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Boundedness of solutions of difference systems with delays

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ABSTRACT

We give sufficient conditions for the boundedness of all solutions of some classes of systems of difference equations with delays, by comparison of their norms with the solution of certain auxiliary scalar difference equations.

1. Introduction

Let $\mathbb{R}$ denote the set of real numbers, $\mathbb{Z}$ and $\mathbb{Z}^+$ the set of integers and nonnegative integers, respectively, $\mathbb{N}(n_0) = \{n_0, n_0+1, \ldots\}$, $n_0 \in \mathbb{Z}^+$.

Let k be a positive integer. The set of all k-dimensional column vectors with real components is denoted by $\mathbb{R}^k$ and the set of $k \times k$ matrices with real entries by $\mathbb{R}^{k\times k}$. Let $\|.\|$ denote any norm of a vector or the associated induced norm of a square matrix. The set $\mathbb{R}^{k\times k}$ can be endowed with many norms, but they are all equivalent. The zero matrix in $\mathbb{R}^{k\times k}$ is denoted by 0 and the identity matrix by I.

In this paper, we consider the k-dimensional systems of difference equations

$$\mathbf{x}(n+1) = A(n)\mathbf{x}(n) + \sum_{s=1}^{m} B_s(n)\mathbf{x}(n-h_s(n)), \quad n \ge n_0 \tag{1.1}$$

and

$$\mathbf{x}(n+1) = A(n)\mathbf{x}(n) + f(n, \mathbf{x}(n-h(n))), \quad n \ge n_0 \tag{1.2}$$

$\lim_{n\longrightarrow\infty}(n-h_s(n)) = \infty$, $s = 1, 2, \ldots, m$; $\lim_{n\longrightarrow\infty}(n-h(n)) = \infty$, and $f : \mathbb{Z}^+ \times \mathbb{R}^k \to \mathbb{R}^k$. The coefficients $A(n)$, $B_s(n)$ are $k \times k$ real matrices. It is always assumed that $A(n)$ is a nonsingular matrix for all $n \ge n_0$.

By a solution of Eq. (1.1) (Eq. (1.2)) we mean a sequence $\mathbf{x} := (\mathbf{x}(n))$, $\mathbf{x}(n) \in \mathbb{R}^k$ satisfying Eq. (1.1) (Eq. (1.2)) for any $n \in \mathbb{N}(n_0)$.

Further, we will extensively apply the solution representation formula and properties of the fundamental matrix of the unperturbed system

$$\mathbf{x}(n+1) = A(n)\mathbf{x}(n), \quad n \ge n_0. \tag{1.3}$$

Let X denote the solution of the matrix equation

$$X(n+1) = A(n)X(n), \quad X(n_0) = I. \tag{1.4}$$

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Throughout the paper, we shall assume the following hypothesis.

(H_0) There exist real constants $\alpha \geq 1$ and $0 < \eta < 1$ such that

$$\left\| X(n)X^{-1}(s) \right\| \leq \alpha \eta^{n-s}, \quad \text{for all } n \geq s \geq n_0,$$

i.e. that system (1.3) is uniformly asymptotically stable (for the definition, we refer to [1]).

It is well known that the difference equations appear in the discussion of many problems in numerical analysis, discrete dynamic systems, mathematical biology and economy, as well as many other branches of science. For example, many discrete models in biology can be described by Clark's difference equation

$$x(n+1) = \alpha_1 x(n) + g(x(n-k)), \quad n \geq 0, \tag{1.5}$$

where $\alpha_1 \in [0, 1)$, $k \in 1, 2, \ldots$, and $g : \mathbb{R}_+ \to \mathbb{R}_+$, which is a particular case of Eq. (1.2) (see, e.g., [2–5], Sections 2.5, 4.5–4.7 in monograph [6]). In this model, $x(n)$ represents the number of adult members of the population in the year n, $\alpha \in [0, 1)$ is the survival rate, and g is the recruitment function, which in general is a nonlinear function of the size of the population of adults k years before. Clearly, for this model (H_0) holds.

