Computers and Mathematics with Applications 64 (2012) 2262–2274

Contents lists available at SciVerse ScienceDirect

Computers and Mathematics with Applications

journal homepage: www.elsevier.com/locate/camwa

Sharp algebraic periodicity conditions for linear higher order difference equations

István Győri, László Horváth *

Department of Mathematics, University of Pannonia, 8200 Veszprém, Egyetem u. 10., Hungary

ARTICLE INFO

Keywords:
Periodic solution
Higher order difference equation
Companion matrix

ABSTRACT

In this paper we give easily verifiable, but sharp (in most cases necessary and sufficient) algebraic conditions for the solutions of systems of higher order linear difference equations to be periodic. The main tool in our investigation is a transformation, recently introduced by the authors, which formulates a given higher order recursion as a first order difference equation in the phase space. The periodicity conditions are formulated in terms of the so-called companion matrices and the coefficients of the given higher order equation, as well.

1. Introduction

In this paper we derive new necessary and sufficient, and sufficient algebraic conditions on the periodicity of the solutions of the d-dimensional system of the sth order difference equations

$$x(n) = \sum_{i=1}^{s} A_i(n)x(n-i), \quad n \geq 0, \tag{1}$$

where

(C_1) $s \geq 1$ is a given integer, and $A_i(n) \in \mathbb{R}^{d\times d}$ for every $1 \leq i \leq s$ and $n \geq 0$.

It is clear that the solutions of (1) are uniquely determined by their initial values

$$x(n) = \varphi(n), \quad -s \leq n \leq -1, \tag{2}$$

where $\varphi(n) \in \mathbb{R}^d$. The unique solution of (1) and (2) is denoted by $x(\varphi) = (x(\varphi)(n))_{n\geq -s}$, where the block vector $\varphi := (\varphi(-s), \ldots, \varphi(-1))^T \in V_s$. Here V_s means the sd-dimensional real vector space of block vectors with entries in $\mathbb{R}^d$.

We believe that our results about Eq. (1) are interesting in their own right. Further, we believe that these results offer prototypes toward the development of the theory of the periodic behavior of the solutions of nonlinear higher order difference equations.

This equation and its special cases are studied in many textbooks on difference equations such as [1–6], and so on.

On p. 25 in the book [3], Grove and Ladas put the following two questions:

"What is it that makes every solution of a difference equation periodic with the same period?"

"Is there any easily verifiable test that we can apply to determine whether or not this is true?"

Motivated by the above questions, and the papers [7,8], we worked out an easily verifiable algebraic test that we can apply to determine the p-periodic solutions of a linear higher order difference equation. In our results we obtain precise analysis of the periodicity of the solutions not only for the scalar but for the vector case. Note that for this latter case, to the best knowledge of the authors, there are no similar results in the literature.

* Corresponding author.
E-mail addresses: gyori@almos.uni-pannon.hu (I. Győri), lhorvath@almos.uni-pannon.hu, lhorvath@almos.vein.hu (L. Horváth).

0898-1221/$ – see front matter
doi:10.1016/j.camwa.2012.02.018

As an illustration, we formulate two typical applications of our main results. Consider the special case of (1)

$$x(n) = \sum_{i=1}^{s} A_i x(n-i), \quad n \geq 0, \tag{3}$$

where $A_i \in \mathbb{R}^{d\times d}$ for every $1 \leq i \leq s$.

For the integers $1 \leq p \leq s$, V_s^p denotes the set of all initial vectors $\varphi = (\varphi(-s), \ldots, \varphi(-1))^T \in V_s$ such that $\varphi(i) = \varphi(j)$ if $i \equiv j(\mod p)(i, j = -s, \ldots, -1)$. Of course $V_s^s = V_s$.

If $a \in \mathbb{R}$, then $[a]$ denotes the greatest integer that does not exceed a.

Theorem 1.1. *Suppose $1 \leq p \leq s$ is an integer, let $u := \left[\frac{s}{p}\right]$, and let $v := s - up(0 \leq v \leq p-1)$. Then for every $\varphi \in V_s^p$ the solution of (3) and (2) is p-periodic if and only if one of the following conditions holds:*

(a_1) $v = 0$ *and*

$$\sum_{j=0}^{u-1} A_{jp+i} = O, \quad 1 \leq i < p; \qquad \sum_{j=0}^{u-1} A_{jp+p} = I. \tag{4}$$

(a_2) $0 < v < p-1$, *moreover*

$$\sum_{j=0}^{u} A_{jp+i} = O, \quad 1 \leq i \leq v; \qquad \sum_{j=0}^{u-1} A_{jp+i} = O, \quad v+1 \leq i < p,$$

and

$$\sum_{j=0}^{u-1} A_{jp+p} = I.$$

(a_3) $v = p-1$ *and*

$$\sum_{j=0}^{u} A_{jp+i} = O, \quad 1 \leq i < p; \qquad \sum_{j=0}^{u-1} A_{jp+p} = I.$$

Remark 1.1. If $p = s$, then $v = 0$ and (4) can be written in the form

$$A_i = O, \quad 1 \leq i \leq s-1, \qquad A_s = I.$$

Theorem 1.2. *Suppose $p > s$ is an integer. Then for every $\varphi \in V_s$ the solution of (3) and (2) is p-periodic if and only if the rank of the matrix*

$$\begin{pmatrix} B_p - I & B_{p-1} & \ldots & B_2 & B_1 \\ B_1 & B_p - I & \ldots & B_3 & B_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ B_{p-1} & B_{p-2} & \ldots & B_1 & B_p - I \end{pmatrix} \tag{5}$$

with entries

$$B_i := A_i, \quad 1 \leq i \leq s \quad \text{and} \quad B_i := O, \quad s+1 \leq i \leq p$$

is equal to $d(p-s)$.

Theorems 1.1 and 1.2 are consequences of our Theorems 4.1–4.3. For the definition of the periodic solutions see Section 2 below.

It is known that the system (1) can be reformulated to a ds-dimensional system of first order difference equations in an appropriate sequence space (see e.g. [2]). The matrices of the first order ds-dimensional system are called companion matrices of Eq. (1) and the system itself is called a companion system of (1). It is clear that the companion matrices are block matrices defined by all $A_i(n)$ in Eq. (1). Recently, Győri and Horváth introduced a new transformation which is extremely useful in analyzing the summability of the solutions of higher order difference equations (see e.g. [9,10]). In this paper we show that our transformation is also powerful in studying the periodicity of the solutions of the Eq. (1).

Our paper is essentially subdivided into six parts.

Section 2 is fundamental for our work and contains basic results on our transformation of the Eq. (1) into a first order system with tractable companion matrices.

Section 3 gives necessary and sufficient, and sufficient periodicity conditions via the companion matrices for both time dependent and independent cases.

The main results are stated in Section 4. It is an amazing fact that the algebraic periodicity conditions in the main theorems are surprisingly simple.

