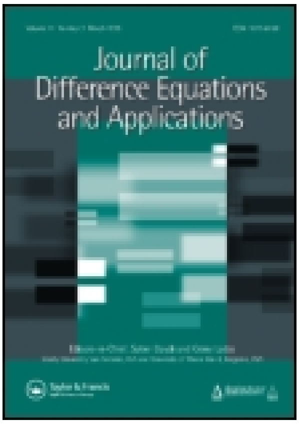

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On admissibility of the resolvent of discrete Volterra equations

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On admissibility of the resolvent of discrete Volterra equations

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Suppose that a pair of sequence spaces is admissible with respect to a discrete linear Volterra operator. This paper gives sufficient conditions for the same pair of spaces to be admissible with respect to the associated resolvent operator. The spaces considered include spaces of weighted bounded and weighted convergent sequences. Classes of discrete kernels are discussed for which the appropriate weight sequences are not purely exponential, but the product of an exponential and a slowly decaying sequence.

Keywords: discrete Volterra equations; admissibility; resolvent; weighted spaces

AMS Subject Classification: 39A11; 39A70; 47B39

1. Introduction

Recently Song and Baker [14] studied admissibility for linear and nonlinear discrete Volterra equations. Also, Appleby et al. [1] examined the asymptotic properties of solutions of discrete Volterra equations using weighted sequence spaces; subsequently the main result in [1] was improved by Györi and Horváth [9]. This paper studies some connections between these works.

As motivation, consider the linear discrete Volterra equation

$$x(n) = h(n) + \sum_{j=0}^{n} H(n,j)x(j), \quad n \in \mathbb{Z}^+, \tag{1}$$

with $H(n, j) = 0$, for $j > n$ and $\det(I - H(n, n)) \neq 0$. Its solutions can be expressed in terms of the resolvent kernel $\{R(n, j)\}$ by

$$x(n) = h(n) - \sum_{j=0}^{n} R(n,j)h(j), \quad n \in \mathbb{Z}^+. \tag{2}$$

Associated with the resolvent kernel is the resolvent operator $\mathcal{R}: \ell \to \ell$ defined by $(\mathcal{R}\phi)(n) = \sum_{j=0}^{n} R(n,j)\phi(j)$ for all sequences $\{\phi(j)\}_{j\geq 0}$ in ℓ. A pair (X,Y) of subspaces of ℓ is *admissible* with respect to $\mathcal{R}$, if $\mathcal{R}(X) \subset Y$. It is plainly desirable to know conditions under which this happens. Application of results in [14] give necessary and sufficient conditions in terms of the resolvent kernel $\{R(n, j)\}$ rather than the kernel $\{H(n, j)\}$ occurring in (1).

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In this paper, the techniques developed in [1,9] are used to obtain new sufficient conditions in terms of $\{H(n, j)\}$ for (X, Y) to be admissible with respect to $\mathcal{R}$. The form of all the results is that if (X, Y) is admissible with respect to $\mathcal{H}$ and stability conditions holds, then (X,Y) is also admissible with respect to $\mathcal{R}$. The results in [14] are stated in terms of norms, while the methods of [1,9] involve manipulating the absolute values of matrices. Therefore we require necessary and sufficient conditions for admissibility in terms of absolute values: in two cases the admissibility conditions of [14] are merely translated, but in one case it is more expedient to use an alternative set of conditions.

Our results are applied to a class of linear nonconvolution equations whose solutions can be weighted by a sequence which is not purely exponential, but the product of an exponential and a slowly decaying sequence.

Nonlinear equations of the form $x(n) = h(n) + \sum_{j=0}^{n} H(n, j) f(j, x(j))$ for $n \geq 0$, are also studied in [14]. Admissibility conditions and fixed-point methods are used to determine qualitative properties of the solutions. However, in some cases a variation of constants procedure should be used to transform the nonlinear equation into one amenable to this approach. Admissibility results for resolvent operators are required.

2. Preliminaries

2.1 Notation

In this paper, $\mathbb{E}$ denotes either $\mathbb{R}$ or $\mathbb{C}$. $\mathbb{E}^{d\times d}$ is the space of all $d \times d$ matrices with entries in $\mathbb{E}$; the zero and identity matrices are denoted by 0 and I, respectively, and the matrix E is defined by $E_{ij} = 1$. $\mathbb{E}^{d\times d}$ can be endowed with many norms, but they are all equivalent. $\|A\|$ usually stands for the Euclidean norm of a matrix $A = (A_{ij}) \in \mathbb{E}^{d\times d}$. This should not be confused with its *absolute value*, which is the matrix in $\mathbb{R}^{d\times d}$ defined by $(|A|)_{ij} = |A_{ij}|$. A matrix $A = (A_{ij})$ in $\mathbb{R}^{d\times d}$ is *nonnegative*, if $A_{ij} \geq 0$, in which case we write $A \geq 0$. A partial ordering is defined on $\mathbb{E}^{d\times d}$ by letting $A \leq B$, if and only if $B - A \geq 0$.

The *spectral radius* of a matrix A is given by $\rho(A) = \lim_{n\to\infty} \|A^n\|^{1/n}$, where $\|\cdot\|$ is any norm on $\mathbb{E}^{d\times d}$. $\rho(A)$ is independent of the norm employed to calculate it, and equals the maximum of the absolute values of the eigenvalues of A. We use the inequalities $\rho(A) \leq \rho(|A|)$ and $\rho(A) \leq \rho(B)$, if $0 \leq A \leq B$.

2.2 Sequence spaces

The linear space of sequences $\phi : \mathbb{Z}^+ \to \mathbb{E}^d$ is denoted by $\ell(\mathbb{E}^d)$. For each $m \geq 0$, let ${}_m\|\phi\| := \max_{0\leq n\leq m} \|\phi(n)\|$, where $\|\cdot\|$ is the Euclidean norm on $\mathbb{E}^d$. Each $\phi \mapsto {}_m\|\phi\|$ is a semi-norm on $\ell(E^d)$, which is a Fréchet space if endowed with the topology generated by this family of semi-norms. A sequence in $\ell(\mathbb{E}^d)$ converges if and only if it converges on every finite subset of $\mathbb{Z}^+$. $\ell^\infty(\mathbb{E}^d)$ is the Banach space consisting of all bounded sequences $\phi \in \ell(\mathbb{E}^d)$, equipped with the norm $\|\phi\|_{\ell^\infty} := \sup_{n\geq 0} \|\phi(n)\|$: then $\ell^\infty(\mathbb{E}^d) \subset \ell(\mathbb{E}^d)$ algebraically and topologically. Two important subspaces of $\ell^\infty(\mathbb{E}^d)$ are $\ell^c(\mathbb{E}^d)$ and $\ell^0(\mathbb{E}^d)$: $\ell^c(\mathbb{E}^d)$ consists of all sequences ϕ in $\ell^\infty(\mathbb{E}^d)$, for which $\lim_{n\to\infty} \phi(n)$ exists (and is finite), and $\ell^0(\mathbb{E}^d)$ consists of all sequences ϕ in $\ell^c(\mathbb{E}^d)$ for which $\lim_{n\to\infty} \phi(n) = 0$. It is well-known that $\ell^0(\mathbb{E}^d) \subset \ell^c(\mathbb{E}^d) \subset \ell^\infty(\mathbb{E}^d)$ both algebraically and topologically.

Let $\{\gamma(n)\}_{n\in\mathbb{Z}^+}$ be a non-vanishing sequence in $\mathbb{E}$; i.e. $\gamma(n) \neq 0$ for all $n \geq 0$. Then, $\ell_\gamma(\mathbb{E}^d)$ denotes the space of all sequences ϕ in $\mathbb{E}^d$ with $\sup_{n\geq 0} \|\phi(n)\|/|\gamma(n)| < \infty$. Hence, $\ell^\infty(\mathbb{E}^d) = \ell_\gamma(\mathbb{E}^d)$ where $\gamma(n) \equiv 1$. Also, let $\ell_\gamma^c(\mathbb{E}^d)$ denote the subspace of sequences ϕ in $\ell_\gamma(\mathbb{E}^d)$ such that $\lim_{n\to\infty} \phi(n)/\gamma(n)$ exists.

