The Behavior of Solutions of Linear Volterra Difference Equations with Infinite Delay

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Abstract—A class of linear Volterra difference equations with infinite delay is considered. A basic theorem on the behavior of solutions is established and, as a corollary, a useful exponential estimate for solutions is obtained and a stability criterion is derived. These results are achieved via an appropriate positive root of the corresponding characteristic equation. © 2004 Elsevier Ltd. All rights reserved.

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1. INTRODUCTION

Within the past two decades, the theory of difference equations has occupied a central position in the development of mathematics. One reason for this is its natural connectivity with other areas of mathematics. Another one is its widespread applicability in the sciences. For the general background of difference equations, one can refer to the books by Agarwal [1], Elaydi [2], Kelley and Peterson [3], Lakshmikantham and Trigiante [4], and Mickens [5].

Some difference equations of particular interest are the so-called Volterra difference equations (of convolution type), which may be considered as discrete analogues of the famous Volterra integrodifferential equations. Volterra integrodifferential and difference equations have been widely used as mathematical models in population dynamics. For this kind of difference equation, see [2, pp. 239–250]. Although the bibliography on the Volterra integrodifferential equations is quite extended, however there has not yet been analogously much work on the Volterra difference equations. We choose to refer here to the papers by Jaros and Stavroulakis [6], Kiventidis [7], Kordonis and Philos [8], Ladas, Philos and Sficas [9], and Philos [10] for some results concerning the existence and/or the nonexistence of positive solutions of certain linear Volterra difference equations. Also, for some results on the stability of Volterra difference equations, we typically refer to [11,12] (see, also, [2, pp. 239–250]).

This paper deals with the behavior of solutions of linear Volterra difference equations with infinite delay. The results obtained are the discrete analogues of some recent ones given by
Kordonis and Philos [13] on the behavior of solutions of linear integrodifferential equations with unbounded delay. The results in [13] are motivated by the ones due to Driver, Sasser and Slater [14] (see, also, [15]) concerning the behavior of the solutions of first-order linear autonomous delay differential equations. Let us notice that, motivated by the very significant results in [14], some asymptotic and stability results have been obtained by Graef and Qian [16], Kordonis, Niyianni and Philos [17], Philos [18], and Philos and Purnaras [19] for linear delay differential equations and also by Driver, Ladas and Vlahos [20], Kordonis and Philos [21], Kordonis, Philos and Purnaras [22], and Pituk [23] for linear delay difference equations. For some related results we refer to the recent paper by Pituk [24].

Throughout the paper, \( \mathbb{N} \) stands for the set of all nonnegative integers and \( \mathbb{Z} \) stands for the set of all integers. Also, the set of all nonpositive integers will be denoted by \( \mathbb{Z}^- \). Moreover, the forward difference operator \( \Delta \) will be considered to be defined as usual, i.e.,

\[
\Delta s_n = s_{n+1} - s_n, \quad n \in \mathbb{N},
\]

for any sequence \( (s_n)_{n \in \mathbb{N}} \) of real numbers.

Consider the Volterra difference equation with infinite delay

\[
\Delta x_n = ax_n + \sum_{j=-\infty}^{n} K_{n-j} x_j,
\]

where \( a \) is a real number, and \( (K_n)_{n \in \mathbb{N}} \) is a sequence of real numbers which is assumed to be not eventually identically zero.

By a solution of the Volterra difference equation (E), we mean a sequence \( (x_n)_{n \in \mathbb{Z}} \) of real numbers which satisfies (E) for all \( n \in \mathbb{N} \).

In what follows, by \( S \) we will denote the nonempty set of all sequences \( \phi = (\phi_n)_{n \in \mathbb{Z}^-} \) of real numbers such that, for each \( n \in \mathbb{N} \),

\[
\Phi_n \equiv \sum_{j=-\infty}^{0} K_{n-j}\phi_j \text{ exists as a real number.}
\]

It is clear that, for any given initial sequence \( \phi = (\phi_n)_{n \in \mathbb{Z}^-} \) in \( S \), there exists a unique solution \( (x_n)_{n \in \mathbb{Z}} \) of the difference equation (E) which satisfies the initial condition

\[
x_n = \phi_n, \quad \text{for } n \in \mathbb{Z}^-;
\]

this solution \( (x_n)_{n \in \mathbb{Z}} \) is said to be the solution of the initial problem (E),(C) or, more briefly, the solution of (E),(C). In the sequel, for any \( \phi = (\phi_n)_{n \in \mathbb{Z}^-} \) in \( S \), the solution of (E),(C) will be denoted by \( (x_n(\phi))_{n \in \mathbb{Z}} \).