In Section 5 some illustrative examples are given to show the effectiveness of our theory for higher order equations.

Some preliminary results and the proofs of the main results can be found in Section 6.

2. Definitions and a phase space transformation

Definition 2.1. (a) The zero matrix and the identity matrix in $\mathbb{R}^{d\times d}$ are denoted by O and I, respectively.
(b) $\mathcal{O}$ and $\mathcal{I}$ mean the zero matrix and the identity matrix in the real vector space of $p \times p (p \geq 1)$ block matrices with entries in $\mathbb{R}^{d\times d}$, respectively.

Let $(x(n))_{n\geq -s}$ be a given sequence in $\mathbb{R}^d$. Then for any fixed $n \geq 0$ we introduce an sd-dimensional state vector $x_n = (x_n(-s), \ldots, x_n(-1))^T \in V_s$, where the entries $x_n(i)$'s are defined by $x_n(i) := x(n+i)(-s \leq i \leq -1)$.

Based on the state vector notation Győri and Horváth [9] introduced a new transformation to rewrite a higher order system of difference equations into an sd-dimensional system of first order difference equations.

In the state vector notation, we transcribe (1) as

$$x_{ks} = \mathcal{C}^{(k)} x_{(k-1)s}, \quad k \geq 1, \tag{6}$$

with initial condition

$$x_0 = \varphi, \tag{7}$$

where $x_{ks} = (x_{ks}(-s), \ldots, x_{ks}(-1))^T = (x(ks-s), \ldots, x(ks-1))^T \in V_s$.

The block matrix

$$\mathcal{C}^{(k)} := \begin{pmatrix} \mathcal{C}^{(k)}(-s,-s) & \ldots & \mathcal{C}^{(k)}(-s,-1) \\ \vdots & \ddots & \vdots \\ \mathcal{C}^{(k)}(-1,-s) & \ldots & \mathcal{C}^{(k)}(-1,-1) \end{pmatrix}, \quad k \geq 1 \tag{8}$$

is a ds by ds matrix, where the entries are d by d matrices. $\mathcal{C}^{(k)}$ is called the companion matrix of Eq. (1) and it is defined by the formula

$$\mathcal{C}^{(k)} = \left(\mathcal{I} - \mathcal{L}^{(k)}\right)^{-1} \mathcal{U}^{(k)}, \tag{9}$$

where

$$\mathcal{L}^{(k)} := \begin{pmatrix} O & \ldots & O & O \\ \mathcal{L}^{(k)}(-s+1,-s) & \ldots & O & O \\ \vdots & \ddots & \ddots & \vdots \\ \mathcal{L}^{(k)}(-1,-s) & \ldots & \mathcal{L}^{(k)}(-1,-2) & O \end{pmatrix}, \tag{10}$$

and

$$\mathcal{U}^{(k)} := \begin{pmatrix} \mathcal{U}^{(k)}(-s,-s) & \mathcal{U}^{(k)}(-s,-s+1) & \ldots & \mathcal{U}^{(k)}(-s,-1) \\ O & \mathcal{U}^{(k)}(-s+1,-s+1) & \ldots & \mathcal{U}^{(k)}(-s+1,-1) \\ \vdots & \vdots & \ddots & \vdots \\ O & O & \ldots & \mathcal{U}^{(k)}(-1,-1) \end{pmatrix}, \tag{11}$$

for any $k \geq 1$. The elements of the above block matrices are defined as follows:

$$\mathcal{L}^{(k)}(i,j) = \begin{cases} O, & -s \leq i \leq j \leq -1 \\ A_{i-j}(ks+i), & -s \leq j < i \leq -1 \end{cases}, \tag{12}$$

and

$$\mathcal{U}^{(k)}(i,j) = \begin{cases} A_{s+i-j}(ks+i), & -s \leq i \leq j \leq -1 \\ O, & -s \leq j < i \leq -1 \end{cases} \tag{13}$$

for any $k \geq 1$.

Remark 2.1. The matrix $\mathcal{L}^{(k)}$ is nilpotent, since $\left(\mathcal{L}^{(k)}\right)^s = \mathcal{O}$. Therefore

$$\left(\mathcal{I} - \mathcal{L}^{(k)}\right)^{-1} = \mathcal{I} + \mathcal{L}^{(k)} + \cdots \left(\mathcal{L}^{(k)}\right)^{s-1}, \quad k \geq 1.$$

The next theorem gives the relations between the solutions of the higher order system (1) and the related first order system (6).

Theorem 2.1. *Assume* (C_1). *Then*

(a) *For any* $\varphi = (\varphi(-s), \ldots, \varphi(-1))^T \in V_s$, $x(\varphi) = (x(\varphi)(n))_{n\geq -s}$ *is the solution of* (1) *and* (2) *exactly if* $(x_{ks}(\varphi))_{k\geq 1} = ((x_{ks}(\varphi)(-s), \ldots, x_{ks}(\varphi)(-1))^T)_{k\geq 1}$ *is the solution of* (6) *and* (7), *that is*

$$x_{ks}(\varphi)(i) = \sum_{j=-s}^{-1} \mathcal{C}^{(k)}(i,j) x_{(k-1)s}(\varphi)(j) \quad k \geq 1, \ -s \leq i \leq -1. \tag{14}$$

(b) *The explicit form of the solution* $(x_{ks}(\varphi))_{k\geq 1}$ *of* (6) *and* (7) *is*

$$x_{ks}(\varphi)(i) = x(\varphi)(ks+i) = \left(\mathcal{C}^{(k)} \ldots \mathcal{C}^{(1)}\varphi\right)(i), \quad k \geq 1, \ -s \leq i \leq -1,$$

or shortly written

$$x_{ks}(\varphi) = \mathcal{C}^{(k)} \ldots \mathcal{C}^{(1)}\varphi, \quad k \geq 1.$$

(c) *The explicit form of the solution* $x(\varphi)$ *of* (1) *and* (2) *is*

$$\begin{aligned} x(\varphi)(n) &= x_{([\frac{n}{s}]+1)s}(\varphi)\left(n - \left(\left[\frac{n}{s}\right]+1\right)s\right) \\ &= \left(\mathcal{C}^{([\frac{n}{s}]+1)} \ldots \mathcal{C}^{(1)}\varphi\right)\left(n - \left(\left[\frac{n}{s}\right]+1\right)s\right), \quad n \geq 0. \end{aligned}$$

We illustrate the transformation by two examples.

Example 2.1. Consider the second order difference equation

$$x(n) = A_1(n)x(n-1) + A_2(n)x(n-2), \quad n \geq 0, \tag{15}$$

where $A_1(n), A_2(n) \in \mathbb{R}^{d\times d}$ for every $n \geq 0$.

