Continuity of convex functions

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March 28, 2019

Definition 1 *Given an arbitrary set $X \subset \mathbb{R}^n$, we define its convex hull as the set*

$$\operatorname{conv}(X) = \{\alpha_1 x^1 + \cdots + \alpha_r x^r \mid x^i \in X,\ \alpha_i \geq 0,\ \alpha_1 + \cdots + \alpha_r = 1,\ r \in \mathbb{N}\}.$$

It can be proved that this set is indeed convex and it is the intersection of all convex sets that contain X. Figures 1 and 2 show some sets and their corresponding convex hulls.

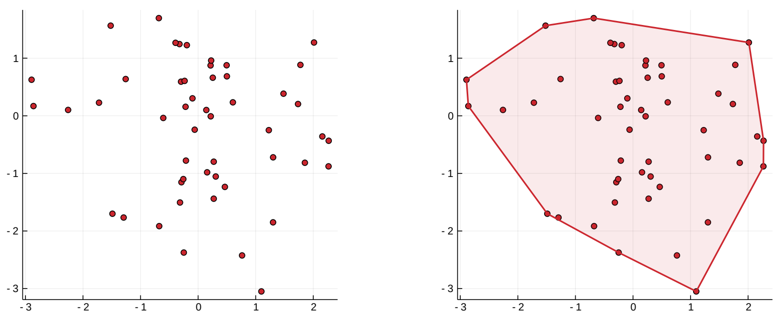

Figure 1: On the left, a finite set with 50 random points in $\mathbb{R}^2$. On the right, its convex hull.

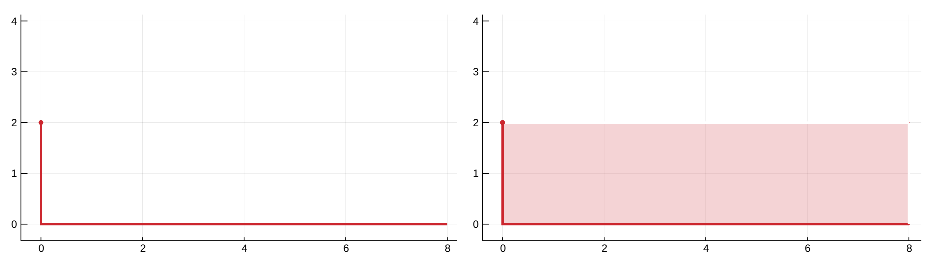

Figure 2: The set $X = \{x \in \mathbb{R}^2 \mid x_1 \geq 0,\ 0 \leq x_2 \leq 2,\ x_1 x_2 = 0\}$ and its convex hull. This figure illustrates the fact that the convex hull of a closed set is not necessarily closed.

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Lemma 2 *Let $A \subset \mathbb{R}^n$ and $B \subset \mathbb{R}^m$ be arbitrary sets. Then*

$$\operatorname{conv}(A \times B) = \operatorname{conv}(A) \times \operatorname{conv}(B).$$

Proof. Note first that if $\begin{pmatrix} x \\ y \end{pmatrix} \in \operatorname{conv}(A \times B)$, then

$$\begin{pmatrix} x \\ y \end{pmatrix} = \sum_{i=1}^{k} \gamma_i \begin{pmatrix} a^i \\ b^i \end{pmatrix} = \begin{pmatrix} \sum \gamma_i a^i \\ \sum \gamma_i b^i \end{pmatrix},$$

with $\gamma_i \geq 0$, $\sum \gamma_i = 1$, $a^i \in A$ and $b^i \in B$. This means that $\begin{pmatrix} x \\ y \end{pmatrix} \in \operatorname{conv}(A) \times \operatorname{conv}(B)$.

Conversely, if $\begin{pmatrix} x \\ y \end{pmatrix} \in \operatorname{conv}(A) \times \operatorname{conv}(B)$, we have

$$x = \sum_{i=1}^{p} \alpha_i a^i \quad \text{and} \quad y = \sum_{j=1}^{q} \beta_j b^j$$

with $\alpha_i, \beta_j \geq 0$, $\sum \alpha_i = \sum \beta_j = 1$, $a^i \in A$ and $b^j \in B$. Therefore,

$$\sum_{i=1}^{p} \sum_{j=1}^{q} \alpha_i \beta_j \begin{pmatrix} a^i \\ b^j \end{pmatrix} = \sum_{i=1}^{p} \alpha_i \left(\sum_{j=1}^{q} \begin{pmatrix} \beta_j a^i \\ \beta_j b^j \end{pmatrix} \right) = \sum_{i=1}^{p} \alpha_i \begin{pmatrix} a^i \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$$

and

$$\sum_{i=1}^{p} \sum_{j=1}^{q} \alpha_i \beta_j = \sum_{i=1}^{p} \alpha_i \sum_{j=1}^{q} \beta_j = 1.$$

Since $\alpha_i \beta_j \geq 0$, we have $\begin{pmatrix} x \\ y \end{pmatrix} \in \operatorname{conv}(A \times B)$. □

Corollary 3 *Considering the sets $A = \{-1, 1\}$ and $C = \{u \in \mathbb{R}^n \mid u_j \in A\} = A \times \cdots \times A$, there holds* $\operatorname{conv}(C) = B_{\|\cdot\|_\infty}[0, 1]$.

