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# Periodic solutions of non-linear discrete Volterra equations with finite memory

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ABSTRACT

In this paper we discuss the existence of periodic solutions of *discrete* (and *discretized*) non-linear Volterra equations with finite memory. The literature contains a number of results on periodic solutions of non-linear Volterra *integral equations* with finite memory, of a type that arises in biomathematics. The "summation" equations studied here can arise as discrete models in their own right but are (as we demonstrate) of a type that arise from the discretization of such integral equations. Our main results are in two parts: (i) results for discrete equations and (ii) consequences for quadrature methods applied to integral equations. The first set of results are obtained using a variety of fixed-point theorems. The second set of results address the preservation of properties of integral equations on discretizing them. The effect of weak singularities is addressed in a final section. The detail that is presented, which is supplemented using appendices, reflects the differing prerequisites of functional analysis and numerical analysis that contribute to the outcomes.



## 1. Introduction

The equation

$$\mathrm{x}(t) = \int_{t-\tau}^{t} \mathrm{k}(t,s)\mathrm{f}(s,\mathrm{x}(s))\mathrm{d}s, \quad \text{for } t \in \mathbb{R} \text{ with } \tau > 0,\ \mathrm{x}(t) \in \mathbb{R}, \tag{1.1}$$

is an integral equation of Volterra type, studied [1] as a model of certain epidemic problems. We study possible periodic solutions of an analogous *discrete* system

$$x(n) = \sum_{j=n-N}^{n} k(n,j) f(j, x(j)), \quad N \in \mathbb{N},\ x(n) \in \mathbb{R}, \tag{1.2}$$

given $N$. We expect to satisfy (1.2) for $n \in \mathbb{Z}$, under certain conditions on $\{k(\cdot,\cdot)\}$ and $\{f(\cdot,\cdot)\}$ (see below). Here, $\mathbb{R} = (-\infty,\infty)$, $\mathbb{R}_+ = [0,\infty)$, $\mathbb{N}$ denotes the natural numbers (positive integers), $\mathbb{Z}$ denotes the integers; $\mathbb{Z}_+$ will denote the set of non-negative integers, $\mathbb{Q}_+$ the positive rationals (*quotients* $m/n$, with $m, n \in \mathbb{N}$).

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**Remark 1.1.** If $\mathsf{k}(t,s)\equiv 1$, then (1.1) reduces to the integral equation

$$\mathsf{x}(t)=\int_{t-\tau}^{t}\mathsf{f}(s,\mathsf{x}(s))\mathrm{d}s. \tag{1.3}$$

Examples in the literature also include those of the form $\mathsf{x}(t)=\int_{t-\tau}^{t}\mathsf{k}(t,s)\mathsf{g}(s)\{\mathsf{x}(s)\}^{\gamma}\mathrm{d}s$ and, more generally,

$$\mathsf{x}(t)=\int_{t-\tau}^{t}\mathsf{k}(t,s)\mathsf{g}(s)[\{\mathsf{x}(s)\}^{\alpha}+\{\mathsf{x}(s)\}^{\beta}]\mathrm{d}s \tag{1.4}$$

subject to certain conditions [2]. Eq. (1.3) was discussed, in [1], as a model for the spread of certain infectious diseases with periodic contact rate that varies seasonally ($\tau>0$ is the length of time an individual remains infectious), and it is assumed that there exists $\varpi>0$ with $\mathsf{f}(t+\varpi,u)=\mathsf{f}(t,u)$ for all $t\in\mathbb{R}$ and $u\in\mathbb{R}_+$ (see [1]). Using Krasnosel'skiĭ's fixed-point theorem in a cone [3, p. 137], Cooke and Kaplan [1] established the existence, if $\tau$ is sufficiently large, of a nontrivial periodic non-negative solution to (1.3) with period $\varpi$. Later, Leggett and Williams [4] generalized the results in [1]. We shall give some results for discrete versions of (1.4) (with or without $\beta=0$) in the present paper.

### *1.1. On particular solutions*

Given the terminology "integral equation" for (1.1), it seems appropriate to term (1.2) a "summation equation" (of Volterra type). Eq. (1.2) has also been called a recurrence relation; some authors call it a difference equation. Solution of (1.2) can be seen as a ('constructive') equation-solving issue. We examine the problem from that perspective in this subsection, but we subsequently adopt an existentialist viewpoint.

Suppose we are given $\varphi$ and

$$x(n):=\varphi(n)\quad \text{for } n\in\{n_0-N,\ldots,n_0-1\}. \tag{1.5}$$

If we suppose (1.2) holds for $n\geq n_0$ $(n\in\mathbb{N})$ then we seek a *forward solution* defined by (1.5) as a sequence $\{x(n)\}_{n\geq n_0}$ satisfying (1.5) and

$$x(n)-k(n,n)f(n,x(n))=\sum_{j=n-N}^{n-1}k(n,j)f(j,x(j)),\quad n\geq n_0. \tag{1.6a}$$

If, in addition to the above, (1.2) holds for $n\leq n_0-N-1$ $(n\in\mathbb{N})$ then Eq. (1.2) gives us equations to be satisfied by a *backward solution* (assume for simplicity that $k(n,n)\neq 0$ for $n\leq n_0-N-1$) of

$$k(n,n)f(n,x(n))=x(n+N)-\sum_{j=n+1}^{n+N}k(n,j)f(j,x(j)),\quad (n\leq n_0-N-1). \tag{1.6b}$$

Consequently, the values $x(n)=\varphi(n)$ in (1.5) satisfy (1.2) for $n=n_0-N,\ldots,n_0-1$.

Conditions (on $k$ and $f$) are required for the solvability of each set of implicit equations in (1.6) to determine a solution, given $\varphi$. The Eqs. (1.6) both have the format $F(n,x(n))=v_n$ with $F(n,u)=u-k(n,n)f(n,u)$ and $F(n,u)=k(n,n)f(n,u)$ (respectively, and with differing $v_n$) and solvability conditions can be found from applications of the implicit function theorem. Without further conditions, any forward solution $\{x(n)\}_{n>n_0}$ may not be unique, or there may be an integer $N^0$ such that $\{x(n)\}_{n>N^0}$ is undefined and similar observations apply to the backward solution. If, given $\varphi$, the values $\{\ldots,x(n_0-M),x(n_0-M+1),\ldots,x(n_0-1),x(n_0),x(n_0+1),\ldots,x(n_0+M),\ldots\}$ *exist* and are *uniquely defined* then one can speak of *the unique solution* $\{x(\varphi;n)\}$.

Clearly, any solution of (1.2) for $n\in\mathbb{Z}$ defines for arbitrary $n_0\in\mathbb{Z}$ the sequence $\varphi$ in (1.5); we do not require (1.5) to discuss existence of solutions in general. There remains the issue of whether, for any solution, $\sup_{n\in\mathbb{Z}}|x(n)|<\infty$ or, for some $p\in(0,\infty)$, $\sum_{n\in\mathbb{Z}}|x(n)|^p<\infty$ and whether $\{\ldots x(n_0-M),x(n_0-M+1),\ldots,x(n_0-1),x(n_0),x(n_0+1),x(n_0+2),\ldots\}$ is, for example, positive, or, periodic (or almost periodic).

### *1.2. Practical motivation*

We referred to previous work on integral equations. From the perspective of mathematical modelling in biomathematics, involving (say) models of population growth, it is widely recognized that *when population sizes are small*, models based on a continuous or differentiable function defined on $\mathbb{R}$ may be less suited than models based on discretely-defined functions (where values may be observable only at discrete times). *The construction of discrete models in (e.g.) biomathematics is governed by an attempt to model the scientific understanding.* Periodicity of solutions (or almost periodicity) is a matter of scientific interest.

Where continuous models are to be preferred, on grounds of realism, over discrete models, the scientifically faithful forms of such continuous models rarely have closed-form solutions. Where practically useful insights into solutions are sought, one may turn to numerical methods, applied to realistic models, to provide approximate values. In this situation, one seeks 'appropriate' numerical formulae; we are once more led to consider discrete (summation) equations.

It is natural to assume continuity of $\mathsf{k}$ in the light of familiar models arising in biomathematics. We concentrate on that case. However, results for the smooth case can be modified (see below) for a class of weakly-singular kernels (those of Abel type) and modified discretization methods.

### 1.3. Discrete equations

Eq. (1.2) is a discrete analogue of (1.1) and (1.2) arises in the numerical analysis of (1.1) (see below, and [5–7]), as well as in discrete models. It appears that little work has been done on questions of periodic solutions of the *implicit*, *non-linear*, *finite-memory* discrete system[1] (1.2). This motivates us to investigate periodic solutions of (1.2), and to parallel some results for (1.1) stated in [2]. We establish existence results for periodic solutions of (1.2), *via* a variety of fixed-point theorems (*cf.* [12]). In Section 2, we give some basic results and recall several fixed-point theorems, and give our main results in Section 3. In Section 4, we discuss quadrature methods for (1.1) and demonstrate the application of the general results of Section 3.

**Example 1.1.** Suppose $\tau > 0$, and $\mathfrak{h} > 0$ is chosen with $N_\tau \in \mathbb{N}$ and $\mathfrak{h} = \tau/N_\tau$, and suppose that $\vartheta \in [0, 1]$. Then the "*composite*" or "*repeated*" version (Section 4.1) of the $\vartheta$-rule applied to (1.1) yields (1.2) with

$$k(n, j) = \mathfrak{h}\mathsf{k}(n\mathfrak{h}, j\mathfrak{h}) \quad \text{for } j \notin \{n - N_\tau, n\}, \tag{1.7a}$$

$$k(n, n - N_\tau) = (1 - \vartheta)\mathfrak{h} \times \mathsf{k}(n\mathfrak{h}, (n - N_\tau)\mathfrak{h}), \qquad k(n, n) = \vartheta\mathfrak{h} \times \mathsf{k}(n\mathfrak{h}, n\mathfrak{h}),$$

$$f(n, v) = \mathsf{f}(n\mathfrak{h}, v). \tag{1.7b}$$

By assumption, we can write $\mathfrak{h} \in \mathfrak{H}_\tau$ if we define (for any $\sigma \in \mathbb{R}_+$)

$$\mathfrak{H}_\sigma := \{\sigma/N_\sigma \text{ for some } N_\sigma \in \mathbb{N}\}. \tag{1.8}$$

In the case $\vartheta = 0$, we obtain equations $x(n) = \sum_{j=n-N_\tau}^{n-1} \mathfrak{h}\mathsf{k}(n\mathfrak{h}, j\mathfrak{h})\mathsf{f}(j\mathfrak{h}, x(j))$. These relations are *explicit* and are special cases (for $N = N_\tau$) of the form[2] $x(n) = \sum_{j=n-N}^{n-1} k(n, j)f(j, x(j))$. However, the cases $\vartheta = \frac{1}{2}$ and $\vartheta = 1$ yield *implicit* recurrence relations and it is known that these relations display, in many cases, better stability properties than obtained if $\vartheta = 0$. Given smoothness conditions, the convergence rates of $x(\varphi; n)$ to $\mathsf{x}(\boldsymbol{\phi}; t)$, as $\mathfrak{h} \searrow 0$, with $n\mathfrak{h} = t$ $(n \to \infty)$, with $\varphi(j) = \boldsymbol{\phi}(j\mathfrak{h})$ for $j\mathfrak{h} \in [-\tau, 0]$, are optimal if $\vartheta = \frac{1}{2}$.

As noted earlier, under certain conditions a particular solution $\{x(\varphi; n)\}$ of (1.2) for $n \in \mathcal{Z} \subseteq \mathbb{Z}$ corresponds to a choice $x(\varphi; n) := \varphi(n)$ for $n \in \{-N, \dots, -1\}$; $x(\varphi; n)$ is an analogue of a solution $\mathsf{x}(\boldsymbol{\phi}; t)$ of (1.1) for $t \geq 0$, wherein $\mathsf{x}(\boldsymbol{\phi}; t) = \boldsymbol{\phi}(t)$ for $t \in [-\tau, 0]$. In the current paper, we shall have no need to consider the choice of $\varphi$ when addressing the existence of solutions with certain properties. Existing results appear as theorems; our own results (presented as propositions) are in two parts: (i) results for discrete equations (*per se*) and (ii) consequences for quadrature methods applied to integral equations. One aim of the paper is to demonstrate that it is possible by picking appropriate families of quadrature rules to preserve appropriate periodicity displayed in the integral equation model. One might suggest that seeking to preserve periodicity displays some parallels with the search for symplectic numerical methods for certain classes of differential equations, a topic that is seen of some significance in numerical analysis.

