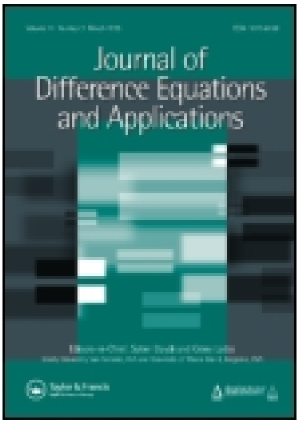

Journal of Difference Equations and Applications
Publication details, including instructions for authors and subscription information:
http://www.tandfonline.com/loi/gdea20

Non-negative convergent solutions of discrete Volterra equations

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Version of record first published: 17 Feb 2010.

To cite this article: Yihong Song , Jingsong Ni & Christopher T.H. Baker (2011): Non-negative convergent solutions of discrete Volterra equations, Journal of Difference Equations and Applications, 17:03, 423-439

To link to this article: http://dx.doi.org/10.1080/10236190903022766

Journal of Difference Equations and Applications
Vol. 17, No. 3, March 2011, 423–439

Non-negative convergent solutions of discrete Volterra equations

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(*Received 7 July 2008; final version received 5 May 2009*)

The existence of non-negative convergent solutions of discrete Volterra equations is obtained, by using a variety of fixed-point theorems. Examples are given to illustrate the results.

Keywords: fixed-point theorems; discrete Volterra equations; admissibility; non-negative convergent solutions

AMS Subject Classification: 39A10; 39A11; 39A12; 39A70

1. Introduction

In this paper, we study the *implicit discrete Volterra equations*

$$x(n) = \sum_{j=0}^{n} B(n,j) f(j, x(j)), \quad n \in \mathbb{Z}^+, \tag{1}$$

with

$$B(n,j) \in \mathbb{R} \quad \text{for each } j \le n \in \mathbb{Z}^+ \text{ and } f : \mathbb{Z}^+ \times \mathbb{R} \to \mathbb{R}.$$

(We write $\mathbb{Z}^+ = \{0, 1, \ldots, \}$, $\mathbb{R} = (-\infty, \infty)$ and $\mathbb{R}^+ = [0, \infty)$.) Values $B(n,j)$ with $j > n$ do not enter the summation equations above and are to be set to zero.

Here, $x = \{x(n)\}_{n \in \mathbb{Z}^+}$ is a sequence, with $x(n) \in \mathbb{R}$, that is to be determined. Thus in the study of (1), the first issue to address is the existence of a solution $x(n)$ to the equation

$$x(n) - B(n,n) f(n, x(n)) = \sum_{j=0}^{n-1} B(n,j) f(j, x(j)) \ \text{(for each } n = 0, 1, 2, \ldots),$$

given, for $n \ge 1$, the 'preceding' values $x(0), \ldots, x(n-1)$.

Such a solution $\{x(0), x(1), x(2), \ldots\}$ of (1) is *non-negative*, if $x(j) \ge 0$ for $j \in \mathbb{Z}^+$ and is termed a *convergent solution*, if $\lim_{j \to \infty} x(j)$ exists and is finite. We continue the work started in [30], in order to establish existence results for non-negative convergent solutions

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ISSN 1023-6198 print/ISSN 1563-5120 online

DOI: 10.1080/10236190903022766
http://www.informaworld.com

of (1); we use admissibility theory for discrete equations [30] and a variety of fixed-point theorems. We also give examples to illustrate the results.

Remark 1. The theory of difference equations (see, for example [1,17]) has gained widespread attention. For recent work related to discrete Volterra equations such as (1), see (for example) [1,3,4,6–8,14–19,21,25,27–31].

The results in this paper are theoretical, but they are in principle applicable. In practice, discrete equations of the form (1) may arise in their own right as discrete models or from certain discretization procedures in the numerical solution of Volterra integral or functional equations. See, for example, [5,9–11,23] and references therein. Integral equations can sometimes be reduced to ordinary differential equations, integro-differential equations, or delay-differential equations; likewise, if $B(n,j) = 0$ when $n - j > k$ then our summation equations reduce to finite-term recurrence relations, and if $\{B(n,j)\}$ satisfy an appropriate recurrence relation then the summation equations can be reduced to related difference equations.

Our paper is organized as follows. In Section 2, we present some basic results and recall several fixed-point theorems; we give our main results in Section 3, which contains separate subsections corresponding to each of the fixed-point theorems that are employed to deduce results.

2. Preliminaries and basic results

In the next subsections, we consider some basic infrastructure.

2.1 *Some Banach spaces whose elements are sequences*

We need to consider various spaces of sequences (equivalent to spaces of functions defined on $\mathbb{Z}^+$). We denote by $\ell(\mathbb{R})$, the linear space

$$\ell(\mathbb{R}) = \left\{u : u = \{u(n)\}_{n=0}^{\infty}, u(n) \in \mathbb{R}\right\}. \tag{2}$$

When endowed with a metric induced by a choice of norm (for example, $\|u\| := \sup_{j\in\mathbb{Z}^+}|u(j)|$), $\ell(\mathbb{R})$ is a topological space. For any *positive* sequence $g = \{g(n)\}_{n\in\mathbb{Z}^+} \in \ell(\mathbb{R})$ (i.e. $g(n) > 0$ for all $n \in \mathbb{Z}^+$), we will denote by $\ell_g(\mathbb{R})$, the Banach space of all sequences of $\ell(\mathbb{R})$ such that $\sup_{n\in\mathbb{Z}^+}|x(n)|/|g(n)| < \infty$ with the norm $\|\cdot\|_{\ell_g}$ defined by

$$\|u\|_{\ell_g} = \sup_{n\in\mathbb{Z}^+}\frac{|u(n)|}{|g(n)|}, \tag{3}$$

for all $u \in \ell_g(\mathbb{R})$. For $\{g(n)\}_{n\in\mathbb{Z}^+}$ with $|g(n)| = 1$ ($n \in \mathbb{Z}^+$), the space $\ell_g(\mathbb{R})$ becomes the well-known space $\ell^\infty(\mathbb{R})$ with norm $|\cdot|_\infty$ given by

$$|u|_\infty = \sup_{n\geq 0}|u(n)|. \tag{4}$$

The opportunities for the choice of $\{g(n)\}_{n\in\mathbb{Z}^+}$ provide a large variety of spaces consisting of sequences with a required behaviour. One subspace of $\ell^\infty(\mathbb{R})$ is needed in the sequel.

We denote by $\ell^{\infty}_{\rm cvg}(\mathbb{R})$, the space of convergent sequences, namely

$$\ell^{\infty}_{\rm cvg}(\mathbb{R}) = \Big\{u | u \in \ell^{\infty}(\mathbb{R});\ \lim_{n\to\infty} u(n)\,\text{exists}\Big\}, \tag{5}$$

with the norm (4) inherited from $\ell^{\infty}(\mathbb{R})$. A set S of sequences, with $\mathsf{S} \subset \ell^{\infty}_{\rm cvg}(\mathbb{R})$, is termed *equi-convergent* if, given $\varepsilon > 0$, there corresponds $N(\varepsilon) > 0$ such that (where $u(\infty) := \lim_{n\to\infty} u(n)$), we have $|u(n) - u(\infty)| < \varepsilon$ for any $n \geq N(\varepsilon)$ and for all $u \in \mathsf{S}$.