Where it is clear from the context, we suppress the dependence on the codomain of the sequences, and for example write ℓ^∞ for $\ell^\infty(\mathbb{E}^d)$.

2.3 Resolvent kernels and operators

Throughout the paper $H : \mathbb{Z}^+ \times \mathbb{Z}^+ \to \mathbb{E}^{d\times d}$ satisfies $H(n, j) = 0$, for $j > n$. Corresponding to the sequence $\{H(n, j)\}$ is a continuous operator $\mathcal{H} : \ell(\mathbb{E}^d) \to \ell(\mathbb{E}^d)$ defined by

$$(\mathcal{H}\phi)(n) = \sum_{j=0}^{n} H(n,j)\phi(j), \quad n \in \mathbb{Z}^+. \tag{3}$$

The equation (1) has a unique solution $\{x(n)\} \in \ell$ for each $\{h(n)\} \in \ell$, provided

$$\det(I - H(n,n)) \neq 0, \quad n \in \mathbb{Z}^+. \tag{4}$$

If (4) holds, the resolvent kernel $\{R(n, m)\}$ is defined to be the solution of

$$R(n,k) = \sum_{j=k}^{n} H(n,j)R(j,k) - H(n,k), \quad 0 \leq k \leq n, \tag{5}$$

with $R(n, k) = 0$ for all $k > n \geq 0$. It can be shown that $\{R(n, m)\}$ also obeys

$$R(n,k) = \sum_{j=k}^{n} R(n,j)H(j,k) - H(n,k), \quad 0 \leq k \leq n.$$

The unique solution of (1) can be represented in terms of the resolvent as in (2). The resolvent operator $\mathcal{R} : \ell(\mathbb{E}^d) \to \ell(\mathbb{E}^d)$ is defined by

$$(\mathcal{R}\phi)(n) = \sum_{j=0}^{n} R(n,j)\phi(j), \quad n \in \mathbb{Z}^+. \tag{6}$$

For further information about the resolvents of discrete Volterra operators, see Vecchio [15].

One motivation for studying the admissibility of the resolvent operator is Theorem 4.2 of [14], which is stated here for ease of reference.

THEOREM 2.1. *Suppose that $H(n, j) = 0$ for $j > n$ and* (4) *holds. Let X be one of the spaces $\ell_\gamma(\mathbb{E}^d)$, $\ell^\infty(\mathbb{E}^d)$, $\ell^c(\mathbb{E}^d)$, $\ell^0(\mathbb{E}^d)$. Then the following statements are equivalent.*

(1) *For each $\{h(n)\}_{n\geq 0}$ in X, the solution $\{x(n)\}_{n\geq 0}$ of* (1) *is in X.*
(2) *The pair (X,X) is admissible with respect to the operator $\mathcal{R}$ given by* (6).

2.4 Admissibility

The theory of admissibility for ordinary differential equations was developed by Perron, Bellman, and Massera and Schäffer [10]. The theory for Volterra integral operators was begun by Corduneanu [3], and advanced by Miller, Cushing and others. Different but related concepts of admissibility have been employed in different works; so for example that used in [10] differs from employed in [3]. Corduneanu [4,5] give informative and comprehensive accounts of the theory up to 1991.

The theory of admissibility for discrete Volterra equations was studied in [14]. It is clear from Section 4 of [14] that admissibility of the resolvent is an important ingredient in using fixed point theorems to prove results about nonlinear perturbations of (1). The work of Medina on h-stability of Volterra difference equations in [11] is also connected to admissibility of pairs of weighted spaces with respect to resolvent operators. One of the motivations of h-stability was to obtain estimates on solutions decaying to either an exponentially stable or weakly stable equilibrium.

Definition 2.2. *Let X and Y be two subspaces of* $\ell(\mathbb{E}^d)$. *The pair* (X,Y) *is said to be admissible with respect to* $\mathcal{K} : \ell(\mathbb{E}^d) \to \ell(\mathbb{E}^d)$ *if* $\mathcal{K}(X) \subset Y$.

Since $\ell(\mathbb{E}^d)$ is a Fréchet space, the following is a statement of a standard result (e.g., Lemma 1.1 of Chapter 2 in [4], Lemma 2.6 in [14]).

Proposition 2.3. *If* (X,Y) *is admissible with respect to* $\mathcal{K}$, *and the topologies of both X and Y are stronger than that of* $\ell(\mathbb{E}^d)$, *then the restriction* $\tilde{\mathcal{K}} = \mathcal{K}|_X^Y : X \to Y$ *is continuous.*

3. Admissibility for discrete Volterra operators

Three theorems in [14] are central to our paper. They provide necessary and sufficient conditions for the pairs $(\ell_\gamma, \ell_\delta)$, (ℓ^c, ℓ^c) and (ℓ_γ, ℓ^c) to be admissible with respect to the operator $\mathcal{H}$ defined by (3). In order to reveal the connections between the results in [1] and [14], two of these three key theorems are reformulated in terms of absolute values of matrices rather than norms.

Firstly, we reformulate Theorem 3.1 of [14].

Theorem 3.1. *Let* γ *and* δ *be non-vanishing sequences in* $\ell(\mathbb{E})$. *Then* $(\ell_\gamma, \ell_\delta)$ *is admissible with respect to* $\mathcal{H}$, *if and only if*

$$\sup_{n\geq 0}\sum_{j=0}^{n} \frac{|H(n,j)\gamma(j)|}{|\delta(n)|} < \infty. \tag{7}$$

Theorem 3.1 of [14] in fact uses the condition

$$\sup_{n\geq 0}\sum_{j=0}^{n} \frac{\|H(n,j)\gamma(j)\|}{|\delta(n)|} < \infty, \tag{8}$$

where $\|\cdot\|$ denotes the Euclidean norm on $\mathbb{E}^{d\times d}$. By considering the maximum norm on $\mathbb{E}^{d\times d}$, it is easily shown that (8) is equivalent to (7).

The sufficiency of (8) appears in Gol'dengeršel' [6] in the case that $d = 1$ and $\gamma(n) = \delta(n) = r^n$ with $r > 0$. Though no proofs are given, details of proofs for corresponding results for Volterra integral equations can be found in Gol'dengeršel' [7].

The following is an alternative formulation of Theorem 3.7 of [14].

THEOREM 3.2. *Suppose that*

$$\lim_{n\to\infty} H(n,j) = H_\infty(j) \text{ exists.} \tag{9}$$

Then (ℓ^c, ℓ^c) *is admissible with respect to* $\mathcal{H}$ *if and only if* H *satisfies both*

$$\sup_{n\geq 0} \sum_{j=0}^{n} |H(n,j)| < \infty, \tag{10}$$

$$\lim_{n\to\infty} \sum_{j=0}^{n} H(n,j) \text{ exists.} \tag{11}$$

If these conditions hold, for every $\phi \in \ell^c$ *with* $\phi(\infty) = \lim_{n\to\infty}\phi(n)$,

$$\lim_{n\to\infty} (\mathcal{H}\phi)(n) = V_H\phi(\infty) + \sum_{j=0}^{\infty} H_\infty(j)\phi(j), \tag{12}$$

where

$$V_H = \lim_{n\to\infty} \sum_{j=0}^{n} H(n,j) - \sum_{j=0}^{\infty} H_\infty(j). \tag{13}$$

The next result assures us that the second term in (12) is well-defined and finite.