Equations of type (1.1) and their special cases (especially if $B_s(n)$, $s = 1, \ldots, m$ are identically zero matrices) were considered by many authors. For example, Abu-Saris et al. in [7] gave extension of Poincare theorem to the system of difference equations

$$\mathbf{x}(n+1) = (A + B(n))\,\mathbf{x}(n).$$

In [8], Pituk described, in terms of the initial condition, the asymptotic behavior of the solutions of equation

$$\mathbf{x}(n+1) = (A + B(n))\,\mathbf{x}(n) + g(n),$$

in the case when A has a simple dominant eigenvalue λ_0, $\sum_{n=0}^{\infty} \|B(n)\| < \infty$ and $\sum_{i=0}^{\infty} |\lambda_0|^{-n} \|g(n)\| < \infty$. Several asymptotic results for solutions of the perturbed linear diagonal system

$$\mathbf{x}(n+1) = (\Lambda(n) + R(n))\,\mathbf{x}(n),$$

where $\Lambda(n) = (\lambda_1(n), \ldots, \lambda_k(n))$ are obtained by Ren et al. in [9]. Čermák [10] established an asymptotic bound of solutions of system

$$\mathbf{x}(n+1) = A(n)\mathbf{x}(n) + \sum_{k=0}^{p} B_k(n)\mathbf{x}(\alpha(n) + k), \quad n \geq n_0,$$

where $\alpha(n) + k < n$, $n \in \mathbb{N}(n_0)$. This bound is expressed via a solution of some scalar difference inequality assuming that this solution admits certain properties. Stability of system (1.1) has been investigated, for example in [11–13].

For nonlinear systems of type (1.2) there are, to the best of our knowledge, relatively few papers (e.g. [14,15]).

The results obtained in this paper are motivated by the results of two papers by Čermák [10], and Crisci et al. [16]. The main goal of this paper is to present sufficient conditions for the boundedness of all solutions of the system of linear equations with several delays (1.1) and for the system of nonlinear equation (1.2). The results are obtained via a solution of certain auxiliary scalar difference equations by a suitable choice of coefficients appearing in them, which will allow us, in a simple manner, to approximate the solutions of the considered systems in an explicit form.

2. System of linear equations with several delays

In this section, we consider the linear system of difference equations

$$\mathbf{x}(n+1) = A(n)\mathbf{x}(n) + \sum_{s=1}^{m} B_s(n)\mathbf{x}(n - h_s(n)), \quad n \geq n_0 \tag{2.1}$$

with the initial conditions

$$\mathbf{x}(n) = \varphi(n) \quad \text{for } n \in E_0,$$

where $E_0 = \bigcup_{i=1}^{m} E_i \cup \{n_0\}$, $E_i = \{n \in \mathbb{Z} : \; n = l - h_i(l) \leq n_0, \; l \geq n_0\}$, $\varphi : E_0 \to \mathbb{R}^k$. Here $A(n)$, $B_s(n)$, $s = 1, 2, \ldots, m$; $n \in \mathbb{N}(n_0)$, are $k \times k$ real matrices, $h_s : \mathbb{Z}^+ \to \mathbb{Z}^+$ and $\lim_{n \to \infty}(n - h_s(n)) = \infty$ $(s = 1, 2, \ldots, m)$.

Theorem 2.1. *Suppose condition* (H_0) *holds. Assume the following.*

(H_1) $\|B_i(n)\| \leq b_i$, $1 \leq i \leq m$, $n \geq n_0$, $\sum_{i=1}^{m} b_i = b$.

(H_2) *There exists a constant* $q > 0$ *such that* $\eta < \frac{q}{\tau(n)} \leq 1$,

$$q^{\tau(n)} \left(\frac{q}{\tau(n)} - \eta \right) \prod_{s=n-\tau(n)}^{n-1} \frac{1}{\tau(s)} \geq \alpha b \quad \textit{for all } n - \tau(n) \geq n_0,$$

where $\tau(n) = \max\left\{1, \max_{1 \leq i \leq m} h_i(n)\right\}$, *for* $n \geq n_0$.

