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Asymptotic properties of solutions of difference equations with several delays and Volterra summation equations

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ABSTRACT

We study a scalar linear difference equation with several delays by transforming it to a system of Volterra equations without delays. The results obtained for this system are then used to establish oscillation criteria and asymptotic properties of solutions of the considered equation.

1. Introduction

Let $\mathbb{R}$ denote the set of real numbers, $\mathbb{Z}$ and $\mathbb{Z}^+$ the set of integers and nonnegative integers, respectively, $\mathbb{N}(n_0) = \{n_0, n_0+1, \ldots\}, n_0 \in \mathbb{Z}^+$.

In this paper we consider a scalar linear difference equation with several delays

$$\Delta x(n) = \sum_{i=0}^{m} a_i(n)x(h_i(n)) + f(n), \quad n \geqslant n_0 \tag{1.1}$$

where $a_i,\ f : \mathbb{N}(n_0) \to \mathbb{R},\ h_i : \mathbb{N}(n_0) \to \mathbb{Z},\ h_0(n) = n,\ h_i(n) \leqslant n$ for $i = 1,2,\ldots,m$ and $\lim_{n\to\infty} h_i(n) = \infty$ for $i = 0,1,2,\ldots,m$.

By a solution of Eq. (1.1) we mean a sequence $x := (x(n))$ satisfying (1.1) for any $n \in \mathbb{N}(n_0)$. A solution x of (1.1) is said to be oscillatory if the terms $x(n)$ of the sequence are neither eventually all positive nor all negative. Otherwise, the solution is called nonoscillatory.

Currently, the problem of oscillation and nonoscillation of solutions of delay difference equations is receiving much attention, see the monographs by Agarwal et al. [1] and Győri and Ladas [9]. Nonoscillation of difference equations is less studied compared to sufficient oscillation conditions. The well known result ([9], Theorem 7.8.2) for an equation of type (1.1) with several constant delays

$$\Delta x(n) = \sum_{i=1}^{m} p_i(n)x(n-k_i), \quad n \geqslant n_0, \tag{1.2}$$

states that if $0 \leqslant k_1 \leqslant k_2 \leqslant \ldots \leqslant k_m$ and

$$\sum_{i=1}^{s} p_i(n) \geqslant 0 \quad \text{for } s = 1,2,\ldots,m \quad \text{and } n \geqslant n_0,$$

then (1.2) has a positive nondecreasing solution.

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Zhou in [21] obtained oscillation and nonoscillation results for Eq. (1.1) with negative coefficients. Existence of nonoscillatory solutions of (1.1) where the coefficients are positive, negative or of arbitrary signs was studied by Berezansky et al. in [3] and by Berezansky and Braverman in [4].

The aim of this paper is to obtain asymptotic properties of the solutions of Eq. (1.1). It is known that difference equations can be transformed in different ways to difference equations of Volterra type. Transforming further the summation Volterra equation with delays obtained from (1.1) to the system of Volterra equations without delays of the form

$$y(n) = p(n) + \sum_{s=n_0}^{n-1} Q(n, s+1) y(s), \quad n \geqslant n_0,$$

we get various results on the asymptotic behaviour of solutions for this system. These results are then used to establish some properties of the solutions of Eq. (1.1). We provide some examples to illustrate the results.

During the last few years, asymptotic properties (stability, oscillation) of Volterra difference equations and discrete Volterra systems has been investigated in a number of papers, for example, in Applelby et al. [2], Choi [5], Crisci et al. [6], Diblik et al. [7], Győri and Horvath [10], Győri and Reynolds [11], Kolmanovskii [12,14], Medina [15], Morchało [16–18], Song and Baker [19,20]; see also the references cited therein.

Let k be a positive integer. The set of all k-dimensional column vectors with real components is denoted by $\mathbb{R}^k$ and the set of $k \times k$ matrices with real entries by $\mathbb{R}^{k\times k}$. Let $\|.\|$ denote any norm of a vector or the associated induced norm of a square matrix. The set $\mathbb{R}^{k\times k}$ can be endowed with many norms, but they are all equivalent. The identity matrix is denoted by I. A matrix $A = (A_{ij})$ in $\mathbb{R}^{k\times k}$ is nonnegative if $A_{ij} \geqslant 0$, in which case we write $A \geqslant 0$. A partial ordering is defined on $\mathbb{R}^{k\times k}$ by letting $A \leqslant B$ if and only if $B - A \geqslant 0$, which is equivalent to $A_{ij} \geqslant B_{ij}$ for all $1 \leqslant i \leqslant k$ and $1 \leqslant j \leqslant k$. The absolute value of A is the matrix $|A|$ defined by $(|A|)_{ij} = |A_{ij}|$ for all $1 \leqslant i \leqslant k$ and $1 \leqslant j \leqslant k$.