Throughout the paper, it will be supposed that there exists a positive real number \( \gamma \) such that

\[
\sum_{j=0}^{\infty} \gamma^{-j} |K_j| < \infty.
\]

Under this assumption, it is clear that the sequence \( \phi_n = \gamma^n \) for \( n \in \mathbb{Z}^- \) belongs to the set \( S \).

With the difference equation (E) we associate its characteristic equation

\[
\lambda - 1 = a + \sum_{j=0}^{\infty} \lambda^{-j} K_j, \quad (*)
\]

which is obtained by looking for a solution of (E) of the form \( x_n = \lambda^n \) for \( n \in \mathbb{Z} \), where \( \lambda \) is a positive number. It is noteworthy that

\[
0 < \sum_{j=0}^{\infty} \lambda^{-j} |K_j| < \infty, \quad \text{for all } \lambda \geq \gamma.
\]
Our purpose in the present paper is to establish a useful inequality for solutions of the Volterra difference equation (E) and, as a consequence, to obtain an estimate for solutions of the difference equation (E), which leads to a stability criterion for (E). Our results are derived by the use of the unique root of the characteristic equation (*) in the interval $(\gamma, \infty)$, under an appropriate hypothesis on the coefficient $a$, the number $\gamma$, and the kernel $(K_n)_{n \in \mathbb{N}}$. The main results of the paper are stated in Section 2, while their proofs are given in Section 3.

2. STATEMENT OF THE MAIN RESULTS

In order to state our results, let us introduce the following hypothesis:

$$\sum_{j=0}^{\infty} \gamma^{-j} K_j > \gamma - 1 - a \quad \text{and} \quad \sum_{j=1}^{\infty} \gamma^{-j} |K_j| \leq \gamma. \quad (H)$$

We have the next simple result: if (H) holds, then the characteristic equation (*) has a unique root $\lambda_0$ in the interval $(\gamma, \infty)$; this root is such that

$$\frac{1}{\lambda_0} \sum_{j=1}^{\infty} \lambda_0^{-j} |K_j| < 1. \quad (2.1)$$

To show this result, let us define

$$F(\lambda) = \lambda - 1 - a - \sum_{j=0}^{\infty} \lambda^{-j} K_j, \quad \text{for} \ \lambda \geq \gamma.$$ 

From the first inequality of (H) it follows that $F(\gamma) < 0$. Moreover, for every $\lambda \geq \gamma$, we have

$$F(\lambda) \geq \lambda - 1 - a - \sum_{j=0}^{\infty} \lambda^{-j} |K_j| \geq \lambda - 1 - a - \sum_{j=0}^{\infty} \gamma^{-j} |K_j|,$$

which ensures that $F(\infty) = \infty$. Furthermore, by using the second inequality of (H), we obtain for $\lambda > \gamma$

$$F'(\lambda) = 1 + \sum_{j=1}^{\infty} \lambda^{-j-1} j K_j = 1 + \frac{1}{\lambda} \sum_{j=1}^{\infty} \lambda^{-j} j K_j \geq 1 - \frac{1}{\lambda} \sum_{j=1}^{\infty} \lambda^{-j} j |K_j| + 1 - \frac{1}{\gamma} \sum_{j=1}^{\infty} \gamma^{-j} j |K_j| \geq 0,$$

and so $F$ is strictly increasing on $(\gamma, \infty)$. Hence, in the interval $(\gamma, \infty)$, the equation $F(\lambda) = 0$ has a unique root $\lambda_0$. Finally, from the second inequality of (H) it follows that

$$\sum_{j=1}^{\infty} \lambda_0^{-j} j |K_j| < \sum_{j=1}^{\infty} \gamma^{-j} j |K_j| \leq \gamma < \lambda_0,$$

which means that $\lambda_0$ satisfies (2.1). So, the proof is complete.