Then the solution $\mathbf{x}$ *of system* (2.1) *with the initial conditions* $\mathbf{x}(n)=\varphi(n)$ *for* $n\in E_0$, *has the property*

$$\|\mathbf{x}(n)\|\le c_0q^n\prod_{s=n_0}^{n-1}\frac{1}{\tau(s)},\quad n\ge n_0, \tag{2.2}$$

for every $c_0>\max\left\{\alpha\,\|\varphi(n_0)\|\,,\max_{n\in E_0}\frac{\|\varphi(n)\|}{q^n}\right\}$.

Proof. By the variation of constants formula (see [1], Theorem 3.17), the solution of system (2.1) satisfies, for $n\ge n_0$, the following equation

$$\mathbf{x}(n)=X(n)\varphi(n_0)+\sum_{j=n_0}^{n-1}X(n)X^{-1}(j+1)\sum_{s=1}^{m}B_s(j)\mathbf{x}(j-h_s(j)). \tag{2.3}$$

Thus, by virtue of the assumptions (H_0) and (H_1), we have

$$\|\mathbf{x}(n)\|\le\alpha\eta^{n-n_0}\,\|\varphi(n_0)\|+\alpha\sum_{j=n_0}^{n-1}\eta^{n-j-1}\sum_{s=1}^{m}b_s\,\|\mathbf{x}(j-h_s(j))\|\,. \tag{2.4}$$

Let us consider the scalar equation

$$y(n+1)=\eta y(n)+p(n)y(n-\tau(n)), \tag{2.5}$$

where $p(n)=q^{\tau(n)}\left(\frac{q}{\tau(n)}-\eta\right)\prod_{s=n-\tau(n)}^{n-1}\frac{1}{\tau(s)}$, q is a positive constant. Then the solution of Eq. (2.5) is of the form

$$y(n)=c_0q^n\prod_{s=n_0}^{n-1}\frac{1}{\tau(s)},\quad c_0=\text{const.}>0. \tag{2.6}$$

Moreover, by virtue of (H_2), $p(n)>0$ for all $n\in\mathbb{N}(n_0)$ and the sequence y is positive and nonincreasing. We can also write Eq. (2.5) in the summation form

$$y(n)=c_0\eta^{n-n_0}+\sum_{j=n_0}^{n-1}\eta^{n-j-1}p(j)y(j-\tau(j)),\quad c_0=\text{const.}>0. \tag{2.7}$$

Let us denote $\|\mathbf{x}(n)\|-y(n)=z(n)$. Then, by (2.4) and (2.7), we get

$$z(n)\le(\alpha\|\varphi(n_0)\|-c_0)\eta^{n-n_0}+\alpha\sum_{j=n_0}^{n-1}\eta^{n-j-1}\left[\sum_{s=1}^{m}b_s\,\|\mathbf{x}(j-h_s(j))\|-\frac{p(j)}{\alpha}y(j-\tau(j))\right].$$

From this, using the fact that $p(n)\ge b\alpha$, we obtain

$$z(n)\le(\alpha\,\|\varphi(n_0)\|-c_0)\eta^{n-n_0}+\alpha\sum_{j=n_0}^{n-1}\eta^{n-j-1}\left[\sum_{s=1}^{m}b_s(y(j-h_s(j))-y(j-\tau(j)))+\sum_{s=1}^{m}b_sz(j-h_s(j))\right]. \tag{2.8}$$

Since the function y is nonincreasing, we have

$$y(n-h_s(n))-y(n-\tau(n))\le 0\quad\text{for all } n\in\mathbb{N}(n_0).$$

Let c_0 be such that both the following inequalities are simultaneously satisfied:

$$\alpha\,\|\varphi(n_0)\|<c_0\quad\text{and}\quad\|\varphi(n)\|<y(n)=c_0q^n\quad\text{for } n\in E_0, \tag{2.9}$$

i.e. $c_0>\max\left\{\alpha\,\|\varphi(n_0)\|\,,\max_{n\in E_0}\frac{\|\varphi(n)\|}{q^n}\right\}$. Then, by (2.8) we have

$$z(n)<\alpha\sum_{j=n_0}^{n-1}\eta^{n-j-1}\sum_{s=1}^{m}b_sz(j-h_s(j)). \tag{2.10}$$