In the future we assume any product which does not involve any factors is equal to one, and any sum which does not include any terms is equal to zero.

2. Preliminaries

In this section we will transform the scalar difference Eq. (1.1) to the system of Volterra equations without delays. Together with Eq. (1.1) we will also consider the following equation

$$\Delta x(n) = b(n)x(n), \quad n \geqslant n_0, \tag{2.1}$$

where $b : \mathbb{N}_0 \to \mathbb{R},\ b(n) \neq -1$. The solution $X(n,k)$ of the problem

$$\Delta x(n) = b(n)x(n), \quad n \geqslant k,\ x(k) = 1,$$

is called the fundamental function of Eq. (2.1).

Let us note that the fundamental function (solution) of the linear difference equation

$$x(n+1) = (1 + b(n))x(n), \quad n \geqslant n_0,$$

can be easily computed

$$X(n,k) = \prod_{j=k}^{n-1}(1 + b(j)), \quad k \geqslant n_0.$$

Let us write Eq. (1.1) in the form

$$x(n+1) = (1 + b(n))x(n) + [a_0(n) - b(n)]x(n) + \sum_{i=1}^{m} a_i(n)x(h_i(n)) + f(n)$$

or in the equivalent form, using the function $X(n,k)$ (see [8], Theorem 3.17)), i.e.

$$x(n) = \left(\prod_{j=n_0}^{n-1}(1 + b(j))\right)\left[x_0 + \sum_{k=n_0}^{n-1}\left(\prod_{j=n_0}^{k}(1 + b(j))\right)^{-1} f(k)\right] + \sum_{k=n_0}^{n-1}\prod_{j=k+1}^{n-1}(1 + b(j))\left\{[a_0(k) - b(k)]x(k) + \sum_{i=1}^{m} a_i(k)x(h_i(k))\right\}. \tag{2.2}$$

where $x_0 = x(n_0)$. Hence

$$x(n) = X(n, n_0)v(n, n_0) + \sum_{k=n_0}^{n-1} X(n, k+1)\left\{[a_0(k) - b(k)]x(k) + \sum_{i=1}^{m} a_i(k)x(h_i(k))\right\}, \tag{2.3}$$

where

$$X(n,s) = \prod_{j=s}^{n-1}(1 + b(j)) \tag{2.4}$$

and

$$v(n, n_0) = x_0 + \sum_{k=n_0}^{n-1} \left(\prod_{j=n_0}^{k} (1 + b(j)) \right)^{-1} f(k). \tag{2.5}$$

Replacing n by $h_q(n)$ in (2.3) we get

$$x(h_q(n)) = X(h_q(n), n_0)v(h_q(n), n_0) + \sum_{k=n_0}^{h_q(n)-1} X(h_q(n), k+1) \left\{ [a_0(k) - b(k)]x(k) + \sum_{i=1}^{m} a_i(k)x(h_i(k)) \right\}. \tag{2.6}$$

Let us denote

$$y_q(n) = x(h_q(n)), \quad q = 0, 1, \dots, m, \quad n \geqslant n_0, \tag{2.7}$$

$$p_q(n) = \begin{cases} X(h_q(n), n_0)v(h_q(n), n_0) & \text{if } h_q(n) > n_0 \\ x(h_q(n)) & \text{if } h_q(n) \leqslant n_0 \end{cases} \tag{2.8}$$

and

$$D_q = \{(n, s) : n \geqslant n_0, \ n_0 \leqslant s \leqslant h_q(n) - 1\}$$

for $q = 0, 1, \dots, m, \ n \geqslant n_0$. Moreover, let

$$Q_{q0}(n, k+1) = \begin{cases} X(h_q(n), k+1)[a_0(k) - b(k)] & (n, k) \in D_q \\ 0 & (n, k) \notin D_q \end{cases} \tag{2.9}$$

and

$$Q_{qi}(n, k+1) = \begin{cases} X(h_q(n), k+1)a_i(k) & (n, k) \in D_q \\ 0 & (n, k) \notin D_q \end{cases} \tag{2.10}$$

for $q = 0, 1, \dots, m$ and $i = 1, 2, \dots, m$. Then, Eq. (2.6) takes the form