PROPOSITION 3.3. *Suppose that* H *obeys* (9) *and* (10). *Then*

$$\sum_{j=0}^{\infty} |H_\infty(j)| \leq \limsup_{n\to\infty} \sum_{j=0}^{n} |H(n,j)| \leq \sup_{n\geq 0} \sum_{j=0}^{n} |H(n,j)|. \tag{14}$$

Proof. The argument used to establish ([14], equation (3.41)) is modified by changing norms to absolute values. For every $k \geq 0$,

$$\sum_{j=0}^{k} |H_\infty(j)| = \lim_{i\to\infty} \sum_{j=0}^{k} |H(i,j)| = \lim_{i\to\infty} \sup_{n\geq i} \sum_{j=0}^{k} |H(n,j)|$$
$$\leq \lim_{i\to\infty} \sup_{n\geq i} \sum_{j=0}^{n} |H(n,j)| = \limsup_{n\to\infty} \sum_{j=0}^{n} |H(n,j)| \leq \sup_{n\geq 0} \sum_{j=0}^{n} |H(n,j)|.$$

By taking the limit superior as $k \to \infty$, we obtain (14). □

The matrix V_H occurs in several places with different assumptions in force. For the sake of clarity, we state the following.

PROPOSITION 3.4. *Suppose that* H *obeys* (9) *and* (10). *Then the following are equivalent:*

(i) $\lim_{n\to\infty}\sum_{j=0}^{n} H(n,j)$ *exists;*
(ii) $V_H := \lim_{m\to\infty}\lim_{n\to\infty}\sum_{j=m}^{n} H(n,j)$ *exists and,*
(iii) $\lim_{m\to\infty}\limsup_{n\to\infty}\left|V_H - \sum_{j=m}^{n} H(n,j)\right| = 0$ *for some* V_H *in* $\mathbb{E}^{d\times d}$.

It is not hard to demonstrate that (i) ⇒ (ii) ⇒ (iii) ⇒ (i). In examples, it is often easier to verify (iii) than the other conditions.

Finally, in this section, we give necessary and sufficient conditions for (ℓ_γ, ℓ^c) to be admissible with respect to $\mathcal{H}$. Theorem 3.4 and Remark 3.5 of [14] stated that the two conditions

$$\sup_{n\geq 0}\sum_{j=0}^{n} \|H(n,j)\|\,|\gamma(j)| < \infty, \tag{15}$$

$$\lim_{n\to\infty}\sum_{j=0}^{n} \|H(n,j)\|\,|\gamma(j)| = \sum_{j=0}^{\infty} \|H_\infty(j)\|\,|\gamma(j)|, \tag{16}$$

taken together, are necessary and sufficient for this. Other variants are also discussed. However, our proofs are more succinct if we employ an alternative pair of conditions.

THEOREM 3.5. *Let* $\{\gamma(n)\}$ *be a non-vanishing sequence in* $\ell(\mathbb{E})$. *Suppose that H satisfies* (9). *Then* (ℓ_γ, ℓ^c) *is admissible with respect to* $\mathcal{H}$ *if and only if H obeys*

$$\sup_{n\geq 0}\sum_{j=0}^{n} |H(n,j)|\,|\gamma(j)| < \infty, \tag{17}$$

$$\lim_{m\to\infty}\limsup_{n\to\infty}\sum_{j=m}^{n} |H(n,j)|\,|\gamma(j)| = 0. \tag{18}$$

If these conditions hold, $\lim_{n\to\infty}(\mathcal{H}\phi)(n) = \sum_{j=0}^{\infty} H_\infty(j)\phi(j)$ *for every* $\phi \in \ell_\gamma$.

Proof. We prove that (15) and (16) are equivalent to (15) and

$$\lim_{m\to\infty}\limsup_{n\to\infty}\sum_{j=m}^{n} \|H(n,j)\|\,|\gamma(j)| = 0. \tag{19}$$

Suppose that (15) and (16) hold. Let $n > m > 0$. By taking the limit as $n \to \infty$ in

$$\sum_{j=0}^{n} \|H(n,j)\|\,|\gamma(j)| = \sum_{j=0}^{m-1} \|H(n,j)\|\,|\gamma(j)| + \sum_{j=m}^{n} \|H(n,j)\|\,|\gamma(j)|, \tag{20}$$

we deduce using (9) that

$$\lim_{n\to\infty}\sum_{j=0}^{n} \|H(n,j)\|\,|\gamma(j)| = \sum_{j=0}^{m-1} \|H_\infty(j)\|\,|\gamma(j)| + \lim_{n\to\infty}\sum_{j=m}^{n} \|H(n,j)\|\,|\gamma(j)|,$$

and therefore that $\lim_{n\to\infty}\sum_{j=m}^{n}\|H(n,j)\|\,|\gamma(j)| = \sum_{j=m}^{\infty}\|H_\infty(j)\|\,|\gamma(j)|$, which in turn implies (19).

Next, we suppose that (15) and (19) are true. By taking the limit superior as $n \to \infty$ of both sides of (20), we obtain

$$\limsup_{n\to\infty}\sum_{j=0}^{n}\|H(n,j)\||\gamma(j)| \le \sum_{j=0}^{m-1}\|H_\infty(j)\||\gamma(j)| + \limsup_{n\to\infty}\sum_{j=m}^{n}\|H(n,j)\||\gamma(j)|.$$

Therefore, (19) implies that

$$\limsup_{n\to\infty}\sum_{j=0}^{n}\|H(n,j)\||\gamma(j)| \le \sum_{j=0}^{\infty}\|H_\infty(j)\||\gamma(j)|.$$

The reverse inequality follows from the argument showing ([14], equation (3.41)): for every $k \ge 0$,

$$\begin{aligned}\sum_{j=0}^{k}\|H_\infty(j)\||\gamma(j)| &= \lim_{i\to\infty}\sum_{j=0}^{k}\|H(i,j)\||\gamma(j)| = \lim_{i\to\infty}\sup_{n\ge i}\sum_{j=0}^{k}\|H(n,j)\||\gamma(j)| \\ &\le \lim_{i\to\infty}\sup_{n\ge i}\sum_{j=0}^{n}\|H(n,j)\||\gamma(j)| = \limsup_{n\to\infty}\sum_{j=0}^{n}\|H(n,j)\||\gamma(j)|.\end{aligned}$$

By taking the limit superior of each side as $k \to \infty$, we get the desired inequality.

Finally, we observe that the equivalence between (15) and (19), and the pair (17) and (18), can be concluded using the equivalence of the Euclidean and the maximum norms on $\mathbb{E}^d$. □

4. Asymptotic behaviour of solutions of discrete Volterra equations

A variant of Theorem 5.1 of [1] will be used to establish that the solutions of various equations are bounded.

THEOREM 4.1. *Suppose that* $H(n, j) \ge 0$ *for all* $(n, j) \in \mathbb{Z}^+ \times \mathbb{Z}^+$, *and that*

$$W_H := \lim_{m\to\infty}\limsup_{n\to\infty}\sum_{j=m}^{n}H(n,j) < \infty, \quad \rho(W_H) < 1, \tag{21}$$

$$\sup_{n\ge j}H(n,j) < \infty \quad \text{for each } j \ge 0. \tag{22}$$

Suppose also that $h = \{h(n)\}_{n\ge0}$ *and* $y = \{y(n)\}_{n\ge0}$ *are sequences of nonnegative vectors in* $\mathbb{E}^d$ *such that h is bounded and*

$$y(n) \le h(n) + \sum_{j=0}^{n}H(n,j)y(j), \quad n \ge 0. \tag{23}$$

Then y is also bounded.