From (2.9), for $n\in E_0$ we have $y(n)>\|\varphi(n)\|$, hence $z(n)<0$ for $n\in E_0$. We will show that $z(n)<0$ for all $n>n_0$. Let us choose $n_1>n_0$ such that $z(n_1)\ge 0$ and $z(n)<0$ for $n_0<n<n_1$. Then, by (2.10) we get

$$z(n_1)<\alpha\sum_{j=n_0}^{n_1-1}\eta^{n_1-j-1}\sum_{s=1}^{m}b_sz(j-h_s(j)).$$

Therefore, by the assumption $z(n) < 0$ for $n_0 < n < n_1$, we get a contradiction with $z(n_1) \geq 0$. Hence $z(n) < 0$ for all $n \in \mathbb{N}(n_0)$. Thus $\|\mathbf{x}(n)\| \leq y(n)$ for all $n \in \mathbb{N}(n_0)$ and therefore, by (2.6) we get condition (2.2). This completes the proof of the theorem. □

Note, that by (H_2) from (2.2) we have

$$\|\mathbf{x}(n)\| \leq c_0 q^{n_0} \prod_{s=n_0}^{n-1} \frac{q}{\tau(s)} \leq c_0 q^{n_0}. \tag{2.11}$$

Hence, if the assumptions of Theorem 2.1 are satisfied then all solutions of system (1.1) are bounded.

Example 1. Consider the two-dimensional linear system

$$\begin{cases} x_1(n+1) = \frac{1}{2}x_1(n) + \frac{1}{2}\sin\left(n\frac{\pi}{2}\right)x_2(n) + \frac{1}{2}x_1(n-2) \\ x_2(n+1) = \frac{1}{2}\cos\left(n\frac{\pi}{2}\right)x_2(n) + \frac{1}{2}x_2(n-2), \quad n \geq 3, \end{cases} \tag{2.12}$$

with the initial conditions

$$\varphi(1) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad \varphi(2) = \begin{pmatrix} 2 \\ 2 \end{pmatrix}, \qquad \varphi(3) = \begin{pmatrix} \frac{1}{2} \\ 1 \end{pmatrix}. \tag{2.13}$$

It is easy to check that $\left\|X(n)X^{-1}(s)\right\|_1 \leq \prod_{i=s}^{n-1} \|A(i)\|_1 \leq \left(\frac{1}{2}\right)^{n-s}$. Hence, the nonperturbed system corresponding to system (2.12) is uniformly asymptotically stable. For the matrix $B_1 = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$ we have $\|B\|_1 = \frac{1}{2} = b$. Moreover, $\tau(n) \equiv \max\{1, 2\} = 2$. Let us take $\alpha = 1$ and $\eta = \frac{1}{2}$. Then, for $q = 2$ condition (H_2) is satisfied. Moreover, we have

$$\max\left\{\alpha\,\|\varphi(n_0)\|_1, \max_{n\in E_0} \frac{\|\varphi(n)\|_1}{q^n}\right\} = \max\left\{\frac{3}{2}, \frac{1}{2}, 1, \frac{3}{16}\right\} = \frac{3}{2}.$$

Hence, the solution $\mathbf{x}$ of system (2.12)–(2.13) has the property

$$\|\mathbf{x}(n)\| < \frac{3}{2} \cdot 2^n \prod_{s=3}^{n-1} \frac{1}{2} = 12, \quad \text{for all } n \geq 2.$$

As a consequence of Theorem 2.1 we get the following global asymptotic stability conditions for system (1.1).

Corollary 1. *Suppose that conditions* (H_0)–(H_2) *hold. If* $q < 1$ *then any solution* $\mathbf{x}$ *of system* (1.1) *tends to zero as* $n \to \infty$.