In what follows, provided that (H) holds, if $\lambda_0$ is the unique root of (*) in the interval $(\gamma, \infty)$, we will denote by $S(\lambda_0)$ the nonempty subset of $S$, which contains all the sequences $\phi = (\phi_n)_{n \in \mathbb{Z}^-}$ in $S$ such that the sequences $(\lambda_0^{-n} \phi_n)_{n \in \mathbb{Z}^-}$ are bounded. It is obvious that the sequence $\phi_n = \lambda_0^{-n}$ for $n \in \mathbb{Z}^-$ is an element of the set $S(\lambda_0)$.

The basic result of this paper is the following theorem.
THEOREM. Assume that (H) holds and let \( \lambda_0 \) be the unique root of (*) in the interval \((\gamma, \infty)\).

Then, for any \( \phi = (\phi_n)_{n \in \mathbb{Z}} \) in \( S(\lambda_0) \), it holds

\[
\left| \lambda_0^{-n} x_n(\phi) - \frac{L(\phi)}{1 + (1/\lambda_0) \sum_{j=1}^{\infty} \lambda_0^{-j} K_j} \right| \leq M(\phi) \left[ \frac{1}{\lambda_0} \sum_{j=1}^{\infty} \lambda_0^{-j} |K_j| \right], \quad \text{for all } n \in \mathbb{N}, \tag{2.2}
\]

where

\[
L(\phi) = \phi_0 + \frac{1}{\lambda_0} \sum_{j=1}^{\infty} \lambda_0^{-j} K_j \left( \sum_{r=-j}^{-1} \lambda_0^{-r} \phi_r \right) \tag{2.3}
\]

and

\[
M(\phi) = \sup_{n \in \mathbb{Z}^-} \left| \lambda_0^{-n} \phi_n - \frac{L(\phi)}{1 + (1/\lambda_0) \sum_{j=1}^{\infty} \lambda_0^{-j} K_j} \right|. \tag{2.4}
\]

NOTE. Inequality (2.1) guarantees that \( 1 + (1/\lambda_0) \sum_{j=1}^{\infty} \lambda_0^{-j} K_j \) is a positive real number. Moreover, by the definition of \( S(\lambda_0) \), from (2.1) it follows that \( L(\phi) \) is a real number. Furthermore, by the definition of \( S(\lambda_0) \), \( M(\lambda_0) \) is finite.

An interesting consequence of the above theorem is a corollary given below. Before stating this corollary, we need two well-known definitions.

The trivial solution of (E) is said to be stable (at 0) if for every \( \epsilon > 0 \) there exists a number \( \delta = \delta(\epsilon) > 0 \) such that, for any initial sequence \( \phi = (\phi_n)_{n \in \mathbb{Z}} \) in \( S \) with

\[
\|\phi\| \equiv \sup_{n \in \mathbb{Z}^-} |\phi_n| < \delta,
\]

it holds

\[
|x_n(\phi)| < \epsilon, \quad \text{for all } n \in \mathbb{Z}.
\]

Moreover, the trivial solution of (E) is called asymptotically stable (at 0) if it is stable (at 0) in the above sense and, in addition, there exists a number \( \delta_0 > 0 \) such that, for any initial sequence \( \phi = (\phi_n)_{n \in \mathbb{Z}} \) in \( S \) with \( \|\phi\| < \delta_0 \), it holds

\[
\lim_{n \to \infty} x_n(\phi) = 0.
\]

COROLLARY. Assume that (H) holds and let \( \lambda_0 \) be the unique root of (*) in the interval \((\gamma, \infty)\).

Then, for any \( \phi = (\phi_n)_{n \in \mathbb{Z}} \) in \( S(\lambda_0) \), it holds

\[
|x_n(\phi)| \leq \Theta N(\phi) \lambda_0^n, \quad \text{for all } n \in \mathbb{N}, \tag{2.5}
\]

where

\[
\Theta = \frac{\left[ 1 + (1/\lambda_0) \sum_{j=1}^{\infty} \lambda_0^{-j} |K_j| \right]^2}{1 + (1/\lambda_0) \sum_{j=1}^{\infty} \lambda_0^{-j} K_j} + \frac{1}{\lambda_0} \sum_{j=1}^{\infty} \lambda_0^{-j} |K_j| \tag{2.6}
\]

and

\[
N(\phi) = \sup_{n \in \mathbb{Z}^-} [\lambda_0^{-n} |\phi_n|]. \tag{2.7}
\]
Moreover, the trivial solution of (E) is stable (at 0) if \( \lambda_0 = 1 \) and it is asymptotically stable (at 0) if \( \lambda_0 < 1 \).

