathbb{E}^d)$  (see [3]) given by

$$(B\phi)(n) = \sum_{j=0}^n B(n,j)\phi(j), \quad n \in \mathbb{Z}^+, \quad \phi \in \ell(\mathbb{E}^d). \quad (2.9)$$

Therefore, to investigate the behaviour of solutions of equations (2.7) and (2.8), we first study the operator  $B$  given by equation (2.9) in various sequence spaces. Without risk of confusion, we employ the same notation  $B$  for the operators, for the individual matrices, and (we define  $B(n,j) \equiv 0$  when  $j > n$ ) the matrix sequence  $B = \{B(n,m)\}_{m,n \in \mathbb{Z}^+}$  (and, similarly, the notation  $x$  for the function from  $\mathbb{Z}^+$  to  $\mathbb{E}^d$ , the vectors  $x(n)$  and the sequence  $x = \{x(n)\}_{n \in \mathbb{Z}^+}$ ).

For a detailed discussion of Volterra summation operators on  $\ell(\mathbb{E}^d)$ , see [3]. We need the following definition.

**DEFINITION 2.4** *Let  $X$  and  $Y$  be two sequence spaces; the pair  $(X, Y)$  is called admissible with respect to the operator  $B$  if  $Bx \in Y$  for any  $x = \{x(n)\}_{n \in \mathbb{Z}^+} \in X$ .*

**LEMMA 2.5** *Suppose that  $\{B(n,m)\}$  is a  $d \times d$  matrix sequence. Then the discrete Volterra operator  $B$ , defined by equation (2.9), is continuous from  $\ell(\mathbb{E}^d)$  to  $\ell(\mathbb{E}^d)$ .*

*Proof.* Let  $\phi^m = \{\phi^m(n)\}_{n \in \mathbb{Z}^+} \in \ell(\mathbb{E}^d)$  be a sequence converging to  $\phi = \{\phi(n)\}_{n \in \mathbb{Z}^+}$ . This implies that  $\phi^m(n) \rightarrow \phi(n)$  as  $m \rightarrow \infty$  for each  $n \in \mathbb{Z}^+$ . Since  $(B\phi^m)(n) - (B\phi)(n) = \sum_{j=0}^n B(n,j)(\phi^m(j) - \phi(j))$  and  $B(n,j)$  are bounded for  $0 \leq j \leq n$ , it follows that  $B\phi^m \rightarrow B\phi$  in  $\ell(\mathbb{E}^d)$  as  $m \rightarrow \infty$ .  $\square$

Since complete metric spaces (and Banach spaces) are Hausdorff with the natural topologies, the next lemma applies for such cases. We use it to investigate the continuity of linear operators (see, e.g. [6]), which plays an important rôle in our discussion.

**LEMMA 2.6** *Let  $X$  and  $Y$  be two linear topological Hausdorff spaces and assume  $\mathcal{L} : X \rightarrow Y$  is a continuous linear mapping. Assume further that  $X_1 \subset X$  and  $Y_1 \subset Y$  are Fréchet spaces whose topologies are stronger than the topologies of  $X$  and  $Y$ , respectively. If  $\mathcal{L}X_1 \subset Y_1$ , then  $\mathcal{L}$  is continuous from  $X_1$  to  $Y_1$ .*

Thus, for any two complete metric subspaces  $X_1 \subset \ell(\mathbb{E}^d)$  and  $Y_1 \subset \ell(\mathbb{E}^d)$  whose topologies are stronger than the topologies of  $\ell(\mathbb{E}^d)$  and  $\ell(\mathbb{E}^d)$ , respectively, if  $BX_1 \subset Y_1$ , then  $B : X_1 \rightarrow Y_1$  is continuous, where  $B$  is given by equation (2.9).

### 3. Admissibility of linear discrete Volterra operators

#### 3.1 Admissibility of the pair $(\ell_g, \ell_g)$ with respect to discrete Volterra operators

We are in a position to prove the following result.

**THEOREM 3.1** Suppose that  $\{B(n, m)\}$  is a  $d \times d$  matrix sequence. Then, a necessary and sufficient condition for the admissibility of the pair  $(\ell_g, \ell_g)$  with respect to the operator  $B$  given by equation (2.9) is that

$$\left\{ \sum_{j=0}^n |B(n, j)| |g(j)| \right\}_{n \in \mathbb{Z}^+} \in \ell_g(\mathbb{E}). \quad (3.1)$$

*Proof.* Condition (3.1) ensures that there exists a positive constant  $A$  such that

$$\sum_{j=0}^n |B(n, j)| |g(j)| \leq A |g(n)|, \quad n \in \mathbb{Z}^+. \quad (3.2)$$

Thus, for  $\phi = \{\phi(n)\}_{n \in \mathbb{Z}^+} \in \ell_g(\mathbb{E}^d)$ , we have

$$\left| \sum_{j=0}^n B(n, j) \phi(j) \right| \leq \sum_{j=0}^n |B(n, j)| |g(j)| (|\phi(j)| / |g(j)|) \leq A |g(n)| \|\phi\|_{\ell_g}, \quad n \in \mathbb{Z}^+,$$

which shows that  $B\phi \in \ell_g$ .

To prove the necessity of condition (3.2), we first consider the case  $d = 1$  (thus,  $\{B(n, j)\}$  is either a real- or complex-number sequence) and suppose that condition (3.2) does not hold. We shall prove that the result

$$B\ell_g \subset \ell_g \quad (3.3)$$

is, with this assumption, impossible. In fact, if equation (3.2) is not satisfied, then there exists an integer sequence  $n_m$ ,  $n_m \rightarrow \infty$  as  $m \rightarrow \infty$ , such that

$$\sum_{j=0}^{n_m} |B(n_m, j)| |g(j)| \geq m |g(n_m)|, \quad m \geq 1. \quad (3.4)$$

For fixed  $n > 0$  and  $\mathbb{E} = \mathbb{R}$ , define  $f^{n_m} \in \ell_g$  to be<sup>†</sup>

$$f^{n_m}(j) = \begin{cases} \text{sign}\{B(n_m, j)\} |g(j)| & \text{if } 0 \leq j \leq n_m, \\ 0 & \text{if } j > n_m \end{cases} \quad (3.5)$$

or define  $f^{n_m} \in \ell_g$  to be

$$f^m(j) = \begin{cases} 0 & \text{if } 0 \leq j \leq n_m \text{ and } |B(n_m, j)| = 0, \\ \overline{B(n_m, j)} |g(j)| / (|B(n_m, j)|) & \text{if } 0 \leq j \leq n_m \text{ and } |B(n_m, j)| \neq 0, \\ 0 & \text{if } j > n_m \end{cases} \quad (3.6)$$

<sup>†</sup>For  $u \in \mathbb{R}$ ,  $\text{sign}(u) = 1$  if  $u \geq 0$ ,  $\text{sign}(u) = -1$  if  $u < 0$ .

if  $\mathbb{E} = \mathbb{C}$ , where  $\bar{a}$  denotes the conjugate of the number  $a$ . It is clear that in both cases  $\|f^{n_m}\|_{\ell_g} \leq 1$  for  $m = 0, 1, \dots$ , and

$$\sum_{j=0}^{n_m} B(n_m, j) f^{n_m}(j) = \sum_{j=0}^{n_m} |B(n_m, j)| |g(j)| \geq m |g(n_m)|. \quad (3.7)$$

If equation (3.3) is true, it follows from Remark 2.2, Lemmas 2.5 and 2.6 that  $B : \ell_g \rightarrow \ell_g$  is continuous. Consequently,  $\{Bf^{n_m}\}$  should be bounded in  $\ell_g$ . This contradicts equation (3.7). Thus, the hypothesis that equation (3.3) does not hold implies that  $B\ell_g \subsetneq \ell_g$ .

We now consider the general case where  $B(n, m) = [b_{r,s}(n, m)]$  ( $r, s = 1, 2, \dots, d$ ) is a matrix kernel. Observe that  $\phi = \{\phi(n)\}_{n \in \mathbb{Z}^+} \in \ell_g(\mathbb{E}^d)$  if and only if each coordinate of  $\phi$  belongs to  $\ell_g(\mathbb{E})$ . This follows easily from the definition of the space  $\ell_g(\mathbb{E}^d)$ . Generally, we can write  $\phi = (\phi_1, \dots, \phi_d)$ , where  $\phi_q \in \ell_g(\mathbb{E})$ . Now take  $\phi \in \ell_g(\mathbb{E}^d)$  such that  $\phi_q \equiv 0$  on  $\mathbb{Z}^+$  for  $q \neq s$  where  $s$  is fixed. Since  $B\phi \in \ell_g(\mathbb{E}^d)$ ,

$$\left\{ \sum_{k=0}^n b_{r,s}(n, k) \phi_s(k) \right\}_{n \in \mathbb{Z}^+} \in \ell_g(\mathbb{E}), \quad (3.8)$$

for any  $r \in \{1, 2, \dots, d\}$ , and the fixed  $s$ . Since  $\phi_s$  is arbitrary in  $\ell_g(\mathbb{E})$  we have, from the discussion above for the scalar case,

$$\left\{ \sum_{k=0}^n |b_{r,s}(n, k)| |g(k)| \right\}_{n \in \mathbb{Z}^+} \in \ell_g(\mathbb{E}), \quad \text{for } r = 1, 2, \dots, d. \quad (3.9)$$

As  $r$  can be chosen arbitrarily from the set  $\{1, 2, \dots, d\}$ , it follows from equation (3.9) that  $\{\sum_{k=0}^n |B(n, k)| |g(k)|\}_{n \in \mathbb{Z}^+} \in \ell_g(\mathbb{E}^d)$ . This completes the proof of Theorem 3.1.  $\square$

From Lemma 2.6 and Theorem 3.1 we obtain the following result.