Proof. For $q < 1$, by (2.2) we have

$$\lim_{n\longrightarrow\infty} \|\mathbf{x}(n)\| \leq \lim_{n\longrightarrow\infty} c_0 q^n = 0.$$

This completes the proof. □

Example 2. Consider the scalar difference equation

$$x(n+1) = \frac{1}{3n}x(n) + \frac{3n-2}{12n}x(n-1), \quad n \geq 2. \tag{2.14}$$

Here, $a(n) = \frac{1}{3n}$, $b(n) = \frac{3n-2}{12n}$, $\tau(n) \equiv 1$. We have

$$\left\|X(n)X^{-1}(s)\right\| = \prod_{i=s}^{n-1} \frac{1}{3i} \leq \left(\frac{1}{6}\right)^{n-s} \quad \text{for } n \geq s \geq 2,$$

and $\|b(n)\| = \frac{3n-2}{12n} \leq \frac{1}{4}$ for $n \geq s \geq 2$. Let $\alpha = 1$, $\eta = \frac{1}{6}$ and $q = \frac{5}{6}$. Then all assumptions of Corollary 1 are satisfied. So, all solutions of Eq. (2.14) tend to zero. One such solution is $x(n) = \frac{1}{2^n}$.

The local stability analysis for biological models which are described by some scalar delay difference equation of the form

$$x(n+1) + px(n) + qx(n-k) = 0, \quad n = 0, 1, \ldots$$

where $p, q \in \mathbb{R}$, $k \in 0, 1, \ldots$ were presented by Clark [2]. For this special case of Eq. (1.5), conditions (H_1) and (H_2) are satisfied.

3. System of nonlinear equations

In this section, we consider the nonlinear system

$$\mathbf{x}(n+1) = A(n)\mathbf{x}(n) + f(n, \mathbf{x}(n-h(n))), \quad n \geq n_0 \tag{3.1}$$

with the initial conditions

$$\mathbf{x}(n) = \varphi(n) \quad \text{for } n \in E_0,$$

where $A(n)$ $(n = n_0, n_0, +1, \dots)$ are $k \times k$ real matrices, $h : \mathbb{N}(n_0) \to \mathbb{N}(n_0)$ and $\lim_{n \longrightarrow \infty}(n - h(n)) = \infty$, $f : \mathbb{N}_0 \times \mathbb{R}^k \to \mathbb{R}^k$,

$$E_0 = \{n \in \mathbb{Z} : n = s - h(s) \leq n_0,\ s \geq n_0\} \cup \{n_0\}, \quad \varphi : E_0 \to \mathbb{R}^k.$$

Theorem 3.1. *Suppose condition* (H_0) *holds. Assume the following.*
(H_3) $\|f(n, \mathbf{x})\| \leq \beta \|\mathbf{x}\|^\gamma$, $\mathbf{x} \in \mathbb{R}^k$, $\gamma > 1$, $\beta > 0$, $n \geq n_0$.
(H_4) *There exists a nonincreasing sequence* $\tau : \mathbb{N}_0 \to \mathbb{N}_0$ *such that* $h(n) \leq \tau(n)$ *and* $\frac{\tau(n)}{n} < \frac{\gamma-1}{\gamma}$ *for all* $n > n_0$.
(H_5) *There exists a positive constant* m_1 *such that* $\max_{n \in E_0} \|\varphi(n)\| < \frac{m_1}{\alpha}$ *and* $\alpha\beta(m_1)^{\gamma-1} < 1$.
Then there exists an integer $n^* > n_0$ *such that the solution* $\mathbf{x}$ *of system* (3.1) *with the initial conditions* $\mathbf{x}(n) = \varphi(n)$ *for* $n \in E_0$*, has the property*

$$\|\mathbf{x}(n)\| \leq m_1 \lambda^{-(n-n_0)}, \quad \text{for } n \geq n^*,$$

where λ *is a constant such that* $1 < \lambda < \frac{1}{\eta}$.

