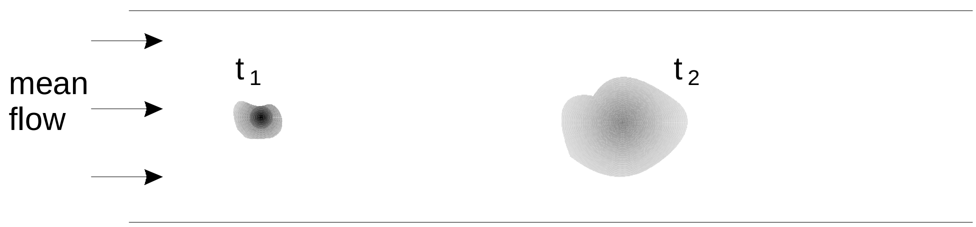
MASS TRANSPORT PROCESSES

When a tracer cloud is introduced into a fluid flow (e.g. river) two processes are visible:

- The cloud is carried away from the point of discharge by the mean current.
- The cloud spreads in all directions due to irregular motions and grows in size.

Advection - mass transport by mean velocity field

Diffusion - mass transport by irregular fluctuations ("random movements")



PASSIVE MIXING PROCESSES - advection + diffusion as they exist in environment (passive source)

ACTIVE MIXING PROCESSES - generation of mean and random velocity field by source - momentum or buoyancy

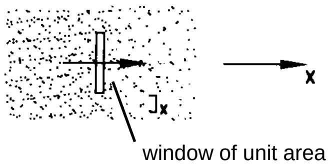
passive mixing
(weak source)

Active mixing
(strong source e.g. high velocity)

MOLECULAR DIFFUSION

– molecular diffusion of dissolved or suspended matter is caused by random movement through molecular motions of individual molecules

$$\frac{\partial c}{\partial x} < 0$$



$$J_x = \text{diffusive mass flux} = \frac{\text{mass}}{\text{area, time}} = \left[\frac{M}{L^2, T}\right]$$

Fick's Law of Diffusion:

– rate of transport is proportional to the spatial concentration gradient $\frac{\partial c}{\partial x}$

$$\boxed{J_x = -D_{AB}\frac{\partial c}{\partial x}}$$ Fick´s law (1-D)

$$D_{AB} = \text{coefficient of molecular diffusivity} = \left[\frac{L^2}{T}\right] = f \;\; \text{(solvent, solute, temperature...)}$$

Typical values for molecular diffusion:

Dissolved matter in water: **$D \sim 2 \times 10^{-5}$ cm²/s**
Gases in air: **$D \sim 2 \times 10^{-1}$ cm²/s**

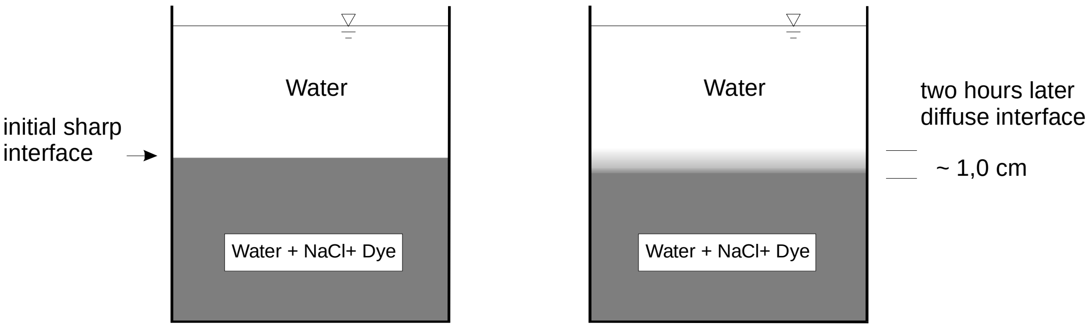
Generalization:

$$\vec{J} = (J_x, J_y, J_z)$$

$$\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$$ divergence vector

$$\boxed{\vec{J} = -D\,\nabla C}$$ Fick´s law (3-D)

Example: Salt diffusion in a vessel

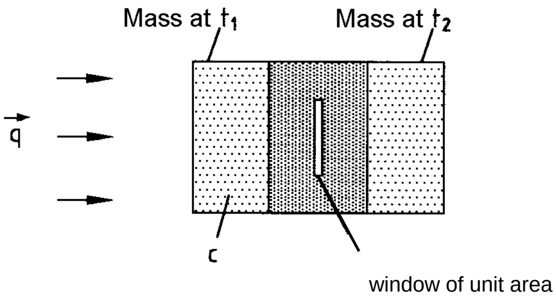


ADVECTIVE MASS FLUX

– transport of matter due to mean motion (current)

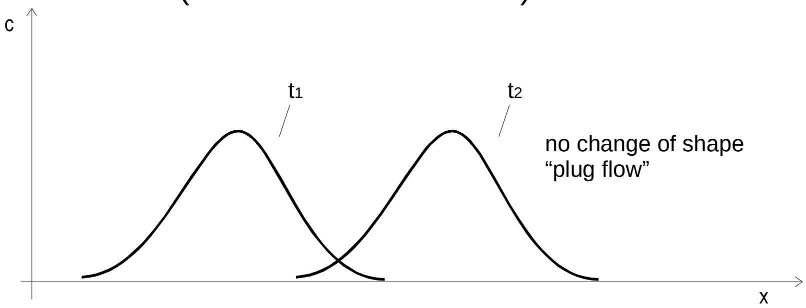
$\vec{q} = (u, v, w) =$ hydrodynamic velocity

$c\,\vec{q} = \left[\dfrac{M}{L^3}\dfrac{L}{T}\right] = \left[\dfrac{M}{L^2, T}\right] =$ advective mass flux



Example:

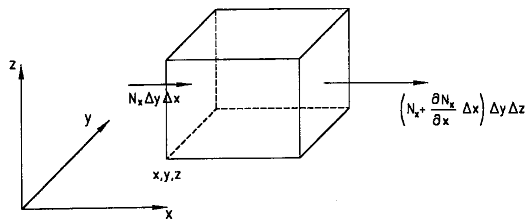
Pure advection in x-direction (no diffusion no reaction)



TOTAL MASS FLUX: Advection + diffusion

$$\vec{N} = (N_x, N_y, N_z) = c\,\vec{q} + \vec{J} = c\,\vec{q} - D\nabla C$$

MASS CONSERVATION PRINCIPLE



R = rate of production or decay per unit volume = $\left[\frac{M}{L^3, T}\right]$

$$\begin{bmatrix}\text{net flux of mass} \\ \text{(In – Out)}\end{bmatrix} + \begin{bmatrix}\text{Rate of production/} \\ \text{decay of mass} \\ \text{(chemical, physical,} \\ \text{biological...)}\end{bmatrix} = \begin{bmatrix}\text{Rate of mass} \\ \text{accumulation} \\ \text{within element}\end{bmatrix}$$

$$-\frac{\partial N_x}{\partial x}\Delta x\Delta y\Delta z - \frac{\partial N_y}{\partial y}\Delta x\Delta y\Delta z - \frac{\partial N_z}{\partial z}\Delta x\Delta y\Delta z + R\,\Delta x\Delta y\Delta z = \frac{\partial c}{\partial t}\Delta x\Delta y\Delta z$$

$$\frac{\partial c}{\partial t} + \frac{\partial N_x}{\partial x} + \frac{\partial N_y}{\partial y} + \frac{\partial N_z}{\partial z} = \frac{\partial c}{\partial t} + \nabla \cdot \vec{N} = \frac{\partial c}{\partial t} + \nabla \cdot c\vec{q} - \nabla \cdot \left(D \nabla c\right) = R$$

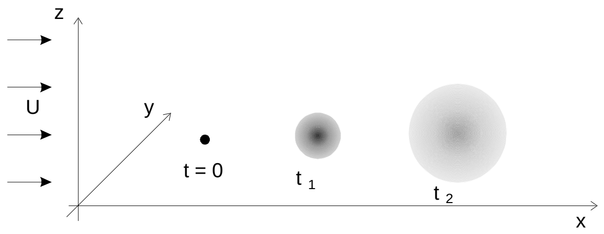
$$\boxed{\frac{\partial c}{\partial t} + \vec{q} \cdot \nabla c = D \nabla^2 c + R}$$

Convective Diffusion Equation

$$\underbrace{\frac{\partial c}{\partial t}}_{\text{Temporal change}} + \underbrace{u\frac{\partial c}{\partial x} + v\frac{\partial c}{\partial y} + w\frac{\partial c}{\partial z}}_{\text{Advective transport}} = \underbrace{D\left(\frac{\partial^2 c}{\partial x^2} + \frac{\partial^2 c}{\partial y^2} + \frac{\partial^2 c}{\partial z^2}\right)}_{\text{Diffusive transport}} + \underbrace{R}_{\text{Reaction}}$$

Solution of the convective diffusion equation:

1. Basic solution: Instantaneous point source (IPtS) in uniform current and infinite space



$\vec{q} = (U, 0,0)$, (uniform velocity field)
$R = -kc$ (first order decay)

$$\frac{\partial c}{\partial t} + U\frac{\partial c}{\partial x} = D\left(\frac{\partial^2 c}{\partial x^2} + \frac{\partial^2 c}{\partial y^2} + \frac{\partial^2 c}{\partial z^2}\right) - kc$$

Initial Conditions (I.C.) $t < 0$, $c = 0$ $t = 0$ Sudden mass input M on (x_1, y_1, z_1)

Boundary Conditions (B.C.) $c \to 0$ as $(x, y, z) \to \infty$

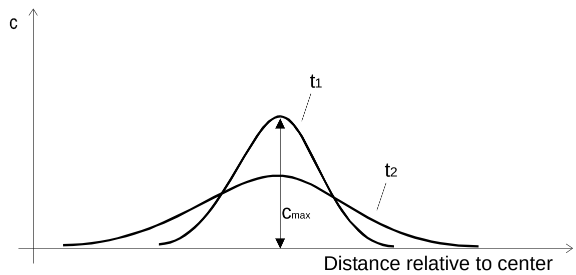
Solution:

$$c = \underbrace{\frac{M}{(4\pi D t)^{3/2}}}_{c_{max}} \underbrace{e^{-\frac{[(x-x_1)-Ut]^2 + (y-y_1)^2 + (z-z_1)^2}{4Dt}}}_{\text{spatial distribution}} \underbrace{e^{-kt}}_{\text{non-conservative (reaction) effect}}$$

1. Position of cloud center:
   $(x_1 + Ut, y_1, z_1)$

2. Cloud size:
   $4Dt = 2\sigma^2$, σ = variance
   $\sigma = \sqrt{2Dt}$ = standard deviation ~ $t^{0,5}$

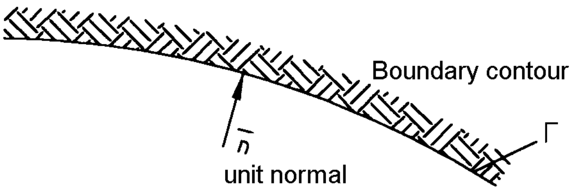
3. Maximum concentration
   $c_{max} \sim t^{-3/2}$



2. Other solutions

Boundary conditions (finite space):

on Γ: c = specified



Ex: c = 0 Absorption condition (Dirichlet B.C.)