DEFINITION 2.1. *Let $X_0\,X_1$, and X_2 denote (normed) linear spaces and suppose an operator $\mathfrak{T}$ maps X_0 onto a set $R_0 \subseteq X_2$ and $R_0 = \mathfrak{T}(X_0)$. We use the same notation for the operator obtained by restricting $\mathfrak{T}$ to $X_1 \subset X_0$. If we seek to emphasize that an operator $\mathfrak{T}$ is acting on a space X_1 and its range is $R_1 \subseteq X_2$, we write $\mathfrak{T} \in (X_1 \to X_2)$ or, with greater precision, $\mathfrak{T} \in (X_1 \to R_1)$. The notation $C(X_1 \to X_2)$ denotes the set of maps that are continuous from X_1 into X_2.*

Suppose that $X_{\natural}, X^{\natural}$ are normed linear spaces. The pair $(X_{\natural}, X^{\natural})$ is termed *admissible* with respect to the operator $\mathfrak{T}$, if $\mathfrak{T}(X_{\natural}) \subseteq X^{\natural}$.

2.2 Admissibility of the pair ($\ell_{\rm g}(\mathbb{R})$, $\ell^{\infty}_{\rm cvg}(\mathbb{R})$) and related results

For a given equation, solution properties, such as convergence, boundedness and positivity, are fundamental properties to be studied. As observed in ([13], §5.5), admissibility concepts are related to stability in various senses. Convergent solutions of Volterra integral equations and related causal problems have been discussed in [12] (Chapter 2), [13] (pp. 119–122) and [22], by using admissibility theory for continuous Volterra operators, contraction mapping theorems and topological degree, respectively. Below, we investigate convergent solutions of (1) by employing admissibility theory for discrete Volterra operators [30] and various fixed-point theorems.

We employ the notation

$$(Bu)(n) = \sum_{j=0}^{n} B(n,j)u(j), \quad \text{where } u \in \ell(\mathbb{R}),\ n \in \mathbb{Z}^{+}, \tag{6}$$

($u(j) \in \mathbb{R}$ for $j \in \mathbb{Z}^{+}$). Associated with (6) is a 'discrete Volterra operator' B defined on the linear space $\ell(\mathbb{R})$; we use the same notation for the corresponding operator defined on any normed linear subspace of $\ell(\mathbb{R})$, such as $\ell_{\rm g}(\mathbb{R})$. We adopt the following hypotheses throughout this paper. By definition, the pair $(\ell_{\rm g}(\mathbb{R}), \ell^{\infty}_{\rm cvg}(\mathbb{R}))$ is admissible with respect to the operator B defined by (6), if $B(\ell_{\rm g}(\mathbb{R})) \subseteq \ell^{\infty}_{\rm cvg}(\mathbb{R})$.

*Hypothesis H*1. The pair $(\ell_{\rm g}(\mathbb{R}), \ell^{\infty}_{\rm cvg}(\mathbb{R}))$ is admissible with respect to the operator B defined by (6), $B(n,j) \geq 0$ for all $0 \leq j \leq n$, $n \in \mathbb{Z}^{+}$, and

$$\lim_{n\to\infty} B(n,j) = \beta(j) < \infty, \quad \text{for each } j \in \mathbb{Z}^{+}. \tag{7}$$

Theorem 2.2(a) (see [30], Theorem 3.4) establishes conditions that ensure the admissibility needed in Hypothesis H1 once (7) is known to hold.

THEOREM 2.2.

(a) *Given the discrete Volterra operator B given by* (6), *suppose that* (7) *holds. Then a necessary and sufficient condition for the admissibility of the pair* $(\ell_{\mathrm{g}}(\mathbb{R}), \ell^{\infty}_{\mathrm{cvg}}(\mathbb{R}))$ *with respect to B is that:*

$$\sum_{j=0}^{\infty} |\beta(j)||g(j)| < \infty, \tag{8}$$

$$\lim_{n\to\infty} \sum_{j=0}^{n} |B(n,j)||g(j)| = \sum_{j=0}^{\infty} |\beta(j)||g(j)|. \tag{9}$$

(b) *In addition, if* $(\ell_{\mathrm{g}}(\mathbb{R}), \ell^{\infty}_{\mathrm{cvg}}(\mathbb{R}))$ *is admissible with respect to B, then for any* $u = \{u(n)\}_{n\geq 0} \in \ell_{\mathrm{g}}(\mathbb{R})$,

$$\lim_{n\to\infty} (Bx)(n) = \sum_{j=0}^{\infty} \beta(j)x(j).$$

Remark 2. If the kernel $B(n,j)$ in (6) is of convolution type $B(n,j) = b(n-j)$, then $\lim_{n\to\infty} b(n-j) = \lim_{n\to\infty} b(n)$ for any $j \geq 0$. In this case, it follows from (8) and (9) that Theorem 2.2 holds either for $\lim_{n\to\infty} b(n) = 0$ and any sequence $\{g(n)\}_{n\in\mathbb{Z}^+}$ with $g(n) \neq 0$ $(n \in \mathbb{Z}^+)$ or $\lim_{n\to\infty} b(n) \neq 0$ and $\sum_{j=0}^{\infty} |g(j)| < \infty$. Thus the results in this paper include the convolution kernel as a special case.

Remark 3. The choice of 'quadrature weights' used in quadrature discretization of

$$y(t) = \int_0^t k(t-s)f(s, y(s))\mathrm{d}s,$$

provides a source of examples in which $B(n,j)$ has the form $b(n-j)$ for integer j with $j_0 \leq j \leq n$ (compare the preceding Remark). As a very simple example, using a trapezoidal rule with $h > 0$, we obtain $B(n,j) = W(n,j)k([n-j]h)$ for j, $n \in \mathbb{Z}^+$, where $W(0,0) = 0$, $W(n,0) = W(n,n) = 1/2h$ $(n > 0)$, $W(n,j) = h$ for $0 < j < n$, $W(n,j) = 0$ for $j > n$. See also [24].

We list some results to be used in our discussion and omit details (see [26]). The first concerns a compactness criterion in $\ell^{\infty}_{\mathrm{cvg}}(\mathbb{R})$.

Observation 2.3. Let $\mathsf{S} \subset \ell^{\infty}_{\mathrm{cvg}}(\mathbb{R})$ satisfy the following conditions: (i) S is bounded in $\ell^{\infty}_{\mathrm{cvg}}(\mathbb{R})$ and (ii) the sequences in S are equi-convergent. Then S is compact in $\ell^{\infty}_{\mathrm{cvg}}(\mathbb{R})$.

This observation leads to the following result (see [26], Lemma 6) concerning discrete Volterra operators acting from $\ell_{\mathrm{g}}(\mathbb{R})$ to $\ell^{\infty}_{\mathrm{cvg}}(\mathbb{R})$.

LEMMA 2.4. *Suppose that Hypothesis H*1 *holds for the discrete Volterra operator* $B \in (\ell_{\mathrm{g}}(\mathbb{R}) \to \ell^{\infty}_{\mathrm{cvg}}(\mathbb{R}))$ *given by* (6). *Then B is completely continuous* (*the linear operator B is continuous and compact*).

Remark 4. We recall that an operator $\mathfrak{T} \in (X_{\#} \to X^{\#})$ where $X_{\#}$ is a topological space is called a *compact operator* when it maps an arbitrary bounded subset of $X_{\#}$ into a corresponding set that is relatively compact in $X^{\#}$. (A subset S of a topological space $X^{\#}$ is called relatively compact, if its closure $\bar{S}$ is compact.) A continuous compact operator is called *completely continuous*.