Proof. Let us consider the auxiliary scalar equation

$$y(n+1) = \eta y(n) + p(n) y^\gamma(n - \tau(n)), \quad n \geq n_0, \tag{3.2}$$

where $p(n) = (\lambda^{-1} - \eta)\lambda^{(\gamma-1)(n-n_0)}\lambda^{-\gamma\tau(n)}$, $1 < \lambda < \frac{1}{\eta}$. Then $p(n) > 0$. It is easy to check that the solution of Eq. (3.2) is of the form

$$y(n) = \lambda^{-(n-n_0)}.$$

Now, let us write Eqs. (3.1) and (3.2) in the equivalent summation forms

$$\mathbf{x}(n) = X(n)\varphi(n_0) + \sum_{j=n_0}^{n-1} X(n)X^{-1}(j+1) f\,(j, \mathbf{x}j - h(j)) \tag{3.3}$$

and

$$y(n) = \eta^{n-n_0} + \sum_{j=n_0}^{n-1} \eta^{n-j-1} p(j) y^\gamma\left(j - \tau(j)\right), \tag{3.4}$$

where X is the solution of the matrix equation (1.4).

Let $\mathbf{x}$ be a solution of Eq. (3.1). We put, in Eq. (3.3), a new unknown function $\mathbf{x}(n) = m_1 \mathbf{x}_1(n)$, $m_1 > 0$. Then

$$\mathbf{x}_1(n) = \frac{1}{m_1} X(n)\varphi(n_0) + \frac{1}{m_1} \sum_{j=n_0}^{n-1} X(n)X^{-1}(j+1) f\left(j, m_1 \mathbf{x}_1(j - h(j))\right),$$

and by assumptions (H_0) and (H_4) we have

$$\|\mathbf{x}_1(n)\| \leq \frac{\alpha}{m_1} \eta^{n-n_0} \|\varphi(n_0)\| + \frac{\alpha}{m_1} \beta m_1^\gamma \sum_{j=n_0}^{n-1} \eta^{n-j-1} \|\mathbf{x}_1(j - h(j))\|^\gamma. \tag{3.5}$$

Therefore, by (H_5) we get

$$\|\mathbf{x}_1(n)\| < \eta^{n-n_0} + \sum_{j=n_0}^{n-1} \eta^{n-j-1} \|\mathbf{x}_1(j - h(j))\|^\gamma.$$

Hence, by virtue of (3.4) and (3.5), we have

$$\|\mathbf{x}_1(n)\| - y(n) < \sum_{j=n_0}^{n-1} \left[\|\mathbf{x}_1(j - h(j))\|^\gamma - p(j) y^\gamma(j - \tau(j))\right] \eta^{n-j-1}. \tag{3.6}$$

The solution y is decreasing, hence

$$y(j - h(j)) < y(j - \tau(j))$$

and

$$p(j)y^{\gamma}\left(j-h(j)\right) < p(j)y^{\gamma}\left(j-\tau(j)\right).$$

Then, from (3.6) we have

$$\|\mathbf{x}_1(n)\| - y(n) < \sum_{j=n_0}^{n-1}\left[\|\mathbf{x}_1\left(j-h(j)\right)\|^{\gamma} - p(j)y^{\gamma}\left(j-h(j)\right)\right]\eta^{n-j-1}.$$

By (H_4) the sequence $(p(n))$ is increasing for all $n > n_0$. Hence, there exists $n^* > n_0$ such that $p(n) > 1$ for $n \geq n^*$. Then

$$\|\mathbf{x}_1(n)\| - y(n) < \sum_{j=n^*}^{n-1}\left[\|\mathbf{x}_1\left(j-h(j)\right)\|^{\gamma} - y^{\gamma}\left(j-h(j)\right)\right]\eta^{n-j-1}. \tag{3.7}$$