NOTE. Inequality (2.1) guarantees that \( \Theta \) is a real number greater than 1. Also, by the definition of \( S(\lambda_0) \), \( N(\phi) \) is finite.

Before closing this section, we notice that the Volterra difference equation (E) can be considered as the discrete version of the famous Volterra integrodifferential equation with unbounded delay

\[
x'(t) = ax(t) + \int_{-\infty}^{t} K(t-s)x(s) \, ds,
\]

where \( K \) is a continuous real-valued function on the interval \([0, \infty)\) which is not eventually identically zero. The results obtained in this paper should be looked upon as the discrete analogues of the ones given by Kordonis and Philos [13] for solutions of the integrodifferential equation \((E^*)\).

3. PROOFS OF THE MAIN RESULTS

PROOF OF THEOREM. Consider an arbitrary initial sequence \( \phi = (\phi_n)_{n \in \mathbb{Z}} \) in \( S(\lambda_0) \) and define

\[
y_n = \lambda_0^{-n} x_n(\phi), \quad \text{for } n \in \mathbb{Z}.
\]

Then, for every \( n \in \mathbb{N} \), we have

\[
\Delta x_n(\phi) - ax_n(\phi) - \sum_{j=-\infty}^{n} K_{n-j} x_j(\phi) \equiv \Delta x_n(\phi) - ax_n(\phi) - \sum_{j=0}^{\infty} K_j x_{n-j}(\phi) \\
= \Delta(\lambda_0^n y_n) - a\lambda_0^n y_n - \lambda_0^n \sum_{j=0}^{\infty} K_j \lambda_0^{-j} y_{n-j} \\
= \lambda_0^n [\lambda_0 \Delta y_n + (\lambda_0 - 1)y_n] - a\lambda_0^n y_n - \lambda_0^n \sum_{j=0}^{\infty} \lambda_0^{-j} K_j y_{n-j} \\
= \lambda_0^n \left[ \lambda_0 \Delta y_n + (\lambda_0 - 1 - a)y_n - \sum_{j=0}^{\infty} \lambda_0^{-j} K_j y_{n-j} \right].
\]

So, by taking into account the fact that \( \lambda_0 \) is a (positive) root of the characteristic equation \((*)\), we obtain for \( n \in \mathbb{N} \)

\[
\Delta x_n(\phi) - ax_n(\phi) - \sum_{j=-\infty}^{n} K_{n-j} x_j(\phi) = \lambda_0^n \left[ \lambda_0 \Delta y_n + \left( \sum_{j=0}^{\infty} \lambda_0^{-j} K_j \right) y_n - \sum_{j=0}^{\infty} \lambda_0^{-j} K_j y_{n-j} \right] \\
= \lambda_0^n \left[ \lambda_0 \Delta y_n + \sum_{j=0}^{\infty} \lambda_0^{-j} K_j (y_n - y_{n-j}) \right] \\
= \lambda_0^n \left[ \lambda_0 \Delta y_n + \sum_{j=1}^{\infty} \lambda_0^{-j} K_j (y_n - y_{n-j}) \right].
\]

Thus, \( (x_n(\phi))_{n \in \mathbb{Z}} \) is a solution of (E) if and only if \( (y_n)_{n \in \mathbb{Z}} \) satisfies

\[
\Delta y_n = \frac{1}{\lambda_0} \sum_{j=1}^{\infty} \lambda_0^{-j} K_j (y_n - y_{n-j}), \quad \text{for all } n \in \mathbb{N}.
\]

Moreover, we observe that the initial condition (C) can be written in the following equivalent form:

\[
y_n = \lambda_0^{-n} \phi_n, \quad \text{for } n \in \mathbb{Z}^-.
\]

(3.2)
Furthermore, by using (2.3) and (3.2), we can easily verify that (3.1) is equivalent to
\[ y_n = \frac{1}{\lambda_0} \sum_{j=1}^{\infty} \lambda_0^{-j} K_j \left( \sum_{r=n-j}^{n-1} y_r \right) + L(\phi), \quad \text{for } n \in \mathbb{N}. \] (3.3)