Then it is easy to see that

$$\mathcal{L}^{(k)} = \begin{pmatrix} O & O \\ A_1(2k-1) & O \end{pmatrix}, \quad k \geq 1,$$

and

$$\mathcal{U}^{(k)} = \begin{pmatrix} A_2(2k-2) & A_1(2k-2) \\ O & A_2(2k-1) \end{pmatrix}, \quad k \geq 1.$$

Thus the companion matrices of (15) are given as follows:

$$\begin{aligned} \mathcal{C}^{(k)} &= \left(\mathcal{I} - \mathcal{L}^{(k)}\right)^{-1} \mathcal{U}^{(k)} \\ &= \begin{pmatrix} A_2(2k-2) & A_1(2k-2) \\ A_1(2k-1)A_2(2k-2) & A_1(2k-1)A_1(2k-2) + A_2(2k-1) \end{pmatrix}, \quad k \geq 1. \end{aligned}$$

Example 2.2. Consider the difference equation

$$x(n) = x(n-1) + A(n)x(n-s), \quad n \geq 0, \tag{16}$$

where $s \geq 2$. This is a special case of Eq. (1) with the matrices $A_1(n) = I$, $A_2(n) = \cdots = A_{s-1}(n) = O$ and $A_s(n) = A(n)$.

In this case Lemma 20 in Győri and Horváth [10] can be used. It follows that all the entries of $\mathcal{L}^{(k)} (k \geq 1)$ except the entries $\mathcal{L}^{(k)}(i, i-1) (-s+1 \leq i \leq -1)$ are O, and

$$\mathcal{L}^{(k)}(i, i-1) = I, \quad -s+1 \leq i \leq -1.$$

Similarly, all the entries of $\mathcal{U}^{(k)} (k \geq 1)$ except the entries $\mathcal{U}^{(k)}(i, i) (-s \leq i \leq -1)$ and $\mathcal{U}^{(k)}(-s, -1)$ are O, and

$$\mathcal{U}^{(k)}(i, i) = A(ks+i), \quad -s \leq i \leq -1, \qquad \mathcal{U}^{(k)}(-s, -1) = I.$$

With the help of (9) these yield that $\mathcal{C}^{(k)} = \mathcal{C}_1^{(k)} + \mathcal{C}_2^{(k)}$ $(k \geq 1)$, where for all $-s \leq i, j \leq -1$

$$\mathcal{C}_1^{(k)}(i,j) = \begin{cases} O, & \text{if } i < j \\ A(ks+i), & \text{if } i = j \\ A(ks+j), & \text{if } i > j \end{cases}$$

and

$$\mathcal{C}_2^{(k)}(i,j) = \begin{cases} O, & \text{if } j \leq -2 \\ I, & \text{if } j = -1. \end{cases}$$

We close this section some basic definitions about periodicity.

Definition 2.2. Assume (C_1). Let $\varphi = (\varphi(-s), \ldots, \varphi(-1))^T \in V_s$ be a given initial vector.

(a_1) The solution $x(\varphi) = (x(\varphi)(n))_{n \geq -s}$ of (1) and (2) is called periodic if there exists a positive integer p such that $x(\varphi)(n+p) = x(\varphi)(n)$ for all $n \geq -s$. In this case we say that $x(\varphi)$ is p-periodic.
(a_2) The solution $(x_{ks}(\varphi))_{k \geq 0}$ of (6) and (7) is called periodic if there exists a positive integer q such that $x_{(k+q)s}(\varphi) = x_{ks}(\varphi)$ for all $k \geq 0$. In this case we say that $(x_{ks}(\varphi))_{k \geq 0}$ is q-periodic.
(b_1) φ is said to be a p-periodic initial vector of (1) if the solution $x(\varphi)$ of (1) and (2) is p-periodic.
(b_2) φ is said to be a q-periodic initial vector of (6) if the solution $(x_{ks}(\varphi))_{k \geq 0}$ of (6) and (7) is q-periodic.
(c_1) We say that p is the prime period of the solution $x(\varphi)$ of (1) and (2) if it is p-periodic and p is the smallest positive integer having this property.
(c_2) We say that q is the prime period of the solution $(x_{ks}(\varphi))_{k \geq 0}$ of (6) and (7) if it is q-periodic and q is the smallest positive integer having this property.
(d) A periodic solution of (1) is called nontrivial if it is different from the zero solution.

3. Periodicity conditions via the companion matrices

The next theorem gives a strong connection between the periodicity of the solutions of the higher order difference equation (1) and the related solutions of the first order companion difference equation (6).

The least common multiple of the positive integers u and v will be denoted by $[u, v]$.

Theorem 3.1. *Assume* (C_1). *Let* $\varphi = (\varphi(-s), \ldots, \varphi(-1))^T \in V_s$ *be a given initial vector.*

(a) *If the solution* $x(\varphi)$ *of* (1) *and* (2) *is* p*-periodic, then the solution* $(x_{ks}(\varphi))_{k \geq 0}$ *of* (6) *and* (7) *is* $\frac{[p,s]}{s}$*-periodic.*
(b) *If the solution* $(x_{ks}(\varphi))_{k \geq 0}$ *of* (6) *and* (7) *is* q*-periodic, then the solution* $x(\varphi)$ *of* (1) *and* (2) *is* qs*-periodic.*
(c) *If the solution* $x(\varphi)$ *of* (1) *and* (2) *is periodic with prime period* p*, then the prime period of the solution* $(x_{ks}(\varphi))_{k \geq 0}$ *of* (6) *and* (7) *is* $q := \frac{[p,s]}{s}$.
(d) *The solution* $(x_{ks}(\varphi))_{k \geq 0}$ *of* (6) *and* (7) *is* q*-periodic if and only if*

$$\left(\mathcal{C}^{(lq+i)} \ldots \mathcal{C}^{((l-1)q+i+1)} - \mathcal{I}\right) \mathcal{C}^{(i)} \ldots \mathcal{C}^{(1)} \mathcal{C}^{(0)} \varphi = 0 \tag{17}$$

for every $l \geq 1$ *and* $0 \leq i \leq q-1$*, where* $\mathcal{C}^{(0)} := \mathcal{I}$.

Remark 3.1. (a) Theorem 3.1(a) and (b) imply that the solution $x(\varphi)$ of (1) and (2) is periodic if and only if the solution $(x_{ks}(\varphi))_{k \geq 0}$ of (6) and (7) is periodic.
(b) It is obvious that the solution $x(\varphi)$ of (1) and (2) is ms-periodic with some positive integer m if and only if the solution $(x_{ks}(\varphi))_{k \geq 0}$ of (6) and (7) is m-periodic.
(c) Suppose the solution $x(\varphi)$ of (1) and (2) is periodic. Theorem 3.1(c) shows that the prime period of $(x_{ks}(\varphi))_{k \geq 0}$ is uniquely determined by the prime period of $x(\varphi)$. Conversely, this is not true in general: for example, if $s = 5$ and $q = 2$ in Theorem 3.1(c), then p is either 2 or 10.
(d) If the solution $(x_{ks}(\varphi))_{k \geq 0}$ of (6) and (7) is periodic with prime period q, then Theorem 3.1(c) implies that the prime period p of the solution $x(\varphi)$ of (1) and (2) is a divisor of qs.