The existence of one or more non-negative convergent solutions of (1) will be established via fixed-point methods. To be specific, we will use either Kransnoselskii's fixed-point theorem (see [2], Theorem 2.1.1), the nonlinear alternative (see [2], Theorem 1.2.1) or the Leggett–Williams fixed-point theorem (see [2], Theorem 4.3.1). We now recall these theorems, in Theorems 2.5–2.7, respectively. (For basic properties of cones, and induced partial orderings, see [20].)

THEOREM 2.5 (KRANSNOSELSKII'S FIXED-POINT THEOREM). *Let X be a Banach space with a norm* $\|\cdot\|$ *and let* $W \subset X$ *be a cone in X. Assume* Ω_1 *and* Ω_2 *are open subsets of X with* $0 \in \Omega_1$, $\bar{\Omega}_1 \subset \Omega_2$, *and let* $\mathfrak{T} : W \cap (\bar{\Omega}_2 \backslash \Omega_1) \to W$ *be a completely continuous operator such that either* (*i*) $\|\mathfrak{T}u\| \leq \|u\|$, $u \in W \cap \partial\Omega_1$ *and* $\|\mathfrak{T}u\| \geq \|u\|$, $u \in W \cap \partial\Omega_2$ *or* (*ii*) $\|\mathfrak{T}u\| \geq \|u\|$, $u \in W \cap \partial\Omega_1$ *and* $\|\mathfrak{T}u\| \leq \|u\|$, $u \in W \cap \partial\Omega_2$. *Then* $\mathfrak{T}$ *has a fixed point in* $W \cap (\bar{\Omega}_2 \backslash \Omega_1)$.

THEOREM 2.6 (NONLINEAR ALTERNATIVE). *Let W be a convex subset of a normed linear space E and let U be an open subset of W, with* $p^\star \in U$. *Then every completely continuous map* $\mathfrak{T} : \bar{U} \to W$ *has at least one of the following two properties:* (*i*) $\mathfrak{T}$ *has a fixed point in* $\bar{U}$ *and* (*ii*) *there exists* $u \in \partial U$ *with* $u = (1 - \lambda)p^\star + \lambda\mathfrak{T}u$ *for some* $0 < \lambda < 1$.

THEOREM 2.7 (LEGGETT–WILLIAMS FIXED-POINT THEOREM). *Let* $E = (E, \|\cdot\|)$ *be a Banach space,* $W \subset E$ *a cone in E,* $r_1 > 0$, $r_2 > 0$, $r_1 \neq r_2$ *with* $R = \sup\{r_1, r_2\}$ *and* $r = \min\{r_1, r_2\}$. *Define* $\bar{W}_\eta = \{u \in W : \|u\| \leq \eta\}$ *and* $S_\eta = \{u \in W : \|u\| = \eta\}$. *Let* $N : \bar{W}_R \to W$ *be a completely continuous map such that* (*i*) *there exists* $u_0 \in W\backslash\{0\}$ *with* $Nu \not\leq u$ *for* $u \in S_r \cap W(u_0)$; *here,* $W(u_0) = \{u \in W : \text{there exists } \lambda > 0 \text{ with } u \geq \lambda u_0\}$ *and* (*ii*) $\|Nu\| \leq \|u\|$ *for* $u \in S_R$. *Then N has at least one fixed point* $x \in W$ *with* $r \leq \|x\| \leq R$.

Here, we start by studying the Nemytskii operator F, associated with f in (1) and defined by

$$F(u)(n) = f(n, u(n)) \quad \text{where } f : \mathbb{Z}^+ \times \mathbb{R} \to \mathbb{R}. \tag{10a}$$

We make one of the following assumptions:

Hypothesis H2. $f : \mathbb{Z}^+ \times \mathbb{R} \to \mathbb{R}^+$ and $f(n, \cdot)$ is continuous for each $n \in \mathbb{Z}^+$.

Hypothesis H2$^+$. $f : \mathbb{Z}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ and $f(n, \cdot)$ is continuous for each $n \in \mathbb{Z}^+$.

Clearly, the definition of the operator F requires us to specify the space in which u is required to lie (and the range of possible arguments n – the domain of u). (We use the same notation F whatever its domain; see Remark 2.1.) We always require $u \in \ell(\mathbb{R})$, but in general, we ask that

$$u \in \ell^\infty_{\text{cvg}}(\mathbb{R}), \quad n \in \{0, 1, 2, \dots\}, \tag{10b}$$

Thus $F \in (\ell^{\infty}_{\text{cvg}}(\mathbb{R}) \to \ell(\mathbb{R}))$. We shall wish to strengthen this by showing that for given g, $F(\ell^{\infty}_{\text{cvg}}(\mathbb{R})) \subseteq \ell_{\text{g}}(\mathbb{R}) \subset \ell(\mathbb{R})$ and that F has additional properties. The following observations follow immediately from appropriate definitions.

Observations 2.8.

(i) The Nemytskii operator F on $\ell^{\infty}_{\text{cvg}}(\mathbb{R})$ defined in (10) transforms bounded subsets of $\ell^{\infty}_{\text{cvg}}(\mathbb{R})$ into bounded subsets of $\ell_{\text{g}}(\mathbb{R})$, if and only if

$$\sup_{n\geq 0,|u|\leq k} \frac{|f(n,u)|}{g(n)} =: M_k < \infty \quad \text{for each } k > 0.$$

(ii) To verify that the Nemytskii operator F defined by (10) is a continuous map of $\ell^{\infty}_{\text{cvg}}(\mathbb{R})$ into $\ell_{\text{g}}(\mathbb{R})$, it is necessary and sufficient to establish (first) that $F(\ell^{\infty}_{\text{cvg}}(\mathbb{R})) \subseteq \ell_{\text{g}}(\mathbb{R})$ and also that

$$\lim_j \sup_{n\geq 0} \frac{|f(n,u_j(n)) - f(n,u_0(n))|}{g(n)} = 0,$$

when $\lim_{j\to\infty} u_j = u_0 \in \ell^{\infty}_{\text{cvg}}(\mathbb{R})$.

Suppose f, g are as above, then

$$f(n,v) = g(n)p(n,v) \quad \text{for all } n \geq 0 \text{ if } p(n,v) = \frac{f(n,v)}{g(n)} \quad (v \in \mathbb{R}). \tag{11a}$$

In our examples, we shall always discuss functions f with the related form

$$f(n,v) = g(n)p(n,v). \tag{11b}$$

Clearly, the properties of the Nemytskii operator F can be characterized by those of $p(n,u)$. Thus it is readily seen from Observations 2.8 that $F \in (\ell^{\infty}_{\text{cvg}}(\mathbb{R}) \to \ell_{\text{g}}(\mathbb{R}))$ defined by (10) is a continuous map, if f has the property $|f(n,v) - f(n,v')| \leq D_n g(n)|v - v'|$ for all $n \geq 0$ and any v, $v' \in \mathbb{R}$, where $D_n \geq 0$ and $\sup_{n\geq 0} D_n < \infty$.