$$y(n) = p(n) + \sum_{k=n_0}^{n-1} Q(n, k+1)y(k), \quad n \geqslant n_0, \tag{2.11}$$

where

$$Q(n, k) = \begin{pmatrix} Q_{00}(n, k) & Q_{01}(n, k) & \dots & Q_{0m}(n, k) \\ Q_{10}(n, k) & Q_{11}(n, k) & \dots & Q_{1m}(n, k) \\ \dots & & & \\ Q_{m0}(n, k) & Q_{m1}(n, k) & \dots & Q_{mm}(n, k) \end{pmatrix}$$

and

$$y(n) = \begin{pmatrix} y_0(n) \\ y_1(n) \\ \dots \\ y_m(n) \end{pmatrix}, \quad p(n) = \begin{pmatrix} p_0(n) \\ p_1(n) \\ \dots \\ p_m(n) \end{pmatrix}.$$

3. Main results

We start our main results with establishing conditions for the boundedness and oscillation of solutions of the Volterra system (2.11). We will use the following definition.

Definition 3.1. We say that a solution $y = [y_0, \dots, y_m]^T$ of Eq. (2.11) oscillates if for some $i = 0, 1, \dots, m$, and for every integer $n_1 \geqslant 0$ there exists $n \geqslant n_1$ such that $y_i(n)y_i(n+1) \leqslant 0$. Otherwise, the solution is said to be nonoscillatory (all its components are either eventually positive or eventually negative).

In the proof of the next theorem, the following lemma, which is a small modification of Lemma 2.1 in [18], will be needed.

Lemma 3.1. *Let $q : \mathbb{N}(n_0) \to \mathbb{R}^+$ and $L(n, s) \in \mathbb{N}(n_0) \times \mathbb{N}(n_0) \to \mathbb{R}^+, L(n, s) = 0$ for $n < s, \ L(n, s)$ is nonincreasing in $n \in \mathbb{N}(n_0)$ and y is a sequence of positive real numbers such that*

$$y(n) \leqslant q(n) + \sum_{s=0}^{n-1} L(n, s+1)y(s),$$

holds for all $n \geqslant n_0$. *Then*

$$y(n) \leqslant Q(n)\left\{1 + \sum_{s=n_0}^{n-1} L(s+1,s+1)\exp\left(\sum_{l=s+1}^{n-1} L(l+1,l+1)\right)\right\}.$$

for $n \in \mathbb{N}(n_0)$, *where* $Q(n) = \max_{0\leqslant s\leqslant n} q(s)$.

Proof. The proof is analogous to the proof of Lemma 2.1 from [18]. □

Theorem 3.1. *Assume the following:*

1. *For all* $(n,j) \in \mathbb{Z}^+ \times \mathbb{Z}^+, Q(n,j)$ *is nonnegative if* $n_0 \leqslant j \leqslant n$ *and* $Q(n,j) = 0$ *if* $j > n$.
2. $Q(n,j)$ *is nonincreasing in* $n \in \mathbb{Z}^+$ *for every* $j \in \mathbb{Z}^+$.
3. $\lim_{n\to\infty} \sum_{s=n_0}^{n-1} \|Q(s,s)\| < \infty$.
4. $\lim_{n\to\infty} \max_{0\leqslant j\leqslant n} \|p(j)\| < \infty$.

Then all solutions of Eq. (2.11) are bounded.

Proof. Let y be a solution of Eq. (2.11). Then

$$\|y(n)\| \leqslant \|p(n)\| + \sum_{s=n_0}^{n-1} \|Q(n,s+1)\|\|y(s)\|, \quad n \geqslant n_0.$$

Now, using Lemma 3.1, we get

$$\|y(n)\| \leqslant \max_{n_0<j\leqslant n} \|p(j)\| \left\{1 + \sum_{s=n_0}^{n-1} \|Q(s+1,s+1)\| \exp\left(\sum_{l=s+1}^{n-1} \|Q(l+1,l+1)\|\right)\right\}.$$

Hence, by assumptions 3 and 4 we obtain that y is bounded. This completes the proof. □

Example 3.1. Consider the linear Volterra difference equation

$$y(n) = -\frac{1}{n(n+1)} - \frac{2}{(n+1)(n+2)}\left(1 - \frac{1}{2^n}\right) + \sum_{s=0}^{n-1} \frac{(s+1)2^{-s}}{(n+1)(n+2)} y(s). \tag{3.1}$$

It is easy to see that the assumptions of Theorem 3.1 are satisfied. So, all solutions of Eq. (3.1) are bounded. One such solution is $y(n) = \frac{1}{n+1}$.