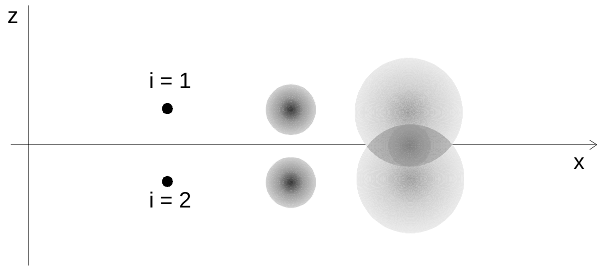
$$\frac{\partial c}{\partial n} = \text{specified}$$

Ex: $\frac{\partial c}{\partial n} = 0$ Diffusion barrier (Reflection condition)

(Neumann B.C.)

Multiple sources → superposition, due to linearity of governing equation
→ image sources

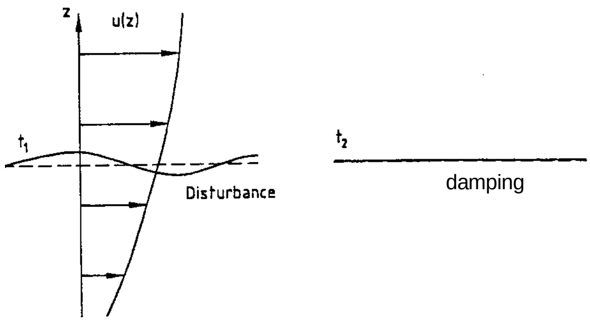
$$c = \sum_{i=1}^{k} c_i$$



Time dependence
Initial condition:

- Instantaneous sources
- Continuous sources

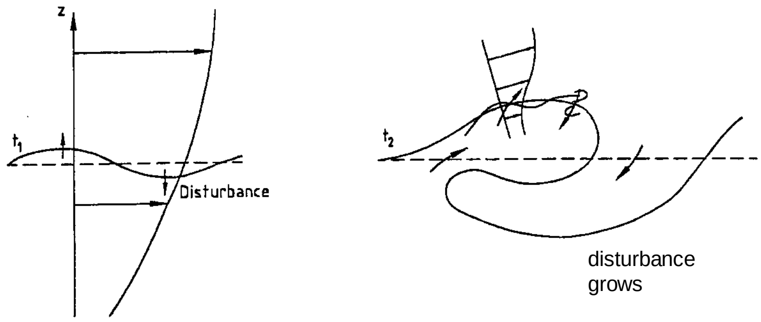
TURBULENCE - an instability of a base flow (shear flow)



Possibility 1:

- damping of disturbance (viscosity)
- stable flow

LAMINAR FLOW



Possibility 2:

- insufficient damping
- large eddies
- secondary eddies

TURBULENT FLOW

SPECTRUM OF EDDIES

Large eddies: Integral length scale ℓ_I Integral velocity scale u_I

=> Integral time scale $t_I = \dfrac{\ell_I}{u_I}$

Extraction of energy from mean flow: $\varepsilon \sim \dfrac{K.E.}{time} \sim \dfrac{u_I^{\,2}}{\ell_I / u_I} = \dfrac{u_I^{\,3}}{\ell_I} = \left[\dfrac{L^2}{T^3}\right]$

Energy cascade (transfer of energy from larger to smaller scales)

Smallest eddies: at some scale (Kolmogorov length scale) the eddy size will be small enough so that viscous damping will prevent further breakdown

Dimensionality: viscosity $\nu = \left[\dfrac{L^2}{T}\right]$, energy dissipation rate $\varepsilon = \left[\dfrac{L^2}{T^3}\right]$

Kolmogorov length scale $\eta = \dfrac{\nu^{3/4}}{\varepsilon^{1/4}}$

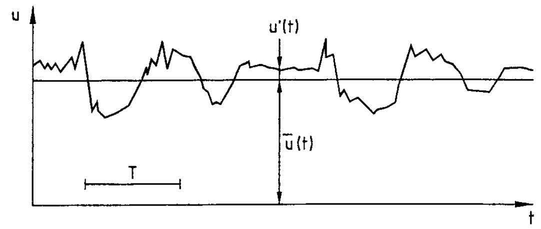
Turbulence spectrum: $\ell_I \xrightarrow[\text{"equilibrium subrange"}]{\varepsilon} \eta$

Overall criterion: Reynolds number $Re = \dfrac{UL}{\nu} > 100 \text{ to } 1000$

U ~ base velocity, L ~ base flow length

TURBULENT DIFFUSION

Random motion due to turbulent velocity fluctuations cause additional transport ("small scale advection").



$u'(t) =$ fluctuating velocity

$\overline{u}(t) = \frac{1}{T}\int_{t}^{t-T} u(t')\,dt' =$ mean velocity

$\overline{u'} = \frac{1}{T}\int_{t}^{t-T} u'(t')\,dt' = 0$

$u = \overline{u} + u'$ $\quad w = \overline{w} + w'$

$v = \overline{v} + v'$ $\quad c = \overline{c} + c'$

Averaging process:
Reynolds decomposition

$\overline{uc} = \overline{\overline{u}\,\overline{c}} + \overline{\overline{u}c'} + \overline{u'\overline{c}} + \overline{u'c'}$

$$\frac{1}{T}\int_{t}^{t+T}\left[\frac{\partial c}{\partial t} + \frac{\partial uc}{\partial x} + \frac{\partial vc}{\partial y} + \frac{\partial wc}{\partial z} = D\left(\frac{\partial^2 c}{\partial x^2} + \frac{\partial^2 c}{\partial y^2} + \frac{\partial^2 c}{\partial z^2}\right) + R\right]dt$$

$$\frac{\partial \overline{c}}{\partial t} + \overline{u}\frac{\partial \overline{c}}{\partial x} + \overline{v}\frac{\partial \overline{c}}{\partial y} + \overline{w}\frac{\partial \overline{c}}{\partial z} = -\frac{\partial}{\partial x}\overline{u'c'} - \frac{\partial}{\partial y}\overline{v'c'} - \frac{\partial}{\partial z}\overline{w'c'} + D\left(\frac{\partial^2 \overline{c}}{\partial x^2} + \frac{\partial^2 \overline{c}}{\partial y^2} + \frac{\partial^2 \overline{c}}{\partial c^2}\right) + \overline{R} \quad \downarrow$$