Proof. It is a consequence of (21) that there are positive integers M_0 and N_0, with $N_0 > M_0$, such that

$$\rho\left(\sup_{n\geq N_0}\sum_{j=M_0}^{n} H(n,j)\right) < 1. \tag{24}$$

Then (23) implies that for $n \geq M_0$,

$$y(n) \leq a(n) + \sum_{j=M_0}^{n} H(n,j)y(j), \quad n \geq M_0, \tag{25}$$

where $a(n) = h(n) + \sum_{j=0}^{M_0-1} H(n,j)y(j)$. The boundedness of h and (22) force $\{a(n)\}_{n\geq M_0}$ to be bounded. Let $k > N_0$ be an integer. Then taking the maximum of each side of (25) over $M_0 \leq n \leq k$,

$$\max_{M_0\leq n\leq k} y(n) \leq \max_{M_0\leq n\leq k} a(n) + \max_{M_0\leq n\leq k}\sum_{j=M_0}^{n} H(n,j)y(j).$$

The last term can be bounded in the following manner:

$$\begin{aligned}\max_{M_0\leq n\leq k}\sum_{j=M_0}^{n} H(n,j)y(j) &\leq \max_{M_0\leq n< N_0}\sum_{j=M_0}^{n} H(n,j)y(j) + \max_{N_0\leq n\leq k}\sum_{j=M_0}^{n} H(n,j)y(j)\\ &\leq \max_{M_0\leq n< N_0}\sum_{j=M_0}^{n} H(n,j)\max_{M_0\leq j< N_0} y(j) + \sup_{n\geq N_0}\sum_{j=M_0}^{n} H(n,j)\max_{M_0\leq j\leq k} y(j).\end{aligned}$$

It follows that

$$\max_{M_0\leq n\leq k} y(n) \leq p + \sup_{n\geq N_0}\sum_{j=M_0}^{n} H(n,j)\max_{M_0\leq n\leq k} y(j),$$

where p is the nonnegative vector

$$p = \sup_{n\geq M_0} a(n) + \max_{M_0\leq n< N_0}\sum_{j=M_0}^{n} H(n,j)\max_{M_0\leq j< N_0} y(j).$$

Due to (24), $(I - \sup_{n\geq N_0}\sum_{j=M_0}^{n} H(n,j))^{-1}$ exists and is nonnegative. Hence, for every $k > N_0$,

$$\max_{M_0\leq n\leq k} y(n) \leq \left(I - \sup_{n\geq N_0}\sum_{j=M_0}^{n} H(n,j)\right)^{-1} p,$$

establishing that $\sup_{n\geq M_0} y(n)$ is finite. Hence, $\{y(n)\}_{n\geq 0}$ is bounded. □

PROPOSITION 4.2. *Suppose that* $H(n,j) \geq 0$ *for all* $(n,j) \in \mathbb{Z}^+ \times \mathbb{Z}^+$ *and* (21) *holds. Then* $\sup_{n\geq 0}\sum_{j=0}^{n} H(n,j)$ *is finite if and only if* (22) *holds.*

Proof. Assume that (22) is true. Let $N_0 > M_0 > 0$ be as in (24). Then

$$\sup_{n\geq 0}\sum_{j=0}^{n} H(n,j) \leq \sup_{n\geq N_0}\sum_{j=M_0}^{n} H(n,j) + \sup_{n\geq M_0}\sum_{j=0}^{M_0} H(n,j) + \max_{M_0\leq n\leq N_0}\sum_{j=M_0}^{n} H(n,j) + \max_{0\leq n\leq M_0}\sum_{j=0}^{n} H(n,j).$$

The sum on the right hand side is finite, the last two terms being finite sums. Conversely, if $\sup_{m\geq 0}\sum_{k=0}^{m} H(m,k) < \infty$, $H(n,j) \leq \sup_{m\geq 0}\sum_{k=0}^{m} H(m,k)$ for all $n \geq j$. □

Our next result gives a condition for the bounded solutions of (23) to obey an *asymptotic inequality*.

THEOREM 4.3. *Suppose that the hypotheses of Theorem* 4.1 *hold. Let* $\limsup_{n\to\infty} H(n,j) =: \bar{H}(j)$ *for all* $j \in \mathbb{Z}^+$. *Then*

$$\limsup_{n\to\infty} y(n) \leq (I - W_H)^{-1}\left(\limsup_{n\to\infty} h(n) + \sum_{j=0}^{\infty}\bar{H}(j)y(j)\right). \tag{26}$$

Proof. By Proposition 4.2, $\sup_{n\geq 0}\sum_{j=0}^{n} H(n,j)$ is finite, and the argument used in Proposition 3.3 demonstrates that $\sum_{j=0}^{\infty}\bar{H}(j)$ is finite; hence $\sum_{j=0}^{\infty}\bar{H}(j)y(j)$ is finite. Let $n \geq m > 0$. The inequality (23) can be expressed as

$$y(n) \leq h(n) + \sum_{j=0}^{m-1}\bar{H}(j)y(j) + \sum_{j=0}^{m-1}(H(n,j) - \bar{H}(j))y(j) + \sum_{j=m}^{n} H(n,j)y(j). \tag{27}$$

Since h and y are bounded, $\limsup_{n\to\infty} h(n)$ and $\sup_{j\geq m} y(j)$ are both finite. Moreover,

$$\limsup_{n\to\infty}\sum_{j=0}^{m-1}(H(n,j) - \bar{H}(j))y(j) \leq \sum_{j=0}^{m-1}\limsup_{n\to\infty}(H(n,j) - \bar{H}(j))y(j) = 0.$$

By taking the limit superior of each side of (27) as $n \to \infty$,

$$\limsup_{n\to\infty} y(n) \leq \limsup_{n\to\infty} h(n) + \sum_{j=0}^{m-1}\bar{H}(j)y(j) + \limsup_{n\to\infty}\sum_{j=m}^{n} H(n,j)\sup_{j\geq m} y(j).$$

If the limit superior as $m \to \infty$ is taken of each side, we obtain that

$$\limsup_{n\to\infty} y(n) \leq \limsup_{n\to\infty} h(n) + \sum_{j=0}^{\infty}\bar{H}(j)y(j) + W_H\limsup_{j\to\infty} y(j),$$

where W_H is given by (21). Since $(I - W_H)^{-1} \geq 0$, we obtain (26). □

Theorem 3.1 of [9] established an asymptotic representation for solutions of the linear Volterra difference equation $z(n+1) = g(n) + \sum_{j=0}^{n} G(n,j)z(j)$, subject to an initial condition $z(0) = z_0$. This result simplified and generalized Theorem 3.1 of [1]. The solution of the above initial-value problem satisfies a particular form of (1). We now give a variant of Theorem 3.1 of [9] for (1). Its statement is tailored for application to the problems for which some admissibility conditions are known.

THEOREM 4.4. *Assume that H obeys* (4) *and* (9). *Suppose also that H satisfies the conditions*

$$W_H = \lim_{m\to\infty} \limsup_{n\to\infty} \sum_{j=m}^{n} |H(n,j)| < \infty, \tag{28a}$$

$$\rho(W_H) < 1, \tag{28b}$$

and

$$\lim_{m\to\infty} \limsup_{n\to\infty} \left| \sum_{j=m}^{n} H(n,j) - V_H \right| = 0 \quad \textit{for some } V_H. \tag{29}$$

If $\{h(n)\}$ *is in* ℓ^c *with* $h(\infty) = \lim_{n\to\infty} h(n)$, *then the solution* $\{x(n)\}$ *of* (1) *is in* ℓ^c *and*

$$\lim_{n\to\infty} x(n) = (I - V_H)^{-1} \left[h(\infty) + \sum_{j=0}^{\infty} H_\infty(j) x(j) \right]. \tag{30}$$

Proof. Observing firstly that (9) implies (22), we deduce from Proposition 4.2 that (10) is true. Therefore, by Proposition 3.3, $\sum_{j=0}^{\infty} |H_\infty(j)|$ is finite. Note also that it is a consequence of Proposition 3.4 and (28) that $\rho(|V_H|) \le \rho(W_H) < 1$, and hence $\rho(V_H) < 1$ and $I - V_H$ is invertible. Taking absolute values of each side of (1),

$$|x(n)| \le |h(n)| + \sum_{j=0}^{n} |H(n,j)||x(j)|.$$

By Theorem 4.1, x is bounded.