Now we are in the position to state a theorem which gives explicit conditions in term of the companion matrices for a solution of Eq. (1) to be periodic.

Theorem 3.2. *Assume* (C_1). *Let* $\varphi = (\varphi(-s), \ldots, \varphi(-1))^T \in V_s$ *be a given initial vector.*

(a) *Let* $p := ms$ *with some positive integer* m*. The solution* $x(\varphi)$ *of* (1) *and* (2) *is* p*-periodic if and only if*

$$\left(\mathcal{C}^{(lm+i)} \ldots \mathcal{C}^{((l-1)m+i+1)} - \mathcal{I}\right) \mathcal{C}^{(i)} \ldots \mathcal{C}^{(1)} \mathcal{C}^{(0)} \varphi = 0$$

for every $l \geq 1$ *and* $0 \leq i \leq m-1$.

(b) *If the solution $x(\varphi)$ of (1) and (2) is p-periodic and $q := \frac{[p,s]}{s}$, then*

$$\left(\mathcal{C}^{(lq+i)} \dots \mathcal{C}^{((l-1)q+i+1)} - \mathcal{I}\right) \mathcal{C}^{(i)} \dots \mathcal{C}^{(1)} \mathcal{C}^{(0)} \varphi = 0$$

for every $l \geq 1$ and $0 \leq i \leq q-1$.

The following result, which is a corollary of the previous theorem is an important periodicity theorem for higher order difference equations with time independent companion matrices.

Theorem 3.3. *Assume* (C_1). *Let $\varphi = (\varphi(-s), \dots, \varphi(-1))^T \in V_s$ be a given initial vector. If*

$$\mathcal{C}^{(k)} = \mathcal{C}, \quad k \geq 1,$$

then

(a) *Let $p := ms$ with some positive integer m. The solution $x(\varphi)$ of (1) and (2) is p-periodic if and only if*

$$\mathcal{C}^m \varphi = \varphi. \tag{18}$$

(b) *If the solution $x(\varphi)$ of (1) and (2) is p-periodic and $q := \frac{[p,s]}{s}$, then*

$$\mathcal{C}^q \varphi = \varphi.$$

4. Main results: simple necessary and sufficient algebraic conditions

Theorem 3.2(a) and (9) give a surprisingly simple way to determine the existence of s-periodic solutions of Eq. (1). The conditions are formulated in terms of the coefficient matrices of (1).

Theorem 4.1. *Assume* (C_1).

(a) *Let $\varphi = (\varphi(-s), \dots, \varphi(-1))^T \in V_s$ be a given initial vector. The solution $x(\varphi)$ of (1) and (2) is s-periodic if and only if*

$$\left(\mathcal{U}^{(l)} + \mathcal{L}^{(l)}\right) \varphi = \varphi \tag{19}$$

for every $l \geq 1$, where

$$\mathcal{U}^{(l)} + \mathcal{L}^{(l)} = \begin{pmatrix} A_s(ls-s) & A_{s-1}(ls-s) & \dots & A_2(ls-s) & A_1(ls-s) \\ A_1(ls-s+1) & A_s(ls-s+1) & \dots & A_3(ls-s+1) & A_2(ls-s+1) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ A_{s-1}(ls-1) & A_{s-2}(ls-1) & \dots & A_1(ls-1) & A_s(ls-1) \end{pmatrix}$$

for all $l \geq 1$.

(b) *The Eq. (1) has a nontrivial s-periodic solution if and only if 1 is an eigenvalue of all the matrices $\mathcal{U}^{(l)} + \mathcal{L}^{(l)} (l \geq 1)$ and they have a common eigenvector corresponding to 1.*

The following results show that if $p \neq s$, we can consider a new homogeneous linear difference equation

$$z(n) = \sum_{i=1}^{p} B_i(n) z(n-i), \quad n \geq 0, \tag{20}$$

where $B_i(n) \in \mathbb{R}^{d \times d}$ for every $1 \leq i \leq p$ and $n \geq 0$, which allows the problem of the existence of p-periodic solutions of Eq. (1) to translate into an equivalent problem about the Eq. (20). The definition of the coefficients $B_i(n)$ in (20) depends on the order between s and p.

Definition 4.1. Assume (C_1), and let p be a positive integer.

(a) Suppose $p < s$, and let $u := \left[\frac{s}{p}\right]$ and $v := s - up (0 \leq v \leq p-1)$.

If $v = 0$, define

$$B_i(n) := \sum_{j=0}^{u-1} A_{jp+i}(n), \quad 1 \leq i \leq p, \tag{21}$$

while if $v \neq 0$, define

$$B_i(n) := \begin{cases} \displaystyle\sum_{j=0}^{u} A_{jp+i}(n), & 1 \leq i \leq v \\ \displaystyle\sum_{j=0}^{u-1} A_{jp+i}(n), & v+1 \leq i \leq p. \end{cases} \tag{22}$$

(b) If $p > s$, define

$$B_i(n) := \begin{cases} A_i(n), & 1 \le i \le s \\ O, & s+1 \le i \le p. \end{cases} \tag{23}$$

Our approaches will use that Theorem 4.1 gives a simple necessary and sufficient condition for the existence of p-periodic solutions of the Eq. (20) by using the $\mathcal{U}^{(k)}$ and $\mathcal{L}^{(k)} (k \ge 1)$ matrices belonging to (20). The analogue of the matrices $\mathcal{U}^{(k)}$ and $\mathcal{L}^{(k)} (k \ge 1)$ for Eq. (20) will be denoted by $\mathcal{U}^{(k)}_{p,s}$ and $\mathcal{L}^{(k)}_{p,s} (k \ge 1)$. Then by (10) and (11),

$$\mathcal{U}^{(l)}_{p,s} + \mathcal{L}^{(l)}_{p,s} = \begin{pmatrix} B_p(lp-p) & B_{p-1}(lp-p) & \dots & B_2(lp-p) & B_1(lp-p) \\ B_1(lp-p+1) & B_p(lp-p+1) & \dots & B_3(lp-p+1) & B_2(lp-p+1) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ B_{p-1}(lp-1) & B_{p-2}(lp-1) & \dots & B_1(lp-1) & B_p(lp-1) \end{pmatrix}$$

for all $l \ge 1$.

We begin with the case $1 \le p < s$.

Theorem 4.2. *Assume* (C_1). *Suppose $1 \le p < s$ is an integer. We consider Eq. (20) with the coefficients either (21) or (22).*

(a) *If $\varphi = (\varphi(-s), \dots, \varphi(-1))^T \in V_s$ is a p-periodic initial vector of (1), and $\psi := (\varphi(-p), \dots, \varphi(-1))^T \in V_p$, then*

$$\left(\mathcal{U}^{(l)}_{p,s} + \mathcal{L}^{(l)}_{p,s}\right) \psi = \psi \tag{24}$$

for every $l \ge 1$.