As a consequence of Theorem 3.1 we get the following result for Eq. (1.1).

Theorem 3.2. *Let* $b : \mathbb{N}(n_0) \to \mathbb{R}$, $-1 < b(n) < 0$ *and* $a_0(n) \geqslant b(n)$ *for all* $n \geqslant n_0$. *Assume the following:*

1. $a_i(n) \geqslant 0$ *for all* $i = 1, 2, \ldots, m$ *and* $n \geqslant n_0$.
2. *For every* $i = 1, \ldots, m$, $\sum_{n=n_0}^{\infty} a_i(n) < \infty$ *and* $\sum_{n=n_0}^{\infty} (a_0(n) - b(n)) < \infty$.
3. *The sequences* b *and* f *are such that for every* $q = 0, 1, \ldots, m$, $\lim_{n\to\infty} \max_{n_0<s\leqslant n} |p_q(s)| < \infty$, *where* p_q *are defined in (2.8).*
4. *There exists* $q \in \{0, 1, \ldots, m\}$ *such that* $h_q(n) = n - r_q$ *where* $r_q \in \mathbb{N}_0$.

Then all solutions of Eq. (1.1) are bounded.

Proof. Suppose that x is an unbounded solution of Eq. (1.1). Take $q \in \{0, 1, \ldots, m\}$ such that $h_q(n) = n - r_q$. Since $h_q(\mathbb{N}(r_q)) = \mathbb{Z}^+$, by (2.7) the sequence y_q is unbounded.

On the other hand, by the assumption $-1 < b(n) < 0$ it follows that $X(h_q(n), j)$ is positive and nonincreasing in $n \in \mathbb{N}_0$ for every $j \in \mathbb{N}_0$. Hence, by (2.9) and (2.10), $Q(n,j)$ is nonnegative and nonincreasing in $n \in \mathbb{Z}^+$ for every $j \in \mathbb{Z}^+$, too. So, all hypotheses of Theorem 3.1 are satisfied. Hence, from Theorem 3.1 it follows that y_q is bounded. This contradiction completes the proof. □

Example 3.2. Consider the difference equation

$$\Delta x(n) = -\frac{2n}{(n-1)(n+1)^2} x(n) + \frac{1}{n^2} x(n-1), \quad n \geqslant 2. \tag{3.2}$$

It is clear that this equation is a particular case of Eq. (1.1), where $a_0(n) = -\frac{2n}{(n-1)(n+1)^2}$, $a_1(n) = \frac{1}{n^2}$, $h_0(n) = n$, $h_1(n) = n-1$ and $f(n) \equiv 0$. Let $b(n) = -\frac{2}{n^2}$. It is easy to see that

$$a_0(n) - b(n) = \frac{2n^2 - 2n - 2}{(n-1)n^2(n+1)^2} \geqslant 0,$$

$$\lim_{n\to\infty}\left(\max_{2<s\leqslant n}|p_0(s)|\right) = |x_0|\lim_{n\to\infty}\left(\max_{2<s\leqslant n}\prod_{j=2}^{s-1}\left(1-\frac{1}{j^2}\right)\right) < \infty$$

and

$$\lim_{n\to\infty}\left(\max_{2<s\leqslant n}|p_1(s)|\right) = |x_0|\lim_{n\to\infty}\left(\max_{2<s\leqslant n}\prod_{j=2}^{s-2}\left(1-\frac{1}{j^2}\right)\right) < \infty.$$

Hence, all assumptions of Theorem 3.2 are satisfied. So, all solutions of Eq. (3.2) are bounded. One such solution is $x(n) = 1 + \frac{1}{n}$.

Theorem 3.3. *Let $\gamma \geqslant 1$. Assume the following:*

1. *For all $(n,j) \in \mathbb{Z}^+ \times \mathbb{Z}^+$, $Q(n,j)$ is nonnegative if $0 \leqslant j \leqslant n$ and $Q(n,j) = 0$ if $j > n$.*
2. *$Q(n,j)$ is nonincreasing in $n \in \mathbb{Z}^+$ for every $j \in \mathbb{Z}^+$.*
3. *For every $q = 0,1,\ldots,m$, $\lim_{n\to\infty}\sum_{s=n_0}^{n}\sum_{k=0}^{m} s^\gamma Q_{qk}(s,s) < \infty$.*
4. *For every $q = 0,1,\ldots,m$, $\limsup_{n\to\infty} p_q(n) = \infty$, $\liminf_{n\to\infty} p_q(n) = -\infty$.*

Then every solution y of Eq. (2.11) with the property $y(n) = O(n^\gamma)$ for all $n \geqslant n_0$ is oscillatory.