3. Main results

Now, we return to equation (1), that is,

$$x(n) = \sum_{j=0}^{n} B(n,j)f(j,x(j)), \quad n \in \mathbb{Z}^+. \tag{12}$$

We define a corresponding operator T on $\ell(\mathbb{R})$ by $Tu = B(Fu)$ (we write $T = B \circ F$);

$$(Tu)(n) = \sum_{j=0}^{n} B(n,j)f(j,u(j)), \quad \text{for } n \in \mathbb{Z}^+ \text{ for any } u = \{u(n)\}_{n\in\mathbb{Z}^+} \in \ell(\mathbb{R}). \tag{13}$$

A fixed-point x of T (with $x = Tx$) is a solution of (12).

3.1 Positive convergent solutions via Krasnoselskii's fixed-point theorem

We now apply Krasnoselskii's fixed-point theorem to establish a result for (12).

THEOREM 3.1. *Suppose that Hypotheses H*1 *and H*2^+ *hold. If the following conditions are satisfied:*

Condition 3.1-1. $F \in C(\ell^\infty_{\rm cvg}(\mathbb{R}) \to \ell_{\rm g}(\mathbb{R}))$ (*the Nemytskii operator F is a continuous mapping of* $\ell^\infty_{\rm cvg}(\mathbb{R})$ *into* $\ell_g(\mathbb{R})$).

Condition 3.1-2. *There exists a non-decreasing function* ψ *and a sequence* q *with the following properties:* $\psi \in (\mathbb{R}^+ \to \mathbb{R}^+)$, $q = \{q(n)\}_{n\geq 0} \in \ell^\infty(\mathbb{R})$, $q(n) \geq 0$ *for all* $n \in \mathbb{Z}^+$ *and*

$$A_0 q(n)\psi(v) \leq \frac{f(n,v)}{g(n)} \leq q(n)\psi(v),$$

for all $v \in \mathbb{R}^+$ *and* $n \in \mathbb{Z}^+$, *where* A_0 *is a constant with* $0 < A_0 \leq 1$.

Condition 3.1-3. *There exists a function* $\phi \in ((0,1) \to (0,\infty))$ *such that for any* $0 < \tau < 1$ *and* $v \geq 0$, *we have*

$$\psi(\tau v) \geq \phi(\tau)\psi(v),$$

Condition 3.1-4.

$$K_2 = \inf_{n\geq 0}\sum_{j=0}^{n} q(j)B(n,j)g(j) > 0.$$

Condition 3.1-5. *There exists* $0 < M < 1$ *with*

$$\frac{M}{\phi(M)} \leq A_0\frac{K_2}{K_1}, \quad \text{where } K_1 = \sup_{n\geq 0}\sum_{j=0}^{n} q(j)B(n,j)g(j).$$

Condition 3.1-6. *There exists* $\alpha > 0$ *with* $\alpha > K_1\psi(\alpha)$.

Condition 3.1-7. *There exists* $\beta > 0$, $\beta \neq \alpha$, *with* $\beta < A_0K_2\psi(M\beta)$.

Then (12) has at least one positive convergent solution $x = \{x(n)\}_{n\in\mathbb{Z}^+} \in \ell^\infty_{\rm cvg}(\mathbb{R})$ *and either (A)* $0 < \alpha < |x|_\infty < \beta$ *and* $x(n) \geq M\alpha$ *for* $n \in \mathbb{Z}^+$, *if* $\alpha < \beta$ *or (B)* $0 < \beta < |x|_\infty < \alpha$ *and* $x(n) \geq M\beta$ *for* $n \in \mathbb{Z}^+$, *if* $\beta < \alpha$.

Remark 5. To obtain required properties of solutions of (12) by use of a particular theorem, one requires corresponding conditions on $B(n,j)$ and $f(j,u)$. Thus, Conditions 3.1-1 to 3.1-7 guarantee that we can apply Kransnoselskii's fixed-point theorem to deduce the existence of positive convergent solutions of (12). Similar conditions were given for obtaining positive periodic (or almost periodic) solutions of non-linear Volterra integral equations in [2] (Chapter 4) and constant-sign periodic and almost periodic solutions of a system of difference equations in [3].

Note that K_1 in Condition 3.1-5 is finite due to Hypothesis H1 and the fact that $q = \{q(n)\}_{n\geq 0} \in \ell^\infty(\mathbb{R})$. Indeed, (8) and (9) in Theorem 2.2 guarantee that

$$0 \leq \sup_{n\geq 0}\sum_{j=0}^{n} B(n,j)g(j) < \infty.$$

Proof. Let $X = (\ell^{\infty}_{\mathrm{cvg}}(\mathbb{R}), |\cdot|_{\infty})$ and

$$W = \left\{ u = \{u(n)\}_{n\in\mathbb{Z}^+} \in \ell^{\infty}_{\mathrm{cvg}}(\mathbb{R}) : u(n) \geq M|u|_{\infty} \quad \text{for } n \in \mathbb{Z}^+ \right\},$$

where M is defined in Condition 3.1-5. It is obvious that W is a cone in $\ell^{\infty}_{\mathrm{cvg}}(\mathbb{R})$. The operator $T = B \circ F$ is defined by

$$(Tu)(n) = \sum_{j=0}^{n} B(n,j)f(j,u(j)), \quad n \in \mathbb{Z}^+.$$

We shall prove that $T(W) \subseteq W$ and then that T has a fixed point in $\ell^{\infty}_{\mathrm{cvg}}(\mathbb{R})$. Since the Nemytskii operator F associated with f is an operator from $\ell^{\infty}_{\mathrm{cvg}}(\mathbb{R})$ into $\ell_{\mathrm{g}}(\mathbb{R})$ by Condition 3.1-1 and since the pair $(\ell_{\mathrm{g}}(\mathbb{R}), \ell^{\infty}_{\mathrm{cvg}}(\mathbb{R}))$ is admissible with respect to the operator B, we have $(B \circ F)(W) \subseteq \ell^{\infty}_{\mathrm{cvg}}(\mathbb{R})$, that is $T(W) \subseteq \ell^{\infty}_{\mathrm{cvg}}(\mathbb{R})$. We first show that

$$T \in (W \to W). \tag{14}$$

Let $u = \{u(n)\}_{n\in\mathbb{Z}^+} \in W$. Then Condition 3.1-2 implies that for $n \in \mathbb{Z}^+$,

$$|(Tu)(n)| \leq \psi(|u|_{\infty}) \sup_{n\geq 0} \sum_{j=0}^{n} q(j)B(n,j)g(j) = K_1 \psi(|u|_{\infty}). \tag{15}$$

On the other hand, since $u \in W$, we have $u(n) \geq M|u|_{\infty}$ for $n \in \mathbb{Z}^+$, and therefore Conditions 3.1-2 to 3.1-3, (15) and Condition 3.1-5 give

$$\begin{aligned}(Tu)(n) &\geq A_0 \sum_{j=0}^{n} q(j)B(n,j)g(j)\psi(u(j)) \geq A_0 \psi(M|u|_{\infty}) \sum_{j=0}^{n} q(j)B(n,j)g(j) \\ &\geq A_0 \phi(M)\psi(|u|_{\infty}) \sum_{j=0}^{n} q(j)B(n,j)g(j) \geq K_2 A_0 \phi(M)\psi(|u|_{\infty}) \geq \frac{K_2}{K_1} A_0 \phi(M)|Tu|_{\infty} \\ &\geq M|Tu|_{\infty},\end{aligned}$$

for $n \in \mathbb{Z}^+$. Thus $Tu \in W$ and (14) holds. Next, Condition 3.1-2 implies that the Nemytskii operator F associated with the function f transforms bounded subsets of $\ell^{\infty}_{\mathrm{cvg}}(\mathbb{R})$ into bounded subsets of $\ell_{\mathrm{g}}(\mathbb{R})$ by Observations 2.8. It follows that T is a completely continuous map of $\ell^{\infty}_{\mathrm{cvg}}(\mathbb{R})$ into $\ell^{\infty}_{\mathrm{cvg}}(\mathbb{R})$ because $B \circ F$ is the composition of F, which is continuous and transforms bounded subsets of $\ell^{\infty}_{\mathrm{cvg}}(\mathbb{R})$ into bounded subsets of $\ell_{\mathrm{g}}(\mathbb{R})$, while B is completely continuous by Lemma 2.4.