Conversely, if $\psi = (\psi(-p), \dots, \psi(-1))^T \in V_p$ such that (24) is satisfied for every $l \ge 1$, then $\varphi = (\varphi(-s), \dots, \varphi(-1))^T \in V_s$, where $\varphi(i) := \psi(j)$ if $i \equiv j (\mathrm{mod}\ p) (i = -s, \dots, -1)$, is a p-periodic initial vector of (1).

(b) *The Eq. (1) has a nontrivial p-periodic solution if and only if there exists a $\psi \in V_p \setminus \{0\}$ such that (24) is satisfied for every $l \ge 1$.*

(c) *The Eq. (1) has a nontrivial p-periodic solution if and only if 1 is an eigenvalue of all the matrices $\mathcal{U}^{(l)}_{p,s} + \mathcal{L}^{(l)}_{p,s} (l \ge 1)$ and they have a common eigenvector corresponding to 1.*

We turn now to the case $p > s$.

Theorem 4.3. *Assume* (C_1). *Suppose $p > s$ is an integer. We consider Eq. (20) with the coefficients (23).*

(a) *If $\varphi = (\varphi(-s), \dots, \varphi(-1))^T \in V_s$ is a p-periodic initial vector of (1), and*

$$\psi := (x(\varphi)(0), \dots, x(\varphi)(p-s-1), \varphi(-s), \dots, \varphi(-1))^T \in V_p,$$

then (24) holds for every $l \ge 1$.

Conversely, if $\psi := (\psi(-p), \dots, \psi(-1))^T \in V_p$ such that (24) is satisfied for every $l \ge 1$, then $\varphi = (\psi(-s), \dots, \psi(-1))^T \in V_s$ is a p-periodic initial vector of (1).

(b) *The Eq. (1) has a nontrivial p-periodic solution if and only if there exists a $\psi \in V_p \setminus \{0\}$ such that (24) is satisfied for every $l \ge 1$.*

(c) *The Eq. (1) has a nontrivial p-periodic solution if and only if 1 is an eigenvalue of all the matrices $\mathcal{U}^{(l)}_{p,s} + \mathcal{L}^{(l)}_{p,s} (l \ge 1)$ and they have a common eigenvector corresponding to 1.*

Remark 4.1. (a) If we confine our attention to higher order difference equations with time independent coefficients, then the equation can be handled easily, since $\mathcal{U}^{(k)}_{p,s}$ and $\mathcal{L}^{(k)}_{p,s} (k \ge 1)$ do not depend on k. In this case we have to study only the equation

$$\left(\mathcal{U}_{p,s} + \mathcal{L}_{p,s}\right) \psi = \psi,$$

where

$$\mathcal{U}_{p,s} + \mathcal{L}_{p,s} = \begin{pmatrix} B_p & B_{p-1} & \dots & B_2 & B_1 \\ B_1 & B_p & \dots & B_3 & B_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ B_{p-1} & B_{p-2} & \dots & B_1 & B_p \end{pmatrix}.$$

(b) It should be stressed that the matrices $\mathcal{U}^{(k)}_{p,s}$ and $\mathcal{L}^{(k)}_{p,s} (k \ge 1)$ do not depend on k not only in the case when the coefficients in Eq. (1) are time independent, but also in the case when the coefficients are p-periodic.

5. Discussion and applications

In order to give examples as transparent as possible, we suppose in the further part of this chapter that the coefficients in the studied equations are real numbers that is $d = 1$.

Now we apply the results in the introduction for some equations.

Example 5.1. (a) Consider the equation

$$x(n) = \sum_{i=1}^{5} A_i x(n-i), \quad n \geq 0, \tag{25}$$

and suppose that $p = 3$. Then, by Theorem 1.1, for every $\varphi \in V_5^3$, that is

$$\varphi(-5) = \varphi(-2), \qquad \varphi(-4) = \varphi(-1),$$

the solution of (25) corresponding to φ is 3-periodic if and only if

$$A_3 = 1, \qquad A_4 = -A_1, \qquad A_5 = -A_2.$$

(b) Consider the equation

$$x(n) = A_1 x(n-1) + A_2 x(n-2), \quad n \geq 0, \tag{26}$$

and suppose that $p = 5$. Some easy calculation shows that the rank of the matrix (5) is 3 if and only if

$$\left.\begin{array}{l} A_1^3 A_2 + 2A_1 A_2^2 = 1 \\ A_1^4 + 3A_1^2 A_2 + A_2^2 = 0 \end{array}\right\}, \tag{27}$$

and therefore Theorem 1.2 implies that every solution of Eq. (26) is 5-periodic if and only if (27) is satisfied.

(27) has two real solutions

$$A_1 = \frac{\sqrt{5}-1}{2}, \qquad A_2 = -1 \quad \text{and} \quad A_1 = \frac{1+\sqrt{5}}{2}, \qquad A_2 = -1.$$

In the next example we apply Theorem 3.3.

Example 5.2. Consider the equation

$$x(n) = A_1(n)x(n-1) + A_2(n)x(n-2) + A_3(n)x(n-3), \quad n \geq 0, \tag{28}$$

where

$$\left.\begin{array}{ll} A_1(3l-3) = A_2(3l-3) = A_3(3l-3) = 0 & \\ A_3(3l-2) = a, \qquad A_2(3l-2) = b & \\ A_1(3l-1) = c, \qquad A_3(3l-1) = d & \end{array}\right., \quad l \geq 1.$$

Here a, b, c and d are real numbers. In this case

$$\mathcal{C}^{(l)} = \mathcal{C} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & a & b \\ 0 & ac & bc+d \end{pmatrix}, \quad l \geq 1.$$

We stress that $\mathcal{C}^{(l)}$ does not depend on l in spite of $A_1(3l-2)$ and $A_2(3l-1)$ being permitted to depend on $l (l \geq 1)$.

By Theorem 3.3, Eq. (28) has a 6-periodic solution different from zero if and only if

$$\det(\mathcal{C}^2 - \mathcal{I}) = 0,$$

which is equivalent with

$$a^2d^2 - a^2 - 2abc - b^2c^2 - 2bcd - d^2 + 1 = 0. \tag{29}$$

If $b \neq 0$, $a, b \in \mathbb{R}$, then (29) gives that

$$c = -\frac{1}{b}(a + d + ad + 1) \quad \text{or} \quad c = -\frac{1}{b}(a + d - ad - 1). \tag{30}$$

Choose $a = 2$, $b = 1$ and $d = 1$. Then (30) insures that $c = -6$. Now some easy computation shows that there are 6-periodic solutions with prime period 6, and the 6-periodic initial vectors belonging to these solutions are

$$(0, \alpha, -3\alpha), \quad \alpha \neq 0.$$

Example 5.3. Consider the equation

$$x(n) = A_1(n)x(n-1) + A_2(n)x(n-2), \quad n \geq 0, \tag{31}$$

where $A_1(n), A_2(n) \in \mathbb{R}$ for every $n \geq 0$.