## 2. Preliminaries to the fixed-point analysis

For basic functional analysis see, *e.g.*, [14]; we here recall a few essentials. Suppose that $X$ is a linear space; if $\|\cdot\|$ is a norm on $X$ we denote by $X_{\|\cdot\|}$ the corresponding normed linear space and if $X_{\|\cdot\|}$ is a Banach space we denote it by $\mathcal{X}$ or $\{X; \|\cdot\|\}$. The closure of a set $S$ is denoted $\overline{S}$, and the boundary of a set $S$ is denoted $\partial S$. For basic properties of *convex sets* and *cones*[3] (needed below) see [3,15]. Denote by

$$\ell(\mathbb{Z}) := \{x \mid x = \{x(n)\}_{n\in\mathbb{Z}},\ x(n) \in \mathbb{R}\} \tag{2.1}$$

the linear space whose elements are sequences with real $x(n)$ (we have the obvious definitions of addition and scalar multiplication, and $x$ or $\{x(n)\}_{n\in\mathbb{Z}}$ denotes $\{\dots, x(-2), x(-1), x(0), x(1), x(2), \dots\}$).

**Definition 2.1.** Let $\omega \in \mathbb{N}$ be a given positive integer. Then $x = \{x(n)\}_{n\in\mathbb{Z}}$ is an *$\omega$-periodic sequence* if $x(n + \omega) = x(n)$ for all $n \in \mathbb{Z}$. $A_\omega = A_\omega(\mathbb{Z})$ denotes the (finite-dimensional) subspace of $\ell(\mathbb{Z})$ consisting of all $\omega$-periodic sequences. We define the norm $|\cdot|_\omega$ on $A_\omega(\mathbb{Z})$ by setting $|x|_\omega = \sup_{n\in\mathbb{Z}} |x(n)| = \max_{1\leq n\leq\omega} |x(n)|$ for $x = \{x(n)\}_{n\in\mathbb{Z}} \in A_\omega(\mathbb{Z})$. A solution $x$ of (1.2) is called periodic if it satisfies (1.2) for $n \in \mathbb{Z}$ and $x \in A_\omega$ for some integer $\omega \in \mathbb{N}$.

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[1] For periodic solutions of *explicit* discrete systems, see, *e.g.*, [8,9]. For periodic solutions of *implicit* discrete systems with *unbounded memory*, see [10,11].

[2] Periodic solutions of such an equation, with $f(j, x(j)) = x(j)$, were discussed in [13].

[3] $C \subset X$ is a cone in $X_{\|\cdot\|}$ if it is a closed convex set and (i) for any non-zero $v \in C$ and any $\lambda \geq 0$, we have $\lambda v \in C$, and (ii) if $v \in X$ is non-zero then at least one of the pair $\{v, -v\}$ does not lie in $C$.

**Table 3.1**
Assumptions in the propositions.

| Assumption | Proposition | | | |
|---|---|---|---|---|
| | 3.1 | 3.2 | 3.3 | 3.4 |
| Eq. (3.2) | ✓ | ✓ | ✓ | ✓ |
| Eq. (3.3) | ✓ | ✓ | ✓ | ✓ |
| Eq. (3.4) | ✓ | ✓ | (a–b) | (a–b) |
| Eq. (3.5) | ✓ | ✓ | × | × |
| Eq. (3.6) | × | × | × | ✓ |

**Definition 2.2.** (i) A subset of a normed linear space $X_{\|\cdot\|}$ is called relatively compact if its closure is compact. (ii) A (linear or non-linear) operator $T : X_{\|\cdot\|} \to X_{\|\cdot\|}$ is called a compact operator when it maps an arbitrary bounded subset of $X_{\|\cdot\|}$ into a corresponding set that is relatively compact in $X_{\|\cdot\|}$. A continuous compact operator is called completely continuous.

**Remark 2.1.** (a) Obviously, an $\omega$-periodic sequence is $\omega_\star$-periodic where $\omega_\star/\omega \in \mathbb{N}$. For given $\omega \in \mathbb{N}$, $\mathcal{A}_\omega = \{A_\omega(\mathbb{Z}); |\cdot|_\omega\}$ is an $\omega$-dimensional Banach space. (b) Let $D \subseteq \mathcal{A}_\omega$. If $T : D \to \mathcal{A}_\omega$ is continuous it follows that it is completely continuous. (From the definition, if $T : D \to \mathcal{A}_\omega$ is continuous and maps bounded sets into bounded sets, then $T$ is completely continuous. However, since $D \subseteq \mathcal{A}_\omega$ and $\mathcal{A}_\omega$ is finite-dimensional $T$ maps bounded sets into bounded sets whenever it is continuous.)

The existence of one or more periodic solutions of (1.2) will be established *via* fixed-point theorems used in [2] to discuss (1.1). We associate with (1.2) an operator $T$ on $\ell(\mathbb{Z})$ (or a subspace), with $(Tx)(n) := \sum_{j=n-N}^{n} k(n,j)f(j,x(j))$, and we identify a solution of (1.2) as a fixed point of $T$. We shall refer to Brouwer's fixed-point theorem (Theorem A.1), but in the main we will use (citing, and paraphrasing, [2]) either *Krasnosel'skiĭ's fixed-point theorem* (stated here as Theorem 2.1), the *"non-linear alternative"* (Theorem 2.2), or the *Leggett–Williams fixed-point theorem* (Theorem 2.3). These results have previously been used in discussions of (1.1), and one of our objectives is to show that results for (1.2) can be found to parallel those for (1.1).

**Theorem 2.1** (*Krasnosel'skiĭ, see [2, Theorem 2.1.1]*). *Let $\mathcal{X} \equiv \{X; \|\cdot\|\}$ be a Banach space and let $C \subset \mathcal{X}$ be a cone in $\mathcal{X}$. Assume $\Omega_1, \Omega_2$ are bounded open subsets of $\mathcal{X}$ with $0 \in \Omega_1$, $\overline{\Omega}_1 \subseteq \Omega_2$, and let $T : C \cap (\overline{\Omega}_2 \setminus \Omega_1) \to C$ be a completely continuous operator such that either* (i) *$\|Tu\| \le \|u\|$, $u \in C \cap \partial\Omega_1$ and $\|Tu\| \ge \|u\|$, $u \in C \cap \partial\Omega_2$ or* (ii) *$\|Tu\| \ge \|u\|$, $u \in C \cap \partial\Omega_1$ and $\|Tu\| \le \|u\|$, $u \in C \cap \partial\Omega_2$ is true. Then $T$ has a fixed point in $C \cap (\overline{\Omega}_2 \setminus \Omega_1)$.*

**Theorem 2.2** (*Non-linear Alternative, see [2, Theorem 1.2.1]*). *Suppose that $K$ is a convex subset of a normed linear space $X_{\|\cdot\|}$, and let $U$ be an open subset of $K$, with $p^\star \in U$. Then every completely continuous map $T_\natural : \overline{U} \to K$ has at least one of the following two properties:* (i) *$T_\natural$ has a fixed point in $\overline{U}$;* (ii) *there is an $x_\lambda \in \partial U$ with $x_\lambda = (1-\lambda)p^\star + \lambda T_\natural x_\lambda$ for some $0 < \lambda < 1$.*

Note that (i) and (ii) are not mutually exclusive, but if conclusion (ii) is shown to be false then conclusion (i) holds. In applications, $K$ can be a cone. Given $r_1, r_2 \in (0,\infty)$, $r_1 \ne r_2$ the following result can be applied with $R = \max\{r_1, r_2\}$ and $r = \min\{r_1, r_2\}$.

**Theorem 2.3** (*Leggett–Williams, see [2, Theorem 4.3.1]*). *Let $X_{\|\cdot\|}$ define a Banach space $\mathcal{X}$, $C \subset X$ a cone in $\mathcal{X}$, and $0 < r < R$. Define $C_\eta = \{x \in C : \|x\| < \eta\}$ ($\partial C_\eta = \{x \in C \mid \|x\| = \eta\}$, $\overline{C}_\eta = \{x \in C \mid \|x\| \le \eta\}$). Let $T : \overline{C}_R \to C$ be a continuous, compact map such that* (i) *there exists $u_0 \in C \setminus \{0\}$ with $Tu \not\le u$ for $u \in \partial C_r \cap C(u_0)$ where $C(u_0) = \{u \in C \mid \exists\lambda > 0 \text{ with } u \geqslant \lambda u_0\}$; and* (ii) *$\|Tu\| \le \|u\|$ for $u \in \partial C_R$. Then $T$ has at least one fixed point $x \in C$ with $r \le \|x\| \le R$.*

To supplement the preceding results we emphasize a consequence of Remark 2.1(b) by stating it as a lemma.

**Lemma 2.1.** *Suppose that in Theorems 2.1, 2.2, 2.3 the underlying linear space $X$ is finite-dimensional; then the condition that $T$ is completely continuous is satisfied when $T$ is continuous.*

## 3. Main results on discrete equations

In our discussion of discrete equations, we rely on assumptions that parallel those made in discussions [2] of the continuous case (1.1). The restrictions imposed by these assumptions are no more than one expects from the existing literature concerning (1.1). For each of the propositions in this section, we adopt corresponding Assumptions from the list in Section 3.1 (as indicated in Table 3.1). Any additional hypotheses are given in the statements of the propositions, but the table gives an indication of the differing assumptions.

Associated with (1.2), we define (for suitable $k, f$) an operator $T$ on $\ell(\mathbb{Z})$ by

$$(Tx)(n) = \sum_{j=n-N}^{n} k(n,j)f(j,x(j)), \quad n \in \mathbb{Z}. \tag{3.1}$$

A fixed point of $T$ is a solution of (1.2). We have $\omega \in \mathbb{N}$. One may replace $\omega$ by $\omega_\star = \widehat{\jmath} \times \omega$ with given $\widehat{\jmath} \in \mathbb{N}$.

### *3.1. Assumptions*

**Assumption 1.**

$$k : \mathbb{Z} \times \mathbb{Z} \to \mathbb{R}_+ \quad \text{and} \quad k(n+\omega, j+\omega) = k(n,j) \quad \text{for every } n, j \in \mathbb{Z}. \tag{3.2}$$

**Assumption 2.** (a)

$$f : \mathbb{Z} \times \mathbb{R}_+ \to \mathbb{R}_+ \tag{3.3a}$$

(b) For each $n \in \mathbb{Z}$,

$$f(n, \cdot) \text{ is continuous on } \mathbb{R}_+. \tag{3.3b}$$

(c) For all $n \in \mathbb{Z}$ and $u \in \mathbb{R}_+$,

$$f(n+\omega, u) = f(n, u). \tag{3.3c}$$

**Assumption 3.** (a) The function $\psi$ is a non-decreasing continuous map

$$\psi : \mathbb{R}_+ \to \mathbb{R}_+; \quad \psi(u_1) \ge \psi(u_2) \quad \text{if } u_1 \ge u_2 \in \mathbb{R}_+. \tag{3.4a}$$

(b) The functions $f$, $\psi$ in (3.4a), and $q \in \mathcal{A}_\omega$ satisfy

$$f(n, u) \le q(n)\psi(u) \quad \text{for all } u \in \mathbb{R}_+ \text{ and } n \in \mathbb{Z}. \tag{3.4b}$$

(c) There exists a constant $a_0 \in (0, 1]$, such that $f$, $\psi$ in (3.4a), and $q \in \mathcal{A}_\omega$ satisfy

$$a_0 q(n)\psi(u) \le f(n, u) \quad \text{for all } u \in \mathbb{R}_+ \text{ and } n \in \mathbb{Z}. \tag{3.4c}$$

(d) There exists a function $\xi : (0, 1) \to \mathbb{R}_+ \setminus \{0\}$ such that, for $\psi$ in (3.4a),

$$\psi(\mu\upsilon) \ge \xi(\mu)\psi(\upsilon), \quad \text{for any } 0 < \mu < 1,\ \upsilon \ge 0. \tag{3.5}$$

**Assumption 4.** With given $f$, $\psi$, $q$, there exists a continuous function $\chi : \mathbb{R}_+ \to \mathbb{R}_+$ with

$$\chi(u)q(n) \le f(n, u) \le q(n)\psi(u) \quad \text{for all } u \in \mathbb{R}_+ \text{ and } n \in \mathbb{Z}. \tag{3.6a}$$

(The second inequality in (3.6a) arises in (3.4b).) In Proposition 3.4 we require (3.6a) with

$$\chi(u)/u \text{ non-increasing for } u \text{ in an interval } (0, r]. \tag{3.6b}$$

A few further assumptions are stated, later, in terms of $\kappa_{\max,\min}(q_\natural)$ which are defined for any $q_\natural \in A_\omega$ by

$$\kappa_{\max}(q_\natural) := \max_{1\le n\le\omega} \sum_{j=n-N}^{n} k(n,j)q_\natural(j), \qquad \kappa_{\min}(q_\natural) := \min_{1\le n\le\omega} \sum_{j=n-N}^{n} k(n,j)q_\natural(j). \tag{3.7}$$

Of course, regarding Assumptions 1–4, it is reasonable to suppose that $f$ is defined on $\mathbb{Z} \times \mathbb{R}$ and that its restriction to $\mathbb{Z} \times \mathbb{R}_+$ (also denoted by $f$) satisfies the above assumptions.