We will show that $\|\mathbf{x}_1(n)\| - y(n) < 0$ for all $n \geq n^*$. Let us choose $n_1 > n^*$ such that $\|\mathbf{x}_1(n_1)\| - y(n_1) \geq 0$ and $\|\mathbf{x}_1(n)\| - y(n) < 0$ for $n^* \leq n < n_1$. Then, $\|\mathbf{x}_1(n)\|^{\gamma} < y^{\gamma}(n)$ for $n^* \leq n < n_1$ and by (3.7) we get

$$\|\mathbf{x}_1(n_1)\| - y(n_1) < \sum_{j=n^*}^{n_1-1}\left[\|\mathbf{x}_1\left(j-h(j)\right)\|^{\gamma} - y^{\gamma}\left(j-h(j)\right)\right]\eta^{n-j-1} < 0$$

which contradicts $\|\mathbf{x}_1(n_1)\| - y(n_1) \geq 0$. Since $\mathbf{x}(n) = m_1\mathbf{x}_1(n)$ we have

$$\|\mathbf{x}(n)\| \leq m_1 y(n) = m_1\lambda^{-(n-n_0)}, \quad n \geq n^*.$$

This completes the proof. □

Example 3. Consider the nonlinear difference equation

$$x(n+1) = \frac{1}{3n}x(n) + \frac{n-1}{n^2}x^2(n-h(n)), \quad n \geq 2 \tag{3.8}$$

where $h(n) = \frac{1+(-1)^n}{2}$, with the initial conditions $x(1) = 0, x(2) = \frac{1}{2}$. The condition ($H_0$) holds with $\alpha = 1$ and $\eta = \frac{1}{6}$. Since,

$$|f(n, x(n))| \leq \frac{n-1}{n^2}\left|x^2(n)\right| \leq \frac{1}{4}|x(n)|^2, \quad n \geq 2,$$

$\beta = \frac{1}{4}$ and $\gamma = 2$. Let $\tau(n) \equiv 1, m_1 = 1, \lambda = 5, n^* = 7$. Then all assumptions of Theorem 3.1 are satisfied. Hence, the solution x of Eq. (3.8) with the given initial conditions has the property $|x(n)| \leq \frac{1}{5^{n-2}}$ for $n \geq 7$.

Remark 1. Note, that since $\lambda > 1$, if the assumptions of Theorem 3.1 are satisfied, then solutions of system (3.1) tend to zero as n tends to infinity.

The following two models are well-known examples of Eq. (1.5).

- The discrete Nicholson's blowflies difference equation [4], where $g(u) = pu(\varpi)^{au}$, $\varpi = \frac{1}{e}$, $a, p \in (0, \infty)$.
- The Beddington–May model [5], where $g(u) = u\left[1 + q\left(1 - (\frac{u}{K})^z\right)\right]_+$, $q, K, z > 0$, $[u]_+ = \max\{u, 0\}$.

For these models conditions (H_3)–(H_5) are satisfied.

4. Remarks

If the matrix $A(n)$ can be represented in one of the following forms:

1. $A(n) = A_1 + A_2(n)$, where $A_1, A_2(n), n = n_0, n_0+1, \ldots$ are $k \times k$ matrices,
2. $A(n) = B_1(n) + B_2(n)$, where $B_1(n) = \mathrm{diag}(\lambda_1(n), \ldots, \lambda_k(n))$, $\lambda_i(n) \neq 0$ for all $n \geq n_0 \geq 0$, $1 \leq i \leq k$ and $B_2(n)$ is the off diagonal part of $A(n)$

and there appear difficulties to estimate the matrix $X(n)X^{-1}(s)$, then one can verify the condition (H_0) by using the following propositions.

Proposition 1. *Let $A(n) = A_1 + A_2(n)$, where $A_1, A_2(n), n = n_0, n_0+1, \ldots$ are $k \times k$ matrices. Assume that the system*

$$Y(n+1) = A_1 Y(n)$$

is uniformly asymptotically stable and

$$\sum_{i=0}^{\infty} \|A_2(i)\| < \infty. \tag{4.1}$$

Then condition (H_0) *holds.*

Proof. Let $\Phi(\cdot,\cdot)$ and $\Psi(\cdot,\cdot)$ denote the state transition matrices of the equations