For any x^∞ in $\mathbb{E}^d$, (1) can be expressed as

$$\begin{aligned} x(n) = h(n) &+ \sum_{j=0}^{m-1} H_\infty(j) x(j) + V_H x^\infty + \sum_{j=0}^{m-1} [H(n,j) - H_\infty(j)] x(j) \\ &+ \left(\sum_{j=m}^{n} H(n,j) - V_H \right) x^\infty + \sum_{j=m}^{n} H(n,j)[x(j) - x^\infty]. \end{aligned} \tag{31}$$

We apply this in the case that x^∞ solves the equation

$$x^\infty = h(\infty) + \sum_{j=0}^{\infty} H_\infty(j) x(j) + V_H x^\infty, \tag{32}$$

which has the unique solution $x^\infty = (I - V_H)^{-1} \left[h(\infty) + \sum_{j=0}^{\infty} H_\infty(j) x(j) \right]$. By subtracting

(32) from (31), and taking the absolute value of each side of the resulting equation, we deduce that

$$|x(n) - x^{\infty}| \leq |h(n) - h(\infty)| + \left|\sum_{j=m}^{\infty} H_{\infty}(j)x(j)\right| + \sum_{j=0}^{m-1} |H(n,j) - H_{\infty}(j)||x(j)| + \left|\sum_{j=m}^{n} H(n,j) - V_H\right||x^{\infty}| + \sum_{j=m}^{n} |H(n,j)||x(j) - x^{\infty}|. \tag{33}$$

Since $\{x(n)\}$ is bounded, $\sup_{j\geq m}|x(j) - x^{\infty}|$ and $\lim\sup_{n\to\infty}|x(n) - x^{\infty}|$ are both finite. By taking the limit superior of each side of (33) as $n \to \infty$,

$$\limsup_{n\to\infty}|x(n) - x^{\infty}| \leq \left|\sum_{j=m}^{\infty} H_{\infty}(j)x(j)\right| + \limsup_{n\to\infty}\left|\sum_{j=m}^{n} H(n,j) - V_H\right||x^{\infty}| + \limsup_{n\to\infty}\sum_{j=m}^{n} |H(n,j)|\sup_{j\geq m}|x(j) - x^{\infty}|.$$

Since (29) holds, the limit superior as $m \to \infty$ can be taken of each side of this inequality, to obtain

$$\limsup_{n\to\infty}|x(n) - x^{\infty}| \leq W_H \limsup_{j\to\infty}|x(j) - x^{\infty}|,$$

where W_H is given by (28). Since $(I - W_H)^{-1} \geq 0$, it follows that $\lim\ \sup_{n\to\infty}|x(n) - x^{\infty}| = 0$ and hence $x(n) \to x^{\infty}$ as $n \to \infty$. □

The limit formula (30) is the counterpart of ([1], equation (10)) or ([9], equation (3.5)). The stability condition (28) is also an hypothesis in [6], in the case of geometric weight functions and $d = 1$. Assumptions like (28) also appear in the theory of Volterra integral equations; for example, see [7] and Gripenberg [8].

Remark 1. If (28) holds with $W_H = 0$, then (29) is true with $V_H = 0$.

5. General results and discussion

Theorems 3.1, 3.7 and 4.10 of [14] can be applied to the resolvent kernel to obtain three sets of necessary and sufficient conditions for the pairs $(\ell_{\gamma}, \ell_{\gamma})$, (ℓ^c, ℓ^c) and (ℓ_{γ}, ℓ^c), respectively, to be admissible with respect to the resolvent operator. However these conditions would be in terms of the resolvent kernel $\{R(n, j)\}$ rather than $\{H(n, j)\}$, which is not desirable.

Reference [14] contains some results which give sufficient conditions in terms of $\{H(n, j)\}$ for admissibility conditions to hold. Lemma 4.18 of [14] states that if

$$\sup_{n\geq 0}\sum_{j=0}^{n} \|H(n,j)\| < 1, \tag{34}$$

then $\sup_{n\geq 0}\sum_{j=0}^{n}\|R(n,j)\| < \infty$: moreover, if $\lim_{n\to\infty}H(n,j) = 0$ for all $j \in \mathbb{Z}^+$, then $\lim_{n\to\infty}R(n,j) = 0$. The proofs can be found in Song [12,13]. We now present some improvements of these results.

Firstly, we state a sufficient condition for the pair $(\ell_\gamma, \ell_\delta)$ to be admissible with respect to $\mathcal{R}$. The notation $H_\delta(n, j) = H(n, j)\delta(j)/\delta(n)$ is sometimes used.

THEOREM 5.1. *Let γ and δ be non-vanishing sequences in $\ell(\mathbb{E})$. Suppose that H satisfies* (4) *and*

$$W_{H_\delta} := \lim_{m\to\infty} \limsup_{n\to\infty} \sum_{j=m}^{n} \frac{|H(n,j)\delta(j)|}{|\delta(n)|} < \infty, \tag{35a}$$

$$\rho(W_{H_\delta}) < 1. \tag{35b}$$

If $(\ell_\gamma, \ell_\delta)$ is admissible with respect to $\mathcal{H}$, then $(\ell_\gamma, \ell_\delta)$ is also admissible with respect to $\mathcal{R}$.

It is convenient to state a preliminary result.

LEMMA 5.2. *Let $\{\delta(n)\}$ be a non-vanishing sequences in $\ell(\mathbb{E})$, and $\phi \in \ell(\mathbb{E}^d)$. Suppose that H satisfies* (4)*. Let $h = \{h(n)\}$ and $x = \{x(n)\}$ be the sequences given by*

$$x(n) = \sum_{j=0}^{n} \frac{R(n,j)}{\delta(n)}\phi(j), \quad h(n) = -\sum_{j=0}^{n} \frac{H(n,j)}{\delta(n)}\phi(j). \tag{36}$$

Then x satisfies the equation

$$x(n) = h(n) + \sum_{j=0}^{n} \frac{H(n,j)\delta(j)}{\delta(n)}x(j), \quad n \in \mathbb{Z}^+. \tag{37}$$

Proof. Let $\phi \in \ell(\mathbb{E}^d)$. By premultiplying $\phi(k)$ by each side of (5), summing over $0 \le k \le n$ and interchanging the order of summation in the double sum, we obtain

$$\sum_{k=0}^{n} R(n,k)\phi(k) = -\sum_{k=0}^{n} H(n,k)\phi(k) + \sum_{j=0}^{n} H(n,j)\sum_{k=0}^{j} R(j,k)\phi(k), \tag{38}$$

and therefore

$$\sum_{k=0}^{n} \frac{R(n,k)}{\delta(n)}\phi(k) = -\sum_{k=0}^{n} \frac{H(n,k)}{\delta(n)}\phi(k) + \sum_{j=0}^{n} \frac{H(n,j)\delta(j)}{\delta(n)} \sum_{k=0}^{j} \frac{R(j,k)}{\delta(j)}\phi(k).$$

Equation (37) is an immediate consequence of this and the definitions in (36). □

Theorem 5.1 is now proved: Theorem 4.1 is not used in the proof.

Proof. Let $\phi \in \ell_\gamma(\mathbb{E}^d)$, and h and x the sequences defined in (36). By hypothesis h is bounded. Equation (37) is satisfied, and hence

$$|x(n) \leq |h(n)| + \sum_{j=0}^{n} \frac{|H(n,j)\delta(j)|}{|\delta(n)|}|x(j)|, \quad j \in Z^+. \tag{39}$$

Due to (35), $|H_\delta|$ obeys (21). Let $j \geq 0$ be fixed. Define a sequence $\{\phi(n)\}$ in ℓ_γ by $\phi(j) = \gamma(j)$ and $\phi(n) = 0$ for all $n \neq j$. By our admissibility assumption,

$$\frac{|H(n,j)\delta(j)|}{|\delta(n)|} = \frac{|\delta(j)|}{|\gamma(j)|}\frac{|H(n,j)\gamma(j)|}{|\delta(n)|} \leq \frac{|\delta(j)|}{|\gamma(j)|}\frac{|(H\phi)(n)|}{|\delta(n)|}$$

is uniformly bounded for all $n \geq j$, and therefore $|H_\delta|$ satisfies (22). We conclude from Theorem 4.1 that x must be bounded. □

Because of Theorem 3.1, Theorem 5.1 can be restated.