**COROLLARY 3.2** *The discrete Volterra operator  $B$  defined by equation (2.9) is continuous from  $\ell_g(\mathbb{E}^d) \rightarrow \ell_g(\mathbb{E}^d)$  if and only if the condition (3.1) holds.*

The next result follows from particular choices of  $g = \{g(n)\}_{n \in \mathbb{Z}^+}$  and  $g = \{g(n)\}_{n \in \mathbb{Z}^+}$ .

**COROLLARY 3.3** *Consider the operator  $B$  given by equation (2.9). Then (i) the necessary and sufficient condition for the admissibility of the pair  $(\ell_g, \ell^\infty)$  with respect to the operator  $B$  is  $\sum_{j=0}^n |B(n, j)| |g(j)| \leq M$ ,  $n \in \mathbb{Z}^+$ ; (ii) the necessary and sufficient condition for the admissibility of the pair  $(\ell^\infty, \ell^\infty)$  with respect to the operator  $B$  is  $\sum_{j=0}^n |B(n, j)| \leq M$ ,  $n \in \mathbb{Z}^+$ , where  $M$  is a constant; (iii) the necessary and sufficient condition for the admissibility of the pair  $(\ell^\infty, \ell_g)$  with respect to the operator  $B$  is  $\sum_{j=0}^n |B(n, j)| \leq M |g(n)|$ ,  $n \in \mathbb{Z}^+$ , where  $M$  is a constant.*

### 3.2 Admissibility of the pair $(\ell_g, \ell_c^\infty)$

A convergent solution of equations (2.7) and (2.8) is of interest in applications. This motivates us to consider the problem of admissibility of the pair  $(\ell_g, \ell_c^\infty)$  with respect to the operator  $B$  in equation (2.9).

We make the following hypotheses throughout the remainder of this section.



**HYPOTHESIS H1** The limit  $\lim_{n \rightarrow \infty} B(n, j) = B^*(j)$  exists for each  $j \in \mathbb{Z}^+$ , where  $\{B(n, j)\}$  is the  $d \times d$  matrix sequence defining the operator  $B$  in equation (2.9).

We note that  $\{B^*(n)\}_{n \in \mathbb{Z}^+}$  is a  $d \times d$  matrix sequence.

**THEOREM 3.4** Suppose that Hypothesis H1 holds. Then a necessary and sufficient condition for the admissibility of the pair  $(\ell_g, \ell_c^\infty)$  with respect to the discrete Volterra operator  $B$  in equation (2.9) is that

$$\sum_{j=0}^{\infty} |B^*(j)| |g(j)| < \infty, \quad (3.10)$$

$$\lim_{n \rightarrow \infty} \sum_{j=0}^n |B(n, j)| |g(j)| = \sum_{j=0}^{\infty} |B^*(j)| |g(j)|. \quad (3.11)$$

In addition, if  $(\ell_g, \ell_c^\infty)$  is admissible with respect to  $B$ , then, for any  $\phi \in \ell_g(\mathbb{E}^d)$ ,

$$\lim_{n \rightarrow \infty} (B\phi)(n) = \sum_{j=0}^{\infty} B^*(j) \phi(j). \quad (3.12)$$

*Proof.* Suppose conditions (3.10) and (3.11) hold. It is clear that if equation (3.12) holds for any  $\phi = \{\phi(n)\}_{n \in \mathbb{Z}^+} \in \ell_g(\mathbb{E}^d)$ , then  $(\ell_g, \ell_c^\infty)$  is admissible with respect to  $B$ . Thus, we just need to prove that equation (3.12) holds.

Note that for any  $\phi = \{\phi(n)\}_{n \in \mathbb{Z}^+} \in \ell_g(\mathbb{E}^d)$ , there is a positive constant  $A_\phi$  such that  $|\phi(n)| \leq A_\phi |g(n)|$  for  $n \in \mathbb{Z}^+$ . It follows from equation (3.10) that the right-hand side of equation (3.12) make sense for any  $\phi = \{\phi(n)\}_{n \in \mathbb{Z}^+} \in \ell_g(\mathbb{E}^d)$  and

$$\lim_{n \rightarrow \infty} \sum_{j=0}^n B^*(j) \phi(j) = \sum_{j=0}^{\infty} B^*(j) \phi(j). \quad (3.13)$$

On the other hand, since equation (3.12) can be written as

$$\lim_{n \rightarrow \infty} \sum_{j=0}^n [B(n, j) - B^*(j)] \phi(j) = 0, \quad (3.14)$$

the result (3.12) will follow from

$$\lim_{n \rightarrow \infty} \sum_{j=0}^n |B(n, j) - B^*(j)| |g(j)| = 0, \quad (3.15)$$

and we shall now prove equation (3.15) by using equations (3.10), (3.11) and Hypothesis H1. For any  $\varepsilon > 0$ , it follows from equations (3.10) and (3.11) that there exists an integer  $N > 0$  such that

$$\sum_{j=N+1}^{\infty} |B^*(j)| |g(j)| < \varepsilon \quad (3.16)$$

and

$$\sum_{j=0}^n |B(n,j)||g(j)| < \sum_{j=0}^{\infty} |B^*(j)||g(j)| + \varepsilon \quad (3.17)$$

whenever  $n \geq N$ . For  $n > N$ , we can write

$$\begin{aligned} \sum_{j=0}^n |B(n,j) - B^*(j)||g(j)| &\leq \sum_{j=0}^N |B(n,j) - B^*(j)||g(j)| \\ &+ \sum_{j=N+1}^n |B(n,j)||g(j)| + \sum_{j=N+1}^n |B^*(j)||g(j)|. \end{aligned} \quad (3.18)$$

For the given  $\varepsilon > 0$ , it follows from Hypothesis H1 that there exists an integer  $N_1 > N$  such that

$$-\sum_{j=0}^N |B(n,j)||g(j)| < -\sum_{j=0}^N |B^*(j)||g(j)| + \varepsilon \quad (3.19)$$

and

$$\sum_{j=0}^N |B(n,j) - B^*(j)||g(j)| < \varepsilon \quad (3.20)$$

whenever  $n > N_1$ . From equations (3.16), (3.19) and (3.20), we have

$$\sum_{j=N+1}^n |B^*(j)||g(j)| \leq \sum_{j=N+1}^{\infty} |B^*(j)||g(j)| < \varepsilon \quad (3.21)$$

for  $n > N_1$  and

$$\begin{aligned} \sum_{j=N+1}^n |B(n,j)||g(j)| &\leq \sum_{j=0}^n |B(n,j)||g(j)| - \sum_{j=0}^N |B(n,j)||g(j)| \\ &< \sum_{j=0}^{\infty} |B^*(j)||g(j)| + \varepsilon - \sum_{j=0}^N |B^*(j)||g(j)| + \varepsilon \\ &= \sum_{j=N+1}^{\infty} |B^*(j)||g(j)| + 2\varepsilon < 3\varepsilon. \end{aligned} \quad (3.22)$$

Thus, from equations (3.18), (3.20)–(3.22), we have  $\sum_{j=0}^n |B(n,j) - B^*(j)||g(j)| \leq 5\varepsilon$ ,  $n > N_1$ , which implies that equation (3.15) holds and  $(\ell_g, \ell_c^\infty)$  is admissible with respect to  $B$ .

To prove the necessity of the condition, we first note that  $B\phi \in \ell_c^\infty(\mathbb{E}^d)$  implies  $B\phi \in \ell_c^\infty(\mathbb{E}^d)$ . From Corollary 3.3 one obtains

$$\sum_{j=0}^n |B(n,j)||g(j)| \leq M, \quad n \in \mathbb{Z}^+, \quad (3.23)$$

where  $M$  is a positive constant. Since for any fixed integer  $n \in \mathbb{Z}^+$  and  $i \geq n$ ,

$$\begin{aligned} \sum_{j=0}^n |B^*(j)|g(j)| &= \lim_{i \rightarrow \infty} \sum_{j=0}^n |B(i,j)|g(j)| = \liminf_{i \rightarrow \infty} \left\{ \sum_{j=0}^n |B(m,j)|g(j)| \right\} \\ &\leq \liminf_{i \rightarrow \infty} \left\{ \sum_{j=0}^m |B(m,j)|g(j)| \right\} \leq M, \end{aligned}$$

we have

$$\sum_{j=0}^{\infty} |B^*(j)|g(j)| \leq \liminf_{i \rightarrow \infty} \left\{ \sum_{j=0}^m |B(m,j)|g(j)| \right\} \leq M, \quad (3.24)$$

which implies that equation (3.10) holds. To prove the necessity of equation (3.11), we first consider the special case  $k = d = 1$ . It is clear from equation (3.24) that

$$\begin{aligned} L = \sum_{j=0}^{\infty} |B^*(j)|g(j)| &\leq \liminf_{i \rightarrow \infty} \left\{ \sum_{j=0}^m |B(m,j)|g(j)| \right\} \\ &\leq \limsup_{i \rightarrow \infty} \left\{ \sum_{j=0}^m |B(m,j)|g(j)| \right\} = L' \leq M. \end{aligned} \quad (3.25)$$

Note that equation (3.11) holds if  $L' = L$ . Therefore, it suffices to prove that  $L' > L$  will result in a contradiction. To this end, we note that there exists an integer  $n_0 > 0$  such that

$$\sum_{j=n_0}^n |B^*(j)|g(j)| < M_1 < (L' - L)/3, \quad n \geq n_0. \quad (3.26)$$

From

$$\sum_{j=n_0}^n |B(n,j)|g(j)| = \sum_{j=0}^n |B(n,j)|g(j)| - \sum_{j=0}^{n_0-1} |B(n,j)|g(j)|, \quad (3.27)$$

one obtains

$$\limsup_{i \rightarrow \infty} \left\{ \sum_{j=n_0}^n |B(n,j)|g(j)| \right\} \geq L' - \sum_{j=0}^{n_0} |B^*(j)|g(j)| \geq L' - L > 3M_1. \quad (3.28)$$