Then by Example 2.1 and by Theorem 4.1, Eq. (31) has a nontrivial 2-periodic solution if and only if there exists $\varphi = (\varphi(-2), \varphi(-1))^T \in \mathbb{R}^2 \setminus \{(0,0)\}$ such that

$$\begin{aligned} &(A_2(2l-2)-1)\,\varphi(-2) + A_1(2l-2)\varphi(-1) = 0 \\ &A_1(2l-1)\varphi(-2) + (A_2(2l-1)-1)\,\varphi(-1) = 0 \end{aligned}$$

for all $l \geq 1$.

For example, if

$$\frac{A_1(2l-2)}{A_2(2l-2)-1} = \frac{A_2(2l-1)-1}{A_1(2l-1)} = \alpha \in \mathbb{R}, \quad l \geq 1,$$

then $\varphi = (-\alpha t, t)^T \in \mathbb{R}^2$ is a 2-periodic initial vector of (31) for every $t \in \mathbb{R}$.

We give a concrete equation: Let

$$\begin{matrix} A_1(2l-2) = 2l-2, & A_2(2l-2) = l \\ A_1(2l-1) = 1, & A_2(2l-1) = 3 \end{matrix}, \quad l \geq 2,$$

and assume

$$A_1(0) := 0, \qquad A_1(1) := 1 \quad \text{and} \quad A_2(1) := 3.$$

In this case, if $A_2(0) := 1$, then $x(\varphi)$ is a 2-periodic solution of (31) with initial vector $\varphi = (2t, -t)^T \in \mathbb{R}^2$ for every $t \in \mathbb{R}$, but if $A_2(0) \neq 1$, then Eq. (31) does not have any nontrivial 2-periodic solution.

Example 5.4. We consider the Eq. (31). Suppose $p > 2$ is an integer. By Theorem 4.3(a), Eq. (31) has a nontrivial p-periodic solution if and only if there exists a $\psi \in \mathbb{R}^p \setminus \{0\}$ such that

$$\left(\mathcal{U}_{p,2}^{(l)} + \mathcal{L}_{p,2}^{(l)}\right)\psi = \psi,$$

where

$$\mathcal{U}_{p,2}^{(l)} = \begin{pmatrix} 0 & \dots & 0 & A_2(lp-p) & A_1(lp-p) \\ 0 & \dots & 0 & 0 & A_2(lp-p+1) \\ 0 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \dots & 0 & 0 & 0 \end{pmatrix}$$

and

$$\mathcal{L}_{p,2}^{(l)} = \begin{pmatrix} 0 & 0 & \dots & 0 & 0 & 0 \\ A_1(lp-p+1) & 0 & \dots & 0 & 0 & 0 \\ A_2(lp-p+2) & A_1(lp-p+2) & \dots & 0 & 0 & 0 \\ 0 & A_2(lp-p+3) & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & A_2(lp-1) & A_1(lp-1) & 0 \end{pmatrix}$$

for all $l \geq 1$.

Suppose $p = 3$. Then Eq. (31) has a 3-periodic solution if and only if there exists a $(\psi(-3), \psi(-2), \psi(-1))^T \in \mathbb{R}^3$ such that

$$\begin{aligned} &-\psi(-3) + A_2(3l-3)\psi(-2) + A_1(3l-3)\psi(-1) = 0 \\ &A_1(3l-2)\psi(-3) - \psi(-2) + A_2(3l-2)\psi(-1) = 0 \\ &A_2(3l-1)\psi(-3) + A_1(3l-1)\psi(-2) - \psi(-1) = 0 \end{aligned}$$

for all $l \geq 1$.

For example, $(2, 0) \in \mathbb{R}^2$ is a 3-periodic initial vector of Eq. (31), if

$$\begin{matrix} A_1(3l-3) = l, & A_2(3l-3) = -\dfrac{1}{2} \\ A_1(3l-2) = -2, & A_2(3l-2) = l^2 \\ A_1(3l-1) = l, & A_2(3l-1) = 2l \end{matrix}, \quad l \geq 1.$$

Example 5.5. Consider the equation

$$x(n) = A_1(n)x(n-1) + A_2(n)x(n-2) + A_3(n)x(n-3), \quad n \geq 0, \tag{32}$$

where $A_1(n), A_2(n), A_3(n) \in \mathbb{R}$ for every $n \geq 0$. By Theorem 4.2(a), Eq. (32) has a nontrivial 2-periodic solution if and only if there exists a $\psi = (\psi(-2), \psi(-1))^T \in \mathbb{R}^2 \setminus \{(0,0)\}$ such that

$$\begin{aligned}
&(A_2(2l-2) - 1)\,\psi(-2) + (A_1(2l-2) + A_3(2l-2))\,\psi(-1) = 0\\
&(A_1(2l-1) + A_3(2l-1))\,\psi(-2) + (A_2(2l-1) - 1)\,\psi(-1) = 0
\end{aligned}$$

for all $l \geq 1$.

For example, $(1, -1, 1) \in \mathbb{R}^3$ is a 2-periodic initial vector of Eq. (32), if

$$\begin{matrix} A_1(2l-2) = -l, & A_2(2l-2) = 1-2l, & A_3(2l-2) = -l \\ A_1(2l-1) = l^2 - 1, & A_2(2l-1) = 2l^2, & A_3(2l-1) = l^2 \end{matrix}, \quad l \geq 1.$$

Example 5.6. We consider the equation

$$x(n) = x(n-1) + Ax(n-5), \quad n \geq 0, \tag{33}$$

where $A \in \mathbb{R}$. (33) is a special case of (16).

If $1 \leq p \leq 4$, then the existence of p-periodic solutions can be examined by Theorem 4.2(a):

(a) If $A = 0$, then $\varphi = (\alpha, \alpha, \alpha, \alpha, \alpha)$ is a 1-periodic initial vector for every $\alpha \in \mathbb{R}$. If $A \neq 0$, then $\varphi = (0,0,0,0,0)$ the only 1-periodic initial vector.
(b) There exists a 2-periodic solution with prime period 2 if and only if $A = -2$. The 2-periodic initial vectors belonging to these solutions are $\varphi = (\alpha, -\alpha, \alpha, -\alpha, \alpha)$, where $\alpha \neq 0$.
(c) There are neither 3-periodic nor 4-periodic solutions with prime period 3 or 4.

By Theorem 4.1, there are no 5-periodic solutions with prime period 5.

By Theorem 4.3(a), there are 6-periodic solutions with prime period 6 if and only if $A = 1$. The 6-periodic initial vectors belonging to these solutions are $\varphi = (\alpha, \beta, \beta - \alpha, -\alpha, -\beta)$, where $\alpha\beta \neq 0$.