Let $\Omega_{\alpha} = \{u \in \ell^{\infty}_{\mathrm{cvg}}(\mathbb{R}) : |u|_{\infty} < \alpha\}$ and $\Omega_{\beta} = \{u \in \ell^{\infty}_{\mathrm{cvg}}(\mathbb{R}) : |u|_{\infty} < \beta\}$. To apply Krasnoselskii's fixed-point theorem (Theorem 2.5), we shall show that

$$|Tu|_{\infty} < |u|_{\infty} \quad \text{for } u \in W \cap \partial\Omega_{\alpha} = S_{\alpha}, \tag{16}$$

and

$$|Tu|_{\infty} > |u|_{\infty} \quad \text{for } u \in W \cap \partial\Omega_{\beta} = S_{\beta}. \tag{17}$$

Notice that conditions (16) and (17) imply naturally that the operator T satisfies the conditions in Theorem 2.5.

To establish (16), let $u \in W \cap \partial\Omega_\alpha = S_\alpha$. In this case, $|u|_\infty = \alpha$ and $u(n) \geq M\alpha$ for all $n \in \mathbb{Z}^+$. Now, for $n \in \mathbb{Z}^+$, we have

$$|(Tu)(n)| \leq \psi(|u|_\infty)\sum_{j=0}^{n} q(j)B(n,j)g(j) \leq \psi(\alpha)K_1.$$

This, together with Condition 3.1-6, yields $|(Tu)|_\infty \leq \psi(\alpha)K_1 < \alpha = |u|_\infty$. Thus (16) is satisfied.

Let $u \in W \cap \partial\Omega_\beta = S_\beta$. Thus, $|u|_\beta = \beta$ and $M\beta \leq u(n) \leq \beta$ for all $n \in \mathbb{Z}^+$. Now, for $n \in \mathbb{Z}^+$, it follows from Condition 3.1-2 that

$$|(Tu)(n)| \geq A_0\sum_{j=0}^{n} q(j)B(n,j)g(j)\psi(u(j)) \geq A_0K_2\psi(M\beta),$$

which together with Condition 3.1-7, yields $(Tu)(n) \geq A_0K_2\psi(M\beta) > \beta = |u|_\infty$ for $n \geq 0$, and thus $|Tu|_\infty > |u|_\infty$, that is, (17) holds.

Applying Krasnoselskii's fixed-point theorem (Theorem 2.5), we conclude that (12) has a solution $x = \{x(n)\}_{n\in\mathbb{Z}^+} \in \ell^\infty_{\mathrm{cvg}}(\mathbb{R})$ with $x \in W \cap (\overline{\Omega_\alpha}\backslash\Omega_\beta)$ if $\beta < \alpha$, whereas $x \in W \cap (\overline{\Omega_\beta}\backslash\Omega_\alpha)$ if $\alpha < \beta$. Finally, it follows from (16) and (17) that the solution satisfies $|x|_\infty \neq \alpha$ and $|x|_\infty \neq \beta$. This completes the proof. □

Remark 6. Since the proof of Theorem 3.1 is based on Krasnoselskii's fixed-point theorem (Theorem 2.5), Condition 3.1-1 can be replaced by the following less restrictive one (Condition 3.1-1*, say): $F \in C(\bar{\Omega}_\beta\backslash\Omega_\alpha \to \ell_{\mathrm{g}}(\mathbb{R}))$, or $F \in C(\bar{\Omega}_\alpha\backslash\Omega_\beta \to \ell_{\mathrm{g}}(\mathbb{R}))$. (The Nemytskii operator F is a continuous mapping of $W \cap (\bar{\Omega}_\beta\backslash\Omega_\alpha)$ or of $W \cap (\bar{\Omega}_\alpha\backslash\Omega_\beta)$ into $\ell_{\mathrm{g}}(\mathbb{R})$.)

Example 3.2. Consider the nonlinear system

$$x(n) = \sum_{j=0}^{n} B(n,j)g(j)[x(j)]^\gamma, \quad \text{for } n \in \mathbb{Z}^+, \tag{18}$$

with $0 < \gamma < 1$ and Hypothesis H1 holds where $\{g(n)\}$ is a positive sequence defining the space $\ell_{\mathrm{g}}(\mathbb{R})$. In addition, assume that

$$\inf_{n\geq 0}\sum_{j=0}^{n} B(n,j)g(j) > 0. \tag{19}$$

Then (18) has at least one positive convergent solution $x = \{x(n)\}_{n\in\mathbb{Z}^+} \in \ell^\infty_{\mathrm{cvg}}(\mathbb{R})$ with

$$\beta = \frac{1}{2}M^{\gamma/(1-\gamma)}(K_2)^{1/(1-\gamma)} < |x|_\infty < 2(K_1)^{1/(1-\gamma)} = \alpha \text{ and } x(n) \geq M\beta \text{ for } n \in \mathbb{Z}^+,$$

here $M = (1/2)(K_2/K_1)^{1/(1-\gamma)}$ with

$$K_2 = \inf_{n\in\mathbb{Z}^+}\sum_{j=0}^{n} B(n,j)g(j) \text{ and } K_1 = \sup_{n\in\mathbb{Z}^+}\sum_{j=0}^{n} B(n,j)g(j).$$

To see that the result in Example 3.2 is correct, we will apply Theorem 3.1 with

$$f(n,u) = g(n)u^{\gamma}, \quad \psi(v) = v^{\gamma}, \quad q \equiv 1, \quad A_0 = 1 \quad \text{and} \quad \phi(v) = v^{\gamma}.$$

Noting Remark 6, we shall first show that the Nemytskii operator F for $f(n,u(n)) = g(n)[u(n)]^{\gamma}$ is a continuous mapping of $W \cap (\bar{\Omega}_{\alpha}\backslash\Omega_{\beta})$ into $\ell_{g}(\mathbb{R})$. To this end, let $\lim_{j\to\infty}u_j = u_0$ in $W \cap (\bar{\Omega}_{\alpha}\backslash\Omega_{\beta})$. Then, $\beta \leq |u_j|_{\infty} \leq \alpha$ for all $j \geq 0$ and $M\beta \leq u_j(n) \leq \alpha$ for all $j \geq 0$ and $n \geq 0$. Now, for any $j > 0$ and $n \geq 0$ with $u_j(n) \neq u_0(n)$, it follows from the mean-value theorem that

$$[u_j(n)]^{\gamma} - [u_0(n)]^{\gamma} = \gamma\frac{1}{\mu^{1-\gamma}}[u_j(n) - u_0(n)],$$

where $\mu \in (u_j(n), u_0(n))$ (or $\mu \in (u_0(n), u_j(n))$), and hence

$$|[u_j(n)]^{\gamma} - [u_0(n)]^{\gamma}| = \frac{\gamma}{\mu^{1-\gamma}}|u_j(n) - u_0(n)| \leq \frac{\gamma}{(M\beta)^{1-\gamma}}|u_j(n) - u_0(n)|. \tag{20}$$

Notice that (20) also holds, if $u_j(n) = u_0(n)$. Then, we obtain

$$|F(u_j)(n) - F(u_0)(n)| \leq g(n)\frac{\gamma}{(M\beta)^{1-\gamma}}|u_j(n) - u_0(n)|, \tag{21}$$

which implies that the Nemytskii operator F corresponding to $f(n,u(n)) = g(n)[u(n)]^{\gamma}$ is a continuous mapping of $W \cap (\bar{\Omega}_{\alpha}\backslash\Omega_{\beta})$ into $\ell_{g}(\mathbb{R})$ by Observations 2.8 and (11).