Thus, there exists an increasing integer sequence  $\{n_m\}$ , with  $\lim_{m \rightarrow \infty} n_m = \infty$ , such that

$$\sum_{j=n_0}^{n_m} |B(n_m,j)|g(j)| > 3M_1, \quad m \geq 1. \quad (3.29)$$

From equation (3.26) one obtains

$$\sum_{j=n_0}^{n_m} |B^*(j)|g(j)| < M_1, \quad m \geq 1. \quad (3.30)$$

We can assume, without loss of generality, that the sequence  $\{n_m\}$  also satisfies

$$\sum_{j=n_0}^{n_m} |B(n_{m+1},j)|g(j)| < M_1, \quad m \geq 1. \quad (3.31)$$

Indeed, it follows from Hypothesis H1 that

$$\lim_{n \rightarrow \infty} \sum_{j=n_0}^{n_1} |B(n, j)| |g(j)| = \sum_{j=n_0}^{n_1} |B^*(j)| |g(j)| < M_1$$

Hence, there exists a first  $n_q > n_1$  in the sequence  $\{n_m\}$  such that  $\sum_{j=n_0}^{n_1} |B(n_q, j)| |g(j)| < M_1$ . We can omit all the terms of the sequence that lie between  $n_1$  and  $n_q$  and denote  $n_q$  by  $n_2$ . Thus equation (3.31) holds for  $m = 1$ . Starting now from  $\lim_{n \rightarrow \infty} \sum_{j=n_0}^{n_2} |B(n, j)| |g(j)| = \sum_{j=n_0}^{n_2} |B^*(j)| |g(j)| < M_1$ , we can construct  $n_3$  such that  $\sum_{j=n_0}^{n_2} |B(n_3, j)| |g(j)| < M_1$ . Of course,  $n_3$  will also be chosen from amongst the terms of the sequence  $\{n_m\}$ . Therefore, we can assume that the sequence  $\{n_m\}$  satisfies both conditions (3.30) and (3.31). The next step is to construct a sequence  $\phi \in \ell_g(\mathbb{E})$  such that  $B\phi \notin \ell_c(\mathbb{E})$ .

For  $m \geq 1$ , we define such  $\phi_* = \{\phi_*(n)\}_{n \in \mathbb{Z}^+}$  as follows.

$$\phi_*(n) = \begin{cases} 0 & \text{if } 0 \leq n \leq n_0, \\ (-1)^{m-1} \text{sign}\{B(n_m, n)\} |g(n)| & \text{if } n_{m-1} + 1 \leq n \leq n_m. \end{cases} \quad (3.32)$$

if  $\mathbb{E} = \mathbb{R}$ , or if  $\mathbb{E} = \mathbb{C}$

$$\phi_*(n) = \begin{cases} 0 & \text{if } 0 \leq n \leq n_0, \\ 0 & \text{if } n_{m-1} + 1 \leq n \leq n_m \text{ and } |B(n_m, n)| = 0, \\ (-1)^{m-1} |g(n)| \frac{|B(n_m, n)|}{|B(n_m, n)|} & \text{if } n_{m-1} + 1 \leq n \leq n_m \text{ and } |B(n_m, n)| \neq 0. \end{cases} \quad (3.33)$$

Obviously,  $|\phi_*(n)| \leq |g(n)|$ . From equations (3.29) and (3.31), we have

$$\begin{aligned} (-1)^{m-1} \sum_{j=n_{m-1}+1}^{n_m} B(n_m, j) \phi_*(j) &= \sum_{j=n_{m-1}+1}^{n_m} |B(n_m, j)| |g(j)| \\ &= \sum_{j=n_0}^{n_m} |B(n_m, j)| |g(j)| - \sum_{j=n_0}^{n_{m-1}} |B(n_m, j)| |g(j)| \\ &\geq 3M_1 - M_1 = 2M_1, \end{aligned}$$

namely,

$$(-1)^{m-1} \sum_{j=n_{m-1}+1}^{n_m} B(n_m, j) \phi_*(j) = \sum_{j=n_{m-1}+1}^{n_m} |B(n_m, j)| |g(j)| > 2M_1. \quad (3.34)$$

It is clear that  $\phi_*$  satisfies

$$(B\phi_*)(n_m) = \sum_{j=0}^{n_m} B(n_m, j) \phi_*(j) = \sum_{j=n_0}^{n_m} B(n_m, j) \phi_*(j).$$

We show now that  $\lim(B\phi_*)(n_m)$  does not exist as  $m \rightarrow \infty$ . In fact, it follows from equations (3.31), (3.34) and  $|\phi_*(n)| \leq |g(n)|$  that

$$\begin{aligned} (-1)^{m-1} \sum_{j=n_0}^{n_m} B(n_m, j) \phi_*(j) &= (-1)^{m-1} \sum_{j=n_0}^{n_{m-1}} B(n_m, j) \phi_*(j) + (-1)^{m-1} \sum_{j=n_{m-1}+1}^{n_m} B(n_m, j) \phi_*(j) \\ &\geq 2M_1 - \sum_{j=n_0}^{n_{m-1}} |B(n_m, j)| |\phi_*(j)| \geq 2M_1 - \sum_{j=n_0}^{n_{m-1}} |B(n_m, j)| |g(j)| > 2M_1 - M_1 = M_1. \end{aligned}$$

Thus, we obtain the inequalities

$$\sum_{j=n_0}^{n_m} B(n_m, j) \phi_*(j) > M_1 \quad \text{if } m = 2p + 1, \quad \sum_{j=n_0}^{n_m} B(n_m, j) \phi_*(j) < -M_1 \quad \text{if } m = 2p.$$

Therefore,  $\lim(B\phi_*)(n_m)$  does not exist as  $m \rightarrow \infty$ , which implies  $L' > L$  should be rejected. Consequently, equation (3.11) is necessary in the scalar case.

For the general case when  $d > 1$ , we note that a vector sequence  $\phi = \{\phi(n)\}_{n \in \mathbb{Z}^+}$  is in  $\ell_c^\infty(\mathbb{E}^d)$  if and only if each coordinate of  $\phi$  belongs to  $\ell_c^\infty(\mathbb{E})$ . By the same procedure of reduction used in the proof of Theorem 3.1, we deduce that it suffices to prove equation (3.11) holds if  $\lim_{n \rightarrow \infty} \sum_{j=0}^n |b_{r,s}(n, j)| |g(j)| = \sum_{j=0}^\infty |b_{r,s}^*(j)| |g(j)|$  for all  $r, s \in \{1, \dots, d\}$ . This follows immediately if we choose the norm of a matrix as the sum of absolute values of its elements. Thus, Theorem 3.4 is established.  $\square$

**Remark 3.5** Condition (3.10) can be replaced by equation (3.23). In fact, from Hypothesis H1 and equation (3.24) we derive equation (3.10). Conversely, from equations (3.10) and (3.11), we obtain the admissibility of pair  $(\ell_g, \ell_c^\infty)$ , this implies equation (3.23).

If  $c = 0$  in  $\ell_c^\infty$ , we obtain the following result.  $\square$

**COROLLARY 3.6** Suppose that the Hypothesis H1 holds. Then a necessary and sufficient condition that the pair  $(\ell_g, \ell_0^\infty)$  be admissible with respect to the discrete Volterra operator  $B$  given by equation (2.9) is that

$$\left\{ \sum_{j=0}^n |B(n, j)| |g(j)| \right\}_{n \in \mathbb{Z}^+} \in \ell_0^\infty(\mathbb{R}). \quad (3.35)$$

*Proof.* It is clear that condition (3.35) is equivalent to the condition

$$\lim_{n \rightarrow \infty} \sum_{j=0}^n |B(n, j)| |g(j)| = 0. \quad (3.36)$$

The sufficiency of equation (3.35) follows from Theorem 3.4 and the Remark 3.5. In this case, both conditions (3.23) and (3.10) are fulfilled, the latter with  $B^*(n) \equiv 0$ .

For the necessity of equation (3.35), we argue as follows. From  $\phi \in \ell_g$ , we have  $B\phi \in \ell_0^\infty \subset \ell_c^\infty$ . Thus, equations (3.11) and (3.12) are necessary. In this case,  $\lim_{n \rightarrow \infty} (B\phi)(n) = 0$  for any  $\phi \in \ell_g$ , which implies that  $\sum_{n=0}^\infty B^*(n) \phi(n) = 0$ . This leads easily to  $B^*(n) \equiv 0$ , and equation (3.11) reduces to equation (3.36).  $\square$

### 3.3 Admissibility of the pair $(\ell_c^\infty, \ell_c^\infty)$

We note that there does not exist a sequence  $g = \{g(n)\}_{n \in \mathbb{Z}^+}$  such that  $\ell_g = \ell_c^\infty$ . Thus we cannot obtain the corresponding admissibility results for the pair  $(\ell_c^\infty, \ell_c^\infty)$  directly from those proved in the above section. But using the same techniques in the previous sections, we can prove the following result.