Finally, by using Theorem 4.2, we construct an equation which has solutions with two different prime periods.

Example 5.7. Consider the equation

$$x(n) = -\frac{2}{3}x(n-1) + \frac{2}{3}x(n-2) + \frac{5}{3}x(n-3) + \frac{4}{3}x(n-4), \quad n \geq 0.$$

It is easy to check that the initial vectors

$$(\alpha, -\alpha, \alpha, -\alpha), \quad \alpha \neq 0$$

generates solutions with prime period 2, while the initial vectors

$$(-\alpha - \beta, \alpha, \beta, -\alpha - \beta), \quad \alpha\beta \neq 0$$

generates solutions with prime period 3.

6. Proofs of the main results

Proof of Theorem 3.1. (a) Let $q := \frac{[p,s]}{s}$. It follows that $qs = pl$ with some positive integer l. Then

$$\begin{aligned}
x_{(k+q)s}(\varphi)(i) &= x(\varphi)(ks + qs + i) = x(\varphi)(ks + pl + i)\\
&= x(\varphi)(ks + i) = x_{ks}(\varphi)(i), \quad -s \leq i \leq -1,
\end{aligned}$$

and therefore $(x_{ks}(\varphi))_{k\geq 0}$ is a q-periodic solution of (6) and (7).

(b) By using Theorem 2.1 (c), we have

$$\begin{aligned}
x(\varphi)(n+qs) &= x_{\left(\left[\frac{n+qs}{s}\right]+1\right)s}(\varphi)\left(n + qs - \left(\left[\frac{n+qs}{s}\right] + 1\right)s\right)\\
&= x_{\left(\left[\frac{n}{s}\right]+q+1\right)s}(\varphi)\left(n + qs - \left(\left[\frac{n}{s}\right] + q + 1\right)s\right)\\
&= x_{\left(\left[\frac{n}{s}\right]+1\right)s}(\varphi)\left(n - \left(\left[\frac{n}{s}\right] + 1\right)s\right) = x(\varphi)(n), \quad n \geq -s,
\end{aligned}$$

which shows that $x(\varphi)$ is a qs-periodic solution of (1) and (2).

(c) It follows from (a) that $(x_{ks}(\varphi))_{k\geq 0}$ is a q-periodic solution of (6) and (7).

It remains to show that q is the prime period of $(x_{ks}(\varphi))_{k\geq 0}$. In case $q=1$, it is obvious. Suppose $q>1$, and suppose that $(x_{ks}(\varphi))_{k\geq 0}$ is k-periodic with some $1\leq k<q$. By the definition of the integer q, there is a nonnegative integer l and an integer $1\leq r<p$ such that $ks=pl+r$. Then Theorem 2.1 (c) implies as in (b) that

$$x(\varphi)(n+r)=x(\varphi)(n),\quad n\geq -s,$$

thus $x(\varphi)$ is an r-periodic solution which is a contradiction.

(d) Suppose first, that the solution $(x_{ks}(\varphi))_{k\geq 0}$ of (6) and (7) is q-periodic. Then

$$x_{(k+q)s}(\varphi)=x_{ks}(\varphi),\quad k\geq 0,$$

and since every nonnegative integer k can be written in the form

$$k=(l-1)q+i\quad \text{with } l\geq 1 \text{ and } 0\leq i\leq q-1, \tag{34}$$

$$x_{ks}(\varphi)=x_{((l-1)q+i)s}(\varphi)=x_{is}(\varphi).$$

With the help of Theorem 2.1(b) these yield that

$$\mathcal{C}^{(k+q)}\ldots\mathcal{C}^{(k+1)}x_{ks}(\varphi)=\mathcal{C}^{(lq+i)}\ldots\mathcal{C}^{((l-1)q+i+1)}x_{is}(\varphi)=x_{is}(\varphi),\quad k\geq 0,$$

and therefore

$$\left(\mathcal{C}^{(lq+i)}\ldots\mathcal{C}^{((l-1)q+i+1)}-\mathcal{I}\right)\mathcal{C}^{(i)}\ldots\mathcal{C}^{(1)}\mathcal{C}^{(0)}\varphi=0$$

for every $l\geq 1$ and $0\leq i\leq q-1$.

Conversely, suppose that (17) holds for every $l\geq 1$ and $0\leq i\leq q-1$. By using the representation (34) of the nonnegative integer k, it is enough to prove that

$$x_{((l-1)q+i)s}(\varphi)=x_{is}(\varphi) \tag{35}$$

for every $l\geq 1$ and $0\leq i\leq q-1$.

The cases $l=1$ and $0\leq i\leq q-1$ are trivial, and we complete the proof by induction on l for each fixed $0\leq i\leq q-1$. Let l be a positive integer such that (35) holds for a fixed $0\leq i\leq q-1$. By (17)

$$\mathcal{C}^{(lq+i)}\ldots\mathcal{C}^{((l-1)q+i+1)}\mathcal{C}^{(i)}\ldots\mathcal{C}^{(1)}\mathcal{C}^{(0)}\varphi=\mathcal{C}^{(i)}\ldots\mathcal{C}^{(1)}\mathcal{C}^{(0)}\varphi,$$

and hence, by the induction hypothesis and by Theorem 2.1(b)

$$\begin{aligned}x_{(lq+i)s}(\varphi)&=\mathcal{C}^{(lq+i)}\ldots\mathcal{C}^{((l-1)q+i+1)}x_{((l-1)q+i)s}(\varphi)\\&=\mathcal{C}^{(lq+i)}\ldots\mathcal{C}^{((l-1)q+i+1)}\mathcal{C}^{(i)}\ldots\mathcal{C}^{(1)}\mathcal{C}^{(0)}\varphi=x_{is}(\varphi).\end{aligned}$$

The proof of the theorem is now complete. □

Proof of Theorem 3.2. (a) It follows from Remark 3.1(b) and Theorem 3.1(d).
(b) It follows from Theorem 3.1(a) and Theorem 3.1(d). □

Proof of Theorem 3.3. (a) By Theorem 3.2(a), the solution $x(\varphi)$ of (1) and (2) is p-periodic if and only if

$$\left(\mathcal{C}^{m}-\mathcal{I}\right)\mathcal{C}^{i}\varphi=0,\quad 0\leq i\leq m-1,$$

which is equivalent with (18).
(b) We can prove as before by applying Theorem 3.2(b). □

Proof of Theorem 4.1. (a) By Theorem 3.2(a), the solution $x(\varphi)$ of (1) and (2) is s-periodic if and only if

$$\left(\mathcal{C}^{(l)}-\mathcal{I}\right)\varphi=0$$

for every $l\geq 1$, which is equivalent with (19) because of (9).
(b) It comes from (a). □

To prove Theorems 4.2 and 4.3 we prepare two lemmas.