Corollary 5.3. *Let γ and δ be non-vanishing sequences in $\ell(\mathbb{E})$. Suppose that H satisfies* (4)*,* (7) *and* (35)*. Then the resolvent of H satisfies*

$$\sup_{n\geq 0}\sum_{j=0}^{n} \frac{|R(n,j)\gamma(j)|}{|\delta(n)|} < \infty. \tag{40}$$

Proof. Due to Theorem 3.1, $(\ell_\gamma, \ell_\delta)$ is admissible with respect to $\mathcal{H}$. Let $v \in \mathbb{E}^d$ be a constant vector, and define $\phi(n) = \gamma(n)v$. If h and x are as in (36), the inequality (39) again holds. The conclusion follows upon again applying Theorem 4.1 to it. Alternatively, the result is a consequence of Theorem 3.1, which says that (40) is necessary and sufficent for $(\ell_\gamma, \ell_\delta)$ to be admissible with respect to $\mathcal{R}$. □

A sufficient condition for the admissibilty of $(\ell^c_\gamma, \ell^c_\delta)$ with respect to $\mathcal{R}$ is now given: Theorem 3.2 is not employed in the proof.

Theorem 5.4. *Let γ and δ be non-vanishing sequences in $\ell(\mathbb{E})$. Suppose that H satisfies* (4) *and* $\lim_{n\to\infty}H(n, j)/\delta(n)$ *exists for all $j \geq 0$. Assume also that H obeys* (35) *and*

$$\lim_{m\to\infty}\limsup_{n\to\infty}\left|\sum_{j=m}^{n} \frac{H(n,j)\delta(j)}{\delta(n)} - V_{H_\delta}\right| = 0 \quad \textit{for some} \ \ V_{H_\delta}. \tag{41}$$

If $(\ell^c_\gamma, \ell^c_\delta)$ is admissible with respect to $\mathcal{H}$, then $(\ell^c_\gamma, \ell^c_\delta)$ is admissible with respect to $\mathcal{R}$.

Proof. Let ϕ be a sequence in ℓ^c_γ, and define the sequences h and x as in (36). By assumption h is in ℓ^c. We conclude from Proposition 3.4, and Theorem 4.4 that x is also convergent. □

A sufficient condition for (ℓ_γ, ℓ^c) to be admissible with respect to $\mathcal{R}$ is now described. Theorem 3.5 is not used in the proof.

Theorem 5.5. *Let γ be a non-vanishing sequence in $\ell(\mathbb{E})$. Suppose that* (4), (9), (28) *and* (29) *hold. If (ℓ_γ, ℓ^c) is admissible with respect to $\mathcal{H}$, then (ℓ_γ, ℓ^c) is also admissible with respect to $\mathcal{R}$.*

Proof. Define $\delta(n) \equiv 1$. Let ϕ be a sequence in ℓ_γ, and define sequences h and x as in (36). By assumption h is in ℓ^c. Proposition 5.2 says that (37) holds. We conclude from Theorem 4.4 that x is also convergent. □

Granted that $(\ell^c_\gamma, \ell^c_\delta)$ [or (ℓ_γ, ℓ^c)] is admissible with respect to $\mathcal{H}$, sufficient conditions have been found for the same pairs to be admissible with respect to $\mathcal{R}$. Our aim is now to establish some properties of the resolvent kernel, including the necessary conditions given in Theorems 3.2 and 3.5. One approach is to invoke those theorems and the following result.

Proposition 5.6. *Assume that the kernel H satisfies* (4) *and* (9). *Suppose also that H obeys* (28) *and* (29). *Then $\lim_{n\to\infty} R(n,k) = R_\infty(k)$ exists for each $k \geq 0$, with*

$$R_\infty(k) = (I - V_H)^{-1}\left[\sum_{j=k}^{\infty} H_\infty(j)R(j,k) - H_\infty(k)\right], \tag{42}$$

where V_H is as in (29).

Proof. Let $k \in \mathbb{Z}^+$ be fixed. We introduce an arbitrary vector $v \neq 0$ in $\mathbb{E}^d$. It follows from (5) that (1) holds with $x(n) = R(n, k)v$ and $h(n) = -H(n, k)v$ for $n \geq k$, and $x(n) = h(n) = 0$ otherwise. Since $h(n) \to -H_\infty(k)v$ as $n \to \infty$, $h \in \ell^c$. Theorem 4.4 says that $R(n, k)v = x(n) \to x(\infty)$ as $n \to \infty$. Since v is arbitrary, $R(n, k) \to R_\infty(k)$ as $n \to \infty$, where $x(\infty) = R_\infty(k)v$. Because $x(\infty)$ satisfies (30),

$$R_\infty(k)v = (I - V_H)^{-1}\left[-H_\infty(k)v + \sum_{j=k}^{\infty} H_\infty(j)R(j,k)v\right],$$

which implies (42). □

Generally, (42) does not give an explicit formula for $\lim_{n\to\infty} R(n, m)$. Comparing our hypotheses to those of Lemma 4.18 of [14], (34) has been replaced by the weaker condition (28) and $H_\infty(j)$ need not be trivial.

It is intended that the next result be compared with Theorem 3.2.

Theorem 5.7. *Suppose that H obeys* (4), (9), (28) *and* (29). *Then $\lim_{n\to\infty}\sum_{j=0}^{n} R(n,j)$ exists, and $V_R := \lim_{n\to\infty}\sum_{k=0}^{n} R(n,k) - \sum_{k=0}^{\infty} R_\infty(k)$ obeys*

$$I - V_R = (I - V_H)^{-1}. \tag{43}$$

Also, for all $\phi \in \ell^c$ with $\phi(\infty) = \lim_{n\to\infty}\phi(n)$,

$$\lim_{n\to\infty}(\mathcal{R}\phi)(n) = V_R\phi(\infty) + \sum_{k=1}^{\infty} R_\infty(k)\phi(k). \tag{44}$$

Proof. Our assumptions imply that the hypotheses of Proposition 5.6 hold, and therefore that $\lim_{n\to\infty} R(n,k) = R_\infty(k)$ exists and obeys (42). Let v in $\mathbb{E}^d$ be an arbitrary vector. By applying Lemma 5.2 in the case that $\phi(n) \equiv v$ and $\delta(n) \equiv 1$, we obtain the equation

$$x(n) = h(n) + \sum_{j=0}^{n} H(n,j)x(j), \quad n \in \mathbb{Z}^+,$$

where h and x are given by $x(n) = \sum_{k=0}^{n} R(n,k)v$ and $h(n) = -\sum_{k=0}^{n} H(n,k)v$. Proposition 3.4 says that (11) is true, and hence $h(n) \to -\lim_{n\to\infty} \sum_{k=0}^{n} H(n,k)v$ as $n \to \infty$: h is in ℓ^c. If we apply Theorem 4.4, we can deduce that $\lim_{n\to\infty} x(n) = \lim_{n\to\infty} \sum_{k=0}^{n} R(n,k)v$ exists. Since v is arbitrary, (30) leads to

$$\lim_{n\to\infty} \sum_{k=0}^{n} R(n,k) = (I - V_H)^{-1} \left[\sum_{j=0}^{\infty} H_\infty(j) \sum_{k=0}^{j} R(j,k) - \lim_{n\to\infty} \sum_{k=0}^{n} H(n,k) \right]. \tag{45}$$

It is a consequence of this and (42) that

$$\lim_{n\to\infty} \sum_{k=0}^{n} R(n,k) - \sum_{k=0}^{\infty} R_\infty(k) = -(I - V_H)^{-1} V_H.$$

(43) follows from this and $(I - V_H)^{-1}(I - V_H) = I$. The last assertion in Theorem 3.2 implies (44). □

Another result of interest is to deduce, for fixed $M \geq 0$, that $\sum_{j=M}^{n} R(n,j)$ and $\sum_{j=0}^{M} R(n,j)$ are both convergent as $n \to \infty$. The convergence of $\sum_{m=0}^{M} R(n,m)$ as $n \to \infty$ assuming (34) is proved in [13].

