Prove that $pG$.

For each $3$. Let $2$. Prove that if $p$.

Let $\alpha, \beta, k \in \mathbb{R}$.

Exercise 3 - 04.04.16

1. Let $p \in S_{1,0}^{m}(\mathbb{R}^n \times \mathbb{R}^n_{\xi})$ and $q \in S_{1,0}^{\ell}(\mathbb{R}^n \times \mathbb{R}^n_{\xi})$ be pseudo-differential symbols, with $\ell, m \in \mathbb{R}$. Prove that $pq \in S_{1,0}^{m+\ell}(\mathbb{R}^n \times \mathbb{R}^n_{\xi})$ and that for every $k \in \mathbb{N}_0$ one has

$$|pq|_{k}^{m+\ell} \leq C_{k} |p|_{k}^{m} |q|_{k}^{\ell},$$

where $C_{k}$ depends only on $k$ and $n$.

2. Prove that if $p \in S_{1,0}^{m}(\mathbb{R}^n \times \mathbb{R}^n_{\xi})$ then $D_x^\alpha D_\xi^\beta p \in S_{1,0}^{m-|\beta|}(\mathbb{R}^n \times \mathbb{R}^n_{\xi})$, for all multi-indices $\alpha$ and $\beta$, and that for every $k \in \mathbb{N}_0$ one has

$$|D_x^\alpha D_\xi^\beta p|_{k}^{m-|\beta|} \leq C |p|_{k+|\alpha|+|\beta|}^{m},$$

where $C$ depends only on $\alpha, \beta, k$ and $n$.

3. Let $p \in S_{1,0}^{m}(\mathbb{R}^n \times \mathbb{R}^n_{\xi})$ and $\phi \in S(\mathbb{R}^n_{\xi})$. Prove that the function

$$r(x, \varphi) = p(x, \varphi) \cdot \varphi(\xi), \quad x, \xi \in \mathbb{R}^n,$

is a symbol in $S_{1,0}^{-\infty}(\mathbb{R}^n \times \mathbb{R}^n_{\xi})$.

4. Prove that $p \in S_{1,0}^{m}$ if and only if $(\xi)^{-m}p \in S_{1,0}^{0}$.

5. For each $G \in L^{\infty}(\mathbb{R}^n)$ set

$$G(D_x)f \doteq \mathcal{F}^{-1}[G(\xi)\hat{f}(\xi)], \forall f \in L^2(\mathbb{R}^n).$$

(a) Prove that $G(D_x) \in \mathcal{L}(L^2(\mathbb{R}^n))$ and the mapping

$$\Phi : L^{\infty}(\mathbb{R}^n) \rightarrow \mathcal{L}(L^2(\mathbb{R}^n))
\quad G \mapsto \Phi(G) \doteq G(D_x)$$

is linear and bounded. Moreover, show that for every $G, H \in L^{\infty}(\mathbb{R}^n)$, we have

$$G(D_x) \circ H(D_x) = (G \cdot H)(D_x).$$

(b) Prove that if $G \in C_{\text{poly}}^{\infty}(\mathbb{R}^n)$, then $G(D_x) : \mathcal{S}(\mathbb{R}^n) \rightarrow \mathcal{S}(\mathbb{R}^n)$ (defined similarly as in (1)), is a bounded operator, and for $G_j \in C_{\text{poly}}^{\infty}(\mathbb{R}^n), j = 1, 2$, Equation (2) holds as well.

(c) Let $p \in C_{\text{poly}}^{\infty}(\mathbb{R}^n)$. Prove that for all $\lambda \in \mathbb{C} \setminus \overline{p(\mathbb{R}^n)}$ we have $(\lambda - p(D_x))^{-1} \in \mathcal{L}(L^2(\mathbb{R}^n))$ and

$$(\lambda - p(D_x))(\lambda - p(D_x))^{-1}f = (\lambda - p(D_x))^{-1}(\lambda - p(D_x))f = f$$

for all $f \in \mathcal{S}(\mathbb{R}^n)$, where $p(D_x)f = \mathcal{F}^{-1}[p(\xi)\hat{f}(\xi)]$ for all $f \in \mathcal{S}(\mathbb{R}^n)$.

(d) For which $\lambda \in \mathbb{C}$ there exists $(\lambda - \Delta)^{-1}$ in the sense above?