Notice that Conditions 3.1-1 to 3.1-3 are obviously satisfied. To verify Condition 3.1-5, note that

$$\frac{M}{\phi(M)} = M^{1-\gamma} = \left(\frac{1}{2}\right)^{1-\gamma}\frac{K_2}{K_1} \leq \frac{K_2}{K_1} = A_0\frac{K_2}{K_1}.$$

Also Condition 3.1-6 is satisfied since $\alpha/(\psi(\alpha)) = \alpha^{1-\gamma} = 2^{1-\gamma}K_1 > K_1$. Finally, Condition 3.1-7 is satisfied since

$$\frac{\beta}{\psi(M\beta)} = \frac{1}{M^{\gamma}}\beta^{1-\gamma} = \frac{1}{M^{\gamma}}\left(\frac{1}{2}\right)^{1-\gamma}M^{\gamma}K_2 = \left(\frac{1}{2}\right)^{1-\gamma}K_2 < K_2 = A_0K_2.$$

Now apply (B) in Theorem 3.1. □

Example 3.3. By taking $B(n,j) = \mathrm{e}^{-(n-j)}$ and $g(n) \equiv 1$ for all $n \geq 0$ in Example 3.2, we consider as a concrete example the equations

$$x(n) = \sum_{j=0}^{n} \mathrm{e}^{-(n-j)}[x(j)]^{\gamma} \quad \text{for } n \in \mathbb{Z}^{+}, \ \text{with } 0 < \gamma < 1. \tag{22}$$

In this case, Hypothesis H1 holds with

$$\lim_{n\to\infty} B(n,j) = \lim_{n\to\infty} \mathrm{e}^{-(n-j)} = 0 \quad \text{for all } j \geq 0,$$

$$K_2 = \inf_{n\geq 0} \sum_{j=0}^{n} B(n,j)g(j) = \inf_{n\geq 0} \sum_{j=0}^{n} \mathrm{e}^{-(n-j)} = 1 > 0,$$

and

$$K_1 = \sup_{n\geq 0} \sum_{j=0}^{n} B(n,j)g(j) = \sup_{n\geq 0} \sum_{j=0}^{n} e^{-(n-j)} = \frac{e}{e-1} < \infty.$$

By Example 3.2, we see that (22) has at least one positive convergent solution $x = \{x(n)\}_{n\in\mathbb{Z}^{+}} \in \ell^{\infty}_{\mathrm{cvg}}(\mathbb{R})$ with

$$\beta = \frac{1}{2} M^{\gamma/(1-\gamma)} < |x|_{\infty} < 2(K_1)^{1/(1-\gamma)} = \alpha \quad \text{and } x(n) \geq M\beta \quad \text{for } n \in \mathbb{Z}^{+},$$

here

$$M = \frac{1}{2}\left(\frac{1}{K_1}\right)^{1/(1-\gamma)} = \frac{1}{2}\left(\frac{e-1}{e}\right)^{1/(1-\gamma)}.$$

With additional conditions on $B(n,j)$ and f in (12), applications of Theorem 3.1 will yield additional positive convergent solutions of (12). For completeness, we provide one multiple solution result.

THEOREM 3.4. *Suppose that Hypothesis H*1 *and Conditions* 3.1-1 *to* 3.1-5 *hold* (*with* $K_{1,2}$ *and* M *as defined therein*). *In addition, suppose there are constants* $\gamma_0 < \gamma_1 < \gamma_2$ *with the following satisfied:*

Condition 3.1-1. $\gamma_0 > 0$ *is such that* $\gamma_0 < A_0 K_2 \psi(M\gamma_0)$.
Condition 3.1-2. $\gamma_1 > 0$ *is such that* $\gamma_1 > K_1 \psi(\gamma_1)$.
Condition 3.1-3. $\gamma_2 > 0$ *is such that* $\gamma_2 < A_0 K_2 \psi(M\gamma_2)$.

Then, (12) *has at least two positive solutions* $x_1 = \{x_1(n)\}_{n\in\mathbb{Z}^{+}}$, $x_2 = \{x_2(n)\}_{n\in\mathbb{Z}^{+}} \in \ell^{\infty}_{\mathrm{cvg}}(\mathbb{R})$ *with* $0 < \gamma_0 < |x_1|_{\infty} < \gamma_1 < |x_2|_{\infty} < \gamma_2$, $x_1(n) \geq M\gamma_0$ *and* $x_2(n) \geq M\gamma_1$ *for* $n \in \mathbb{Z}^{+}$.

Proof. The existence of x_1 follows from (B) in Theorem 3.1 with $\alpha = \gamma_1$ and $\beta = \gamma_0$, and the existence of x_2 follows from (A) in Theorem 3.1 with $\alpha = \gamma_1$ and $\beta = \gamma_2$. □

3.2 Non-negative convergent solutions via the non-linear alternative

Comparing Kransnoselskii's fixed-point theorem (Theorem 2.5) with the non-linear alternative (Theorem 2.6; which we are now about to apply), we note that both theorems guarantee the existence of a fixed point for an operator, but under different conditions. Since the location of a fixed point in Theorem 2.5 is more precise than that in Theorem 2.6, the hypotheses for the completely continuous operator in Theorem 2.5 are stricter than those in Theorem 2.6. It is, therefore, understandable that the assumptions in the following theorem are less strict than those in Theorem 3.1.

THEOREM 3.5. *Suppose that Hypotheses H*1 *and H*2 *hold. In addition, assume the following:*

Condition 3.5-1. $F \in C(\ell^{\infty}_{\text{cvg}}(\mathbb{R}) \to \ell_{\text{g}}(\mathbb{R}))$ (*the Nemytskii operator F is a continuous mapping of* $\ell^{\infty}_{\text{cvg}}(\mathbb{R})$ *into* $\ell_{\text{g}}(\mathbb{R})$).
Condition 3.5-2. *There exists a non-decreasing function* $\psi \in (\mathbb{R}^+ \to \mathbb{R}^+)$ *and* $q \in \ell^{\infty}(\mathbb{R})$ *with* $q(n) \geq 0$ *such that*

$$\frac{f(n,v)}{g(n)} \leq q(n)\psi(v) \quad \text{for all} \;\; v \in \mathbb{R}^+ \;\; \text{and} \;\; n \in \mathbb{Z}^+,$$

Condition 3.5-3. *There exists* $\alpha > K_1\psi(\alpha)$, *where*

$$K_1 = \sup_{n\in\mathbb{Z}^+} \sum_{j=0}^{n} q(j)B(n,j)g(j).$$

Then, (12) *has at least one non-negative convergent solution* $x = \{x(n)\}_{n\in\mathbb{Z}^+} \in \ell^{\infty}_{\text{cvg}}(\mathbb{R})$ *with* $|x|_\infty < \alpha$.