**THEOREM 3.7** *Let Hypothesis H1 be satisfied for  $B$  defined by equation (2.9). Then, a necessary and sufficient condition for the pair  $(\ell_c^\infty, \ell_c^\infty)$  to be admissible with respect to  $B$  is that the conditions of Corollary 3.3 (ii) hold, namely,*

$$\sup_{n \in \mathbb{Z}^+} \sum_{j=0}^n |B(n, j)| = M < \infty \quad (3.37)$$

and the limit

$$\lim_{n \rightarrow \infty} \sum_{j=0}^n B(n, j) \quad (3.38)$$

exists. In addition, if the pair  $(\ell_c^\infty, \ell_c^\infty)$  is admissible with respect to  $B$ , then for any  $\phi = \{\phi(n)\}_{n \in \mathbb{Z}^+} \in \ell_c^\infty$  the following condition holds:

$$\lim_{n \rightarrow \infty} \sum_{j=0}^n B(n, j) \phi(j) = -\phi(\infty) \sum_{j=0}^{\infty} B^*(j) + \sum_{j=0}^{\infty} B^*(j) \phi(j) + \phi(\infty) \lim_{n \rightarrow \infty} \sum_{j=0}^n B(n, j). \quad (3.39)$$

*Proof.* For sufficiency, suppose that equations (3.37) and (3.38) hold. We first show

$$\sum_{j=0}^{\infty} |B^*(j)| \leq M < \infty. \quad (3.40)$$

To this end, we define  $B(n, j) \equiv 0$  when  $n < j$ . Since  $B^*(j) = \lim_{n \rightarrow \infty} B(n, j)$ , we have

$$\begin{aligned} \sum_{j=0}^n |B^*(j)| &= \lim_{i \rightarrow \infty} \sum_{j=0}^n |B(i, j)| = \liminf_{i \rightarrow \infty, m \geq i} \left\{ \sum_{j=0}^n |B(m, j)| \right\} \\ &\leq \liminf_{i \rightarrow \infty, m \geq i} \left\{ \sum_{j=0}^m |B(m, j)| \right\} \leq M \end{aligned} \quad (3.41)$$

for any fixed integer  $n \in \mathbb{Z}^+$ . Thus equation (3.40) holds. It remains to prove condition (3.39). Note that equation (3.39) is equivalent to the following condition:

$$\lim_{n \rightarrow \infty} \sum_{j=0}^n [B(n, j) - B^*(j)][\phi(j) - \phi(\infty)] = 0 \quad (3.42)$$

for  $\phi = \{\phi(n)\}_{n \in \mathbb{Z}^+} \in \ell_c^\infty$  and  $\phi(\infty) = \lim_{n \rightarrow \infty} \phi(n)$ .

To prove equation (3.42), let  $\phi = \{\phi(n)\}_{n \in \mathbb{Z}^+} \in \ell_c^\infty$  and denote  $\phi(\infty) = \lim_{n \rightarrow \infty} \phi(n)$ . For any  $\varepsilon > 0$ , there corresponds an integer  $N > 0$  such that if  $n > N$ , then

$$|\phi(n) - \phi(\infty)| \leq \frac{\varepsilon}{2(\|\phi\|_\infty + M)}, \quad \sum_{j=0}^N |B(n, j) - B^*(j)| < \frac{\varepsilon}{2(M + \|\phi\|_\infty)}. \quad (3.43)$$

Thus for any  $n > N$ , it follows from equations (3.37), (3.40) and (3.43) that

$$\begin{aligned} & \left| \sum_{j=0}^n [B(n, j) - B^*(j)][\phi(j) - \phi(\infty)] \right| \\ & \leq \sum_{j=0}^N |B(n, j) - B^*(j)| |\phi(j) - \phi(\infty)| + \sum_{j=N+1}^n |B(n, j) - B^*(j)| |\phi(j) - \phi(\infty)| \\ & \leq 2\|\phi\|_\infty \sum_{j=0}^N |B(n, j) - B^*(j)| + \frac{\varepsilon 2M}{2(\|\phi\|_\infty + M)} < \varepsilon, \end{aligned}$$

which implies equation (3.42) and thus equation (3.39) holds.

We now consider necessity: suppose that  $B\ell_c^\infty \subset \ell_c^\infty$  and we show that equations (3.37) and (3.38) are satisfied. The latter is the simplest to establish: Take as a special case  $\phi(n) = 1$  and  $\phi = \{\phi(n)\}_{n \in \mathbb{Z}^+}$  for  $n \in \mathbb{Z}^+$ ; then  $\phi \in \ell_c^\infty$ , and  $\lim_{n \rightarrow \infty} \sum_{j=0}^n B(n, j)\phi(j)$  exists as  $n \rightarrow \infty$ , since  $B\ell_c^\infty \subset \ell_c^\infty$ . Since  $\sum_{j=0}^n B(n, j)\phi(j) = \sum_{j=0}^n B(n, j)$ , condition (3.38) is satisfied.

To prove equation (3.37), we use techniques similar to those found in the proof of Theorem 3.1. Without loss of generality, we can consider only the scalar case. Since  $B: \ell \rightarrow \ell$  is continuous by Lemma 2.5 and  $B\ell_c^\infty \subset \ell_c^\infty$ , it follows from Lemma 2.6 that  $B: \ell_c^\infty \rightarrow \ell_c^\infty$  is continuous, which implies the following condition holds.

$$\sup_{\|\phi\|_\infty \leq 1} \left\{ \|B\phi\|_\infty = \sup_{n \in \mathbb{Z}^+} \left| \sum_{j=0}^n B(n, j)\phi(j) \right| \right\} = M_2 < \infty, \quad (3.44)$$

where  $\phi = \{\phi(n)\}_{n \in \mathbb{Z}^+} \in \ell_c^\infty$ . Assume condition (3.37) does not hold. For  $M_2 + 1 > 0$ , there exists an integer  $m > 0$  such that  $\sum_{j=0}^m |B(m, j)| > M_2 + 1$ . We define  $\phi_*$  as follows. Let

$$\phi_*(j) = \begin{cases} \text{sign}\{B(m, j)\} & \text{if } 1 \leq j \leq m, \\ 0 & \text{if } j > m \end{cases} \quad (3.45)$$

if  $\mathbb{E} = \mathbb{R}$  and let

$$\phi_*(j) = \begin{cases} 0 & \text{if } 0 \leq j \leq m \text{ and } |B(m, j)| = 0, \\ \overline{B(m, j)} / (|B(m, j)|) & \text{if } 1 \leq j \leq m \text{ and } |B(m, j)| \neq 0, \\ 0 & \text{if } j > m \end{cases} \quad (3.46)$$

if  $\mathbb{E} = \mathbb{C}$ . It is clear that  $\|\phi_*\|_\infty \leq 1$  and  $\sum_{j=0}^m B(m, j)\phi_*(j) = \sum_{j=0}^m |B(m, j)| > M_2 + 1$ . Hence  $\|B\phi_*\|_\infty = \sup_{n \in \mathbb{Z}^+} |\sum_{j=0}^n B(n, j)\phi_*(j)| > M_2 + 1$ , which contradicts equation (3.44). This completes the proof of Theorem 3.7.  $\square$

**Remark 3.8** Note that  $\phi_*$  in equations (3.45) or (3.46) belongs to  $\ell_0^\infty$ . Therefore, we prove, in fact, that if  $B\ell_0^\infty \subset \ell_0^\infty$ , then  $\sup_{n \in \mathbb{Z}^+} \sum_{j=0}^n |B(n, j)| < \infty$ .  $\square$

**COROLLARY 3.9** Let Hypothesis H1 be satisfied for  $B$  defined by equation (2.9). Then, a necessary and sufficient condition for the pair  $(\ell_0^\infty, \ell_0^\infty)$  be admissible with respect to  $B$  is that

the condition of Corollary 3.3 (ii) holds, namely,

$$\sup_{n \in \mathbb{Z}^+} \sum_{j=0}^n |B(n, j)| = M < \infty \quad (3.47)$$

and

$$B^*(n) \equiv 0, \quad n \in \mathbb{Z}^+. \quad (3.48)$$

*Proof.* For sufficiency: suppose that equations (3.47) and (3.48) hold. From equation (3.42) it follows that

$$\lim_{n \rightarrow \infty} \sum_{j=0}^n B(n, j) \phi(j) = 0 \quad \text{for any } \phi = \{\phi(n)\}_{n \in \mathbb{Z}^+} \in \ell_0^\infty. \quad (3.49)$$

For necessity, suppose that  $B\ell_0^\infty \subset \ell_0^\infty$ . It follows from Remark 3.8 that condition (3.47) holds. Thus condition (3.40) is also satisfied. From equation (3.42), we have

$$\lim_{n \rightarrow \infty} \sum_{j=0}^n B(n, j) \phi(j) = \sum_{j=0}^{\infty} B^*(j) \phi(j) = 0 \quad \text{for any } \phi = \{\phi(n)\}_{n \in \mathbb{Z}^+} \in \ell_0^\infty,$$

so that  $B^*(n) \equiv 0$  for  $n \in \mathbb{Z}^+$ . □

#### 4. Solutions of some discrete Volterra equations

In section 3, we discussed several admissibility properties of the discrete Volterra operator  $B$  in different situations; in this section we consider the existence of solutions of the corresponding Volterra summation equation,

$$x(n) = h(n) + \sum_{j=0}^n B(n, j) f(j, x(j)), \quad n \in \mathbb{Z}^+, \quad (4.1)$$

and investigate the properties of solutions using the notation and concepts above.

##### 4.1 linear discrete Volterra equations

We commence with the linear equations

$$x(n) = h(n) + \sum_{j=0}^n B(n, j) x(j), \quad n \in \mathbb{Z}^+, \quad (4.2)$$

where  $h = \{h(n)\}_{n \in \mathbb{Z}^+}$  is a given sequence in  $\mathbb{E}^d$  and  $B : \mathbb{Z}^+ \times \mathbb{Z}^+ \rightarrow \mathbb{E}^{d \times d}$ , and investigate the relation between the properties of solutions of equation (4.2) and the admissibility theory developed in section 3. A necessary and sufficient condition (see [15]) for the existence and unicity of a solution of equation (4.2) in  $\ell(\mathbb{E}^d)$  is that

$$\det(I - B(n, n)) \neq 0 \quad \text{for all } n \in \mathbb{Z}^+ \quad (4.3)$$



and we always assume that equation (4.3) holds in this subsection. The resolvent  $R : \mathbb{Z}^+ \times \mathbb{Z}^+ \rightarrow \mathbb{E}^{d \times d}$  of the kernel  $B$  in equation (4.2) is defined via the solution of

$$R(n, m) = \sum_{j=m}^n R(n, j)B(j, m) - B(n, m), \quad m \leq n \in \mathbb{Z}^+ \quad (4.4a)$$

or

$$R(n, m) = \sum_{j=m}^n B(n, j)R(j, m) - B(n, m), \quad m \leq n \in \mathbb{Z}^+; \quad (4.4b)$$

$$R(n, m) = 0 \quad \text{for } m > n. \quad (4.4c)$$

We note that both equations (4.4a) and (4.4b) are solvable by equation (4.3). It is known (see [15]) that if  $R(n, m)$  satisfies equation (4.4a), then it also satisfies equation (4.4b) and vice versa. For details of the resolvent and the variation of constants formula (4.5), see [15] whose sign conventions we follow here.<sup>†</sup> The solution  $x = \{x(n)\}$  of equation (4.2) can be expressed as

$$x(n) = h(n) - \sum_{j=0}^n R(n, j)h(j), \quad n \in \mathbb{Z}^+. \quad (4.5)$$