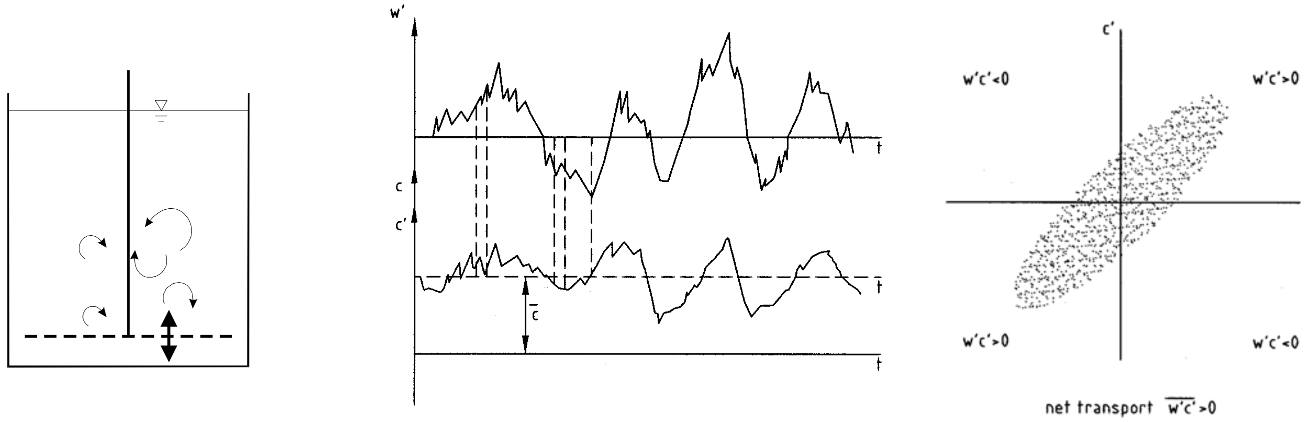
$\downarrow$

time change of mean conc.	advective transport by mean velocity	"small scale advection" ⇓ turbulent mass transport	molecular diffusion ≈ 0	reac-tion

$\vec{J}_t = \left(\overline{u'c'}, \overline{v'c'}, \overline{w'c'}\right) =$ turbulent diffusive mass flux $= \left[\frac{M}{L^2, T}\right]$

$\vec{J}_t$ needs parameterization : analogy to molecular diffusion (Fick's law) ?

Example: Tank with Grid Stirring

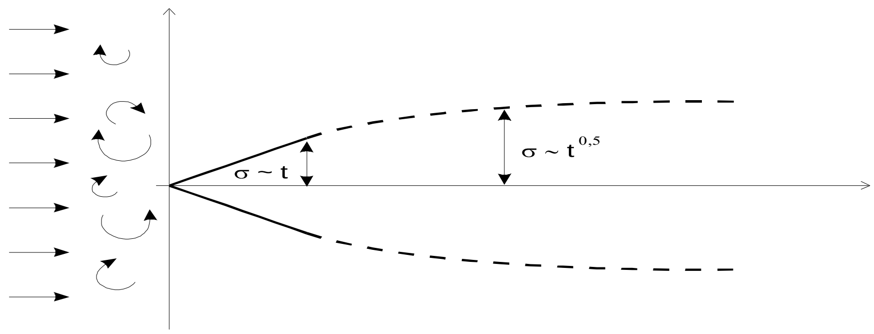


net transport $\overline{w'c'} > 0$

DIFFUSION IN TURBULENT FLOW

→ fluctuating velocity field ⇒ multiple scale eddies (spectrum)!

Ex: Release from POINT SOURCE



a) Short time after release $t<t_I$, $\sigma < \ell_I$
First small eddies, then eddies of increasing size will cause spreading
$\sigma_x = \sqrt{\overline{u'^2}}\; t \sim t^1$ Rapid growth (Taylor theorem, 1921)
$\sqrt{\overline{u'^2}} = \text{rms velocity} \sim u_I$

b) Long time after release $t>t_I$: eddies are independent of each other
Large eddies control spreading => random process
$\sigma_x = \sqrt{2\left(u_I^2 t_I\right)t} \sim t^{1/2}$ "Fickian" behavior: corresponds to spreading with gradient-type diffusion and a constant coefficient

∴ <u>Analogy:</u> $u_I^2 t_I = u_I \ell_I = E_x = \text{turbulent diffusivity} = \left[\frac{L^2}{T}\right]$

$$J_{tx} = \overline{u'c'} = -E_x \frac{\partial \overline{c}}{\partial x}, \qquad J_{ty} = \overline{v'c'} = -E_y \frac{\partial \overline{c}}{\partial y}, \qquad J_{tz} = \overline{w'c'} = -E_z \frac{\partial \overline{c}}{\partial z}$$

PRACTICAL TURBULENT DIFFUSION EQ.: $\bar{c} \rightarrow c$

$$\frac{\partial c}{\partial t} + u\frac{\partial c}{\partial x} + v\frac{\partial c}{\partial y} + w\frac{\partial c}{\partial z} = \underbrace{\frac{\partial}{\partial x}\left(E_x \frac{\partial c}{\partial x}\right) + \frac{\partial}{\partial y}\left(E_y \frac{\partial c}{\partial y}\right) + \frac{\partial}{\partial z}\left(E_z \frac{\partial c}{\partial z}\right)}_{\text{turbulent diffusive transport}} + R + D\nabla^2 c$$

Valid for $t > t_I$; $\sigma > \ell_I$

Diffusivities can have dependence on spatial position and direction:

- $E_x, E_y, E_z = f(x, y, z)$ non-homogeneous

if $E_x, E_y, E_z =$ const. homogeneous $\rightarrow E_x \frac{\partial^2 c}{\partial x^2} + E_y \frac{\partial^2 c}{\partial y^2} + E_x \frac{\partial^2 c}{\partial z^2}$