COROLLARY 5.8. *Suppose that H satisfies the hypotheses of Theorem* 5.7. *Then the limits* $\lim_{n\to\infty} \sum_{j=M}^{n} R(n,j)$ *and* $\lim_{n\to\infty} \sum_{m=0}^{M} R(n,m)$ *both exist. Moreover, a limit formula for each can be derived from* (42) *and* (45).

If our sufficient conditions hold for (ℓ_γ, ℓ^c) to be admissible with respect to $\mathcal{R}$, it is now shown directly that the resolvent kernel $\{R(n,j)\}$ has the same properties as $\{H(n,j)\}$ in Theorem 3.5.

THEOREM 5.9. *Let γ be a non-vanishing sequence in $\ell(\mathbb{E})$. Suppose that H obeys* (4), (9), (17) *and* (18). *Assume also that* (28) *and* (29) *hold. Then (ℓ_γ, ℓ^c) is admissible with respect to $\mathcal{R}$,* $\lim_{n\to\infty} R(n,k) =: R_\infty(k)$ *with $R_\infty(k)$ given by* (42)*, and*

$$\sup_{n\geq 0} \sum_{j=0}^{n} |R(n,j)||\gamma(j)| < \infty, \tag{46}$$

$$\lim_{m\to\infty} \limsup_{n\to\infty} \sum_{j=m}^{n} |R(n,j)||\gamma(j)| = 0. \tag{47}$$

Proof. We conclude from Theorem 3.5 that (ℓ_γ, ℓ^c) is admissible with respect to $\mathcal{H}$. Then Theorem 5.5 establishes that this pair is also admissible with respect to $\mathcal{R}$. We can also observe from Corollary 5.3 that (46) is true. It only remains to prove (47).

Let v be an arbitrary vector in $\mathbb{E}^d$, and $M > 0$ a fixed integer. In a similar fashion to the demonstration of (38) in the case $\phi(n) = \gamma(n)v$, we can deduce that for all $n \geq M$,

$$\sum_{k=M}^{n} |R(n,k)\gamma(k)v| \leq \sum_{k=M}^{n} |H(n,k)\gamma(k)v| + \sum_{j=M}^{n} |H(n,j)| \sum_{k=M}^{j} |R(j,k)\gamma(k)v|.$$

This inequality can be expressed as

$$y_M(n) \leq h_M(n) + \sum_{j=0}^{n} |H(n,j)| y_M(j), \quad n \geq M, \tag{48}$$

if we put $h_M(n) = \sum_{k=M}^{n} |H(n,k)\gamma(k)v|$ and $y_M(n) = \sum_{k=M}^{n} |R(n,k)\gamma(k)v|$ for $n \geq M$, and $h_M(n) = y_M(n) = 0$, for $n < M$. We deduce from (18) that $\limsup_{n\to\infty} \sum_{k=M}^{n} |H(n,k)\gamma(k)v|$ is a finite vector, and hence that $\{h_M(n)\}$ is bounded. Then, Theorem 4.1 says that y_M is bounded as well. Also, $\limsup_{n\to\infty} |H(n,j)| = \lim_{n\to\infty} |H(n,j)| = |H_\infty(j)|$ is finite. Therefore, it can be concluded from Theorem 4.3 that

$$\limsup_{n\to\infty} y_M(n) \leq (I - W_H)^{-1} \left(\limsup_{n\to\infty} h_M(n) + \sum_{j=M}^{\infty} H_\infty(j) y_M(j) \right). \tag{49}$$

But (18) implies that $\limsup_{n\to\infty} h_M(n) \to 0$ as $M \to \infty$. Therefore, taking the limit superior as $M \to \infty$ in (49) demonstrates that $\limsup_{n\to\infty} y_M(n) \to 0$ as $M \to \infty$, from which we infer that (47) holds. □

6. Applications

In [1] a result was proved about the asymptotic behaviour of solutions of the linear convolution equation $z(n+1) = g(n) + \sum_{j=0}^{n} G^\sharp(n-j)z(j)$. An important feature was that the assumptions implied that the *characteristic equation had no roots*. In this section, the result is generalized to linear perturbations of such convolution equations.

In order to precisely state our results, we recall a definition from [1].

Definition 6.1. *Let $r > 0$ be finite. A real-valued sequence $\gamma = \{\gamma(n)\}_{n\geq 0}$ is in $\mathcal{W}(r)$ if $\gamma(n) > 0$ for all $n \geq 0$, and*

$$\lim_{n\to\infty} \frac{\gamma(n-1)}{\gamma(n)} = \frac{1}{r}, \quad \sum_{j=0}^{\infty} \gamma(j) r^{-j} < \infty, \tag{50}$$

$$\lim_{m\to\infty} \left(\limsup_{n\to\infty} \sum_{j=m}^{n-m} \frac{\gamma(n-j)\gamma(j)}{\gamma(n)} \right) = 0. \tag{51}$$

In Proposition 4.4 of [1], it is proved that a positive real-valued sequence $\gamma = \{\gamma(n)\}_{n\geq 0}$ satisfying (50), is in $\mathcal{W}(r)$ if and only if

$$\lim_{n\to\infty} \sum_{j=0}^{n} \frac{\gamma(n-j)\gamma(j)}{\gamma(n)} = 2 \sum_{j=0}^{\infty} \gamma(j) r^{-j}. \tag{52}$$

In [2], results are proved about sequences satisfying (50) and (52). It is shown there that $\gamma(n) = r^n n^{-\alpha}$ for $\alpha > 1$; $\gamma(n) = r^n n^{-\alpha}\exp(-n^{\beta})$ for $\alpha \in \mathbb{R}$, $0 < \beta < 1$; and $\gamma(n) = r^n e^{-n/(\log n)}$ are all sequences in $\mathcal{W}(r)$. However, the sequences defined by $\gamma(n) = r^n$, and $\gamma(n) = r^n n^{-\alpha}$, $\alpha \le 1$ are *not* in $\mathcal{W}(r)$.

We can deduce from Theorem 4.1 a result in the case that γ is in $\mathcal{W}(r)$. It is envisaged that the main applications of this result would be in the case $r = 1$.

THEOREM 6.2. *Let γ be in $\mathcal{W}(r)$ for some $r > 0$. Suppose that*

$$\limsup_{n\to\infty} H(n, n-k) =: H^{\sharp}(k) < \infty, \quad k \in \mathbb{Z}^+, \tag{53}$$

$$\sum_{k=0}^{\infty} |H^{\sharp}(k)| r^{-k} < \infty, \quad \rho\left(\sum_{k=0}^{\infty} |H^{\sharp}(k)| r^{-k}\right) < 1. \tag{54}$$

Assume also that there is a non-negative matrix $\bar{C}$, such that

$$|H(n,j)| \le \bar{C}\gamma(n-j), \quad 0 \le j \le n. \tag{55}$$

If (4) *holds, then* (1) *has a solution x in ℓ_γ for every $h \in \ell_\gamma$*

Proof. We divide (1) by $\gamma(n)$ and obtain

$$\tilde{x}(n) = \tilde{h}(n) + \sum_{j=0}^{n} \tilde{H}(n,j)\tilde{x}(j), \quad n \in \mathbb{Z}^+, \tag{56}$$

where $\tilde{x}(n) = x(n)/\gamma(n)$, $\tilde{h}(n) = h(n)/\gamma(n)$ and

$$\tilde{H}(n,j) = \frac{H(n,j)\gamma(j)}{\gamma(n)}, \quad 0 \le j \le n. \tag{57}$$

We demonstrate that $|\tilde{H}|$ obeys hypotheses (21) and (22) of Theorem 4.1.