### *3.2. From conditions on (1.1) to conditions on (1.2) obtained by discretization*

We assume $\varpi \in \mathbb{R}_+$ and let us suppose that

$$\mathsf{k} \in C(\mathbb{R} \times \mathbb{R} \to \mathbb{R}_+) \quad \text{and, for every } t, s \in \mathbb{R}, \qquad \mathsf{k}(t+\varpi, s+\varpi) = \mathsf{k}(t, s). \tag{3.8a}$$

Further,

$$\mathsf{f} : C(\mathbb{R} \times \mathbb{R}_+ \to \mathbb{R}_+), \quad \text{and} \quad \mathsf{f}(t+\varpi, u) = \mathsf{f}(t, u). \tag{3.8b}$$

**Lemma 3.1.** *With the above assumption, if $\mathfrak{h} > 0$, and $\mathfrak{h}W_\ell \in \mathbb{R}_+$, for $\ell \in \{n-N, n-N+1, \dots, n\}$, $n \in \mathbb{Z}$ and when $k$ and $\mathsf{k}$, $f$ and $\mathsf{f}$ are related by*

$$k(n, j) = \mathfrak{h}W_{n-j}\mathsf{k}(n\mathfrak{h}, j\mathfrak{h}), \quad j \in \{n-N, n-N+1, \dots, n\},\ n \in \mathbb{Z} \tag{3.9}$$

$$f(n, u) = \mathsf{f}(n\mathfrak{h}, u), \quad n \in \mathbb{Z},\ u \in \mathbb{R}, \tag{3.10}$$

*then (3.2) and (3.3) are satisfied with $\varpi = \mathfrak{h} \times \omega$ where $\omega \in \mathbb{N}$. Suppose, also, that*

$$\mathsf{q} \in C(\mathbb{R} \to \mathbb{R}) \quad \text{and} \quad \mathsf{q}(t) = \mathsf{q}(t+\varpi) \quad \text{for } t \in \mathbb{R}. \tag{3.11a}$$

*Then if $q(n) = \mathsf{q}(n\mathfrak{h})$ (for $n \in \mathbb{Z}$) and $\varpi = \mathfrak{h} \times \omega$, it follows that $q \in \mathcal{A}_\omega$. Relations (3.4b) or (3.4c) follow if, respectively,*

$$\mathsf{f}(t, u) \leq \mathsf{q}(t)\psi(u) \quad \textit{for all } u \in \mathbb{R}_+ \textit{ and } t \in \mathbb{R}, \tag{3.11b}$$

*or*

$$a_0\mathsf{q}(t)\psi(u) \leq \mathsf{f}(t, u) \quad \textit{for all } u \in \mathbb{R}_+ \textit{ and } t \in \mathbb{R}. \tag{3.11c}$$

**Remark 3.1.** Conditions on $\mathsf{k}$, $\mathsf{f}$ and $\mathsf{q}$, in the literature on (1.1), include the assumptions (on $\mathsf{k}$, $\mathsf{f}$, $\mathsf{q}$) of Lemma 3.1. Now, for $\varpi \in \mathbb{R}_+$, $\mathfrak{H}_\varpi \subset \mathbb{R}$ is defined as $\{\varpi/N_\varpi \mid N_\varpi \in \mathbb{N}\}$. In view of Lemma 3.1, we later suppose that $\mathfrak{h} \in \mathfrak{H}_\varpi$. Referring to Example 1.1, where $\mathfrak{h} = \tau/N_\tau$, we also ask that $\mathfrak{h} \in \mathfrak{H}_\tau$. This restricts $\tau$ and $\varpi$ (we need $\tau = \{N_\tau/N_\varpi\}\varpi$). This restriction will be overcome; see Proposition 4.1 *et seq*.

### 3.3. Positive periodic solutions via Krasnosel'skiĭ's fixed-point theorem

First, we use Theorem 2.1 to establish a result for (1.2). In Krasnosel'skiĭ's theorem, take $\mathcal{X}$ and $C$ to be

$$\mathcal{X} := \mathcal{A}_\omega \equiv \{A_\omega, |\cdot|_\omega\}, \quad \text{and} \quad C := \{x \in \mathcal{A}_\omega \mid x(n) \geq M|x|_\omega \text{for} n \in \mathbb{Z}\} \tag{3.12}$$

with $M$ from (3.13b): $x \in C$ when $x(n) \geq M|x|_\omega$ for $1 \leq n \leq \omega$. It is readily shown that $C$ is a cone in $\mathcal{A}_\omega$. Let $T : C \to \ell(\mathbb{Z})$ be defined by (3.1). Then, $T : C \to \mathcal{A}_\omega$. Indeed, for any $x \in C \subset \mathcal{A}_\omega$, it follows from (3.2) and (3.3c) that $(Tx)(n+\omega) = (Tx)(n)$, and so $Tx \in \mathcal{A}_\omega$, because

$$\sum_{j=n+\omega-N}^{n+\omega} k(n+\omega, j)f(j, x(j)) = \sum_{j=n-N}^{n} k(n+\omega, j+\omega)f(j+\omega, x(j+\omega)) = \sum_{j=n-N}^{n} k(n, j)f(j, x(j)).$$

Next we require that $T : C \to \mathcal{A}_\omega$ is continuous and compact. Continuity is readily established (it follows from the uniform continuity of $f$ on compact subsets of its domain of definition) and by Lemma 2.1 the required result follows.

**Proposition 3.1.** *Taking the Assumptions indicated in Table 3.1 suppose (where q is the function in (3.4b), and given (3.7)), that*

$$\kappa_{\min}(q) > 0; \tag{3.13a}$$

*there exists $M \in (0, 1)$ with*

$$M/\xi(M) \leq a_0\kappa_{\min}(q)/\kappa_{\max}(q); \tag{3.13b}$$

*there exists $\alpha > 0$ with*

$$\alpha > \kappa_{\max}(q)\psi(\alpha); \tag{3.13c}$$

*and there exists $\beta > 0$, $\beta \neq \alpha$, with*

$$\beta < a_0\kappa_{\min}(q)\psi(M\beta). \tag{3.13d}$$

*Then, (1.2) has at least one positive periodic solution $x \in \mathcal{A}_\omega(\mathbb{Z})$, with*

$$0 < \min\{\alpha, \beta\} < |x|_\omega < \max\{\alpha, \beta\}; \qquad x(n) \geq M\min\{\alpha, \beta\} \quad \textit{for } n \in \mathbb{Z}. \tag{3.14}$$

**Proof.** For any $a > 0$ let $\Omega_a = \{x \in \mathcal{A}_\omega : |x|_\omega < a\}$. To apply Theorem 2.1, we show the following hold:

(a) $T : C \to C$, (b) $|Tx|_\omega \leq |x|_\omega$ for $x \in C \cap \partial\Omega_\alpha = S_\alpha$, (c) $|Tx|_\omega \geq |x|_\omega$ for $x \in C \cap \partial\Omega_\beta = S_\beta$.

(a) Let $x \in C$. Then (3.4a)–(3.4b) imply that, for $n \in \{1, \dots, \omega\}$,

$$|(Tx)(n)| \leq \psi(|x|_\omega) \max_{n\in\{1,\dots,\omega\}} \sum_{j=n-N}^{n} k(n, j)q(j) = \kappa_{\max}(q)\psi(|x|_\omega). \tag{3.15}$$

On the other hand, since $x \in C$, we have $x(n) \geq M|x|_\omega$ for $n \in \mathbb{Z}$, and thus (3.4a), (3.4c), (3.5) and (3.13b) give, for $n \in \{1, \dots, \omega\}$,

$$\begin{aligned}(Tx)(n) &\geq a_0 \sum_{j=n-N}^{n} k(n, j)q(j)\psi(x(j)) \geq a_0\psi(M|x|_\omega) \sum_{j=n-N}^{n} k(n, j)q(j)\\ &\geq a_0\xi(M)\psi(|x|_\omega) \sum_{j=n-N}^{n} k(n, j)q(j) \geq \kappa_{\min}(q)a_0\xi(M)\psi(|x|_\omega)\\ &\geq \{\kappa_{\min}(q)/\kappa_{\max}(q)\}a_0\xi(M)|Tx|_\omega \geq M|Tx|_\omega;\end{aligned}$$

so $Tx \in C$ and (a) holds.

To establish (b), let $x \in C \cap \partial\Omega_\alpha = S_\alpha$. In this case, $|x|_\omega = \alpha$ and $x(n) \ge M\alpha$ for all $n \in \mathbb{Z}$. Now for $n \in \{1, \ldots, \omega\}$, we have $|(Tx)(n)| \le \psi(|x|_\omega) \sum_{j=n-N}^{n} k(n,j)q(j) \le \psi(\alpha)\kappa_{\max}(q)$. This, with (3.13c), yields $|Tx|_\omega \le \psi(\alpha)\kappa_{\max}(q) < \alpha = |x|_\omega$; thus (b) is satisfied.

To establish (c), let $x \in C \cap \partial\Omega_\beta = S_\beta$. Then $|x|_\omega = \beta$ and $M\beta \le x(n) \le \beta$ for all $n \in \mathbb{Z}$. Now, for $n \in \{1, \ldots, \omega\}$, it follows from (3.4a) and (3.4c) that $|(Tx)(n)| \ge a_0 \sum_{j=n-N}^{n} k(n,j)q(j)\psi(x(j)) \ge a_0\kappa_{\min}(q)\psi(M\beta)$, which, together with (3.13d), yields $(Tx)(n) \ge a_0\kappa_{\min}(q)\psi(M\beta) > \beta = |x|_\omega$ for $n \in \{1, \ldots, \omega\}$, and thus $|Tx|_\omega > |x|_\omega$; that is, (c) holds.

Applying Krasnosel'skiĭ's Theorem 2.1, we conclude that (1.2) has a solution $x \in \mathcal{A}_\omega$ with $x \in C \cap (\overline{\Omega_\alpha} \setminus \Omega_\beta)$ if $\beta < \alpha$, or $x \in C \cap (\overline{\Omega_\beta} \setminus \Omega_\alpha)$ if $\alpha < \beta$. Finally, we note that $|x|_\omega \ne \alpha$ and $|x|_\omega \ne \beta$. In fact, if $|x|_\omega = \alpha$, then from $x = Tx$ we have $\alpha = |x|_\omega = |Tx|_\omega \le \psi(\alpha)\kappa_{\max}(q) < \alpha = |x|_\omega$ (which is a contradiction). A similar argument shows that $|x|_\omega \ne \beta$. This completes the proof. □

**Example 3.1.** Consider (see Remark 1.1) the non-linear system

$$x(n) = \sum_{j=n-N}^{n} k(n,j)g(j)[x(j)]^\gamma \quad \text{for } n \in \mathbb{Z}, \tag{3.16}$$

where $0 < \gamma < 1$. Assume that $k$ satisfies (3.2) and, for $\omega \in \mathbb{N}$, that

$$g \in \mathcal{A}_\omega, \quad g(n) \ge 0, \quad \text{and} \quad \kappa_{\min}(g) > 0 \tag{3.17}$$

(with $\kappa_{\min}(g)$, $\kappa_{\max}(g)$ defined by (3.7)). Then (3.16) has at least one positive periodic solution $x \in \mathcal{A}_\omega$, where with $M = \frac{1}{2}(\kappa_{\min}(g)/\kappa_{\max}(g))^{\frac{1}{1-\gamma}}$, $\beta = \frac{1}{2}M^{\frac{\gamma}{1-\gamma}}(\kappa_{\min}(g))^{\frac{1}{1-\gamma}} < |x|_\omega < 2(\kappa_{\max}(g))^{\frac{1}{1-\gamma}} = \alpha$ and $x(n) \ge M\beta$ for $n \in \mathbb{Z}$.