Next, we set
\[ z_n = y_n - \frac{L(\phi)}{1 + \left(1/\lambda_0\right) \sum_{j=1}^{\infty} \lambda_0^{-j} K_j}, \quad \text{for } n \in \mathbb{Z}. \]

Then we can immediately verify that (3.3) reduces to the following equivalent equation:
\[ z_n = -\frac{1}{\lambda_0} \sum_{j=1}^{\infty} \lambda_0^{-j} K_j \left( \sum_{r=n-j}^{n-1} z_r \right), \quad \text{for } n \in \mathbb{N}. \] (3.4)

Moreover, the initial condition (3.2) can equivalently be written
\[ z_n = \lambda_0^{-n} \phi_n - \frac{L(\phi)}{1 + \left(1/\lambda_0\right) \sum_{j=1}^{\infty} \lambda_0^{-j} K_j}, \quad \text{for } n \in \mathbb{Z}. \] (3.5)

Because of the definitions of \((y_n)_{n \in \mathbb{Z}}\) and \((z_n)_{n \in \mathbb{Z}}\), (2.2) can be written as follows:
\[ |z_n| \leq M(\phi) \left[ \frac{1}{\lambda_0} \sum_{j=1}^{\infty} \lambda_0^{-j} K_j \right], \quad \text{for all } n \in \mathbb{N}, \] (3.6)

where \(M(\phi)\) is defined by (2.4). So, the proof of our theorem can be accomplished by showing that (3.6) holds true.

Now, in view of (2.4) and (3.5), we have
\[ |z_n| \leq M(\phi), \quad \text{for } n \in \mathbb{Z}. \] (3.7)

We will show that the constant \(M(\phi)\) is a bound of the sequence \((z_n)_{n \in \mathbb{Z}}\), i.e.,
\[ |z_n| \leq M(\phi), \quad \text{for every } n \in \mathbb{Z}. \] (3.8)

To this end, let us consider an arbitrary number \(\epsilon > 0\). We claim that
\[ |z_n| < M(\phi) + \epsilon, \quad \text{for all } n \in \mathbb{Z}. \] (3.9)

Otherwise, because of (3.7), there exists an integer \(n_0 > 0\) such that
\[ |z_n| < M(\phi) + \epsilon, \quad \text{for } n \in \mathbb{Z}, \quad \text{with } n \leq n_0 - 1, \]

and
\[ |z_{n_0}| \geq M(\phi) + \epsilon. \]

Then, by taking into account (2.1), from (3.4) we obtain
\[ M(\phi) + \epsilon \leq |z_{n_0}| = \frac{1}{\lambda_0} \left| \sum_{j=1}^{\infty} \lambda_0^{-j} K_j \left( \sum_{r=n_0-j}^{n_0-1} z_r \right) \right| \]
\[ \leq \frac{1}{\lambda_0} \sum_{j=1}^{\infty} \lambda_0^{-j} |K_j| \left( \sum_{r=n_0-j}^{n_0-1} |z_r| \right) \]
\[ \leq \left[ \frac{1}{\lambda_0} \sum_{j=1}^{\infty} \lambda_0^{-j} |K_j| \left[ M(\phi) + \epsilon \right] \right]< M(\phi) + \epsilon, \]
which is a contradiction. So, our claim is true. Since (3.9) is satisfied for all numbers \( \epsilon > 0 \), it follows that (3.8) holds true. Finally, in view of (3.8), from (3.4) we derive for all \( n \in \mathbb{N} \)

\[
|z_n| = \frac{1}{\lambda_0} \left| \sum_{j=1}^{\infty} \lambda_0^{-j} K_j \left( \sum_{r=n-j}^{n-1} z_r \right) \right|
\leq \frac{1}{\lambda_0} \sum_{j=1}^{\infty} \lambda_0^{-j} |K_j| \left( \sum_{r=n-j}^{n-1} |z_r| \right)
\leq \left[ \frac{1}{\lambda_0} \sum_{j=1}^{\infty} \lambda_0^{-j} |K_j| \right] M(\phi),
\]
i.e., (3.6) is satisfied. So, the proof of the theorem is complete.

