Pseudo-differential operators

Exercises 4 - 02.05.16

1. Prove that a pseudo-differential operator has a unique formal adjoint.

2. Let $p(x,\xi)$ and $q(x,\xi)$ be any two symbols. Prove that:

   (a) $(p^*(x,D_x))^* = p(x,D_x)$;

   (b) $(p(x,D_x)q(x,D_x))^* = q^*(x,D_x)p^*(x,D_x)$.

3. Let $P(x,D_x) = \sum_{|\alpha|\le m} a_\alpha(x)D^\alpha$ and $Q(x,D_x) = \sum_{|\beta|\le \ell} b_\beta(x)D^\beta$ operators with $a_\alpha, b_\beta \in C_b^\infty(\mathbb{R}^n)$.

   (a) Compute the symbol of product $P(x,D_x)Q(x,D_x)$ directly;

   (b) Compute the symbol of the formal adjoint of $P(x,D)$ directly;

   (c) Compare you answers with the symbols obtained by theorems given in the theory of ΨDO.

4. Let $p \in S^m_{1,0}(\mathbb{R}^n_x \times \mathbb{R}^n_\xi)$, $m \in \mathbb{R}$, be any symbol. Show that for all $x,\xi \in \mathbb{R}^n$ we have

$$e^{-ix\cdot\xi}(p(x,D_x)e^{ix\cdot\xi})(x) = p(x,\xi).$$

5. Fixed $y \in \mathbb{R}^n$, define the operator $\tau_y : C_b^\infty(\mathbb{R}^n) \to C_b^\infty(\mathbb{R}^n)$ by $\tau_y u(x) = u(x-y), \forall x \in \mathbb{R}^n$.

   (a) Use the last exercise to compute the symbol of this operator.

   (b) Does $\tau(x,\xi) \in S^m_{1,0}(\mathbb{R}^n_x \times \mathbb{R}^n_\xi)$ for some $m \in \mathbb{R}$?

6. Let $p \in S^m_{1,0}(\mathbb{R}^n_x \times \mathbb{R}^n_\xi)$ and $v \in \mathcal{S}(\mathbb{R}^n)$. Prove that

$$w(\xi) \doteq \int_{\mathbb{R}^n_x} e^{-ix\cdot\xi}(p(x,\xi)v(x)dx \in \mathcal{S}(\mathbb{R}^n).$$