Proof. Let y be a solution of Eq. (2.11) with the property $y(n) = O(n^\gamma)$. Then, there exists a positive constant C, such that $\max_{0\leqslant k\leqslant m}\frac{|y_k(n)|}{n^\gamma} \leqslant C$ for $n \in \mathbb{Z}^+$. We claim that y is oscillatory. If not, it is nonoscillatory. So, there exists a $n_1 \geqslant n_0$, such that for $n \geqslant n_1$ either, $y_q(n) > 0$ or $y_q < 0$ for every $q = 0,1,\ldots,m$. Let $y_q(n) > 0$ for $n \geqslant n_1$ and some $q = 0,1,\ldots,m$. Then, from (2.11) using assumptions 1 and 2, for $n \geqslant n_1$ we get

$$\begin{aligned} y_q(n) &= p_q(n) + \sum_{s=n_0}^{n_1-1}\sum_{k=0}^{m} Q_{qk}(n,s+1)y_k(s) + \sum_{s=n_1}^{n-1}\sum_{k=0}^{m} Q_{qk}(n,s+1)y_k(s) \\ &\leqslant p_q(n) + \sum_{s=n_0}^{n_1-1}\sum_{k=0}^{m} Q_{qk}(s+1,s+1)y_k(s) + C\sum_{s=n_1}^{n-1}\sum_{k=0}^{m} s^\gamma Q_{qk}(s+1,s+1). \end{aligned}$$

Let $M = \sum_{s=n_0}^{n_1-1}\sum_{k=0}^{m} Q_{qk}(s+1,s+1)y_k(s)$. Hence, by 3 and 4

$$\liminf_{n\to\infty} y_q(n) \leqslant M + \liminf_{n\to\infty} p_q(n) + C\lim_{n\to\infty}\sum_{s=n_1}^{n-1}\sum_{k=0}^{m} s^\gamma Q_{qk}(s+1,s+1) = -\infty.$$

Since $y_q(n) > 0$ for $n \geqslant n_1$ we obtain a contradiction. The proof in case $y_q(n) < 0$ is similar. This completes the proof. □

Example 3.3. Consider the linear Volterra difference equation

$$y(n) = (-1)^n(n+1)^2 + \frac{1}{2n^2}[(-1)^n - 1] + \sum_{s=0}^{n-1}\frac{1}{n^2(s+1)^2}y(s), \quad n \geqslant 1. \tag{3.3}$$

Let $\gamma = 2$. It is easy to see that the assumptions of Theorem 3.3 are satisfied. So, all solutions of Eq. (3.3) with the property $y(n) = O(n^2)$ are oscillatory. One such solution is $y(n) = (-1)^n(n+1)^2$.

As a consequence of Theorem 3.3 we get following result for Eq. (1.1).

Theorem 3.4. *Let $\gamma \geqslant 1$, $b : \mathbb{N}(n_0) \to \mathbb{R}$, $-1 < b(n) < 0$ and $a_0(n) \geqslant b(n)$ for $n \geqslant n_0$. Assume the following:*

1. *$a_i(n) \geqslant 0$ for all $i = 1,2,\ldots,m$ and $n \geqslant n_0$.*
2. *For every $i = 1,\ldots,m$, $\sum_{n=n_0}^{\infty} n^\gamma a_i(n) < \infty$ and $\sum_{n=n_0}^{\infty} n^\gamma (a_0(n) - b(n)) < \infty$.*
3. *The sequences b and f are such that for every $q = 0,1,\ldots,m$, $\limsup_{n\to\infty} p_q(n) = \infty$, $\liminf_{n\to\infty} p_q(n) = -\infty$, where p_q are defined in (2.8).*

Then every solution x of Eq. (1.1) with the property $x(n) = O(n^\gamma)$ for all $n \geqslant n_0$ is oscillatory.

