Pseudo-differential operators

Exercises 2 - 23.03.16

1. Find a function $f \in L^1(\mathbb{R}^n)$ such that $\hat{f} \notin L^1(\mathbb{R}^n)$.

2. Let $p(x) = \sum_{|\alpha|\leq m} c_\alpha x^\alpha$ be a polynomial on $\mathbb{R}^n$. Calculate $\mathscr{F}[p]$ in the sense of $\mathcal{S}'(\mathbb{R}^n)$.

3. For any $x \in \mathbb{R}^n$, let $\delta_x \in \mathcal{S}'(\mathbb{R}^n)$ the delta distribution in x.
   (a) Prove that δ_x is not a regular distribution;
   (b) Show that $\delta_x * g = \tau_{-x} g$, for any $g \in \mathcal{S}(\mathbb{R}^n)$;
   (c) Find $\mathscr{F}[\delta_x]$.

4. Let $f \in \mathcal{S}'(\mathbb{R}^n)$ and $g \in \mathcal{S}(\mathbb{R}^n)$. Prove that:
   (a) $f * g$ is a regular tempered distribution with $f * g \in C^\infty_{poly}(\mathbb{R}^n)$;
   (b) $\mathscr{F}(f * f) = \hat{f} \cdot \hat{g}$;
   (c) $\partial_x^\alpha (f * g) = (\partial_x^\alpha f) * g = f * (\partial_x^\alpha g)$.

5. Consider the PDE $(1 - \Delta)u(x) = f(x)$ in $\mathbb{R}^n$, with $u, f \in \mathcal{S}(\mathbb{R}^n)$.
   (a) Find $p(\xi)$ such that $(1 - \Delta)u(x) = \mathscr{F}^{-1}[p(\xi)\hat{u}(\xi)]$;
   (b) Show that for every $f \in \mathcal{S}(\mathbb{R}^n)$ there is a unique solution $u = (1 - \Delta)^{-1} f \in \mathcal{S}(\mathbb{R}^n)$ of the PDE.
   (c) Show that $(1 - \Delta)^{-1} \in \mathscr{L}(\mathcal{S}(\mathbb{R}^n))$;

6. Let $p : \mathbb{R}^n \to \mathbb{C}$ be a measurable function and set $\Sigma \doteq \overline{p(\mathbb{R}^n)}$. For each $F \in L^\infty(\Sigma)$ let

$$F(p(D_x))f \doteq \mathscr{F}^{-1}[F(p(\xi))\hat{f}(\xi)], \ \forall f \in L^2(\mathbb{R}^n)$$

   Prove that:
   (a) For all $f \in L^\infty(\mathbb{R}^n)$, we have $F(p(D_x))f \in \mathscr{L}(L^2(\mathbb{R}^n))$;
   (b) The mapping $\Phi : L^\infty(\mathbb{R}^n) \to \mathscr{L}(L^2(\mathbb{R}^n))$, defined as above, is linear and

$$F(p(D_x)) \circ G(p(D_x)) = (F \cdot G)(p(D_x)), \ \forall F, G \in L^\infty(\mathbb{R}^n);$$

   (c) For all $\lambda \in \mathbb{C} \setminus \Sigma$ we have $(\lambda - p(D_x))^{-1} \in \mathscr{L}(L^2(\mathbb{R}^n))$

$$(\lambda - p(D_x))(\lambda - p(D_x))^{-1} f = (\lambda - p(D_x))^{-1}(\lambda - p(D_x)) f = f, \ \forall f \in \mathcal{S}(\mathbb{R}^n)$$

   where $p(D_x)f = \mathscr{F}^{-1}[p(\xi)\hat{f}(\xi)]$, for all $f \in \mathcal{S}(\mathbb{R}^n)$
   (d) For which $\lambda \in \mathbb{C}$ there exists $(\lambda - \Delta)^{-1}$ in the sense above?