It follows from (55) that, for $2n > m$,

$$\begin{aligned}\sum_{j=m}^{n} |\tilde{H}(n,j)| &= \sum_{j=m}^{n-m-1} \frac{\gamma(j)\gamma(n-j)}{\gamma(n)} \frac{|H(n,j)|}{\gamma(n-j)} + \sum_{k=0}^{m} \frac{|H(n,n-k)|\gamma(n-k)}{\gamma(n)} \\ &\le \left(\sum_{j=m}^{n-m-1} \frac{\gamma(n-j)\gamma(j)}{\gamma(n)}\right)\bar{C} + \sum_{k=0}^{m} |H(n,n-k)| \frac{\gamma(n-k)}{\gamma(n)}.\end{aligned}$$

Taking the limit superior as $n \to \infty$ of each side and using (50) and (53), we obtain the matrix inequality

$$\limsup_{n\to\infty} \sum_{j=m}^{n} |\tilde{H}(n,j)| \le \limsup_{n\to\infty} \sum_{j=m}^{n-m-1} \frac{\gamma(n-j)\gamma(j)}{\gamma(n)}\bar{C} + \sum_{k=0}^{m} |H^{\sharp}(k)| r^{-k}.$$

By taking the limit superior as $m \to \infty$, we deduce from (51) and (54) that

$$\lim_{m\to\infty} \limsup_{n\to\infty} \sum_{j=m}^{n} |\tilde{H}(n,j)| \le \sum_{k=0}^{\infty} |H^{\sharp}(k)| r^{-k} < \infty. \tag{58}$$

It is then a consequence of the spectral radius condition in (54) that (28) holds with $W_{\tilde{H}} = \sum_{k=0}^{\infty} |H^{\sharp}(k)| r^{-k}$.

The fact that (22) is true follows from

$$|\tilde{H}(n,j)| \le \left(\frac{\gamma(n-j)}{\gamma(n)}\right) \gamma(j) \bar{C}, \quad 0 \le j \le n.$$

Due to (50) the term in parentheses converges to r^{-n} as $n \to \infty$, and hence is uniformly bounded for $n \ge j$. □

The next result gives sufficient conditions for the solution of (1) to be such that $x(n)/\gamma(n)$ converges to a limit as $n \to \infty$.

THEOREM 6.3. *Let γ be in $\mathcal{W}(r)$ for some $r > 0$. Suppose that the limit*

$$\lim_{n\to\infty} H(n, n-k) = H^{\sharp}(k) \quad \textit{exists for all} \;\; k \in \mathbb{Z}^{+}, \tag{59}$$

and $H^{\sharp}$ obeys (54). *Assume also that* (55) *holds, and the limit*

$$\lim_{n\to\infty} \frac{H(n,j)}{\gamma(n-j)} = C_{\infty}(j) \quad \textit{exists for all} \;\; j \in \mathbb{Z}^{+}. \tag{60}$$

If (4) *holds, the solution x of* (1) *is in ℓ^c_γ for every $h \in \ell^c_\gamma$, and*

$$\lim_{n\to\infty} \frac{x(n)}{\gamma(n)} = \left(I - \sum_{j=0}^{\infty} H^{\sharp}(j) r^{-j}\right)^{-1} \left(\lim_{n\to\infty} \frac{h(n)}{\gamma(n)} + \sum_{j=0}^{\infty} C_{\infty}(j) r^{-j} x(j)\right). \tag{61}$$

Proof. It is shown that the hypotheses of Theorem 4.4 hold for $\tilde{H}$ defined by (57). Because, the hypotheses of Theorem 6.3 imply those of Theorem 6.2, the stability condition (28) must hold. The existence of the limit in (9) can be deduced immediately from (50) and (60), since

$$\tilde{H}(n,j) = \frac{H(n,j)}{\gamma(n-j)} \frac{\gamma(n-j)}{\gamma(n)} \gamma(j) \to C_{\infty}(j) r^{-j} \gamma(j) =: \tilde{H}_{\infty}(j) \quad \text{as} \;\; n \to \infty. \tag{62}$$

Next it is proved that $\lim_{m\to\infty}\limsup_{n\to\infty}\left|\sum_{j=m}^{n}\tilde{H}(n,j) - V_{\tilde{H}}\right|$, with $V_{\tilde{H}} = \sum_{k=0}^{\infty}H^{\sharp}(k)r^{-k}$. This series is convergent because of (54). For all $n > 2m > 0$

$$\sum_{j=m}^{n}\tilde{H}(n,j) - \sum_{k=0}^{\infty}H^{\sharp}(k)r^{-k} = \sum_{k=0}^{m}\left\{H(n,n-k)\frac{\gamma(n-k)}{\gamma(n)} - H^{\sharp}(k)r^{-k}\right\} - \sum_{k=m+1}^{\infty}H^{\sharp}(k)r^{-k} + \sum_{j=m}^{n-m-1}\frac{H(n,j)}{\gamma(n-j)}\frac{\gamma(n-j)\gamma(j)}{\gamma(n)}. \quad (63)$$

Because of assumptions (50) and (59),

$$\limsup_{n\to\infty}\sum_{k=0}^{m}\left|H(n,n-k)\frac{\gamma(n-k)}{\gamma(n)} - H^{\sharp}(k)r^{-k}\right| = 0.$$

Therefore, it is a consequence of (63) and (55) that

$$\limsup_{n\to\infty}\left|\sum_{j=0}^{n}\tilde{H}(n,j) - \sum_{k=0}^{\infty}H^{\sharp}(k)r^{-k}\right| \leq \left|\sum_{k=m+1}^{\infty}H^{\sharp}(k)r^{-k}\right| + \limsup_{n\to\infty}\sum_{j=m+1}^{n-m-1}\frac{\gamma(n-j)\gamma(j)}{\gamma(n)}\bar{C}.$$

The right-hand side of this inequality tends to zero as $m\to\infty$ because of (51) and (54).

The hypotheses of Theorem 4.4 have been verified, and therefore $\tilde{x}(n) = x(n)/\gamma(n)$ converges to a limit as $n\to\infty$. This limit satisfies (30), which in this case becomes

$$\lim_{n\to\infty}\frac{x(n)}{\gamma(n)} = (I - V_{\overline{H}})^{-1}\left[\lim_{n\to\infty}\frac{h(n)}{\gamma(n)} + \sum_{j=0}^{n}\tilde{H}_{\infty}(j)\frac{x(j)}{\gamma(j)}\right] = (I - V_{\overline{H}})^{-1}\left[\lim_{n\to\infty}\frac{h(n)}{\gamma(n)} + \sum_{j=0}^{\infty}C_{\infty}(j)r^{-j}\gamma(j)\frac{x(j)}{\gamma(j)}\right],$$

completing the proof of (61). □

Remark 1. If (1) is a convolution equation with $H(n,j) = H^{\sharp}(n-j)$, the hypotheses on H in Theorem 6.3 reduce to

$$\sum_{k=0}^{\infty}|H^{\sharp}(k)|r^{-k} < \infty, \quad \rho\left(\sum_{k=0}^{\infty}|H^{\sharp}(k)|r^{-k}\right) < 1, \quad \lim_{n\to\infty}\frac{H^{\sharp}(n)}{\gamma(n)} \quad \text{exists.}$$

We thereby recover the counterpart of Theorem 3.2 of [1].

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