The operator  $R$  on  $\ell(\mathbb{E}^d)$  corresponds to the resolvent matrices  $\{R(n, m)\}$  associated with the operator  $B$  (also defined on  $\ell(\mathbb{E}^d)$ ) by the matrices  $\{B(n, m)\}$ :

$$(B\phi)(n) = \sum_{j=0}^n B(n, j)\phi(j), \quad (R\phi)(n) = \sum_{j=0}^n R(n, j)\phi(j) \quad (n \in \mathbb{Z}^+, \phi \in \ell(\mathbb{E}^d)). \quad (4.6)$$

The operators  $B$  and  $R$  on  $\ell(\mathbb{E}^d)$  in equation (4.6) are related through the equations

$$(I - B)R = -B, \quad \text{or} \quad R(I - B) = -B. \quad (4.7)$$

**Remark 4.1** From equation (4.3),  $\{R(n, m)\}$  (given by equations (4.4a) or (4.4b)) satisfies  $\det(I - R(n, n)) \neq 0$  for all  $n \in \mathbb{Z}^+$ . If we consider the linear discrete equations

$$y(n) = h(n) + \sum_{j=0}^n R(n, j)y(j), \quad n \in \mathbb{Z}^+, \quad (4.8)$$

where  $h = \{h(n)\}$ , the solution  $y = \{y(n)\}$  of equation (4.8) is given by

$$y(n) = h(n) - \sum_{j=0}^n B(n, j)h(j), \quad n \in \mathbb{Z}^+. \quad (4.9)$$

The equations (4.2), (4.5), (4.8) and (4.9) can be written as  $x = h + Bx$ ,  $x = h - Rh$ ,  $y = h + Ry$  and  $y = h - Bh$ , respectively.  $\square$

We are now in a position to give our main results in this section.

<sup>†</sup>If we sought to preserve the analogy with  $R(t, s)$  in Remark 1.2, we would make a change of sign in equation (4.5) and the associated equations.

**THEOREM 4.2** *Let  $V$  be one of the spaces:  $\ell_g(\mathbb{E}^d)$ ,  $\ell^\infty(\mathbb{E}^d)$ ,  $\ell_c^\infty(\mathbb{E}^d)$  or  $\ell_0^\infty(\mathbb{E}^d)$ . Then the following statements are equivalent. (i) For each  $h \in V$ , the solution  $x = \{x(n)\}_{n \in \mathbb{Z}^+}$  of equation (4.2) on  $\mathbb{Z}^+$  (equivalently, the solution of  $x = h + Bx$ ) belongs to  $V$ . (ii) The pair  $(V, V)$  is admissible with respect to the operator  $R$  given in equation (4.6).*

*Proof.* We show that (i) implies (ii): for each  $h \in V$ , the solution  $x$  of  $x = h + Bx$  belongs to  $V$  and  $x = h - Rh$  by the variation of constants formula (4.5), which implies that for each  $h \in V$  we have  $Rh = h - x \in V$ . It follows from Definition 2.4 that the pair  $(V, V)$  is admissible with respect to the operator  $R$ .

We show that (ii) implies (i): since  $Rh \in V$  for each  $h \in V$ , we have  $h - Rh \in V$ . Set  $x = h - Rh$ . Then  $x = h - Rh$  is a solution of  $x = h + Bx$  and  $x \in V$ .  $\square$

By the same arguments, we obtain the following results.

**THEOREM 4.3** *Let  $V$  be one of the spaces:  $\ell_g(\mathbb{E}^d)$ ,  $\ell^\infty(\mathbb{E}^d)$ ,  $\ell_c^\infty(\mathbb{E}^d)$  or  $\ell_0^\infty(\mathbb{E}^d)$ . Then the following statements are equivalent. (i) For each  $h \in V$ , the solution  $x = \{x(n)\}_{n \in \mathbb{Z}^+}$  of equation (4.8) (or the equation  $x = h + Rx$ ) on  $\mathbb{Z}^+$  belongs to  $V$ . (ii) The pair  $(V, V)$  is admissible with respect to the operator  $B$  given by equation (4.6).*

**Remark 4.4** We see from Theorem 4.3 that the admissibility of the pair  $(V, V)$  with respect to the operator  $B$  given by equation (4.2) does not necessarily relate to the properties of the solutions of equation (4.2).

Let  $V$  be one of the spaces:  $\ell_g(\mathbb{E}^d)$ ,  $\ell^\infty(\mathbb{E}^d)$ ,  $\ell_c^\infty(\mathbb{E}^d)$  or  $\ell_0^\infty(\mathbb{E}^d)$ . The equivalent conditions for the admissibility of the pair  $(V, V)$  with respect to  $B$  (or  $R$ ) can be found in Theorem 3.1, Corollary 3.2, Theorem 3.7 and Corollary 3.9, respectively.  $\square$

## 4.2 Nonlinear discrete Volterra equations

Let us turn to nonlinear equations (4.1), namely,

$$x(n) = h(n) + \sum_{j=0}^n B(n, j)f(j, x(j)), \quad n \in \mathbb{Z}^+. \quad (4.10)$$

We use the notation  $F : \phi \rightarrow F\phi$  to denote the map defined on  $\ell(\mathbb{E}^d)$  by

$$(F\phi)(n) = f(n, \phi(n)) \quad \text{for } \phi \in \ell(\mathbb{E}^d). \quad (4.11)$$

The operator  $F$  is the discrete Niemytzki operator. We employ the same notation  $F$  when the operator acts on  $\ell_g(\mathbb{E}^d)$ , or  $\ell_c(\mathbb{E}^d)$ . A solution of a summation equation is called an  $\ell_g$  solution if the solution belongs to the space  $\ell_g(\mathbb{E}^d)$ .

**4.2.1 Solutions in  $\ell_g$ .** To ensure that a solution  $x$  of equation (4.10) lies in  $\ell_g(\mathbb{E}^d)$ , we shall ask that  $F$  maps  $\ell_g(\mathbb{E}^d)$  into  $\ell_g(\mathbb{E}^d)$ , i.e.

$$\text{for any } \phi \in \ell_g(\mathbb{E}^d), \quad \{(F\phi)(n)\}_{n \in \mathbb{Z}^+} \in \ell_g(\mathbb{E}^d). \quad (4.12)$$

For equation (4.10), we define the linear discrete Volterra operator  $B$  by the relation

$$(B\phi)(n) = \sum_{j=0}^n B(n,j)\phi(j), \quad n \in \mathbb{Z}^+. \quad (4.13)$$

and we show that if  $B\ell_g \subset \ell$  (that is, the pair  $(\ell_g, \ell_g)$  is admissible with respect to  $B$ ), and  $h \in \ell_g(\mathbb{E}^d)$ , then equation (4.10) has unique solution in  $\ell_g$  under certain conditions on  $f$  (to be stated in what follows).

**THEOREM 4.5** *Suppose that (1)  $h(n) \in \ell_g(\mathbb{E}^d)$ ; (2)  $\{B(n,m)\}$  satisfies*

$$\left\{ \sum_{j=0}^n |B(n,j)||g(j)| \right\}_{n \in \mathbb{Z}^+} \in \ell_g(\mathbb{E}); \quad (4.14)$$

(3) the operator  $F$ , defined in equation (4.11), maps  $\ell_g(\mathbb{E}^d)$  to  $\ell_g(\mathbb{E}^d)$  and satisfies

$$\|F\phi - F\psi\|_{\ell_g} \leq \lambda \|\phi - \psi\|_{\ell_g}. \quad (4.15)$$

Then the equation (4.10) has a unique solution in  $\ell_g(\mathbb{E}^d)$  for sufficiently small  $\lambda$ .

*Proof.* We define an operator  $T$  on  $\ell_g(\mathbb{E}^d)$  as follows.

$$(T\phi)(n) = h(n) + \sum_{j=0}^n B(n,j)f(j, \phi(j)), \quad n \in \mathbb{Z}^+, \quad (4.16)$$

for  $\phi \in \ell_g(\mathbb{E}^d)$ . Note that the condition (4.14) is equivalent to

$$\sum_{j=0}^n |B(n,j)||g(j)| \leq M|g(n)|, \quad n \in \mathbb{Z}^+, \quad (4.17)$$

where  $M > 0$  is a constant. Thus for  $\phi \in \ell_g(\mathbb{E}^d)$ , equations (4.16) and (4.17) yield

$$\frac{|(T\phi)(n)|}{|g(n)|} \leq \frac{|h(n)|}{|g(n)|} + \sum_{j=0}^n \frac{|B(n,j)||g(j)|}{|g(n)|} \frac{|f(j, \phi(j))|}{|g(j)|} \leq \|h\|_{\ell_g} + M\|F\phi\|_{\ell_g},$$

which implies that  $T\phi \in \ell_g(\mathbb{E}^d)$ . In addition, for any  $\phi, \psi \in \ell_g(\mathbb{E}^d)$ , we have

$$(T\phi)(n) - (T\psi)(n) = \sum_{j=0}^n B(n,j)[f(j, \phi(j)) - f(j, \psi(j))] \quad (n \in \mathbb{Z}^+) \quad (4.18)$$

and

$$\begin{aligned}
 |(T\phi)(n) - (T\psi)(n)| &= \sum_{j=0}^n |B(n,j)| |f(j, \phi(j)) - f(j, \psi(j))| \\
 &\leq \sum_{j=0}^n |B(n,j)| |g(j)| \frac{|f(j, \phi(j)) - f(j, \psi(j))|}{|g(j)|} \\
 &\leq \sum_{j=0}^n |B(n,j)| |g(j)| \lambda \|\phi - \psi\|_{\ell_g} \leq M |g(n)| \lambda \|\phi - \psi\|_{\ell_g}.
 \end{aligned}$$