**Proof of Corollary.** Let the sequence \( \phi = (\phi_n)_{n\in\mathbb{Z}} \) be an initial sequence in \( S(\lambda_0) \). Then our theorem guarantees that (2.2) is satisfied, where \( L(\phi) \) and \( M(\phi) \) are defined by (2.3) and (2.4), respectively. From (2.2) we immediately obtain

\[
\lambda_0^{-n} |x_n(\phi)| \leq \frac{|L(\phi)|}{1 + (1/\lambda_0) \sum_{j=1}^{\infty} \lambda_0^{-j} j K_j} + M(\phi) \left[ \frac{1}{\lambda_0} \sum_{j=1}^{\infty} \lambda_0^{-j} |K_j| \right], \quad \text{for } n \in \mathbb{N}. \tag{3.10}
\]

On the other hand, (2.4) yields

\[
M(\phi) \leq N(\phi) + \frac{|L(\phi)|}{1 + (1/\lambda_0) \sum_{j=1}^{\infty} \lambda_0^{-j} j K_j},
\]

where \( N(\phi) \) is defined by (2.7). Thus, (3.10) gives

\[
\lambda_0^{-n} |x_n(\phi)| \leq \frac{1 + (1/\lambda_0) \sum_{j=1}^{\infty} \lambda_0^{-j} j |K_j|}{1 + (1/\lambda_0) \sum_{j=1}^{\infty} \lambda_0^{-j} j K_j} |L(\phi)| + N(\phi) \left[ \frac{1}{\lambda_0} \sum_{j=1}^{\infty} \lambda_0^{-j} |K_j| \right], \tag{3.11}
\]

for \( n \in \mathbb{N} \).

But, from (2.3) one can obtain

\[
|L(\phi)| \leq |\phi_0| + \frac{1}{\lambda_0} \sum_{j=1}^{\infty} \lambda_0^{-j} |K_j| \left( \sum_{r=-j}^{-1} \lambda_0^{-r} |\phi_r| \right),
\]

which leads to

\[
|L(\phi)| \leq \left[ 1 + \frac{1}{\lambda_0} \sum_{j=1}^{\infty} \lambda_0^{-j} j |K_j| \right] N(\phi).
\]

Thus, from (3.11) it follows immediately that

\[
\lambda_0^{-n} |x_n(\phi)| \leq \left\{ \frac{1 + (1/\lambda_0) \sum_{j=1}^{\infty} \lambda_0^{-j} j |K_j|}{1 + (1/\lambda_0) \sum_{j=1}^{\infty} \lambda_0^{-j} j K_j} + \frac{1}{\lambda_0} \sum_{j=1}^{\infty} \lambda_0^{-j} |K_j| \right\} N(\phi),
\]

for every \( n \in \mathbb{N} \). This means that (2.5) is always satisfied, where \( \Theta \) is defined by (2.6).
Now, let us suppose that $\lambda_0 \leq 1$. Let $\phi = (\phi_n)_{n \in \mathbb{Z}^-}$ be an arbitrary initial sequence in $S$, which is assumed to be bounded. Define
$$\|\phi\| = \sup_{n \in \mathbb{Z}^-} |\phi_n|.$$ We can immediately see that $\phi \in S(\lambda_0)$ and
$$N(\phi) \leq \|\phi\|.$$ Hence, from (2.5) it follows that
$$|x_n(\phi)| \leq \Theta \|\phi\| \lambda_0^n, \quad \text{for all } n \in \mathbb{N}. \quad (3.12)$$ Since $\lambda_0 \leq 1$, (3.12) guarantees that
$$|x_n(\phi)| \leq \Theta \|\phi\|, \quad \text{for every } n \in \mathbb{N}.$$ So, because of the fact that $\Theta > 1$, we always have
$$|x_n(\phi)| \leq \Theta \|\phi\|, \quad \text{for all } n \in \mathbb{Z}. \quad (3.13)$$ Thus, we have proved that, for any bounded initial sequence $\phi = (\phi_n)_{n \in \mathbb{Z}^-}$ in $S$, (3.12) and (3.13) are satisfied. From (3.13) it follows that the trivial solution of (E) is stable (at 0). Moreover, if $\lambda_0 < 1$, then (3.12) gives
$$\lim_{n \to \infty} x_n(\phi) = 0,$$ which guarantees that the trivial solution of (E) is asymptotically stable (at 0).

The proof of the corollary is now complete.

REFERENCES