To see that the above result is true, we apply Proposition 3.1 with $q = g$, $f(n,u) = q(n)u^\gamma$, $\psi(u) = u^\gamma$, $a_0 = 1$ and $\xi(u) = u^\gamma$. Now, the continuity and periodicity conditions on $k, f$ in Proposition 3.1 and continuity and monotonicity properties of $\psi$ are clearly satisfied; we verify (3.13). Now (3.13a) is satisfied by assumption. To establish (3.13b) for $\xi$, we note that

$$\frac{M}{\xi(M)} = M^{1-\gamma} = \left(\frac{1}{2}\right)^{1-\gamma} \frac{\kappa_{\min}(g)}{\kappa_{\max}(g)} \le \frac{\kappa_{\min}(g)}{\kappa_{\max}(g)} = a_0\frac{\kappa_{\min}(g)}{\kappa_{\max}(g)}.$$

Also (3.13c) holds since $\alpha/(\psi(\alpha)) = \alpha^{1-\gamma} = 2^{1-\gamma}\kappa_{\max}(g) > \kappa_{\max}(g)$. Since $\beta/\psi(M\beta) = \{1/M^\gamma\}\beta^{1-\gamma} = \{1/M^\gamma\}\left(\frac{1}{2}\right)^{1-\gamma} M^\gamma\kappa_{\min}(g) = \left(\frac{1}{2}\right)^{1-\gamma}\kappa_{\min}(g) < a_0\kappa_{\min}(g)$, (3.13d) is also true. Now apply Proposition 3.1.

With additional conditions on $k$ and $f$ in (1.2), applications of Proposition 3.1 will yield additional positive periodic solutions of (1.2). For completeness we provide one result on multiple solutions.

**Proposition 3.2.** *Take the Assumptions indicated in Table 3.1 and suppose also that (3.13b) is satisfied. Also, given (3.7), suppose (where $q \in \mathcal{A}_\omega$ is the function in (3.4b)) that there are constants $0 < \gamma_0 < \gamma_1 < \gamma_2$ with (i) $\gamma_0 < a_0\kappa_{\min}(q)\psi(M\gamma_0)$, with (ii) $\gamma_1 > \kappa_{\max}(q)\psi(\gamma_1)$ and with (iii) $\gamma_2 < a_0\kappa_{\min}(q)\psi(M\gamma_2)$. Then (1.2) has at least two positive periodic solutions $x_1 = \{x_1(n)\}_{n\in\mathbb{Z}}$, and $x_2 = \{x_2(n)\}_{n\in\mathbb{Z}} \in \mathcal{A}_\omega$ with $0 < \gamma_0 < |x_1|_\omega < \gamma_1 < |x_2|_\omega < \gamma_2$, $x_1(n) \ge M\gamma_0$ and $x_2(n) \ge M\gamma_1$ for $n \in \mathbb{Z}$.*

**Proof.** The existence of $x_1$ follows from Proposition 3.1 with $\alpha = \gamma_1$ and $\beta = \gamma_0$, and the existence of $x_2$ follows from Proposition 3.1 with $\alpha = \gamma_1$ and $\beta = \gamma_2$. □

### *3.4. Non-negative periodic solutions via the non-linear alternative*

We use Theorem 2.2 to obtain the following existence result for (1.2).

**Proposition 3.3.** *With the Assumptions indicated in Table 3.1, assume in addition that, with $\kappa_{\max}(q)$ as in (3.7), there exists $\alpha > 0$ with*

$$\alpha > \kappa_{\max}(q)\psi(\alpha). \tag{3.18}$$

*Then (1.2) has a non-negative solution $x \in \mathcal{A}_\omega$ with $|x|_\omega < \alpha$.*

**Proof.** Any non-negative solution (that is, with $x(n) \ge 0$ for $n \in \mathbb{Z}$) of (1.2) is a solution of

$$x(n) = \sum_{j=n-N}^{n} k(n,j)f_\natural(j, x(j)), \quad N \in \mathbb{N},\ x(n) \in \mathbb{R}. \tag{3.19}$$

where

$$f_\natural(n,u) = f(n, |u|). \tag{3.20}$$

By construction, and the assumptions on $f$, $f_\natural(n,\cdot)$ is continuous on $\mathbb{R}$ for each $n \in \mathbb{Z}$ and

$$f_\natural(n+\omega,u) = f_\natural(n,u) \geq 0 \quad \text{for all } n \in \mathbb{Z} \text{ and } u \in \mathbb{R}. \tag{3.21}$$

We apply Theorem 2.2 to (3.19), setting $(T_\natural x)(n) = \sum_{j=n-N}^{n} k(n,j)f_\natural(j,x(j))$ for $n \in \mathbb{Z}$, and taking $X_{\|\cdot\|}$ to be $A_\omega$ with norm $|\cdot|_\omega$ and $U = \{x \in A_\omega \mid |x|_\omega < \alpha\}$. It is easy to see that the operator $T_\natural$ maps $A_\omega$ to $A_\omega$ by conditions (3.2) and by the assumed properties of $f$. In addition, these properties guarantee that $T_\natural : A_\omega \to A_\omega$ is continuous and compact. Let $x_\lambda \in A_\omega$ be any solution of

$$x_\lambda(n) = \lambda \left\{ \sum_{j=n-N}^{n} k(n,j) f_\natural(j, x_\lambda(j)) \right\}, \quad n \in \mathbb{Z},$$

for $0 < \lambda < 1$. Notice that (3.2) and (3.3a) imply $x_\lambda(n) \geq 0$ for all $n \in \mathbb{Z}$. Now for $n \in \{1,\ldots,\omega\}$, we have $|x_\lambda(n)| \leq \sum_{j=n-N}^{n} k(n,j)q(j)\psi(x_\lambda(j)) \leq \psi(|x_\lambda|_\omega)\sum_{j=n-N}^{n} k(n,j)q(j) \leq \kappa_{\max}(q)\psi(|x_\lambda|_\omega)$ and therefore

$$|x_\lambda|_\omega \leq \kappa_{\max}(q)\psi(|x_\lambda|_\omega). \tag{3.22}$$

In addition, (3.18) and (3.22) implies that $|x_\lambda|_\omega \neq \alpha$. Apply Theorem 2.2 (with $p^* = 0$): since we have shown that option (ii) (in the statement of that theorem) cannot occur, we deduce that (1.2) has a solution $x = \{x(n)\}_{n\in\mathbb{Z}} \in \mathcal{A}_\omega$ with $x(n) \geq 0$ for all $n \in \mathbb{Z}$. Further, $|x|_\omega < \alpha$. (We have $|x|_\omega \leq \alpha$ by Theorem 2.2 and $|x|_\omega \neq \alpha$ by an argument similar to that used to show that $|x_\lambda|_\omega \neq \alpha$.) □

If there is $n_0 \in \{1,\ldots,\omega\}$ such that $k(n_0,n_0)f(n_0,0) > 0$, then the *null* periodic sequence, $\{y(n) = 0\}_{n\in\mathbb{Z}}$, is not a solution of (1.2). Then, the non-negative solution $x$ with $|x|_\omega < \alpha$, in Proposition 3.3, satisfies $|x|_\omega > 0$. The next example provides an illustration of the application of Proposition 3.3.

**Example 3.2.** Consider the non-linear system

$$x(n) = \sum_{j=n-N}^{n} k(n,j)g(j)(\mu + |x(j)|^\gamma) \quad \text{for } n \in \mathbb{Z}, \tag{3.23}$$

where $\gamma > 1$ and $\mu > 0$ are given constants. Assume that $k$ satisfies (3.2) and, for $\omega \in \mathbb{N}$, that

$$g \in \mathcal{A}_\omega, \quad g(n) \geq 0, \quad \text{and} \quad \kappa_{\max}(g) > 0 \tag{3.24}$$

(with $\kappa_{\max}(g)$ defined by (3.7)). Then (3.23) has at least a non-negative periodic solution $x \in \mathcal{A}_\omega$ with $0 < |x|_\infty < \alpha$ for any $\alpha > 0$ and $\mu > 0$ satisfying $0 < \mu < \frac{\alpha}{\kappa_{\max}(g)} - \gamma\alpha^\gamma$.

We shall use Proposition 3.3, with

$$q(n) = g(n), \quad n \in \mathbb{Z}, \qquad \psi(u) = \mu + u^\gamma \quad \text{for } u \geq 0,$$

to establish this result: Obviously, $f(n,u) = g(n)(\mu + |u|^\gamma)$ satisfies the conditions on $f$ assumed in Proposition 3.3. It remains to show that Condition (3.18) holds. Indeed, the continuous and differentiable function $h(u) = \{u/\kappa_{\max}(g)\} - \gamma u^\gamma$ satisfies $h(0) = 0$ and $h'(u) = \{1/\kappa_{\max}(g)\} - \gamma^2 u^{\gamma-1} > 0$ for small $u > 0$. Thus, we have $\{u/\kappa_{\max}(g)\} > \gamma u^\gamma$ for small $u > 0$, which implies that we can chose $\alpha > 0$ and $\mu$ such that $0 < \mu < \frac{\alpha}{\kappa_{\max}(g)} - \gamma\alpha^\gamma$, or equivalently, $\gamma\alpha^{\gamma-1} + \frac{\mu}{\alpha} < \frac{1}{\kappa_{\max}(g)}$. By the mean-value theorem, there is a number $0 < \alpha_* < \alpha$ such that

$$\psi(\alpha) = (\psi(\alpha) - \psi(0)) + \psi(0) = \gamma{\alpha_*}^{\gamma-1}\alpha + \mu < \alpha\left(\gamma\alpha^{\gamma-1} + \frac{\mu}{\alpha}\right) < \frac{\alpha}{\kappa_{\max}(g)}.$$

Thus Condition (3.18) holds. We now apply Proposition 3.3 and the result follows.

### *3.5. Non-negative periodic solutions via the Leggett–Williams theorem*

It is possible to use Theorem 2.3 to establish the existence of non-negative periodic solutions of (1.2).

**Proposition 3.4.** *With the Assumptions indicated in Table 3.1, assume also that*

$$\kappa_{\min}(q) > 0; \tag{3.25}$$

$$r < \kappa_{\min}(q)\chi(r), \tag{3.26a}$$
$$R > \psi(R)\kappa_{\max}(q). \tag{3.26b}$$

*Then (1.2) has a non-negative solution $x \in \mathcal{A}_\omega$ with $r \leq |x|_\omega < R$.*

**Proof.** In Theorem 2.3, take $\mathcal{X} = \mathcal{A}_\omega$ and $C = \{x \in \mathcal{A}_\omega \mid x(n) \geq 0\}$ for $n \in \{1, \ldots, \omega\}$, i.e.,$\{x \in \mathcal{A}_\omega \mid x(n) \geq 0 \text{ for } n \in \mathbb{Z}\}$. Clearly, $C$ is a cone in $\mathcal{A}_\omega$. Let $u_0 = \{u_0(n)\}_{n\in\mathbb{Z}}$ with $u_0(n) = 1$ for all $n \in \mathbb{Z}$ and note that $C(u_0) = \{x \in C :$ there exists $\lambda > 0$ with $x(n) \geq \lambda$ for $n \in \{1, \ldots, \omega\}\}$. From (3.2) and (3.3a), it follows that $T : C \to C$ is continuous.