Proof. Let x be a solution of Eq. (1.1) with the property $x(n) = O(n^\gamma)$. Then, by (2.7), for every $q \in \{0, 1, \dots, m\}$, $y_q(n) = O(n^\gamma)$, too. Hence, $y(n) = O(n^\gamma)$. By the assumptions of this theorem it follows that all hypotheses of Theorem 3.3 are satisfied. Therefore, by Theorem 3.3, y is oscillatory. So, there exists $q \in \{0, 1, \dots, m\}$, such that $(y_q(n))$ is oscillatory. Since $\lim_{n \to \infty} h_q(n) = \infty$ the set $h_q(\mathbb{Z}^+)$ is infinite. Then, by (2.7) x is also oscillatory. This completes the proof. □

Remark 3.1. Note, that the assumption $-1 < b(n) < 0$ implies, by (2.4), that $X(n,s)$ is positive. So, if $f(n) \equiv 0$ or f is a nonoscillatory sequence, then by (2.5) and (2.8) it follows that p_q are of constant sign eventually (for every $q \in \{0, 1, \dots, m\}$), and hence the assumption 2 of Theorem 3.3 could not be satisfied.

Example 3.4. Consider the difference equation

$$\Delta x(n) = \frac{1}{n^3} x(n) + \frac{1}{n^2(n-1)} x(n-1) + (-1)^{n+1}(2n+1), \quad n \geqslant 2. \tag{3.4}$$

Let $\gamma = 1$, $b(n) = -\frac{1}{n^3}$. Here $f(n) = (-1)^{n+1}(2n+1)$. It is easy to check that all assumptions of Theorem 3.4 are satisfied. Hence, every solution $(x(n))$ of Eq. (3.4) with the property $x(n) = O(n)$ is oscillatory. One such solution is $x(n) = (-1)^n n$.

Note, that if we assume the following.

$(h1)$ $a_i(n) \geqslant 0$ for $i = 1, 2 \dots, m$, $n \geqslant n_0$,
$(h2)$ there exists $b : \mathbb{N}(n_0) \to \mathbb{R}$ such that $b(n) > -1$ and $a_0(n) \geqslant b(n)$ for all $n \geqslant n_0$,
$(h3)$ $f(n) \geqslant 0$,

then, by (2.2), every solution x of Eq. (1.1) with the initial conditions $x(n) \geqslant 0$ for $n < n_0$ and $x_0 > 0$ is positive and has the property

$$x(n) \geqslant x_0 \prod_{j=n_0}^{n-1} (1 + b(j)) + \sum_{k=n_0}^{n-1} \left(\prod_{j=k+1}^{n-1} (1 + b(j)) \right) f(k). \tag{3.5}$$

Therefore, we get the following corollary.

Corollary 3.1. *Assume that the assumptions $(h1) - (h3)$ hold. Suppose also that*

$$4. \ \liminf_{n \to \infty} \prod_{j=n_0}^{n-1} (1 + b(j)) > 0.$$

Then every solution x of Eq. (1.1) with the initial conditions $x(n) \geqslant 0$ for $n < n_0$ and $x(n_0) = x_0 > 0$ is positive and $\liminf_{n \to \infty} x(n) > 0$. *Moreover, if condition*

$$\sum_{k=n_0}^{\infty} f(k) = \infty,$$

is satisfied, then these solutions have the property $\lim_{n \to \infty} x(n) = \infty$.

The next example shows that the condition 4 in Corollary 3.1 is not necessary.

Example 3.5. Consider the difference equation

$$\Delta x(n) = -\frac{1}{n} x(n) + \frac{1}{(n-\tau)} x(n-\tau) + 1, \quad n > \tau, \tag{3.6}$$

with a certain positive integer τ. Here, $a_0(n) = -\frac{1}{n}$, $a_1(n) = \frac{1}{n-\tau}$ and $f(n) \equiv 1$. Let $b(n) = -\frac{1}{n}$. Since $\lim_{n \to \infty} \left(\prod_{j=\tau}^{n-1} \left(1 - \frac{1}{j}\right) \right) = 0$, assumption 4 of Corollary 3.1 is not satisfied but it is easy to check that $x(n) = n$ is a solution of (3.6), which tends to infinity as n tends to infinity.

In this part of the paper we will consider the asymptotic properties of solutions of Eq. (2.11) using its resolvent matrices of the kernel $Q(n,s)$ in (2.11). Let us find the solution y of Eq. (2.11) as a function of p and auxiliary $(m+1) \times (m+1)$ matrix R, referred to as a resolvent matrix [13]. Let us define

$$Q^{(1)}(n,s+1) = Q(n,s+1),$$

$$Q^{(r)}(n,s+1) = \sum_{l=s+1}^{n-1} Q^{(r-1)}(n,l+1)Q^{(1)}(l,s+1), \quad r = 2,3,\dots$$