- $E_x \neq E_y \neq E_z$ anisotropic

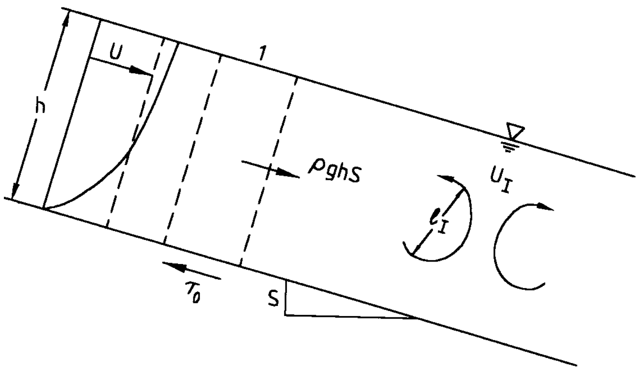
if $E_x = E_y = E_z = E$ isotropic $\rightarrow E\left(\frac{\partial^2 c}{\partial x^2} + \frac{\partial^2 c}{\partial y^2} + \frac{\partial^2 c}{\partial z^2}\right)$

Estimation of turbulent diffusivities:

E = f (base flow characteristics)

Order of magnitude:
$E \sim u_I \ell_I$ single scale only!

TURBULENT CHANNEL FLOW



$Re = \frac{Uh}{\nu} > 500$ for turbulence

$\tau_0 = \rho g h S =$ shear stress

$\sqrt{\frac{\tau_0}{\rho}} = u_* =$ friction (shear) velocity
$u_* = \sqrt{ghS}$

Dominating scales:
$u_I \sim u_*, \ell_I \sim h$

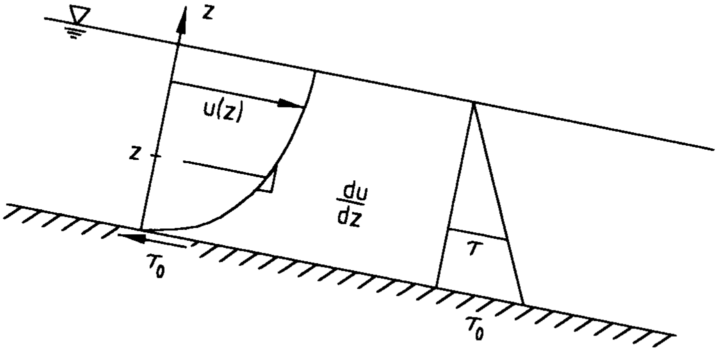
Also: $\tau_0 = \rho \frac{f}{8} U^2$ f = Darcy-Weisbach coefficient, 0.02 .. 0.1 $u_* = \sqrt{\frac{f}{8} U^2}$

Or: $U = \frac{1}{n} h^{2/3} S^{1/2}$ n = Mannings's n, 0.02 ... 0.05 $u_* = \sqrt{\frac{gn^2}{h^{1/3}} U^2}$

Rule of thumb: $u_* \approx (0.05 \text{ to } 0.1)U$

MIXING IN RIVERS

Eddy diffusivity E_z



We expect: $E_z \sim u_* h$

Velocity profile

$$\frac{du}{dz} = f(u_*, z)$$

Dimensional analysis:

$$\frac{du}{dz} \sim \frac{u_*}{z} \quad \text{or} \quad \frac{du}{dz} = \frac{u_*}{\kappa z}$$

Upon integration:

$$\frac{u}{u_*} = \frac{1}{\kappa}\ell nz + C_1 \qquad \text{log-law}$$

κ = von Karman constant = 0.4

C_1 = f (Re, roughness)

$$\tau = \tau_0\left(1 - \frac{z}{h}\right) = \rho\,\varepsilon_z \frac{du}{dz}$$

model for momentum exchange (Boussinesq)

$$\varepsilon_z = \frac{\tau_0\left(1 - \frac{z}{h}\right)}{\rho \frac{du}{dz}} = \kappa\, u_*\left(1 - \frac{z}{h}\right)$$

ε_z = eddy viscosity = $\left[\frac{L^2}{T}\right]$

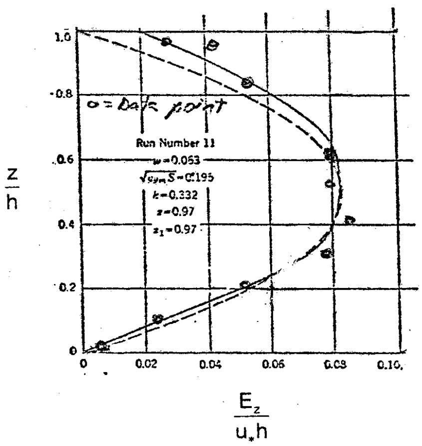
$$\frac{\varepsilon_z}{u_* h} = \kappa \frac{z}{h}\left(1 - \frac{z}{h}\right) \equiv \frac{E_z}{u_* h}$$

Reynolds analogy for turbulent flows

Mass exchange ≈ momentum exchange

Depth averaged:

$$\frac{\overline{E}_z}{u_* h} = \frac{\kappa}{6} = 0.067 \approx 0.1$$



Rules of thumb:

River U, h $\quad u_* \approx 0.1U$

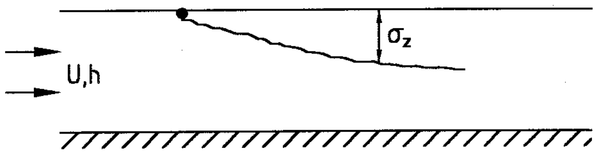
$E_2 \approx 0.1 u_* h \approx 0.01Uh$

Ex:

U = 2 m/s, h = 2m

E_z = 0.04 m²/s = 400 cm²/s >> D = 2 × 10^{-5} cm/s !!