To apply Theorem 2.3, we first show (where $\partial C_\rho := \{x \in C \mid |x|_\omega = \rho\}$, when $\rho \in \mathbb{R}_+$) that

$$|Tx|_\omega \leq |x|_\omega \quad \text{for } x \in \partial C_R. \tag{3.27}$$

If $x \in \partial C_R$, then $|x|_\omega = R$ and, since $\sum_{j=n-N}^{n} k(n,j)q(j)\psi(x(j)) \leq \psi(|x|_\omega) \sum_{j=n-N}^{n} k(n,j)q(j)$, we have

$$(Tx)(n) \leq \psi(R)\kappa_{\max}(q) \tag{3.28}$$

for $n \in \{1, \ldots, \omega\}$. This, together with (3.26b), gives

$$|Tx|_\omega \leq \psi(R)\kappa_{\max}(q) < R = |x|_\omega, \tag{3.29}$$

which implies that (3.27) is true. Next, we show that

$$Tx \nleq x \quad \text{for } x \in \partial C_r \cap C(u_0). \tag{3.30}$$

To show this, let $x = \{x(n)\}_{n\in\mathbb{Z}} \in \partial C_r \cap C(u_0)$, hence $|x|_\omega = r$ and $r \geq x(n) > 0$ for $n \in \{1, \ldots, \omega\}$. Now,

$$(Tx)(n) \geq \sum_{j=n-N}^{n} k(n,j)q(j)\frac{\chi(x(j))}{x(j)}x(j) \geq \frac{\chi(r)}{r} \sum_{j=n-N}^{n} k(n,j)q(j)x(j) \quad \text{for } n \in \{1, \ldots, \omega\}.$$

Let $n_0 \in \{1, \ldots, \omega\}$ be such that $\min_{n\in\{1,\ldots,\omega\}} x(n) = x(n_0)$ and this together with the previous inequality yields, for $n \in \{1, \ldots, \omega\}$,

$$(Tx)(n) \geq \frac{\chi(r)}{r}x(n_0) \sum_{j=n-N}^{n} k(n,j)q(j) \geq \left(\frac{\chi(r)}{r}\kappa_{\min}(q)\right) x(n_0).$$

By (3.26a) we obtain $(Tx)(n) > x(n_0)$ for $n \in \{1, \ldots, \omega\}$ and $(Tx)(n_0) > x(n_0)$, so that (3.30) is true.

Applying Theorem 2.3, we conclude that (1.2) has a non-negative periodic solution $x = \{x(n)\}_{n\in\mathbb{Z}} \in C$ with $r \leq |x|_\omega \leq R$. Note that $|x|_\omega \neq R$, by (3.29). □

**Example 3.3.** Consider the following discrete non-linear system

$$x(n) = \sum_{j=n-N}^{n} k(n,j)g(j)([x(j)]^\alpha + [x(j)]^\beta), \quad n \in \mathbb{Z}, \tag{3.31}$$

with $0 < \alpha < 1$, $\beta \geq 1$ and (3.2) satisfied. In addition assume

$$g \in A_w \quad \text{with } g(n+\omega) = g(n) \geq 0 \quad \text{for all } n \in \mathbb{Z}, \tag{3.32}$$

$$\kappa_{\min}(g) > 0 \quad \text{and} \quad \kappa_{\max}(g) < \frac{1}{2}. \tag{3.33}$$

Then (3.31) has a non-negative solution $x \in \mathcal{A}_\omega$ with $\left(\frac{1}{2}\kappa_{\min}(g)\right)^{\frac{1}{1-\alpha}} \leq |x|_\omega < 1$, in which we have $\kappa_{\min}(g) \equiv \min_{n\in\{1,\ldots,\omega\}} \sum_{j=n-N}^{n} k(n,j)g(j)$.

To establish this, let $f(n,u) = g(n)[u^\alpha + u^\beta]$, $\psi(u) = [u^\alpha + u^\beta]$, $\chi(u) = u^\alpha$, and $q(n) = g(n)$ with $r = \left(\frac{1}{2}\kappa_{\min}(g)\right)^{\frac{1}{1-\alpha}}$ and $R = 1$. Obviously, (3.3a), (3.6a) and (3.25) hold. To establish (3.26a), notice that $r/\chi(r) = r^{1-\alpha} = \frac{1}{2}\kappa_{\min}(g) < \kappa_{\min}(g)$. Also, $\chi(u)/u = 1/u^{1-\alpha}$ is non-increasing on $(0, r]$ as $0 < \alpha < 1$, and finally (3.26b) holds with $R = 1$. We can now apply Proposition 3.4.

Consider (3.33) for varying $\alpha \in (0, 1)$, $\beta \geq 1$. We can denote $x(n)$ by $x_{\alpha,\beta}(n)$, to indicate dependence on the values $\alpha$, $\beta$. The complexity of (3.31) as an equation for $x_{\alpha,\beta}(n)$ when $\{x_{\alpha,\beta}(n-\ell)\}_{\ell=1}^{N}$ are known increases with increasing $\beta$: for example, if $\alpha = \frac{1}{2}$ and $1 < \beta \in \mathbb{N}$ then (3.31) can be viewed as a polynomial equation of degree $2\beta$ for $\sqrt{x(n)} = \sqrt{x_{\alpha,\beta}(n)}$. As our discussion is confined to the unit ball $|x| < 1$, the periodic solution $x_{\alpha,\beta}(n)$ that is addressed in the result is located in this unit ball and satisfies $0 < r \leq |x_{\alpha,\beta}|_\omega < 1$ for any $\alpha \in (0, 1)$, $\beta > 1$.

## 4. Discretization

We develop further the discussion of the discretization of (1.1). We here require continuity of k and f (later, we address the weakly singular case). We first pause to indicate the general nature of results found, concerning (1.1), in the literature and refer to the application of Krasnosel'skiĭ's fixed-point theorem in [2, p. 139]. Thus, with our assumptions, we have the following version of [2, Theorem 4.4.1]:

**Theorem 4.1.** *Given (1.1) with $0 < \tau \in \mathbb{R}_+$, suppose that* (i) k *satisfies (3.8a),* (ii) f *satisfies (3.8b) and* (iii) $\psi$, q, $m$, k *and* f *collectively satisfy (3.11). Suppose that* (iv) *(3.13) hold when* $\kappa_{\max}(q), \kappa_{\min}(q)$ *are replaced by* $\widehat{\kappa}_{\max}(\mathrm{q}) \equiv \widehat{\kappa}^{\tau}_{\max}(\mathrm{q})$ *and* $\widehat{\kappa}_{\min}(\mathrm{q}) \equiv \widehat{\kappa}^{\tau}_{\min}(\mathrm{q})$ *with*

$$\widehat{\kappa}^{\tau}_{\max}(\mathrm{q}) = \inf_{t\in[0,\varpi]} \int_{t-\tau}^{t} \mathrm{k}(t,s)\mathrm{q}(s)\mathrm{d}s, \qquad \widehat{\kappa}^{\tau}_{\min}(\mathrm{q}) = \sup_{t\in[0,\varpi]} \int_{t-\tau}^{t} \mathrm{k}(t,s)\mathrm{q}(s)\mathrm{d}s. \tag{4.1}$$

*Then there exists at least one $\varpi$-periodic solution* x *of (1.1) such that*

$$0 < \min\{\alpha,\beta\} < \sup_t |\mathrm{x}(t)| < \max\{\alpha,\beta\}; \qquad \mathrm{x}(t) \geq M \min\{\alpha,\beta\} \quad \textit{for } t \in \mathbb{R}. \tag{4.2}$$

A $\varpi$-periodic solution x is a solution that satisfies $\mathrm{x}(t+\varpi) = \mathrm{x}(t)$ for all $t \in \mathbb{R}$.

**Proposition 4.1.** *Suppose that $0 < \tau \in \mathbb{R}_+$ and that $\widetilde{\tau} \equiv \widetilde{\tau}(\varepsilon)$ with $0 < \tau - \varepsilon < \widetilde{\tau} \leq \tau$. If the conditions of Theorem 4.1 apply, they apply also when $\tau$ is replaced by $\tilde{\tau}$ provided that $\varepsilon$ is sufficiently small—and the conclusion of Theorem 4.1 applies to a $\varpi$-periodic solution*

$$\tilde{\mathrm{x}}(t) = \int_{t-\tilde{\tau}}^{t} \mathrm{k}(t,s)\mathrm{f}(s,\tilde{\mathrm{x}}(s))\mathrm{d}s. \tag{4.3}$$

*Further, provided that $\varepsilon$ is sufficiently small, if the conditions of Theorem 4.1 apply to (4.3) they establish the existence of a $\varpi$-periodic solution* x *of (1.1).*

**Proof.** We are concerned only with the effect of replacing $\kappa_{\max}(q), \kappa_{\min}(q)$ in (3.13) by $\widehat{\kappa}^{\widetilde{\tau}}_{\max}(q), \kappa^{\widetilde{\tau}}_{\min}(q)$ instead of by $\widehat{\kappa}^{\tau}_{\max}(q), \widehat{\kappa}^{\tau}_{\min}(q)$. Under the assumptions, the integrals $\int_{t'}^{t} \mathrm{k}(t,s)\mathrm{q}(s)\mathrm{d}s$ depend continuously on $t' \in [\tau, \tau+\varepsilon]$, uniformly in $t$. The integral with lower limit $t-\tau$ and that with lower limit $t-\widetilde{\tau}$ are therefore arbitrarily close (uniformly in $t$) for correspondingly small $\varepsilon$. Indeed, $|\int_{t-\tau}^{t} \mathrm{k}(t,s)\mathrm{q}(s)\mathrm{d}s - \int_{t-\tilde{\tau}}^{t} \mathrm{k}(t,s)\mathrm{q}(s)\mathrm{d}s| \leq \varepsilon \sup_{t\in\mathbb{R}} \sup_{s\in[t-\tau,t-\tilde{\tau}]} |\mathrm{k}(t,s)| \sup_{s\in[t-\tau,t-\tilde{\tau}]} |\mathrm{q}(s)|$ and hence is bounded by $\varepsilon \sup_{t\in[0,\varpi]} \sup_{s\in[t-\tau,t-\tilde{\tau}]} |\mathrm{k}(t,s)| \sup_{s\in[t-\tau,t-\tilde{\tau}]} |\mathrm{q}(s)|$. Thus the pair $(\widehat{\kappa}^{\tau}_{\min}(\mathrm{q}), \widehat{\kappa}^{\widetilde{\tau}}_{\min}(\mathrm{q}))$ and the pair $(\widehat{\kappa}^{\tau}_{\max}(\mathrm{q}), \widehat{\kappa}^{\widetilde{\tau}}_{\max}(\mathrm{q}))$ are in each case arbitrarily close for $\tau$ and $\widetilde{\tau}$ sufficiently close. □

We shall exploit the preceding result, which indicates (in broad terms) that one can consider $\tau > 0$ to be replaced by a nearby $\tilde{\tau}$. Assume that

$$\tau - \varepsilon < \tilde{\tau} \leq \tau, \quad \text{and} \quad \tilde{\tau} = N_{\tilde{\tau}}\mathfrak{h} \quad \text{where } N_{\tilde{\tau}} \in \mathbb{N}, \tag{4.4}$$

for sufficiently small $\varepsilon > 0$ that Proposition 4.1 applies. The process for ensuring (4.4) is discussed later (Section 4.4). We consider discretization of

$$\mathrm{x}(t) = \int_{t-\tilde{\tau}}^{t} \mathrm{k}(t,s)\mathrm{f}(s,\mathrm{x}(s))\mathrm{d}s, \quad \text{with } \tilde{\tau} > 0,\ \mathrm{x}(t) \in \mathbb{R}, \tag{4.5}$$

(which is (1.1), with $\tau$ replaced by $\tilde{\tau}$) using quadrature. When we discretize (4.5), we seek (for $N_{\tilde{\tau}} \in \mathbb{Z}_+$, $\mathfrak{h} = \tilde{\tau}/N_{\tilde{\tau}}$) a discrete system of the type

$$\widetilde{x}(n) = \sum_{j-N_{\tilde{\tau}}}^{n} \mathfrak{h} W_{n-j}\mathrm{k}(n\mathfrak{h}, j\mathfrak{h})\mathrm{f}(j\mathfrak{h}, \widetilde{x}(j)), \quad \widetilde{x}(n) \in \mathbb{R}, \tag{4.6}$$

with $\widetilde{x}(j) \approx \mathrm{x}(j\mathfrak{h})$. We shall examine conditions on the integral equation (1.1) which allow the analysis, as in [5], of periodic solutions, and the application to (4.6) of the discrete analysis developed earlier.