Lemma 6.1. *Assume* (C_1). *Suppose* $1\leq p<s$ *is an integer, let* $u:=\left[\frac{s}{p}\right]$, *and let* $v:=s-up\,(0\leq v\leq p-1)$.

(a_1) *If* $\varphi=(\varphi(-s),\ldots,\varphi(-1))^T\in V_s$ *is a p-periodic initial vector of* (1), *then* $\psi:=(\varphi(-p),\ldots,\varphi(-1))^T\in V_p$ *is a p-periodic initial vector of Eq.* (20) *with coefficients either* (21) *or* (22).
(a_2) *If* $\psi=(\psi(-p),\ldots,\psi(-1))^T\in V_p$ *is a p-periodic initial vector of Eq.* (20) *with coefficients either* (21) *or* (22), *then* $\varphi=(\varphi(-s),\ldots,\varphi(-1))^T\in V_s$, *where* $\varphi(i):=\psi(j)$ *if* $i\equiv j(\mathrm{mod}\ p)\,(i=-s,\ldots,-1)$, *is a p-periodic initial vector of* (1).

Proof. (a_1) Suppose φ is a p-periodic initial vector of (1), and let $\psi := (\varphi(-p), \dots, \varphi(-1))^T \in V_p$. Obviously, it is enough to check that

$$z(\psi)(n) = x(\varphi)(n), \quad n \geq -p. \tag{36}$$

Since φ is a p-periodic initial vector of (1),

$$\begin{aligned} x(\varphi)(n) &= \sum_{i=1}^{v}\left(\sum_{j=0}^{u} A_{jp+i}(n)x(\varphi)(n-(jp+i))\right) + \sum_{i=v+1}^{p}\left(\sum_{j=0}^{u-1} A_{jp+i}(n)x(\varphi)(n-(jp+i))\right) \\ &= \sum_{i=1}^{v}\left(\sum_{j=0}^{u} A_{jp+i}(n)x(\varphi)(n-i)\right) + \sum_{i=v+1}^{p}\left(\sum_{j=0}^{u-1} A_{jp+i}(n)x(\varphi)(n-i)\right) \\ &= \sum_{i=1}^{p} B_i(n)x(\varphi)(n-i), \quad n \geq 0, \end{aligned}$$

and hence by the definition of ψ, (36) can be proved by induction on n for $n \geq 0$.

(a_2) Now suppose $\psi = (\psi(-p), \dots, \psi(-1))^T \in V_p$ is a p-periodic initial vector of Eq. (20), and let $\varphi = (\varphi(-s), \dots, \varphi(-1))^T \in V_s$, where $\varphi(i) := \psi(j)$ if $i \equiv j (\mathrm{mod}\ p) (i = -s, \dots, -1)$. Set

$$z(\psi)(i) := \psi(j) \quad \text{if } i \equiv j (\mathrm{mod}\ p),\ i = -s, \dots, -1.$$

By the definition of φ, it is enough to verify that

$$x(\varphi)(n) = z(\psi)(n), \quad n \geq -s.$$

Because ψ is a p-periodic initial vector of Eq. (20),

$$\begin{aligned} z(\psi)(n) &= \sum_{i=1}^{v}\left(\sum_{j=0}^{u} A_{jp+i}(n)\right) z(\psi)(n-i) + \sum_{i=v+1}^{p}\left(\sum_{j=0}^{u-1} A_{jp+i}(n)\right) z(\psi)(n-i) \\ &= \sum_{i=1}^{v}\left(\sum_{j=0}^{u} A_{jp+i}(n)z(\psi)(n-(jp+i))\right) + \sum_{i=v+1}^{p}\left(\sum_{j=0}^{u-1} A_{jp+i}(n)z(\psi)(n-(jp+i))\right) \\ &= \sum_{k=1}^{s} A_k(n)z(\psi)(n-k), \quad n \geq 0, \end{aligned}$$

and therefore we again complete the proof by induction on n for $n \geq 0$.

The proof is completed. □

Lemma 6.2. *Assume* (C_1). *Suppose $p > s$ is an integer.*

(a_1) *If $\varphi = (\varphi(-s), \dots, \varphi(-1))^T \in V_s$ is a p-periodic initial vector of* (1), *then*

$$\psi := (x(\varphi)(0), \dots, x(\varphi)(p-s-1), \varphi(-s), \dots, \varphi(-1))^T \in V_p$$

is a p-periodic initial vector of Eq. (20) *with coefficients* (23).

(a_2) *If $\psi := (\psi(-p), \dots, \psi(-1))^T \in V_p$ is a p-periodic initial vector of Eq.* (20) *with coefficients* (23), *then $\varphi = (\psi(-s), \dots, \psi(-1))^T \in V_s$ is a p-periodic initial vector of* (1).

Proof. This follows by an argument similar to that for the previous lemma, and we omit the details. □

Proof of Theorem 4.2. (a) Theorem 4.1 and Lemma 6.1 can be applied.

(b) and (c) follow from (a). □

Proof of Theorem 4.3. (a) Theorem 4.1 and Lemma 6.2 can be applied.

(b) and (c) follow from (a). □

Proof of Theorem 1.1. It follows from Theorems 4.1 and 4.2. □

Proof of Theorem 1.2. It is easy to check that the first $d(p-s)$ column vectors of the matrix (5) are linearly independent, therefore the result comes from Theorem 4.3. □

Acknowledgments

This work is supported by the Hungarian National Foundations for Scientific Research Grant No. K73274 and TÁMOP-4.2.2/B-10/1-2010-0025 project.

References

[1] R. Agarwal, Difference Equations and Inequalities. Theory, Methods and Applications, Marcel Dekker Inc, New York, 1992.
[2] S. Elaydi, An Introduction to Difference Equations, Springer-Verlag, New York, 1996.
[3] E.A. Grove, G. Ladas, Periodicity in Nonlinear Difference Equations, Chapman and Hall, 2005.
[4] W.G. Kelly, A.C. Peterson, Difference Equations, Academic Press, New York, 1991.
[5] V.L. Kocic, G. Ladas, Global Behaviour of Nonlinear Difference Equations of Higher Order with Applications, Kluwer Academic Publisher, Dordrecht, Holland, 1993.
[6] M.R.S. Kulenovic, G. Ladas, Dynamics of Second Order Rational Difference Equations with Open Problems and Conjectures, Chapman and Hall/CRC, London, 2002.
[7] L. Berg, Nonlinear difference equations with periodic solutions, Rostock. Math. Kolloq. 61 (2006) 13–20.
[8] S. Stević, A note on periodic character of a higher order difference equation, Rostock. Math. Kolloq. 61 (2006) 21–30.
[9] I. Győri, L. Horváth, A new view of the l^p-theory for system of higher order difference equations, Comput. Math. Appl. 59 (2010) 4205–4216.
[10] I. Győri, L. Horváth, l^p-solutions and stability analysis of difference equations using the Kummer's test, Appl. Math. Comput. 217 (2011) 10129–10145.