and

$$R(n,s+1) = \sum_{r=1}^{\infty} Q^{(r)}(n,s+1). \tag{3.7}$$

The double sequence of $(m+1)\times(m+1)$ matrices $R(n,s)$ is called the resolvent kernel associated with the kernel $Q(n,s)$ of Eq. (2.11). Note, that the series $\sum_{r=1}^{\infty} Q^{(r)}(n,s+1)$ is convergent if the kernel $Q(n,s)$ is bounded. It is easy to see that the resolvent $R(n,s)$ satisfies, for any n and $n_0 \leqslant s < n$, the following matrix equations

$$R(n,s+1) = Q(n,s+1) + \sum_{r=s+1}^{n-1} Q(n,r+1)R(r,s+1) \tag{3.8}$$

and

$$R(n,s+1) = Q(n,s+1) + \sum_{r=s+1}^{n-1} R(n,r+1)Q(r,s+1). \tag{3.9}$$

In terms of the resolvent matrix $R(n,s)$ the solution of Eq. (2.11) can be written as

$$y(n) = p(n) + \sum_{k=n_0}^{n-1} R(n,k+1)p(k). \tag{3.10}$$

Multiplying both sides of the equation

$$y(j) = p(j) + \sum_{k=n_0}^{j-1} Q(j,k+1)y(k)$$

by $R(n,j+1)$ on the left and summing with respect to j from n_0 to $n-1$, we obtain

$$\sum_{j=n_0}^{n-1} R(n,j+1)(y(j)-p(j)) = \sum_{j=n_0}^{n-1} R(n,j+1)\sum_{k=n_0}^{j-1} Q(j,k+1)y(k) = \sum_{k=n_0}^{n-1}\sum_{j=n_0}^{k-1} R(n,k+1)Q(k,j+1)y(j).$$

Hence, changing the order of summation, we get

$$\sum_{j=n_0}^{n-1} R(n,j+1)(y(j)-p(j)) = \sum_{j=n_0}^{n-1}\left(\sum_{k=j+1}^{n-1} R(n,k+1)Q(k,j+1)\right)y(j).$$

Then, by (3.9) we get (3.10).

Now, using the form (3.10), we give conditions on p under which the solutions of Eq. (2.11) (and consequently of Eq. (1.1)) tend to zero as n tends to infinity. Note, that the equality (3.10) can be expressed in the form

$$y(n) = P(n,n_0)p(n_0) + \sum_{k=n_0}^{n-1} P(n,k+1)\Delta p(k), \tag{3.11}$$

where $P(n,s) = I + \sum_{l=s}^{n-1} R(n,l+1)$. In fact, we have

$$\begin{aligned} y(n) &= P(n,n_0)p(n_0) + \sum_{k=n_0}^{n-1} P(n,k+1)\Delta p(k) \\ &= \left(I + \sum_{k=n_0}^{n-1} R(n,k+1)\right)p(n_0) + \sum_{k=n_0}^{n-1}\left(I + \sum_{l=k+1}^{n-1} R(n,l+1)\right)\Delta p(k) \\ &= p(n_0) + \sum_{k=n_0}^{n-1} R(n,k+1)p(n_0) + \sum_{k=n_0}^{n-1}\Delta p(k) + \sum_{k=n_0}^{n-1}\sum_{l=k+1}^{n-1} R(n,l+1)\Delta p(k) \\ &= p(n) + \sum_{k=n_0}^{n-1} R(n,k+1)p(n_0) + \sum_{k=n_0}^{n-1} R(n,k+1)\sum_{l=n_0}^{k-1}\Delta p(l) \\ &= p(n) + \sum_{k=n_0}^{n-1} R(n,k+1)p(k). \end{aligned}$$

Therefore, we get the following propositions.

Proposition 3.1. If $p(n) = const$ and $\lim_{n\to\infty} P(n, n_0) = 0$ then the solution $(y(n))$ of Eq. (2.11) satisfies $y(n) \to 0$ as $n \to \infty$.

Proposition 3.2. If

1. $\lim\limits_{n\to\infty} P(n, n_0) = 0$,
2. $\lim\limits_{n\to\infty} \sum_{k=n_0}^{n_1-1} \|P(n, k+1)\| = 0$ for $n_1 > n_0$,
3. $\sum_{k=n_0}^{n-1} \|P(n, k+1)\| \leqslant M$ for all $n > n_0$,
4. $\lim\limits_{n\to\infty} \Delta p(n) = 0$,

then the solution $(y(n))$ of Eq. (2.11) satisfies $y(n) \to 0$ as $n \to \infty$.