Lemma 4 *Let $D \subset \mathbb{R}^n$ be a convex set and let $g : D \to \mathbb{R}$ be a convex function. Then, given $\alpha_i \geq 0$, $i = 1, \ldots, m$, with $\sum \alpha_i = 1$, and $x^i \in D$, we have*

$$g\left(\sum_{i=1}^{m} \alpha_i x^i \right) \leq \sum_{i=1}^{m} \alpha_i g(x^i).$$

Proof. Note first that if $\alpha_1 = 1$, there is nothing to prove. So, assume that $\alpha_1 < 1$ and write

$$\sum_{i=1}^{m} \alpha_i x^i = \alpha_1 x^1 + \beta \sum_{i=2}^{m} \frac{\alpha_i}{\beta} x^i,$$

where $\beta = \sum_{i=2}^{m} \alpha_i = 1 - \alpha_1$. Thus, the convexity of g implies that

$$g\left(\sum_{i=1}^{m} \alpha_i x^i \right) \leq \alpha_1 g(x^1) + \beta g\left(\sum_{i=2}^{m} \frac{\alpha_i}{\beta} x^i \right).$$

An induction argument finishes the proof. □

Theorem 5 *Consider $D = B_{\|\cdot\|_\infty}[0,1] \subset \mathbb{R}^n$ and suppose that $g : D \to \mathbb{R}$ is convex. Then g is bounded and continuous at the origin.*

Proof. Let $M = \max\{g(u) \mid u \in C\}$, where C is the (finite) set given in Corollary 3. Given $y \in D$, this corollary allows us to write $y = \sum \alpha_i u^i$ with $\alpha_i \geq 0$, $\sum \alpha_i = 1$ and $u^i \in C$. So, by Lemma 4 we have

$$g(y) \leq \sum \alpha_i g(u^i) \leq \sum \alpha_i M = M. \tag{1}$$

Moreover, since $0 = \left(1 - \frac{1}{2}\right) y + \frac{1}{2}(-y)$ and $-y \in D$, we conclude that

$$g(0) \leq \left(1 - \frac{1}{2}\right) g(y) + \frac{1}{2} g(-y),$$

which in turn implies that

$$g(y) \geq 2g(0) - g(-y) \geq 2g(0) - M.$$

Thus, using (1), we have the boundedness of g.

Now, consider a sequence $(y^k) \subset D$ such that $y^k \to 0$. Since we want to prove that $g(y^k) \to g(0)$, we may assume, without loss of generality, that $y^k \neq 0$ for all $k \in \mathbb{N}$. So, defining $t_k = \|y^k\|_\infty$, $z^k = \dfrac{y^k}{t_k}$ and $w^k = -\dfrac{y^k}{t_k}$, we have

$$y^k = (1 - t_k)0 + t_k z^k \quad \text{and} \quad 0 = \frac{1}{1 + t_k} y^k + \frac{t_k}{1 + t_k} w^k.$$

By the convexity of g, we obtain

$$g(y^k) \leq (1 - t_k)g(0) + t_k g(z^k) \quad \text{and} \quad g(0) \leq \frac{1}{1 + t_k} g(y^k) + \frac{t_k}{1 + t_k} g(w^k),$$

which implies

$$(1 + t_k)g(0) - t_k g(w^k) \leq g(y^k) \leq (1 - t_k)g(0) + t_k g(z^k).$$

Since $t_k \to 0$, using the boundedness of the sequences $\big(g(z^k)\big)$ and $\big(g(w^k)\big)$ and taking limits, we conclude that $g(y^k) \to g(0)$. □

Theorem 6 *Let $X \subset \mathbb{R}^n$ be a convex set with nonempty interior and let $f : X \to \mathbb{R}$ be a convex function. If $\bar{x} \in \operatorname{int}(X)$, then f is continuous at $\bar{x}$.*

Proof. Consider $\delta > 0$ such that $B_{\|\cdot\|_\infty}[\bar{x}, \delta] \subset X$ and define $g : B_{\|\cdot\|_\infty}[0,1] \to \mathbb{R}$ by $g(y) = f(\bar{x} + \delta y)$. We claim that g is convex. Indeed, given $y, z \in B_{\|\cdot\|_\infty}[0,1]$ and $t \in [0,1]$, we have

$$g\big((1 - t)y + tz\big) = f\big((1 - t)(\bar{x} + \delta y) + t(\bar{x} + \delta z)\big) \leq (1 - t)g(y) + tg(z).$$

Now, take $(x^k) \subset X$ such that $x^k \to \bar{x}$. Assume, without loss of generality, that $x^k \in B_{\|\cdot\|_\infty}[\bar{x}, \delta]$ for all $k \in \mathbb{N}$. So, $y^k = \dfrac{x^k - \bar{x}}{\delta} \in B_{\|\cdot\|_\infty}[0,1]$ and $y^k \to 0$, which by Theorem 5 imply that

$$f(x^k) = f(\bar{x} + \delta y^k) = g(y^k) \to g(0) = f(\bar{x}).$$

□