Proof. Let $E = (\ell^{\infty}_{\text{cvg}}(\mathbb{R}), |\cdot|_\infty)$ and $U = \{u \in \ell^{\infty}_{\text{cvg}}(\mathbb{R}) : |u|_\infty < \alpha\}$. It is easy to see, from the conditions, that the operator T acting on $\ell^{\infty}_{\text{cvg}}(\mathbb{R})$, defined by

$$(Tu)(n) = \sum_{j=0}^{n} B(n,j)f(j,u(j)),$$

for $n \in \mathbb{Z}^+$, maps $\ell^{\infty}_{\text{cvg}}(\mathbb{R})$ to $\ell^{\infty}_{\text{cvg}}(\mathbb{R})$. In addition, Condition 3.5-1 to 3.5-2 and the admissibility of the operator B guarantee that $T \in (\ell^{\infty}_{\text{cvg}}(\mathbb{R}) \to \ell^{\infty}_{\text{cvg}}(\mathbb{R}))$ is completely continuous.

The remainder of the proof proceeds by applying Theorem 3.5 (with $\mathfrak{T}$ replaced by T, etc.), and establishing that the option (ii) therein cannot hold so that option (i) does hold. Let $x_\lambda = \{x_\lambda(n)\}_{n\in\mathbb{Z}^+} \in \ell^{\infty}_{\text{cvg}}(\mathbb{R})$ be any solution of

$$x_\lambda(n) = \lambda \times \sum_{j=0}^{n} B(n,j)f(j,x_\lambda(j)), \quad n \in \mathbb{Z}^+,$$

for $0 < \lambda < 1$. Since $B(n,j) \geq 0$ and $f \in (\mathbb{Z}^+ \times \mathbb{R} \to \mathbb{R}^+)$, the right-hand side of the above equation is non-negative, which implies that $x_\lambda(n) \geq 0$ for all $n \in \mathbb{Z}^+$. Now, for

$n \in \mathbb{Z}^+$, we have

$$|x_\lambda(n)| < \lambda \sum_{j=0}^{n} q(j)B(n,j)g(j)\psi(x_\lambda(j)) \leq \psi(|x_\lambda|_\infty) \sum_{j=0}^{n} q(j)B(n,j)g(j) \leq K_1 \psi(|x_\lambda|_\infty),$$

and therefore

$$|x_\lambda|_\infty \leq K_1 \psi(|x_\lambda|_\infty). \tag{23}$$

In addition, Condition 3.5-3 and (23) implies that $|x_\lambda|_\infty \neq \alpha$. Apply the nonlinear alternative (notice that (ii) cannot occur) to deduce that (12) has a solution $x = \{x(n)\}_{n \in \mathbb{Z}^+} \in \ell^\infty_{\mathrm{cvg}}(\mathbb{R})$ with ($x(n) \geq 0$ for all $n \in \mathbb{Z}^+$) and $|x|_\infty < \alpha$. (Note $|x|_\infty \leq \alpha$ by the nonlinear alternative. If $|x|_\infty = \alpha$, it is readily shown that $\alpha = |x|_\infty \leq K_1 \psi(|x|_\infty) = K_1 \alpha$, which contradicts Condition 3.5-3.) □

Remark 7. If there are positive integers $j_0 \leq n_0$ in $\mathbb{Z}^+$ such that $B(n_0, j_0) f(j_0, 0) > 0$, then the convergent sequence $\{y(n)\}_{n \in \mathbb{Z}^+}$, $y(n) = 0$ for all $n \in \mathbb{Z}^+$, is not a solution of (12). In this case, the non-negative solution $x = \{x(n)\}_{n \in \mathbb{Z}^+} \in \ell^\infty_{\mathrm{cvg}}(\mathbb{R})$ with $|x|_\infty < \alpha$ in Theorem 3.5 satisfies $|x|_\infty > 0$.

3.3 Non-negative convergent solutions via the Leggett–Williams fixed-point theorem

It is possible to use the Leggett–Williams fixed-point theorem to obtain non-negative convergent solutions of (12).

THEOREM 3.6. *Suppose that Hypotheses H*1 *and H*2$^+$ *hold, and let Condition* 3.1-1 *be satisfied* (*the Nemytskii operator F is a continuous mapping of* $\ell^\infty_{\mathrm{cvg}}(\mathbb{R})$ *into* $\ell_{\mathrm{g}}(\mathbb{R})$). *In addition, assume the following conditions are satisfied:*

Condition 3.6-1. *There exist a non-decreasing continuous map* $\psi \in (\mathbb{R}^+ \to \mathbb{R}^+)$, *a continuous map* $\phi \in (\mathbb{R}^+ \to \mathbb{R}^+)$ *and* $q \in \ell^\infty(\mathbb{R})$ *such that*

$$\phi(v)q(n) \leq \frac{f(n,v)}{g(n)} \leq q(n)\psi(v) \quad \textit{for all} \;\; v \in \mathbb{R}^+ \;\; \textit{and} \;\; n \in \mathbb{Z}^+,$$

where $q(n) \geq 0$ *for* $n \in \mathbb{Z}^+$,
Condition 3.6-2. $K_2 > 0$ *where*

$$K_2 = \inf_{n \in \mathbb{Z}^+} \sum_{j=0}^{n} q(j)B(n,j)g(j).$$

Condition 3.6-3. *There exists* $r > 0$ *with* $r < K_2 \phi(r)$.
Condition 3.6-4. $\phi(v)/v$ *is non-increasing on* $(0,r)$.
Condition 3.6-5. *There exists* $R > r$ *with* $R > \psi(R)K_1$, *where*

$$K_1 = \sup_{n \in \mathbb{Z}^+} \sum_{j=0}^{n} q(j)B(n,j)g(j).$$

Then, (12) *has a non-negative convergent solution* $x = \{x(n)\}_{n\in\mathbb{Z}^+} \in \ell^{\infty}_{\mathrm{cvg}}(\mathbb{R})$ *with* $r \leq |x|_{\infty} < R$.

Proof. Let $E = (\ell^{\infty}_{\mathrm{cvg}}(\mathbb{R}), |\cdot|_{\infty})$ and $W = \{u \in \ell^{\infty}_{\mathrm{cvg}}(\mathbb{R}) : u(n) \geq 0 \quad \text{for } n \in \mathbb{Z}^+\}$. Clearly, W is a cone in $\ell^{\infty}_{\mathrm{cvg}}(\mathbb{R})$. To prove our result by applying Theorem 2.7, we shall first chose $u_0 \in W\backslash\{0\}$. We do this by letting $u_0 = \{u_0(n)\}_{n\in\mathbb{Z}^+} \in W\backslash\{0\}$ with $u_0(n) = 1$ for all $n \in \mathbb{Z}^+$ and let

$$W(u_0) = \{u \in W \text{ such that there exists } \lambda > 0 \text{ with } u(n) \geq \lambda u_0(n) = \lambda \text{ for } n \in \mathbb{Z}^+\}.$$