We assume that k satisfies conditions (for example, those in Theorem 4.1) that guarantee the existence of a $\varpi$-periodic solution x of (1.3), on the assumption that $\varepsilon$ is sufficiently small that the conditions continue to be satisfied when $\tau$ is replaced by $\tilde{\tau}$ in (4.4); see Proposition 4.1. To simulate Theorem 4.1 (and similar results) using discrete equations, we present in Section 4.1 (see also [16]) various quadratures that can be used to discretize (1.1), while ensuring the existence of periodic solutions of (4.6).

### 4.1. Quadrature rules: Basic properties and examples

We restrict attention to simple approximations associated with sampling the integrand at *equally-spaced abscissae*. Certain Newton–Cotes rules, the 2-point Radau rule, the composite versions, and classical Romberg and certain Gregory rules provide acceptable examples (see [6,16–18]). Though we shall need to discretize integrals over $[t-\tau, t]$ (first replacing the integral by that over $[t-\tilde{\tau}, t]$), our quadrature rule is defined by the approximation for an integral over [0, 1], of the type

$$\int_0^1 \psi(s)\mathrm{d}s \approx \sum_{\ell=0}^{N} w_{N,\ell}\psi(\ell h) =: \mathcal{Q}_{1/N}(\psi), \quad h = 1/N. \tag{4.7}$$

We restrict the *permitted* quadrature (4.7) to formulae satisfying

$$\sum_{\ell=0}^{N} w_{N,\ell} = 1, \quad \text{and} \quad w_{N,j} \geq 0 \quad \text{for } j \in \{0, 1, \ldots, N\}. \tag{4.8}$$

Since some weights $w_{N,j}$ might vanish, the values of $h$ and $N$ are fixed by requiring that (4.7) with $h \in (0, 1]$ cannot be rewritten using a larger value $h^* > h$ (where $h^* = 1/N^*$), as an approximation $\sum_{\ell=0}^{N^*} w^*_{N,\ell}\psi(\ell h^*)$.

An affine change of variable is used to secure, for arbitrary finite $a < b$, the *induced* approximation

$$\mathcal{Q}_{1/N}^{[a,b]}(\psi) := \sum_{j=0}^{N}\{(b-a)w_{N,j}\}\psi(a + j\mathfrak{h}) \approx \int_a^b \psi(s)\mathrm{d}s, \quad \mathfrak{h} = (b-a)/N \tag{4.9a}$$

($\mathcal{Q}_{1/N}(\psi)$ is $\mathcal{Q}_{1/N}^{[0,1]}(\psi)$). We can use the synonymous notation

$$\mathcal{Q}_{h}^{[a,b]}(\psi) = \sum_{j=0}^{N}\{(b-a)w_{N,j}\}\psi(a + j(b-a)h) \approx \int_a^b \psi(s)\mathrm{d}s, \quad \text{wherein } h = 1/N. \tag{4.9b}$$

For a given interval $[a, b]$ we relate $h$ and $\mathfrak{h}$ as here, and we denote $\{(b-a)h_1, (b-a)h_2, (b-a)h_3, \ldots\}$ by $\mathfrak{H} \equiv (b-a)H$ when $H = \{h_1, h_2, h_3, \ldots\}$, $h_\ell = 1/N_\ell$. $N_\ell \in \mathbb{N}$.

We use the concept of a *family* of quadrature rules. For example, a basic rule generates a family of related *composite* rules: We recall that a family of composite or *m-times repeated* quadrature formulae ($m \in \mathbb{N}$) is based on summation over $\ell$ of the contributions $\mathcal{Q}^{[\ell h,(\ell+1)h]}(\psi)$ (*cf.* (4.9)) to give

$$\int_0^1 \psi(s)\mathrm{d}s = \sum_{\ell=0}^{m-1}\int_{\ell/m}^{(\ell+1)/m} \psi(s)\mathrm{d}s \approx \sum_{\ell=0}^{m-1} \mathcal{Q}_{1/mN}^{[\ell/m,(\ell+1)/m]}(\psi) =: (m \times \mathcal{Q}_{1/N})(\psi) \quad (\text{for } m \in \mathbb{N}). \tag{4.10}$$

The approximation (4.10) is of the generic form (4.7); indeed one might write $(m \times \mathcal{Q}_{1/N})(\psi)$ as $\mathsf{Q}_{1/mN}(\psi)$. The collection $\{\mathsf{Q}_{1/mN}(\psi)\}_{m\in\mathcal{N}}$ where $\mathcal{N} \subseteq \mathbb{N}$ provides an example of a family (denoted, say $\mathfrak{Q}_{\mathrm{rptd}}$) of quadrature rules. Evidently, one obtains from (4.10) $\int_a^b \psi(s)\mathrm{d}s \approx \mathsf{Q}_{1/mN}^{[a,b]}(\psi)$ for $m \in \mathbb{N}$.

### *4.2. Discretization using families of quadrature rules*

We restrict attention to families of quadrature rules (4.7) that correspond to discrete convolutions—that is, where $w_{n,j} = W_{n-j}$ in (4.7).

**Definition 4.1.** A quadrature *family* $\mathfrak{Q}$ (a collection of rules) is defined by a set of formulae (4.7) for $h \in H^{\mathfrak{Q}} \subset (0, 1]$, in which $w_{n,j} = W_{n-j}$. We assume

$$H^{\mathfrak{Q}} := \{h_1, h_2, h_3, \ldots\}, \qquad N^{\mathfrak{Q}} := \left\{N^{[1]}, N^{[2]}, N^{[3]}, \ldots\right\} \quad (\text{in which } h_\ell = 1/N^{[\ell]}), \tag{4.11}$$

are defined by a monotonically decreasing sequence of values $h_\ell$ with $N^{[\ell]} \in N^{\mathfrak{Q}} \subseteq \mathbb{N}$. The choice of $h \in H^{\mathfrak{Q}}$ (or the choice of $N \in N^{\mathfrak{Q}}$) is assumed to define uniquely a particular rule in $\mathfrak{Q}$.

To discretize (1.1) and obtain a discrete system of the type (1.2), we employ

$$\int_{(n-N)\mathfrak{h}}^{n\mathfrak{h}} \psi(s)\mathrm{d}s \approx \sum_{j=n-N}^{n} \mathfrak{h}W_{n-j}\psi(j\mathfrak{h}) \quad \left(\text{with } n \in \mathbb{N}, \sum_{j=0}^{N} W_j = N,\ W_j > 0\right), \tag{4.12}$$

derived from *families of rules* (4.9) on setting $a = (n-N)\mathfrak{h}$ and $b = n\mathfrak{h}$ in corresponding families of rules (4.7), and for the case where $w_{n,j} = W_{n-j}$.

### *4.3. Convergence of quadrature rules as the stepsize tends to 0*

We denote by $R[0, 1]$ the space of *bounded Riemann-integrable functions* on $[0, 1]$.

The types of permitted quadrature rules discussed above provide generic sums, denoted $\{\mathcal{Q}_h(\psi)\}$, that approximate the integral $\int_0^1 \psi(s)\mathrm{d}s$. Thus, the expression $\frac{1}{2}h\psi(0)+h\psi(h)+\cdots+h\psi(1-h)+\frac{1}{2}h\psi(1)$, with $h = 1/m$, $N \in \mathbb{N}$, can be computed for $h = 1, h = \frac{1}{2}, h = \frac{1}{4}, h = \frac{1}{8}, \ldots$, or for $h = 1, \frac{1}{5}, h = \frac{1}{25}, h = \frac{1}{125}, \ldots$ and in general for $h = h_0, h = h_1, h = h_2, \ldots$ where $h_\ell = 1/N_\ell$, $N_\ell \in \mathbb{N}$, and $N_{\ell+1} > N_\ell$. In each case, the elements of the corresponding sequence converge (as $h_\ell \to 0$)

to a limit that coincides with the integral if $\psi \in R[0,1]$. Similar convergence issues arise for a general family of quadratures and a sequence of positive values $h \in H^{\mathfrak{Q}} = \{h_1, h_2, h_3, \ldots\}$ wherein $\lim_{\ell\to\infty} h_\ell = 0$.

**Definition 4.2.** The family $\mathfrak{Q} = \{\mathcal{Q}_h\}_{h\in H^{\mathfrak{Q}}}$ is termed *convergent* if, for any $\psi \in R[0,1]$,

$$\lim_{h\searrow 0,\, h\in H^{\mathfrak{Q}}} \mathcal{Q}_h^{[0,1]}(\psi) = \int_0^1 \psi(s)\mathrm{d}s. \tag{4.13}$$

**Assumption 5.** The family $\mathfrak{Q}$ is assumed to be convergent.

**Theorem 4.2.** *Repeated quadrature rules, classical Romberg rules, and Gregory rules having a fixed number of correction terms, using abscissae at step $h \in H^{\mathfrak{Q}}$, define families that satisfy (4.13).*

**Proof.** The associated quadrature approximations are either Riemann sums, weighted sum of a finite set of Riemann sums, or differ by $\mathcal{O}(h)\|\psi\|_\infty$ from a Riemann sum. The Riemann sums referred to here converge to the required integral as $h \searrow 0$. (See [6, p. 123], [17, Theorem 1], [16].) □

**Theorem 4.3.** *Suppose $\mathfrak{U}$ is a set of functions defined on* $[0,1]$ *that are uniformly bounded and equi-continuous, and (4.13) is valid. Then* $\lim_{h\searrow 0}\sup_{u\in\mathfrak{U}} |\mathcal{Q}_h(u) - \int_0^1 u(s)\mathrm{d}s| = 0$, *where $h \searrow 0$ with $h \in H$.*

**Proof.** If $u \in C[0,1]$, *a fortiori* $u \in R[0,1]$ so $\lim_{h\searrow 0} |\mathcal{Q}_h(u) - \int_0^1 u(s)\mathrm{d}s| = 0$. This convergence is uniform on compact sets, and $\mathfrak{U}$ is compact in $C[0,1]$ by the Arzela–Ascoli theorem. □

The proof of the next result is also straightforward [16].

**Lemma 4.1.** *Let* $\mathsf{k}(t,s)$, $\mathsf{f}(t,u)$ *satisfy (3.8) and let* $\mathsf{q} \in \mathcal{A}_\varpi$. *Then the family of integrands*

$$\{\mathsf{k}\,(t, t+(\sigma-1)\tau)\,\mathsf{f}\,(t+(\sigma-1)\tau, \mathsf{q}(t+(\sigma-1)\tau)) \text{ for } \sigma \in [0,1],\; t \in \mathbb{R}\}$$

*is uniformly bounded and equicontinuous.*

### *4.4. Approximate integration on $[t-\tau, t]$*

For $h \in H^{\mathfrak{Q}}$, $\mathcal{Q}_h \in \mathfrak{Q}$ induces, when $\psi \in R[a,b]$, the approximation $\mathcal{Q}_h^{[a,b]}(\psi)$ in (4.9). By assumption $\lim_{h\searrow 0}\mathcal{Q}_h^{[a,b]}(\psi) = \int_a^b \psi(s)\mathrm{d}s$ the limit being taken with $h \in H^{\mathfrak{Q}}$. Henceforth, we are mainly concerned with integrands $\psi(s)$ of the form $\mathsf{k}(t,s)\mathsf{f}(s,\mathsf{x}(s))$ with $t \in \mathbb{R}$ (integrated for $s \in [t-\tau, t]$).

Given arbitrary $\tau > 0$ and $t \in \mathbb{R}$, a convergent family of quadrature rules $\mathfrak{Q}$ induces corresponding formulae that generate discrete equations obtained from (1.1). For the specific $\tau$ in (1.1), we require quadrature (*cf.* (4.12)) to approximate integrals over $[t-\tau, t]$, in which $t = n\mathfrak{h}$, $\mathfrak{h} \in \mathfrak{H}_\varpi \cap \mathfrak{H}_\tau$. If $\tau/\varpi \notin \mathbb{Q}_+$, this cannot be achieved, so we replace $\tau$ by an approximation $\tilde\tau$ with $\tilde\tau/\varpi \in \mathbb{Q}_+$, and $\mathfrak{h} \in \mathfrak{H}_\varpi \cap \mathfrak{H}_{\tilde\tau}$. The approximation $\tilde\tau \approx \tau$ (with $\tilde\tau \le \tau$) can be made arbitrarily close by taking suitable sufficiently small $\mathfrak{h} \in \mathfrak{H}_\varpi$. To use rules from the family $\mathfrak{Q}$ we also require $\mathfrak{h} \in \tilde\tau\mathfrak{H}^{\mathfrak{Q}}$; this is a refinement of (4.4) requiring $N_{\tilde\tau} \in \mathbb{N}$; now, $N_{\tilde\tau} \in \mathfrak{N}^{\mathfrak{Q}}$. In commonplace families of quadrature, the members $N^{[\ell]}$ in the sequence $\mathfrak{N}^{\mathfrak{Q}}$ form either a increasing arithmetic progression or an increasing geometric progression.