Hence,  $\|T\phi - T\psi\|_{\ell_g} \leq M\lambda \|\phi - \psi\|_{\ell_g}$  and  $T$  is continuous on  $\ell_g(\mathbb{E}^d)$ . If we assume  $\lambda < M^{-1}$ , we conclude that  $T$  is a contraction operator on  $\ell_g(\mathbb{E}^d)$  and the proof of Theorem 4.5 is completed.  $\square$

*Remark 4.6* If we replace the Condition (3) in Theorem 4.5 by the requirement that the mapping  $\phi \rightarrow F\phi$  takes  $\Omega = \{\phi : \phi \in \ell_g(\mathbb{E}^d), \|\phi\|_{\ell_g} \leq \rho\}$  into  $\ell_g(\mathbb{E}^d)$  such that equation (4.15) is satisfied for any  $\phi, \psi \in \Omega$ , then we obtain an existence result in  $\Omega$  provided  $T\Omega \subset \Omega$ . It is readily shown that the inclusion  $T\Omega \subset \Omega$  follows from

$$\|h\|_{\ell_g} + M\|F\theta\|_{\ell_g} \leq \rho(1 - \lambda M), \quad (4.19)$$

where  $\theta$  denotes the null element in  $\ell_g(\mathbb{E}^d)$ .  $\square$

The first corollary of Theorem 4.5 concerns a boundedness result for equation (4.10).

**COROLLARY 4.7** Assume that (1)  $h(n) \in \ell^\infty(\mathbb{E}^d)$

(2)  $\{B(n,m)\}$  satisfies

$$\sum_{j=0}^n |B(n,j)| \leq M \quad n \in \mathbb{Z}^+; \quad (4.20)$$

(3) the mapping  $f : \mathbb{Z}^+ \times \mathbb{E}^d \rightarrow \mathbb{E}^d$  is continuous in the second variable and satisfies

$$|f(n, u) - f(n, u')| \leq \lambda |u - u'| \quad \text{for any } n \in \mathbb{Z}^+ \text{ and all } u, u' \in \mathbb{E}^d. \quad (4.21)$$

Then equation (4.10) has a unique solution in  $\ell^\infty(\mathbb{E}^d)$  for sufficiently small  $\lambda$ .

*Proof.* The proof follows easily from that of Theorem 4.5 if one notes the following circumstances: first, condition (4.20) is a special case of equation (4.17) with  $g(n) = (n) = 1$  for all  $n \in \mathbb{Z}^+$ . Second, condition (3) of Corollary 4.7 implies that the operator  $F$  maps  $\ell^\infty(\mathbb{E}^d)$  to  $\ell^\infty(\mathbb{E}^d)$  and satisfies  $\|F\phi - F\psi\|_{\ell^\infty} \leq \lambda \|\phi - \psi\|_{\ell^\infty}$ .  $\square$

The next corollary relates to the existence of an exponential solution of equation (4.10).

**COROLLARY 4.8** Suppose that the data in equation (4.10) satisfy the following conditions:  
 (1) for  $h : \mathbb{Z}^+ \rightarrow \mathbb{E}^d$ , there exist positive numbers  $h_0$  and  $v \in (0, 1)$ , such that

$$|h(n)| \leq h_0 v^n, \quad n \in \mathbb{Z}^+, \quad (4.22)$$

(2) there exists a positive number  $b_0$  such that, when  $v$  is the value in equation (4.22),

$$|B(n, j)| \leq b_0 v^{n-j}, \quad 0 \leq j \leq n < \infty; \quad (4.23)$$

(3) the mapping  $f : \mathbb{Z}^+ \times \mathbb{E}^d \rightarrow \mathbb{E}^d$  satisfies

$$|f(n, u) - f(n, u')| \leq \lambda \alpha^n |u - u'| \text{ for any } n \in \mathbb{Z}^+ \text{ and all } u, u' \in \mathbb{E}^d \quad (4.24)$$

and  $f(n, 0) \equiv 0$  for  $n \in \mathbb{Z}^+$ , where  $0 < \alpha < v$  is a given constant. Then, provided  $\lambda$  is sufficiently small, equation (4.10) has a unique solution  $x \in \ell^\infty(\mathbb{E}^d)$  such that

$$|x(n)| \leq M_1 v^n, \quad n \in \mathbb{Z}^+ \text{ for some constant } M_1 > 0. \quad (4.25)$$

*Proof.* Condition (4.22) implies  $h \in \ell_g(\mathbb{E}^d)$  with  $g(n) = v^n$  for  $n \in \mathbb{Z}^+$ . We define an operator  $T$  on  $\ell_g(\mathbb{E}^d)$  by setting

$$(T\phi)(n) = h(n) + \sum_{j=0}^n B(n, j) f(j, \phi(j)), \quad n \in \mathbb{Z}^+, \quad (4.26)$$

for  $\phi \in \ell^\infty(\mathbb{E}^d)$ . We conclude that  $T$  maps  $\ell^\infty(\mathbb{E}^d)$  to  $\ell_g(\mathbb{E}^d)$  and  $T$  is also contractive. In fact, if  $\phi, \psi \in \ell^\infty(\mathbb{E}^d)$ , it follows from condition (3) that

$$\begin{aligned} \frac{|(T\phi)(n)|}{v^n} &\leq \frac{|h(n)|}{v^n} + \sum_{j=0}^n \frac{|B(n, j)|}{v^n} |f(j, \phi(j))| \\ &\leq h_0 + b_0 \sum_{j=0}^n v^{n-j} v^{-n} \lambda \alpha^j |\phi(j)| \leq h_0 + b_0 \frac{\lambda v}{v - \alpha} \|\phi\|_\infty < \infty, \text{ for } n \in \mathbb{Z}^+, \end{aligned}$$

and

$$\begin{aligned} |(T\phi)(n) - (T\psi)(n)| &= \sum_{j=0}^n |B(n, j)| |f(j, \phi(j)) - f(j, \psi(j))| \\ &\leq \sum_{j=0}^n b_0 v^{n-j} \lambda \alpha^j |\phi(j) - \psi(j)| \leq b_0 v^n \frac{\lambda v}{v - \alpha} \|\phi - \psi\|_\infty. \end{aligned}$$

Thus  $T\ell^\infty(\mathbb{E}^d) \subset \ell_g(\mathbb{E}^d)$  and  $T$  is a contraction operator if  $\lambda < (v - \alpha/b_0v)$ . Finally,  $T$  has a unique fixed point  $\phi \in \ell^\infty(\mathbb{E}^d)$  such that  $T\phi \in \ell_g(\mathbb{E}^d)$ . This completes the proof.  $\square$

**Remark 4.9** If, for  $g(n) = v^n$  ( $0 < v < 1$ ), the pair  $(\ell_g, \ell_g)$  is admissible with respect to the operator  $B$  in equation (4.13), then condition (4.23) holds. Furthermore, if (with this  $g$ ) we replace the condition (4.23) by the requirement that  $(\ell_g, \ell_g)$  is admissible with respect to  $B$ , the conclusion of Corollary 4.8 holds for  $0 < \alpha \leq v$  in equation (4.24).

### 4.3 Convergent solutions

We again consider the discrete Volterra equation (4.10), namely,

$$x(n) = h(n) + \sum_{j=0}^n B(n, j)f(j, x(j)), \quad n \in \mathbb{Z}^+. \quad (4.27)$$

A solution of a summation or difference equation is called a convergent solution, if the solution belongs to the space  $\ell_c$ . To study convergent solutions of equation (4.27), we assume that  $B(n, j)$  satisfies Hypothesis H1, i.e.

$$\lim_{n \rightarrow \infty} B(n, j) = B^*(j), \quad j \in \mathbb{Z}^+. \quad (4.28)$$

**THEOREM 4.10** Suppose that (i)  $\{h(n)\}_{n \in \mathbb{Z}^+} \in \ell_c(\mathbb{E}^d)$ ; (ii) the pair  $(\ell_g(\mathbb{E}^d), \ell_c(\mathbb{E}^d))$  is admissible with respect to the operator

$$(B\phi)(n) = \sum_{j=0}^n B(n, j)\phi(j), \quad n \in \mathbb{Z}^+, \quad (4.29)$$

namely, conditions (3.10) and (3.11) hold; the operator  $F$  maps  $\ell^\infty(\mathbb{E}^d)$  into  $\ell_g(\mathbb{E}^d)$  and satisfies

$$\|F\phi - F\psi\|_{\ell_g} \leq \lambda \|\phi - \psi\|_\infty \text{ for any } \phi, \psi \in \ell_c. \quad (4.30)$$

Then equation (4.27) has a unique solution  $\{x(n)\}_{n \in \mathbb{Z}^+} \in \ell_c(\mathbb{E}^d)$  for sufficiently small  $\lambda$ .

*Proof.* By the assumptions of Theorem 4.10, one can readily prove that the operator

$$(T\phi)(n) = h(n) + \sum_{j=0}^n B(n, j)f(j, \phi(j)) \quad (n \in \mathbb{Z}^+) \quad (4.31)$$

carries the space  $\ell_c(\mathbb{E}^d)$  into itself. In fact, one can show the sequence  $\{(T\phi)(n)\}_{n \geq 0}$  is a Cauchy sequence in  $\ell^\infty(\mathbb{E}^d)$ . It remains to show that  $T$  is a contraction in the metric of  $\ell^\infty(\mathbb{E}^d)$ .