Proof. For any $\varepsilon > 0$, there exists $n_1 \geqslant n_0$ such that

$$\|P(n, n_0)\| < \frac{\varepsilon}{3\|p(n_0)\|} \quad \text{and} \quad \|\Delta p(k)\| < \frac{\varepsilon}{3M}$$

hold for $n \geqslant n_1$. For this fixed n_1, there exists $n_2 \geqslant n_0$ such that for $n \geqslant n_2$

$$\sum_{k=n_0}^{n_1-1} \|P(n, k+1)\| \|\Delta p(k)\| < \frac{\varepsilon}{3}.$$

Then, for $n \geqslant N = \max\{n_1, n_2\}$, by (3.11) it follows

$$\|y(n)\| \leqslant \|P(n, n_0)\| \|p(n_0)\| + \sum_{k=n_0}^{n_1-1} \|P(n, k+1)\| \|\Delta p(k)\| + \sum_{k=n_1}^{n-1} \|P(n, k+1)\| \|\Delta p(k)\| < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.$$

This completes the proof. □

Theorem 3.5. *Let $z(n) = \sum_{k=n_0}^{n-1} y(k)$ where y is the solution of Eq. (2.11). Then for any $\varepsilon > 0$ there exists $\delta > 0$ such that $\|p(n)\| < \delta$ for $n \geqslant n_0$ imply $\|z(n)\| < \varepsilon$ for all $n \geqslant n_0$ if and only if there exists a constant $C > 0$ such that*

$$\sum_{k=n_0}^{n-1} \|S(n, k+1)\| \leqslant C, \tag{3.12}$$

where $S(n, k+1) = I + \sum_{l=k+1}^{n-1} R(l, k+1)$.

Proof. *Sufficiency*. For any $\varepsilon > 0$, choose $\delta < \frac{\varepsilon}{C}$. From (3.10) we have

$$\begin{aligned} z(n) = \sum_{k=n_0}^{n-1} y(k) &= \sum_{k=n_0}^{n-1} \left(p(k) + \sum_{l=n_0}^{k-1} R(k, l+1) p(l) \right) \\ &= \sum_{k=n_0}^{n-1} p(k) + \sum_{k=n_0}^{n-1} \sum_{l=n_0}^{k-1} R(k, l+1) p(l) \\ &= \sum_{k=n_0}^{n-1} p(k) + \sum_{k=n_0}^{n-1} \sum_{l=k+1}^{n-1} R(l, k+1) p(k) \\ &= \sum_{k=n_0}^{n-1} S(l, k+1) p(k). \end{aligned}$$

Hence, we get

$$\|z(n)\| \leqslant \delta \sum_{k=n_0}^{n-1} \|S(l, k+1)\| \leqslant C\delta < \varepsilon.$$

Necessity. Assume that for any $\varepsilon > 0$ there exists $\delta > 0$ such that $\|p(n)\| < \delta$ for $n \geqslant n_0$ imply $\|z(n)\| < \varepsilon$ for all $n \geqslant n_0$. Suppose $\sum_{k=n_0}^{n-1} \|S(n, k+1)\|$ is unbounded. Then there exists an element $S_{ij}(n, k+1)$ and a number $n_1 \in (n_0, \infty)$ such that $\sum_{k=n_0}^{n_1-1} |S_{ij}(n_1, k+1)| > \frac{\varepsilon}{\delta} + 1$. Let

$$p^*(k) = [p_0^*(k), p_1^*(k), \ldots, p_j^*(k), \ldots, p_m^*(k)]^T,$$

be a vector in $\mathbb{R}^m$, where $p_j^*(k) \neq 0,\ p_i^*(k) = 0$ for $i \neq j,\ i = 0, 1, \ldots, m$ and $p_j^*(k) = \delta \operatorname{sgn} S_{ij}(n_1, k+1)$. Then

$$y_j(n_1) = p_j^*(n_1) + \sum_{k=n_0}^{n_1-1} S_{ij}(n_1, k+1)\delta \operatorname{sgn} S_{ij}(n_1, k+1) \geqslant \varepsilon$$

which is a contradiction. This completes the proof. □

Relationships (2.7), (2.11) and (3.10) can be used to formulate sufficient conditions for stability of Eq. (1.1).
We remark that the results obtained for Eq. (1.1) can be extended analogically for a system of the form

$$\Delta x(n) = \sum_{i=1}^{m} A_i(n) x(h_i(n)) + f(n), \quad n \geqslant n_0$$

where $x(n)$ are d-dimensional column vectors, $A_i(n)$ are $d \times d$ matrices and $f(n)$ are d-dimensional column vectors.

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