$$\mathbf{x}(n+1) = A_1\mathbf{x}(n),$$

and

$$\mathbf{x}(n+1) = (A_1 + A_2(n))\,\mathbf{x}(n).$$

Thus $\Phi(\cdot,\cdot)$ and $\Psi(\cdot,\cdot)$ satisfy

$$\Psi(n,n_0) = \Phi(n,n_0) + \sum_{i=n_0}^{n-1} \Phi(n,i+1)A_2(i)\Psi(i,n_0) \tag{4.2}$$

where $\Phi(n,n_0) = A_1^{n-n_0}$. Since the system $Y(n+1) = A_1Y(n)$ is uniformly asymptotically stable, there exist real scalars $K > 0$ and $0 < \eta < 1$ such that

$$\|\Phi(n,n_0)\| \le K\eta^{n-n_0} \quad \text{for } n \ge n_0 \ge 0.$$

Hence, taking norms of both sides of (4.2), by the discrete Gronwall Inequality (see [1, p.198]), we obtain

$$\|\Psi(n,n_0)\| \le \eta^{n-n_0} \exp\left(K\eta^{-1}\sum_{i=0}^{\infty}\|A_2(i)\|\right).$$

Therefore, by assumption (4.1), we get the condition (H_0).

The idea of the proof of the next proposition is similar to proof of Theorem 4.1 [16] where the authors give conditions on the boundedness of solutions of a linear discrete Volterra system with disturbances using the resolvent of unperturbed system. □

Proposition 2. *Let us consider system* (1.3) *in the form*

$$\mathbf{x}(n+1) = B_1(n)\mathbf{x}(n) + B_2(n)\mathbf{x}(n)$$

where

$$B_1(n) = \mathrm{diag}(\lambda_1(n),\ldots,\lambda_k(n)), \quad \lambda_i(n) \ne 0$$

for all $n \ge n_0 \ge 0$, $1 \le i \le k$ *and* $B_2(n)$ *is the off diagonal part of* $A(n)$. *Let* $\Phi(\cdot,\cdot)$ *be the state transition matrix of the equation*

$$\mathbf{x}(n+1) = B_1(n)\mathbf{x}(n).$$

Assume that

1° $\left\|\Phi^{-1}(n,n_0)\Phi(n,s)\right\| \le K_1 = \text{const.}$, *for all* $n \ge s \ge n_0$,
2° *there exist positive constants* K_2 *and* K_3 *such that*

$$\sum_{i=0}^{\infty}\|B_2(i)\Phi(i,n_0)\| < K_2 \tag{4.3}$$

$$\|\Phi(n,n_0)\| \le K_3\eta^{n-n_0} \quad \textit{for } n \ge s \ge n_0. \tag{4.4}$$

Then condition (H_0) *holds.*

Proof. Let $\Phi(\cdot,\cdot)$ and $\Psi(\cdot,\cdot)$ denote the state transition matrices of the equations

$$\mathbf{x}(n+1) = B_1(n)\mathbf{x}(n),$$

and

$$\mathbf{x}(n+1) = B_1(n)\mathbf{x}(n) + B_2(n)\mathbf{x}(n).$$

Thus $\Phi(\cdot,\cdot)$ and $\Psi(\cdot,\cdot)$ satisfy

$$\Psi(n,n_0) = \Phi(n,n_0) + \sum_{i=n_0}^{n-1}\Phi(n,i+1)B_2(i)\Psi(i,n_0) \tag{4.5}$$

where $\Phi(n, n_0) = \prod_{s=n_0}^{n-1} B_1(s) = \operatorname{diag}\left(\prod_{s=n_0}^{n-1} \lambda_1(s), \ldots, \prod_{s=n_0}^{n-1} \lambda_k(s)\right)$. Hence, analogously as in the proof of Proposition 1, we get

$$\|\Psi(n, n_0)\| \leq \|\Phi(n, n_0)\| K_1 \exp \sum_{i=0}^{\infty} \|B_2(i)\| \, \|\Phi(i, n_0)\| .$$

Then, if $B_2(n)$ and $\Phi(n, n_0)$ satisfy assumptions (4.3) and (4.4), we obtain the condition (H_0). □

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