From Theorem 3.4, it is readily shown that the operator  $B$  given by equation (4.29) is continuous from  $\ell_g(\mathbb{E}^d)$  into  $\ell_c(\mathbb{E}^d)$ . Thus there exists a positive number  $M > 0$  such that  $\|B\phi\|_\infty \leq M\|\phi\|_{\ell_g}$  for any  $\phi \in \ell_g(\mathbb{E}^d)$ . Consequently, for any  $\phi, \psi \in \ell_c$ , we have

$$\|T\phi - T\psi\|_\infty \leq \lambda M \|\phi - \psi\|_\infty,$$

which implies that  $T$  is a contraction if  $\lambda < M^{-1}$ . The proof is completed.  $\square$

**Remark 4.11** By the last statement of Theorem 3.4, it is readily shown that for the unique solution  $\{x(n)\}_{n \in \mathbb{Z}^+}$  of equation (4.27), the value  $x(\infty) = \lim_{n \rightarrow \infty} x(n)$  satisfies  $x(\infty) = h(\infty) + \sum_{j=0}^{\infty} B^*(j)f(j, x(j))$ .  $\square$

**Remark 4.12** As can be shown by attention to the manipulative detail, the preceding results can be extended to the framework indicated in Remark 1.1. However, in what now follows we assume the framework in subsection 1.1.  $\square$

If, in Theorem 4.5,  $F$  is chosen to correspond to  $f(n, \phi(n)) = \phi(n)$  in equation (4.11), then the mapping  $\varphi \rightarrow F\varphi$  becomes the identity mapping. In this case, the crucial condition (3) in Theorem 4.5 is not in general satisfied (equation (4.15) cannot be satisfied with  $\lambda < 1$ ). Hence, Theorem 4.5 cannot be applied to an arbitrary linear equation  $x(n) = h(n) + \sum_{j=0}^n B(n, j)x(j)$ . Similar remarks also apply to all other results in subsection 4.2. However, we have something of a remedy in the following analysis.

**LEMMA 4.13** *Suppose the operator  $F$  in equation (4.11) maps the space  $\ell(\mathbb{E}^d)$  to itself, and suppose that equation (4.3) is satisfied. Then  $x$  is a solution of equation (4.10) if and only if*

$$x(n) = \hat{h}(n) - \sum_{j=0}^n R(n, j)\hat{f}(j, x(j)), \quad n \in \mathbb{Z}^+, \quad (4.32)$$

where, for  $n \in \mathbb{Z}^+$ ,

$$\hat{h}(n) = h(n) - \sum_{j=0}^n R(n, j)h(j) \quad \text{and} \quad \hat{f}(n, x(n)) := f(n, x(n)) - x(n). \quad (4.33)$$

*Proof.* Equation (4.27) can be expressed as  $x = h + BFx$ . Hence,  $x = h + B(F - I)x + Bx$  where  $I$  is the identity map. Since  $R$  exists by virtue of equation (4.3), we deduce that  $x = h + B(F - I)x - R(h + B(F - I)x)$  and hence  $x = (h - Rh) + (B - RB)(F - I)x$  and the result follows because  $B - RB = -R$  and  $\hat{f}(\cdot, \cdot)$  defines the discrete Niemytzki operator  $\hat{F} := F - I$ . We have  $x = \hat{h} - R\hat{F}x$  and all the steps are reversible.  $\square$

In view of the preceding results, we can state analogous results valid under analogous conditions that correspond to the results stated earlier. The analogy is preserved if we introduce  $\hat{B}$ , and regard it as an alias for  $-R$ ; we also invoke the operator  $\hat{F}$  where

$$(\hat{F}\phi)(n) = \hat{f}(n, \phi(n)) \equiv f(n, \phi(n)) - \phi(n) \quad \text{for} \quad \phi \in \ell(\mathbb{E}^d). \quad (4.34)$$

We shall need conditions on  $\hat{h}$ ,  $\hat{f}$  (or  $\hat{F}$ ) and on  $\hat{B}$  (i.e., on  $R$ ), instead of the previous conditions on  $h$ ,  $f$  (or  $F$ ) and  $B$ . At the same time, we can replace  $g$  by  $\hat{g}$ , and replace  $g$  by  $\hat{g}$  in order to obtain the analogous results. If we substitute  $R$  for  $-\hat{B}$ ; the notation  $\hat{x}$  is superfluous as  $\hat{x}$  is  $x$  by Lemma 4.13. Thus, the analogue of Theorem 4.5 now reads:

**THEOREM 4.14** *Suppose that (1)  $\hat{h}(n) \in \ell_{\hat{g}}(\mathbb{E}^d)$ ; (2) the resolvent  $\{R(n, m)\}$  satisfies  $\{\sum_{j=0}^n |R(n, j)|\hat{g}(j)\}_{n \in \mathbb{Z}^+} \in \ell_{\hat{g}}(\mathbb{E})$ ; (3) the operator  $\hat{F}$  maps  $\ell_{\hat{g}}(\mathbb{E}^d)$  to  $\ell_{\hat{g}}(\mathbb{E}^d)$  and satisfies  $\|\hat{F}\phi - \hat{F}\psi\|_{\ell_{\hat{g}}} \leq \hat{\lambda}\|\phi - \psi\|_{\ell_{\hat{g}}}$ . Then equation (4.10) has a unique solution in  $\ell_{\hat{g}}(\mathbb{E}^d)$  for sufficiently small  $\hat{\lambda}$ .*

We can likewise obtain analogues of Corollary 4.8 and Theorem 4.10. The latter requires us to make use of an analogue of Hypothesis H1, and (since  $\hat{B} = -R$ ,  $\hat{B}^* = -R^*$ ) we then request the following Hypothesis.

**HYPOTHESIS H2** *If  $R = \{R(n, j)\}$  is the  $d \times d$  matrix sequence in equation (4.4), the limit  $\lim_{n \rightarrow \infty} R(n, j) = R_*(j)$  exists for each  $j \in \mathbb{Z}^+$ .*

**Remark 4.15** The introduction of  $\hat{g}$  and  $\hat{g}$  serves to emphasize that the new requirements on  $\hat{h}, \hat{f}$  (or  $\hat{F}$ ) and on  $R$  do not in general follow from the previous assumptions on  $h, f$  (or  $F$ ) and  $B$ . Those assumption correspond to the selection of  $g$  to be  $\hat{g}$  and of  $g$  to be  $\hat{g}$ . We then suppose that  $F$  maps  $\ell_g(\mathbb{E}^d)$  to  $\ell_g(\mathbb{E}^d)$ , but cannot automatically conclude that  $\hat{F}$  maps  $\ell_g(\mathbb{E}^d)$  to  $\ell_g(\mathbb{E}^d)$ . ( $\hat{F} = F - I$  and  $I$  maps  $\ell_g(\mathbb{E}^d)$  to  $\ell_g(\mathbb{E}^d)$ .) If  $F$  (or  $f$ ) satisfies conditions (4.15), (4.24), or (4.30), we are unable automatically to conclude that  $\hat{F} = F - I$  satisfies the analogous condition where  $\hat{g}$  is  $g$  and  $\hat{g}$  is  $g$ .  $\square$

The difficulties commented on in Remark 4.15 can be eliminated if we are prepared to limit the choice of underlying spaces. We state without proof the following lemma.

**LEMMA 4.16** *Let  $V$  be one of the spaces  $\ell_g(\mathbb{E}^d)$ ,  $\ell^\infty(\mathbb{E}^d)$ ,  $\ell_c^\infty(\mathbb{E}^d)$  or  $\ell_0^\infty(\mathbb{E}^d)$ . If  $h \in V$  and the pair  $(V, V)$  is admissible with respect to the operator  $R$  in equation (4.6), then  $\hat{h}$ , given by equation (4.33), belongs to  $V$ .*

**Remark 4.17** Observe that Theorem 4.2 provides a condition that ensures that the pair  $(V, V)$  (as above) is admissible with respect to the operator  $R$ .  $\square$

We may like to have conditions expressed in terms of  $h, f$ , and  $B$ . Thus, in place of Hypothesis H2 (for  $R$ ), we may be able to adopt Hypothesis H1 (for  $B$ ) and supplement it with additional conditions. To illustrate this, it has been shown (see [18]) that the following result holds.

**LEMMA 4.18** *If  $\sup_{n \in \mathbb{Z}^+} \sum_{j=0}^n |B(n, j)| < 1$ , then  $\sup_{n \in \mathbb{Z}^+} \sum_{j=0}^n |R(n, j)| < \infty$ . If, in addition,  $\lim_{n \rightarrow \infty} B(n, j) = 0$  for each  $j \geq 0$ , then  $\lim_{n \rightarrow \infty} R(n, j) = 0$  for each  $j \geq 0$ . With the stated conditions, Hypothesis H2 is satisfied.*

Our final two theorems illustrate the type of results that arise from the above analysis. In these results,  $\hat{F}$  is again defined by equation (4.34).