VERTICAL MIXING



$$\sigma_z = \sqrt{2E_z t} = \sqrt{2E_z \frac{x}{U}}$$

Complete vertical mixing: 2,15 $\sigma_z = h$ 10% criterion with Gaussian profile

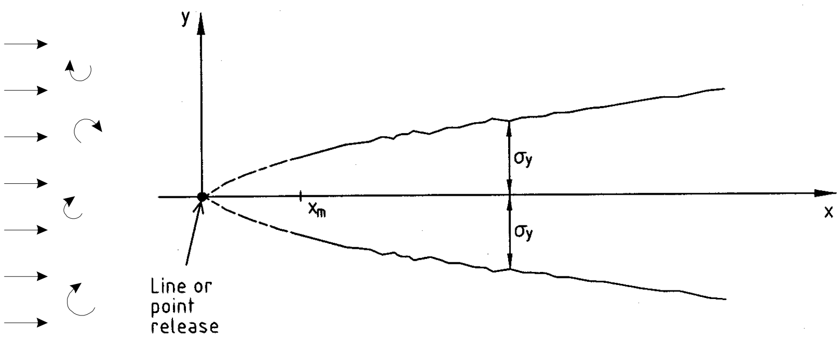
$$\frac{h}{2.15} = \sqrt{2\,E_z \frac{x_m}{U}} \quad \Rightarrow \quad x_m = \left(\frac{h}{2.15}\right)^2 \frac{U}{2E_z}$$

Ex: $x_m = \left(\frac{2}{2.15}\right)^2 \frac{2}{2 \times 0.04} = 25\,\text{m}$ Very fast

Typically: Distance to complete vertical mixing 10 to 20 water depth!

LATERAL MIXING

Wide Channel, Depth h



Conditions:

$x > x_m$
$\sigma_y > h$

Assume isotropy: $E_y \cong \overline{E}_z = 0.067\,u_* h$

Data: $\frac{E_y}{u_* h} \cong 0.15$ to 0.20 straight, uniform channels

Evaluation of field / laboratory tests:

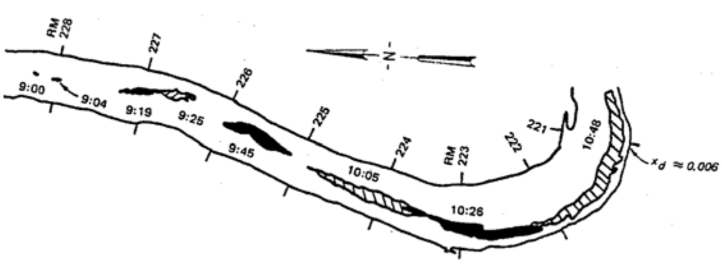
$$\frac{1}{2}\frac{d\sigma_y^{\,2}}{dt} = E_y \cong \frac{1}{2}\frac{\sigma^2{}_y(t_2) - \sigma^2{}_y(t_1)}{t_2 - t_1}$$

or

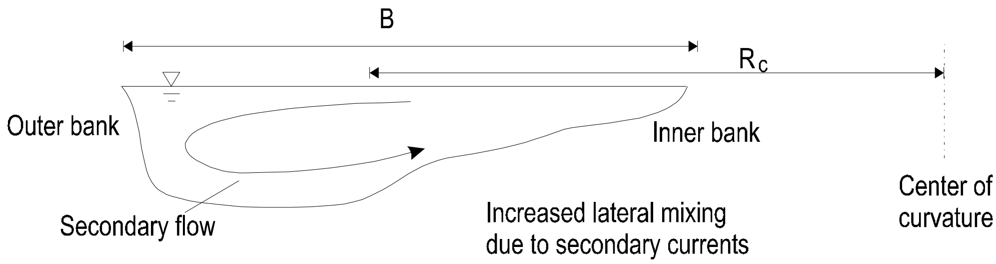
$$\frac{1}{2}\frac{\sigma^2{}_y(x_2) - \sigma^2{}_y(x_1)}{(x_2 - x_1)/U}$$

Outlines of tracer cloud

Data from Mississippi River study on longitudinal mixing (after McQuivey and Keefer 1976b)



NATURAL CHANNELS; IRREGULARITIES

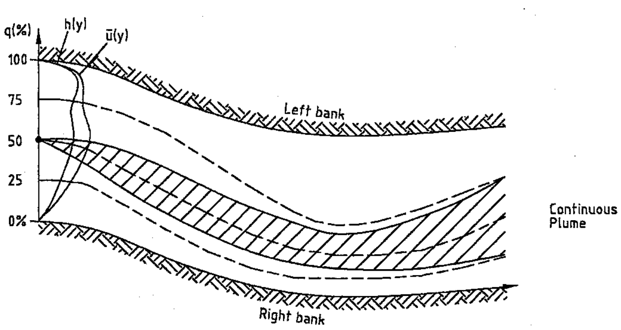


Data: $\dfrac{E_y}{u_* h} = 0.5 \text{ to } 10 = \alpha$ $\quad \alpha = 0.6 \pm 50\%$ (Fischer (1972)

$$\alpha = 0.4\left(\frac{UB}{u_* R_c}\right)^2$$ Yotsukura and Sayre (1976)

Cumulative Discharge Method

Yotsukura and Cobb (1976)



$dq = h\bar{u}dy =$ local discharge $\rightarrow$ from measurement or numerical model

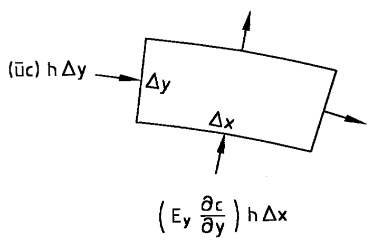
$$q(y) = \int_0^y h\bar{u}dy = \text{cumulative discharge}$$

$$Q = \int_0^B h\bar{u}dy = \text{ total discharge}$$

Approximate local balance:

$$\bar{u}(y)\, h(y)\, \frac{\partial c}{\partial x} = \frac{\partial}{\partial y}\left(E_y(y) h(y) \frac{\partial c}{\partial y}\right)$$