**Assumption 6.** We are given positive $\tau$, $\varpi$, and a family of quadrature rules $\{\mathfrak{Q}_h\}_{h\in\mathfrak{H}^{\mathfrak{Q}}}$ associated with $\mathfrak{N}^{\mathfrak{Q}}$ in (4.11). We assume that, given arbitrary $\varepsilon > 0$, there exists a corresponding $\mathfrak{h}^\star > 0$ such that, whenever $\mathfrak{h} \le \mathfrak{h}_\star$ and $\mathfrak{h} \in \mathfrak{H}_\varpi$, there exist

$$\widetilde\tau \in [\tau-\varepsilon, \tau] \quad \text{with } \widetilde\tau = N_{\widetilde\tau}\mathfrak{h}, \text{ where } N_{\widetilde\tau} \in \mathfrak{N}^{\mathfrak{Q}}.$$

**Remark 4.1.** In Assumption 5, $\mathfrak{h} \in \mathfrak{H}_{\widetilde\tau} \cap \mathfrak{H}_\varpi \cap \widetilde\tau\mathfrak{H}_{\mathfrak{Q}}$ (and this condition is to be satisfied by $\mathfrak{h}^\star$, in the sense that $\mathfrak{h}^\star \in \mathfrak{H}_{\widetilde\tau^\star} \cap \mathfrak{H}_\varpi \cap \widetilde\tau^\star\mathfrak{H}_{\mathfrak{Q}}$). Suppose $\tau/\varpi = \varrho \in \mathbb{R}$; if $\widetilde\tau/\varpi \in \mathbb{Q}_+$ then $\widetilde\tau/\varpi = \mu_1/\mu_2$ with $\mu_{1,2} \in \mathbb{N}$. If $\mathfrak{h} \in \widetilde\tau\mathfrak{H}^{\mathfrak{Q}}$ then $\mathfrak{h} = \widetilde\tau/\widetilde N$ where $\widetilde N \in \mathfrak{N}^{\mathfrak{Q}}$ (then $\widetilde N = \widetilde\tau/\mathfrak{h}$). If $\mathfrak{h} \in \mathfrak{H}_\varpi$; then $N_\varpi := \varpi/\mathfrak{h} \in \mathbb{N}$. Hence, $\mu_1/\mu_2 = \widetilde N/N_\varpi$ where $\widetilde N \in \mathfrak{N}^{\mathfrak{Q}}$ and we need to be able to approximate $\varrho$ arbitrarily closely by fractions of the form $\widetilde N/N_\varpi$.

Given $\varpi$, $\mathfrak{h}$, with $\varpi = N_\varpi\mathfrak{h}$, identify $\omega = \omega(\mathfrak{h})$ with $N_\varpi$, and write $\omega^\star = \omega(\mathfrak{h}^\star)$ for $N_\varpi^\star$; if $\mathfrak{h} < \mathfrak{h}^\star$ then $N_\varpi > N_\varpi^\star$. Select $N_{\tilde\tau}$ so that $N_{\tilde\tau} = \tilde\tau/\mathfrak{h} \in \mathfrak{N}^{\mathfrak{Q}}$ where

$$\tilde\tau = \frac{N_{\tilde\tau}}{N_\varpi}\varpi = \max_{n\in\mathfrak{N}^{\mathfrak{Q}}}\left\{\frac{n}{N_\varpi}\varpi < \tau\right\} \quad \text{and} \quad \tilde\tau^\star = \frac{N_{\tilde\tau^\star}}{N_\varpi^\star}\varpi = \max_{n^\star\in\mathfrak{N}^{\mathfrak{Q}}}\left\{\frac{n^\star}{N_\varpi^\star}\varpi < \tau\right\}.$$

To ensure that $0 \le \tau - \widetilde\tau < \tau - \widetilde\tau^\star$ when $\mathfrak{h} < \mathfrak{h}^\star$, we require $N_{\tilde\tau} > N_{\tilde\tau^\star} \in \mathfrak{N}^{\mathfrak{Q}}$. With $\varrho = \tau/\varpi$, $N_{\tilde\tau}$ must be the largest integer in $\mathfrak{N}^{\mathfrak{Q}}$ such that $N_{\tilde\tau} < \varrho N_\varpi$. The commonplace $\mathfrak{N}^{\mathfrak{Q}}$ give $N^{[\ell]} = \mu + N^{[\ell-1]}$ with $0 < \mu \in \mathbb{N}$ or $N^{[\ell]} = \nu \times N^{[\ell-1]}$ with

$1 < \nu \in \mathbb{N}$, and, for simplicity and convenience, we take the latter case with $N^{[\ell]} = \nu^{\ell}$. Then, $N_{\tilde{\tau}} = \nu^{\tilde{\lambda}}$ with $\tilde{\lambda} = \lfloor \log_{\nu}(\varrho N_{\varpi}) \rfloor$ (and $N_{\tilde{\tau}} > N_{\tilde{\tau}^\star}$).

Our approximate integration proceeds, with a choice of $\mathfrak{h} \in \mathfrak{H}_{\varpi} \subset (0, \tau]$, to determination of $\tilde{\tau}$, to application of a quadrature rule. When

$$\left| \int_{t-\tau}^{t} \psi(s)\mathrm{d}s - \int_{t-\tilde{\tau}}^{t} \psi(s)\mathrm{d}s \right| \le \varepsilon \sup_{s \in [t-\tau, t-\tilde{\tau}]} |\psi(s)|, \tag{4.14}$$

the quadrature involved in (4.7) induces, for $t = n\mathfrak{h}$, and on setting $N$ equal to $N_{\tilde{\tau}}$, the approximation

$$\int_{t-\tilde{\tau}}^{t} \psi(s)\mathrm{d}s \approx \sum_{\ell=0}^{N_{\tilde{\tau}}} \tilde{\tau} w_{\ell}\, \psi(t - \tilde{\tau} + \ell\,\mathfrak{h}) =: \mathcal{Q}_{\mathfrak{h}}^{[t-\tilde{\tau},t]}(\psi), \quad (\mathfrak{h} = \tilde{\tau}/N_{\tilde{\tau}}), \tag{4.15}$$

so the sum in (4.15) approximates the first integral in (4.14) with a sum of the type in (4.12). For examples, see Appendix B.

### 4.5. Dependence on $\tau$

**Remark 4.2.** Prior to discretization, we replaced $\tau$ by a nearby value $\tilde{\tau}$ in the integral equation. To analyze this further, we can appeal to variations of parameters formulae and to inequalities like that of Gronwall [6].

Consider a solution $\mathrm{x}(t) \equiv \mathrm{x}(\tau; t)$ of $\mathrm{x}(\tau; t) = \int_{t-\tau}^{t} \mathrm{k}(t, s)\mathrm{f}(s, \mathrm{x}(\tau; s))\mathrm{d}s$ (that is, of (1.1)). One can see that if $\delta\mathrm{x}(\tau; t) = \mathrm{x}(\tilde{\tau}; t) - \mathrm{x}(\tau; t)$ then

$$\delta\mathrm{x}(\tau; t) = \int_{t-\tau}^{t} \mathrm{k}(t, s)\{\mathrm{f}(s, \mathrm{x}(\tilde{\tau}; s)) - \mathrm{f}(s, \mathrm{x}(\tau; s))\}\mathrm{d}s + \int_{t-\tau}^{t-\tau} \mathrm{k}(t, s)\mathrm{f}(s, \mathrm{x}(\tilde{\tau}; s))\mathrm{d}s. \tag{4.16}$$

Further, if it exists, the first-order sensitivity, can be expressed as $\frac{\partial}{\partial\tau}\mathrm{x}(\tau; t)$. If the derivatives exist,

$$\frac{\partial}{\partial\tau}\mathrm{x}(\tau; t) = \int_{t-\tau}^{t} \left\{ \mathrm{k}(t, s)\mathrm{f}_2(s, \mathrm{x}(\tau; s))\frac{\partial}{\partial\tau}\mathrm{x}(\tau; s) \right\} \mathrm{d}s - \mathrm{k}(t, t-\tau)\mathrm{f}(t-\tau, \mathrm{x}(\tau; t-\tau)). \tag{4.17}$$

The two displayed equations permit further investigation; under mild additional conditions and assumptions, not pursued here, $\delta\mathrm{x}(\tau; t)$ can be related to the size of $\tau - \tilde{\tau}$.

### 4.6. Numerics of integral equations

We can now address the issue of guaranteeing conditions for periodic solutions when we discretize an integral equation (1.1) for which a result like Theorem 4.1 holds. Given a $\varpi$-periodic solution $\mathrm{x}(t)$ of (1.1), the sequence $\{x(n) = \mathrm{x}(n\mathfrak{h})\}_{n\in\mathbb{N}}$ is $\omega$-periodic where $\omega = \varpi/\mathfrak{h}$ and we seek conditions ensuring a $\omega$-periodic solution of our discretized equations. Obviously, $\omega$ is a function of $\mathfrak{h}$, $\omega = \omega(\mathfrak{h})$. We now consider an equation of the type (4.6) but in the form

$$\tilde{x}(n) = \sum_{j=n-N_{\tilde{\tau}}(\mathfrak{h})}^{n} \mathfrak{h} W_{n-j}\mathrm{k}(n\mathfrak{h}, j\mathfrak{h})\mathrm{f}(j\mathfrak{h}, \tilde{x}(j)), \quad \mathfrak{h} \in \mathfrak{H}_{\varpi},\ \tilde{x}(n) \in \mathbb{R}. \tag{4.18}$$

This is of the form (1.2) with $k(n, j) := \mathfrak{h}W_{n-j}\mathrm{k}(n\mathfrak{h}, j\mathfrak{h})$, $f(j, u) := \mathrm{f}(j\mathfrak{h}, u)$. We define (4.18) to be the $\mathcal{Q}_{\mathfrak{h}}$-based discretization of (1.1) and we examine the assumptions made in Lemma 3.1 and Propositions 3.1–3.4 when applied to (4.18). Now if $\varpi = \omega\mathfrak{h}$ for $\omega \in \mathbb{N}$, then $k(n, j) = k(n + \omega, j + \omega)$ when $\mathrm{k}(n\mathfrak{h} + \varpi, j\mathfrak{h} + \varpi) = \mathrm{k}(n\mathfrak{h}, jh)$, and $f(n, u) = f(n + \omega, u)$ when $\mathrm{f}(n\mathfrak{h} + \varpi, u) = \mathrm{f}(n\mathfrak{h}, u)$. The following is typical of the results we seek.

**Proposition 4.2.** *Given* $\mathrm{q} \in \mathcal{A}_{\varpi}$, *and* f *satisfying* (3.8b), *suppose that* $\varpi = \omega\mathfrak{h}$ *and define* $q(n) = \mathrm{q}(n\mathfrak{h})$, $f(n, u) = \mathrm{f}(n\mathfrak{h}, u)$.(a) *Suppose that* $\psi$ *satisfies* (3.4a) *and the functions* f, $\psi$ *and* q *satisfy* $\mathrm{f}(t, u) \le \mathrm{q}(n)\psi(u)$ *for all* $u \in \mathbb{R}_+$ *and* $t \in \mathbb{R}$. *Then* (3.4b) *is satisfied.*(b) *If there exists a constant* $a_0 \in (0, 1)$, *such that* f, $\psi$, *and* q *satisfy* $a_0\mathrm{q}(t)\psi(u) \le \mathrm{f}(t, u)$ *for all* $u \in \mathbb{R}_+$ *and* $t \in \mathbb{R}$, *then* (3.4c) *is satisfied.*

Other assumptions in the discrete case depend on $\kappa_{\min,\max}(q)$ ($q \in \mathcal{A}_{\omega}$, *cf.* (3.7)), which, with $q(n) = \mathrm{q}(n\mathfrak{h})$ compare with $\widehat{\kappa}^{\tilde{\tau}}_{\min,\max}(\mathrm{q})$ in (4.1). Since

$$\int_{t-\tau}^{t} \mathrm{k}(t, s)\mathrm{q}(s)\mathrm{d}s = \int_{0}^{1} \mathrm{k}(t, t + (\sigma - 1)\tau)\mathrm{q}(t + (\sigma - 1)\tau)\mathrm{d}\sigma \approx \sum_{j} w_j\mathrm{k}(t, t + (j/N_{\tau} - 1)\tau)\mathrm{q}(t + (j/N_{\tau} - 1)\tau),$$

Theorem 4.3 and Lemma 4.1 have obvious corollaries that combine (on setting $f(t, u) = u$) to give us Proposition 4.3 below. (The family $\mathfrak{Q}$ of quadratures inducing $\{\mathcal{Q}_{\mathfrak{h}}\}$ is always assumed to be convergent.)