**THEOREM 4.19** *Suppose that (1)  $h(n) \in \ell_g(\mathbb{E}^d)$ ; (2) the resolvent  $\{R(n, m)\}$  satisfies  $\{\sum_{j=0}^n |R(n, j)| |g(j)|\}_{n \in \mathbb{Z}^+} \in \ell_g(\mathbb{E})$ ; (3)  $\hat{F}$  maps  $\ell_g(\mathbb{E}^d)$  to  $\ell_g(\mathbb{E}^d)$  and satisfies  $\|\hat{F}\phi - \hat{F}\psi\|_{\ell_g} \leq \hat{\lambda} \|\phi - \psi\|_{\ell_g}$ . Then equation (4.10) has a unique solution in  $\ell_g(\mathbb{E}^d)$  for sufficiently small  $\hat{\lambda}$ .*

Condition (2) in Theorem 4.19 (which we discuss further, below) is the condition

$$\sup_{n \geq 0} \frac{1}{|g(n)|} \sum_{j=0}^n |R(n, j)| |g(j)| < \infty. \quad (4.35)$$

**THEOREM 4.20** *Suppose that (1)  $g(n) = v^n$  ( $0 < v < 1$ ) for all  $n \geq 0$ ; (2)  $h(n) \in \ell_g(\mathbb{E}^d)$ ; (3) the pair  $(\ell_g, \ell_g)$  is admissible with respect to the resolvent operator  $R$  defined by equation (4.6); (4)  $\hat{F}$ , in equation (4.34) satisfies  $|(\hat{F}\phi)(n) - (\hat{F}\psi)(n)| \leq \hat{\lambda} \hat{\alpha}^n |\phi(n) - \psi(n)|$  for any  $n \in \mathbb{Z}^+$ ,  $\phi, \psi \in \ell_g(\mathbb{E}^d)$  and  $f(n, 0) \equiv 0$  for  $n \in \mathbb{Z}^+$ , where  $\hat{\alpha} \in (0, v]$  is a given constant. Then equation (4.10) has a unique solution in  $\ell_g(\mathbb{E}^d)$  for sufficiently small  $\hat{\lambda}$ .*



In Theorems 4.19 and 4.20, results for the nonlinear case collapse to the linear case (4.2) on setting  $f(n, x(n)) = x(n)$  (equivalently,  $\hat{f}(n, x(n)) = 0$ ,  $\hat{\lambda} = 0$ ).

Let us return to condition (2) in Theorem 4.19 and, for simplicity, take  $d = 1$ ,  $\mathbb{E} = \mathbb{R}$ . We find a condition in terms of  $\{B(n, j)\}$  that guarantees equation (4.35).

We have  $\sum_{m=0}^n |R(n, m)| |g(m)| = \sum_{m=0}^n R(n, m) \sigma_{n,m} |g(m)|$ , with  $\sigma_{n,m} = \text{sign}\{R(n, m)\}$ . Set  $\gamma(n) = \sum_{m=0}^n |R(n, m)| |g(m)|$  (so  $\gamma(n) \geq 0$ ). We can show, using equation (4.4b), that

$$\gamma(n) = \sum_{m=0}^n \sum_{k=0}^n \frac{\sigma_{n,m}}{\sigma_{k,m}} B(n, k) |R(k, m)| |g(m)| - \sum_{m=0}^n \sigma_{n,m} B(n, m) |g(m)| \quad (4.37)$$

(since  $R(k, m) = 0$  if  $k < m$ ) and we obtain the inequality  $|1 - B(n, n)| \gamma(n) \leq \sum_{k=0}^{n-1} |B(n, k)| \gamma(k) + \sum_{m=0}^n |B(n, m)| |g(m)|$ . We deduce (see Remark 4.21) that  $\gamma(n) \leq \gamma^*(n)$  where, with

$$L(n, j) = |B(n, j)| / |1 - B(n, n)| \quad \text{for } j \leq n, \quad L(n, j) = 0 \quad \text{for } j > n, \quad (4.38)$$

the sequence  $\{\gamma^*(n)\}_{n \in \mathbb{Z}^+}$  satisfies

$$\gamma^*(n) = \sum_{j=0}^{n-1} L(n, j) \gamma^*(j) + \nu(n) \quad \text{where } \nu(n) = \sum_{j=0}^n L(n, j) |g(j)|. \quad (4.39)$$

By the linear theory,  $\gamma^*(n) = \sum_{j=0}^{n-1} M(n, j) \nu(j) + \nu(n)$  for an appropriate sequence  $\{M(n, m)\}$  (where  $M(m, n) \geq 0$ ); see below. Satisfaction of condition (4.35) is assured when  $\gamma^* \in \ell_g(\mathbb{E})$ . This, in turn, follows if  $\nu \in \ell_g(\mathbb{R})$  and  $M\nu \in \ell_g(\mathbb{R})$ , which is guaranteed *a fortiori* when  $(\ell_g(\mathbb{R}), \ell_g(\mathbb{R}))$  is admissible with respect to the operator  $L$  defined by the sequence  $\{L(n, m)\}$  given, in terms of  $\{B(n, m)\}$ , by equation (4.38).

**Remark 4.21** We give further detail: denote by  $L$  the matrix of order  $n$  with  $(r, s)$ -th entry  $L_{r,s} = L(r-1, s-1)$  for  $s < r$ ,  $L_{r,s} = 0$  for  $r \geq s$  ( $r, s \in \{1, 2, \dots, n\}$ ). Denote by  $I + M$  the matrix  $[I - L]^{-1}$ , then  $M = L + L^2 + \dots + L^{n-1}$  and (like  $L$ )  $M$  is strictly lower triangular with non-negative entries. When  $[I - L]\gamma = \nu$ ,  $[I - L]\gamma^* = \nu^*$  and  $\nu \leq \nu^*$  (componentwise) it follows that  $\gamma \leq \gamma^*$  (componentwise), so  $\gamma(n) \leq \gamma^*(n)$ . Further,  $M(m, n) = M_{m+1, n+1}$  for  $n < m$ ,  $M(m, n) = 0$  for  $n \geq m$  ( $m, n \in \mathbb{Z}^+$ ).

## 5. Further reading

For a considerable amount of material on recurrence equations, refer to Agarwal [1] and Elaydi [11]. We comment on a selection of the further reading available, as an indication of approaches alternative to our own. The purpose of Cushing, in [8], was to prove an abstract operator version of certain existence theorems for differential and Volterra integral equations which deal with stability properties of solutions. The results unified and generalized many of the basic theorems concerning stability of such equations. Dannan, Elaydi, and Li [9] gave a theory for Volterra difference equations that parallels stability theory of Volterra differential and integro-differential equations. The topics include the  $z$ -transform method, the resolvent matrix (see also [23]) and variation of constants formulae. Kwapisz [16] analyzed the explicit equations  $x(n) = \sum_{i=0}^{n-1} g(n, i, x(i)) + h(n)$  and used weighted norms to find sufficient conditions for all solutions of such equations to be elements of an  $\ell^p$  space. Some

generalizations of  $\ell^p$  spaces were also considered, and the corresponding sufficient conditions established. We also draw further attention to [12,14,15]. Some special cases discussed in [15] provide stimulus for developing the current work further.

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## References

- [1] Agarwal, R.P., 2000, *Difference Equations and Inequalities, Theory, methods, and applications*, 2nd ed. (New York: Marcel Dekker, Inc.).
- [2] Baker, C.T.H., 2000, A perspective on the numerical treatment of Volterra equations. *Journal of Computational and Applied Mathematics*, **125**, 217–249.
- [3] Baker, C.T.H. and Song, Y., 2003, Discrete Volterra operators, fixed point theorems & their application. *Nonlinear Studies*, **10**, 79–101.
- [4] Baker, C.T.H. and Song, Y., 2004, Periodic solutions of discrete Volterra equations. *Mathematics and Computer in Simulation*, **64**, 521–542.
- [5] Corduneanu, C., 1965, Problèmes globaux dans la théorie des équations intégrales de Volterra. *Annali di Matematica Pura ed Applicata*, **62**, 349–363.
- [6] Corduneanu, C., 1973, *Integral Equations and Stability of Feedback Systems* (New York: Academic Press).
- [7] Corduneanu, C., 2002, *Functional Equations with Causal Operators* (London: Taylor & Francis).
- [8] Cushing, J.M., 1976, Admissible operators and solutions of perturbed operator equations. *Funkcialaj Ekvacioj*, **19**, 79–84.
- [9] Dannan, F., Elaydi, S. and Li, P., 2003, Stability theory of Volterra difference equations. *Advances in Stability Theory at the End of the 20th Century, Stability Control Theory Methods Appl.*, 13 (London: Taylor & Francis), pp. 89–106.
- [10] Desor, C.A. and Vidyasager, M., 1975, *Feedback Systems: Input–Output Properties* (New York: Academic Press).
- [11] Elaydi, S., 2005, *An Introduction to Difference Equations*, 2nd ed. (New York: 3rd ed, in press, 2005 Springer-Verlag).
- [12] Ford, N.J. and Baker, C.T.H., 1996, Qualitative behaviour and stability of solutions of discretised nonlinear Volterra integral equations of convolution type. *Journal of Computational and Applied Mathematics*, **66**, 213–225.
- [13] Friedman, A., 1982, *Foundations of Modern Analysis* (New York: Dover).
- [14] Furumochi, T., Murakami, S. and Nagabuchi, Y., 2004, Stabilities in Volterra difference equations on a Banach space. *Difference and Differential Equations*, Fields Institute Communications, **42**, American Math. Soc., Providence, 159–175.
- [15] Kolmanovskii, V.B., Myshkis, A.D. and Richard, J.-P., 2000, Estimate of solutions for some Volterra difference equations, Lakshmikantham's legacy: a tribute on his 75th birthday. *Nonlinear Analysis TMA*, **40**, 345–363.
- [16] Kwapisz, M., 1992, On  $\ell_p$  solutions of discrete Volterra equations. *Aequationes Mathematicae*, **43**, 191–197.
- [17] Massera, J.L. and Schäffer, J.J., 1966, *Linear Differential Equations and Function Spaces* (New York: Academic).
- [18] Song, Y., 2002, Discrete Volterra equations. PhD Thesis, University of Manchester.
- [19] Song, Y., 2006, Almost periodic solutions of discrete Volterra equations. *Journal of Mathematical Analysis and Applications*, **314**, 174–194 (in press).
- [20] Song, Y. and Baker, C.T.H., 2003, Perturbation theory for discrete Volterra equations. *Journal of Difference Equations and Applications*, **10**, 969–987.
- [21] Song, Y. and Baker, C.T.H., 2004, Linearized stability analysis of discrete Volterra equations. *Journal of Mathematical Analysis and Applications*, **294**, 310–333.
- [22] Song, Y. and Baker, C.T.H., 2004, Perturbation of Volterra difference equations. *Journal of Difference Equations and Applications*, **4**, 379–397.
- [23] Vecchio, A., 2001, Volterra discrete equations: summability of the fundamental matrix. *Numerical Mathematics*, **89**, 783–794.