Note that the above $W(u_0)$ can be also defined by $W(u_0) = \{u \in W : \inf_{n\geq 0} u(n) = \underline{u} > 0\}$. From Hypothesis H1, Conditions 3.1-1 and 3.6-1, it follows that $T \in (W \to W)$ is completely continuous, where Tu is defined by

$$(Tu)(n) = \sum_{j=0}^{n} B(n,j)f(j, u(j)) \quad \text{for } n \in \mathbb{Z}^+ \text{ and } u = \{u(n)\}_{n\in\mathbb{Z}^+} \in W.$$

In order to verify the hypotheses of Theorem 2.7, we first show

$$|Tu|_{\infty} \leq |x|_{\infty} \quad \text{for } u \in S_R = \{u \in W : |u|_{\infty} = R\}, \tag{24}$$

where R is given in Condition 3.6-5. In fact, if $u = \{u(n)\}_{n\in\mathbb{Z}^+} \in S_R$, then $|u|_{\infty} = R$. Thus for $n \in \mathbb{Z}^+$, we have

$$(Tu)(n) \leq \sum_{j=0}^{n} q(j)B(n,j)g(j)\psi(u(j)) \leq \psi(|u|_{\infty})\sum_{j=0}^{n} q(j)B(n,j)g(j) \leq \psi(R)K_1. \tag{25}$$

This together with Condition 3.6-5 gives

$$|Tu|_{\infty} \leq \psi(R)K_1 < R = |u|_{\infty}, \tag{26}$$

which implies that (24) holds. Next, for r in Condition 3.6-3, we show that

$$Tu \not\leq u \quad \text{for } u \in S_r \cap W(u_0), \text{ where } S_r = \{u \in W : |u|_{\infty} = r\}. \tag{27}$$

To see this, let $u = \{u(n)\}_{n\in\mathbb{Z}^+} \in S_r \cap W(u_0)$, hence $|u|_{\infty} = r$ and $r \geq u(n) > 0$ for $n \in \mathbb{Z}^+$. Now for $n \in \mathbb{Z}^+$, we have

$$(Tu)(n) \geq \sum_{j=0}^{n} q(j)B(n,j)g(j)\frac{\phi(u(j))}{u(j)}u(j) \geq \frac{\phi(r)}{r}\sum_{j=0}^{n} q(j)B(n,j)g(j)u(j).$$

Let $\underline{u} = \inf_{n\in\mathbb{Z}^+} u(n) > 0$ for $u \in S_r \cap W(u_0)$ and this together with the previous inequality yields

$$(Tu)(n) \geq \frac{\phi(r)}{r}\underline{u}\sum_{j=0}^{n} q(j)B(n,j)g(j) \geq \left(\frac{\phi(r)}{r}K_2\right)\underline{u},$$

for $n \in \mathbb{Z}^+$ and hence,

$$\inf_{n\geq 0}(Tu)(n) \geq \frac{\phi(r)K_2}{r}\underline{u}. \tag{28}$$

Set $p = \phi(r)K_2/r$. Since $p > 1$ by Condition 3.6-3 and $p\underline{u} > \underline{u}$, there is $n_0 \in \mathbb{Z}^+$ such that $p\underline{u} > u(n_0)$, and hence $(Tu)(n_0) \geq p\underline{u} > u(n_0)$ by (28). This implies that (27) is valid.

Applying Theorem 2.7, we conclude that (12) has a non-negative convergent solution $x = \{x(n)\}_{n\in\mathbb{Z}^+} \in W$ with $r \leq |x|_\infty \leq R$. Finally, it follows from (26) that $|x|_\infty \neq R$. □

To illustrate Theorem 3.6 with an example, we consider, a case where $g(n) \equiv 1$ for all $n \geq 0$ – as an example, this simplification involves no essential loss of generality.

Example 3.7. Consider the following discrete nonlinear system

$$x(n) = \sum_{j=0}^{n} B(n,j)h(j)[G(x(j))], \quad n \in \mathbb{Z}^+, \tag{29}$$

where $G(u) = 1/(1 + \ln(1 + u)) + u$, $h = \{h(j)\}_{j\geq 0} \in \ell^\infty_{\mathrm{cvg}}(\mathbb{R})$, $h(j) > 0$ for all $j \in \mathbb{Z}^+$ and Hypothesis H1 is satisfied with $g(n) \equiv 1$ for all $n \geq 0$. In addition, assume

$$K_2 = \inf_{n\in\mathbb{Z}^+}\sum_{j=0}^{n} h(j)B(n,j) > 0, \tag{30}$$

and

$$K_1 = \sup_{n\in\mathbb{Z}^+}\sum_{j=0}^{n} h(j)B(n,j) < \frac{1+\ln 2}{2+\ln 2}, \tag{31}$$

hold. Then (29) has a non-negative solution $x = \{x(n)\}_{n\in\mathbb{Z}^+} \in \ell^\infty_{\mathrm{cvg}}(\mathbb{R})$ with

$$r \leq |x|_\infty < 1 \quad \text{for any } \ 0 < r < 1 \ \text{ with } \ r[1 + \ln(1+r)] < K_2.$$

To see this, let $f(n, v) = h(n)G(v)$, $\psi(v) = G(v)$, $\phi(v) = 1/(1 + \ln(1 + v))$ and $q(n) = h(n)$ with $0 < r(1 + \ln(1 + r)) < K_2$ and $R = 1$. Now, we show that Conditions 3.1-1, 3.6-1 and 3.6-3 to 3.6-5 hold.

It is readily shown that the Nemytskii operator F, defined by $F(\phi)(n) = f(n, \phi(n))$, satisfies condition (i) in Observations 2.8 and thus it transforms bounded subset of $\ell^\infty_{\mathrm{cvg}}(\mathbb{R})$ into bounded subsets of $\ell_{\mathrm{g}}(\mathbb{R})$ by Observations 2.8.

For any $0 \leq u_1 < u_2$, it follows from the mean-value theorem that there exists a v with $v_1 < v < v_2$ such that

$$|f(n,v_2) - f(n,v_1)| = \left|h(n)\left(\frac{-1}{(1+v)[1+\ln(1+v)]^2} + 1\right)(v_2 - v_1)\right| \leq |h|_\infty|v_2 - v_1|,$$

which implies that the Nemytskii operator F is a continuous mapping of $\ell^\infty_{\mathrm{cvg}}(\mathbb{R})$ into $\ell_{\mathrm{g}}(\mathbb{R})$ by Observations 2.8 and (11), and hence Condition 3.1-1 is satisfied.

Obviously, $\psi(v) \geq 0$ for $v \in \mathbb{R}^+$ and increasing on $\mathbb{R}^+$, and hence Condition 3.6-1 is true. Notice that the function $p(v) = v[1 + \ln(1 + v)]$ is continuous and increasing on $\mathbb{R}^+$.

This implies that the function $\phi(v)/u = 1/p(v)$ is non-increasing on $(0, \infty)$. Since $p(0) = 0$ and $p(v) = v[1 + \ln(1 + v)]$ is continuous, we can chose a positive $0 < r < 1$ such that

$$p(r) = \frac{r}{\phi(r)} = r[1 + \ln(1 + r)] < K_2,$$

which implies that Conditions 3.6-3 and 3.6-4 hold.

For $R = 1 > r$, it is clear that $\psi(R)K_1 = (2 + \ln 2)/(1 + \ln 2)K_1 < 1 = R$, and hence Condition 3.6-5 is satisfied. Now apply Theorem 3.6.

Acknowledgements

We are grateful to the editor and referees for their valuable comments on the presentation of the results in this paper. The work of the first author is supported by JGSF of Jiangsu (No. 07KJB110104).

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