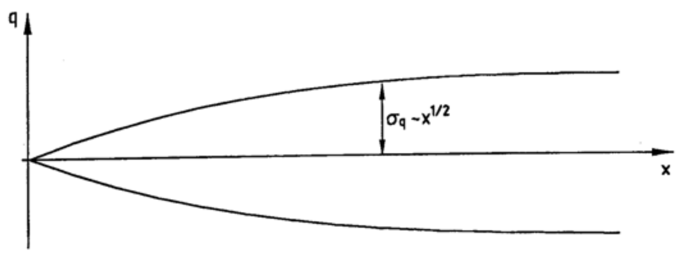
$\downarrow$ depth integrated advection $\quad\quad$ $\downarrow$ depth integrated lateral diffusion

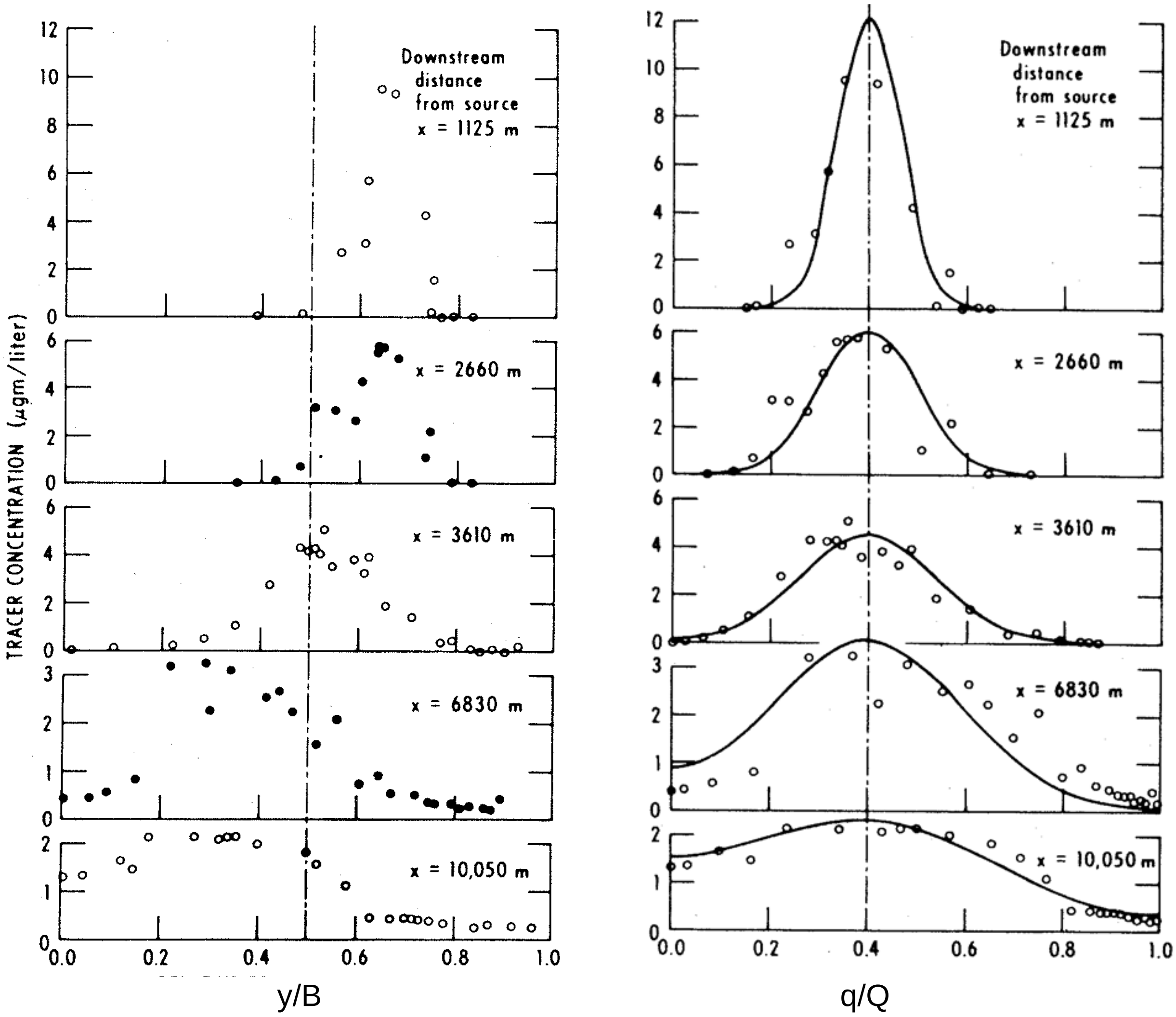


$$\bar{u}h\frac{\partial c}{\partial x} = \bar{u}h\frac{\partial}{\partial q}\left(h^2 E_y \bar{u}\frac{\partial c}{\partial q}\right) \quad \Rightarrow \quad \frac{\partial c}{\partial x} = \frac{\partial}{\partial q}\left(h^2 E_y \bar{u}\frac{\partial c}{\partial q}\right)$$

"Diffusivity" = $\left[\dfrac{L^5}{T^2}\right] \approx$ constant

Understanding of advective velocity field is key to many environmental diffusion problems!