**Proposition 4.3.** *Suppose that* $\mathrm{k}(t, s)$ *satisfies* (3.8a) *in Assumption* 3.2 *and* $\mathrm{q} \in \mathcal{A}_{\varpi}$ *is continuous. Then,* $\sum_{j-N_{\tilde{\tau}}(\mathfrak{h})}^{n} \mathfrak{h}W_{n-j}\mathrm{k}(t, jh)$ $\mathrm{q}(jh) - \int_{t-\tau}^{t} \mathrm{k}(t, s)\mathrm{q}(s)\mathrm{d}s$ *is arbitrarily small (uniformly for* $t$ *in* $\mathbb{R}$*) and* $\widehat{\kappa}^{\tilde{\tau}}_{\min}(\mathrm{q})$, $\widehat{\kappa}^{\tilde{\tau}}_{\max}(\mathrm{q})$ *differ from* $\kappa_{\min}(q)$ *and* $\kappa_{\max}(q)$ *(respectively) by arbitrarily small amounts for sufficiently small* $\mathfrak{h}$.

It now follows that our discrete theory is applicable to the equations obtained by application of convergent quadrature of the type discussed above, given sufficient conditions on the integral equation. The following result is typical:

**Proposition 4.4.** *Suppose the conditions of Theorem 4.1 hold, and consider the $\mathcal{Q}_{\mathfrak{h}}$-discretization (4.18). Then the conditions of Proposition 3.1 apply to (4.18) for all sufficiently small $\mathfrak{h} \in H$.*

## 5. Modifications for weakly singular equations and their discretization

The discussion that we have provided can be adapted to the treatment of a typical class of weakly-singular kernels (kernels of Abel type). We indicate one approach. Consider

$$\mathsf{x}(t) = \int_{t-\tau}^{t} \mathsf{k}^{\sharp}(t,s)\mathsf{f}(s,\mathsf{x}(s))\mathrm{d}s, \quad \text{for } t \in \mathbb{R} \text{ with } \tau > 0,\ \mathsf{x}(t) \in \mathbb{R}. \tag{5.1}$$

**Assumption 7.** Suppose the functions $\mathsf{k}$ and $\mathsf{f}$ satisfy (3.8) and the possibly unbounded kernel $\mathsf{k}^{\natural}$ satisfies

$$\mathsf{k}^{\natural}(t,s) = \mathsf{k}(t,s)/|t-s|^{\nu} \quad \text{with } \nu \in (0,1). \tag{5.2}$$

With this assumption, $\mathsf{k}^{\natural}(t+\varpi, s+\varpi) = \mathsf{k}^{\natural}(t,s)$ for $\varpi \in \mathbb{R}_+$ for every $t, s \in \mathbb{R}$. For every $u \in C(\mathbb{R})$ we use the notation $T^{\sharp}u(t) = \int_{t_\tau}^{t} \mathsf{k}^{\sharp}(t,s)f(s,u(s))\mathrm{d}s$. Without ambiguity, we employ the notation $T^{\sharp}$ for the operator defined on given $D \subseteq \mathcal{A}_{\omega}$, the latter being equipped with the uniform norm. *This operator $T^{\sharp}$ is a compact map from $D \in \mathcal{A}_{\omega}$ to $\mathcal{A}_{\omega}$*, and the abstract functional analysis can be applied to the integral equation case. Preparatory to the numerics, if we define $A(t) = \int_{t_\tau}^{t} (t-s)^{-\nu}\mathrm{d}s = \tau^{1-\nu}/\{1-\nu\}$, then

$$T^{\sharp}u(t) = \int_{t_\tau}^{t} \frac{\mathsf{k}(t,s)f(s,u(s)) - \mathsf{k}(t,t)f(t,u(t))}{(t-s)^{\nu}}\mathrm{d}s + A(t)\mathsf{k}(t,t)f(t,u(t)). \tag{5.3}$$

To obtain a discretized equation we can now apply quadrature to approximate the integral term (the integrand is continuous). As in the case of continuous kernels, we can apply the general discrete theory if we discretize the integral appropriately. The use of quadrature as indicated may give low-order accuracy. More generally, we can adapt other numerical techniques found in the literature [6] for an equation of Abel type.

## 6. Concluding remarks

We are indebted to the referees for their careful reading of the submitted paper and their erudite comments on various details in our approach. In the current work, we have demonstrated (as a first step) the existence of periodic solutions under certain conditions. In our view, a significant next step is to discuss whether such periodic solutions act as attractors to nearby solutions. From the perspective of numerical analysis, it would also be of interest to investigate refinements or modifications of the quadrature methods discussed here, and minimal conditions under which the discretized solutions are accurate.

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## Appendix A. Some background analysis and related observations

Classical fixed-point theorems include those named after Bertus (L.E.J.) Brouwer (1881–1966), and the Polish mathematician Julius Schauder (1899–1943). We first recall Brouwer's theorem in $n$ dimensions:

**Theorem A.1.** *Every continuous map $T$ from the closed unit ball in $\mathbb{R}^n$ to itself has at least one fixed point.*

The usual generalizations of Brouwer's fixed-point theorem to infinite-dimensional spaces all include a compactness assumption of some sort and often, in addition, an assumption of convexity. We recall Schauder's fixed-point theorem.

**Theorem A.2.** *A completely continuous operator that transforms a bounded convex and closed set into itself has at least one fixed point in the given set.*

**Definition A.1.** Let $\mathcal{X} = \{X, \|\cdot\|\}$ be a Banach space (a complete, normed, linear space). (i) A closed, convex, set $C$ in $\mathcal{X}$ is a *(positive) cone* when:

(a) if $u \in C$ then $\lambda u \in C$ for $\lambda \geq 0$; (A.1)

(b) if $u \in C$ and $-u \in C$ then $u = 0$. (A.2)

(ii) A cone $C$ in $\mathcal{X}$ induces a *partial ordering* $\leqslant$ in $\mathcal{X}$ by the definition $u \leqslant v$ if and only if $v - u \in C$. For $u$, $v$ in $C$ such that $v - u \notin C$ we write $u \nleqslant v$. A Banach space with a partial ordering induced by a cone is a *partially ordered Banach space*. (iii) An operator $T$ is a *compression* of a cone $C$ in an ordered Banach space if (a) $T(0) = 0$; (b) there exists $r$, $R$ with $0 < r < R$ such that $T(u) \nleqslant u$ if $u \in C$, $\|u\| \le r$, and $u \neq 0$; and also, (c) for all $\varepsilon > 0$, $(1+\varepsilon)u \nleqslant T(u)$ if $u \in C$, $\|u\| \ge R$.

Using Definition A.1 we can state the "compression of the cone" theorem [3, p. 137] due to Krasnosel'skiĭ (1920–1997). It reads as follows.

**Theorem A.3.** *Let the positive completely continuous operator $T$ be a compression of the cone $C$. Then $T$ has at least one non-zero fixed point on $C$.*

Theorem A.3 can be established using Schauder's theorem. Theorem A.3 was refined by Leggett and Williams [4] who proved the following result, in which the conditions associated with a compression are relaxed.

**Theorem A.4.** *Given a cone $C \in \mathcal{X}$, define $C_\rho := \{v \in C | \|v\| \le \rho\}$ $(\rho \in (0,\infty))$ and $C_\infty = C$. Suppose $u \in C \setminus \{0\}$ and $C[u] := \{v \in C | \alpha v \geqslant u \text{ for some } \alpha > 0\}$. For some $R > 0$, suppose that $T : C_R \to C$ is completely continuous with $T(0) = 0$, and there exists $r$ with $0 < r < R$ such that $T(u) \nleqslant u$ if $u \in C$, $\|u\| = r$ and for all $\varepsilon > 0$, $(1+\varepsilon)u \nleqslant T(u)$ if $u \in C$, $\|u\| = R$. Then $T$ has a fixed point $x \in C$ with $r \le \|x\| \le R$.*

**Remark A.1.** In order to emphasize the connections with theories for (1.1), we have relied on Theorems 2.1–2.3 to establish our main results. We observe, however, that Propositions 3.1 and 3.2 can be deduced by arguments using Brouwer's theorem that by-pass these theorems. Unlike Brouwer's fixed-point theorem (*per se*), Theorems 2.1–2.3 share with our propositions the feature that they provide concrete conditions to be verified to establish the existence of solutions satisfying explicit conditions.

## Appendix B. Examples of quadrature families

For the integral in (4.5) over $[n\mathfrak{h} - \tilde{\tau}, n\mathfrak{h}]$, (4.10) induces the approximation

$$\int_{t-\tilde{\tau}}^{t} \psi(s)\mathrm{d}s \approx \sum_{j=n-N_{\tilde{\tau}}}^{n} \mathfrak{h} W_{n-j}^{(m\times\mathcal{Q})} \psi(j\mathfrak{h}) \quad (t = n\mathfrak{h},\ \mathfrak{h} = \tilde{\tau}/N_{\tilde{\tau}},\ N_{\tilde{\tau}} = mN \in \mathbb{N}),$$

with $W_j^{(m\times\mathcal{Q})} = w_{N,s}$ if $j \equiv s \bmod (m)$ and $s \neq 0$, $W_0^{(m\times\mathcal{Q})} = w_{N,N}$, $W_{rm}^{(m\times\mathcal{Q})} = w_{N,0} + w_{N,N}$ if $rm \notin \{0, N_{\tilde{\tau}}\}$, $W_{N_{\tilde{\tau}}}^{(m\times\mathcal{Q})} = w_{N,0}$. By assumption, $w_{N,j} \ge 0$ for $j \in \{0, 1, \ldots, N\}$; hence, $W_j^{(m\times\mathcal{Q})}$ is non-negative for $j \in \{0, 1, \ldots, N_{\tilde{\tau}}\}$. Likewise, a given rule $R$ from amongst the classical Romberg rules gives an approximation expressible as $\int_0^1 \psi(s)\mathrm{d}s \approx \sum_{j=0}^{2^m} w_{N,j}^{[R]} \psi(j/\{2^m\})$, where $m \in \mathbb{N}$, and where $w_{N,j}^{[R]} = w_{N,N-j}^{[R]} > 0$, and $N = 2^m$. We thus obtain, for $t = n\mathfrak{h}$, formulae of the type

$$\int_{t-\tilde{\tau}}^{t} \psi(s)\mathrm{d}s \approx \sum_{jb=n-2^m}^{n} \mathfrak{h} W_{n-j}^{[R]} \psi(t - \tilde{\tau} + j\mathfrak{h}), \quad \mathfrak{h} = \tilde{\tau}/2^m,\ \text{with } N_{\tilde{\tau}} = 2^m. \tag{B.2}$$

(Note the restricted form of $\mathfrak{h}$.) Further, the $r$th Gregory rule $G_r^{N_{\tilde{\tau}}}$ (using $N_{\tilde{\tau}} + 1$ abscissae, and with $r \le N_{\tilde{\tau}}$) gives for $t = n\mathfrak{h}$ an approximation

$$\int_{t-\tilde{\tau}}^{t} \psi(s)\mathrm{d}s \approx \sum_{j=n-N_{\tilde{\tau}}}^{n} \mathfrak{h} W_{n-j}^{[G]} \psi(j\mathfrak{h}) \quad (t = n\mathfrak{h},\ \tilde{\tau} = N_{\tilde{\tau}}\mathfrak{h},\ G \equiv G_r^{N_{\tilde{\tau}}}) \tag{B.3}$$

in which $W_{n-j}^{[G]} = W_j^{[G]}$, for $j \in \{0, 1, \ldots, N_{\tilde{\tau}}\}$. These rules include the Newton–Cotes case (for $r = N_{\tilde{\tau}}$), so the weights need not be positive, as we require.

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