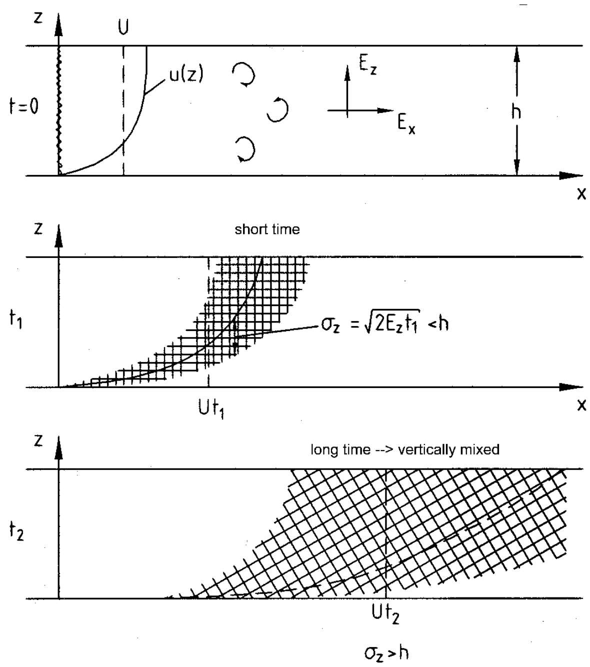


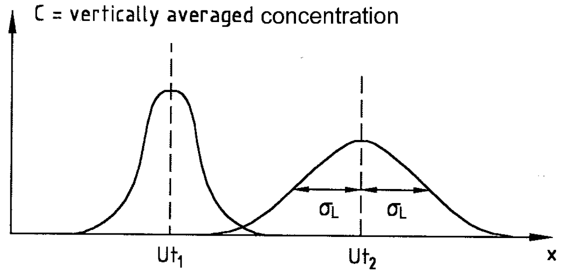


Transverse distributions of dye observed in the Missouri River near Blair, Nebraska, by Yotsukura et al. /1970), plotted (a) versus actual distance across the stream and (b) versus relative cumulative discharge [Yotsukura and Sayre (1976)].

LONGITUDINAL DISPERSION (SHEAR FLOW DISPERSION)

The stretching of a tracer cloud due to vertical and horizontal velocity shear is called longitudinal dispersion. In this case, the velocity differences interact with the transverse diffusion effect and produce an additional transport mechanism





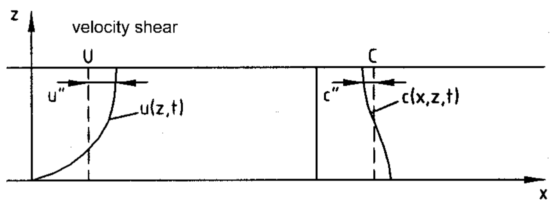
Observations: $t > t_{mixing}$ or $\sigma_z > h$

For long time:

1. Gaussian distribution
2. $\sigma_L \sim \sqrt{t}$

Therefore: behavior is analogous to Fickian diffusion

Analysis:



$$\frac{\partial c}{\partial t} + u(z,t)\frac{\partial c}{\partial x} = \frac{\partial}{\partial x}\left(E_x \frac{\partial c}{\partial z}\right) + \frac{\partial}{\partial z}\left(E_z \frac{\partial c}{\partial z}\right) \qquad \text{2-D}$$

Spatial decomposition: $u = U + u''$, $c = C + c''$, $u'', c'' =$ spatial deviations

Substitute and average $\frac{1}{h}\int_0^h (\ )\,dz$ $\qquad$ U, C = spatial averages

$$\frac{\partial C}{\partial t} + U\frac{\partial C}{\partial x} + \frac{\partial \overline{u''c''}}{\partial x} = \frac{\partial}{\partial x}\left(E_x \frac{\partial C}{\partial x}\right) + \underbrace{\frac{1}{h}\left(E_z \frac{\partial c}{\partial z}\right)\Bigg|_0^h}$$

$$\overline{u''c''} = \frac{1}{h}\int_0^h u''c''\,dz \qquad\qquad = 0 \text{ no flux}$$

$$J_L = \overline{u''c''} = \text{dispersive mass flux} = \left[\frac{M}{L^2,T}\right]$$

$$J_L = \overline{u'' c''} = \text{dispersive mass flux} = \left[\frac{M}{L^2, T}\right]$$

Analogy: $J_L = -E_L \dfrac{\partial C}{\partial x}$ E_L = coeff. of longitudinal dispersion = $\left[\dfrac{L^2}{T}\right]$

– neglect: long. diffusion, $E_x << E_L$

$$\frac{\partial C}{\partial t} + U\frac{\partial C}{\partial x} = E_L \frac{\partial^2 C}{\partial x^2} \qquad \text{1-D}$$

↓ change of mean conc.

↓ advection mean velocity

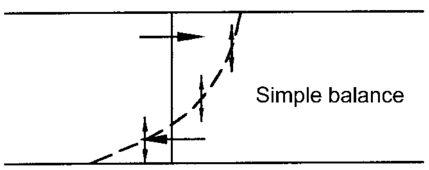
↓ spreading by velocity deviations + transverse diffusion

Local balance as seen by moving observer (U):

$$u''\frac{\partial C}{\partial x} = \frac{\partial}{\partial z}\left(E_z \frac{\partial c''}{\partial z}\right)$$

Taylor (1953, 1954)

↓ transport in x direction by differential velocities ⇔ ↓ diffusion in vertical direction by turbulence



"Stretching" ⇔ "Homogenization"

With given properties: $u''(z), E_z(z)$ one can compute $c'' \sim \dfrac{\partial C}{\partial x}$,

and evaluate $\overline{u'' c''}$

This leads to:

$$E_L = -\frac{1}{h}\int_0^h u'' \int_0^z \frac{1}{E_z}\int_o^z u''\, dz\, dz\, dz$$

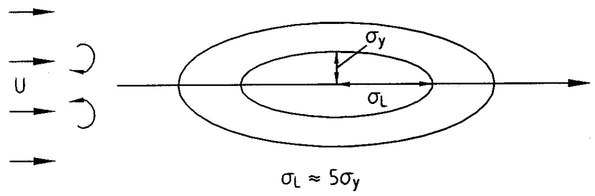
Applications: 2-D Flows

1. Channel flows:

$u'' \sim \log-\text{profile}$

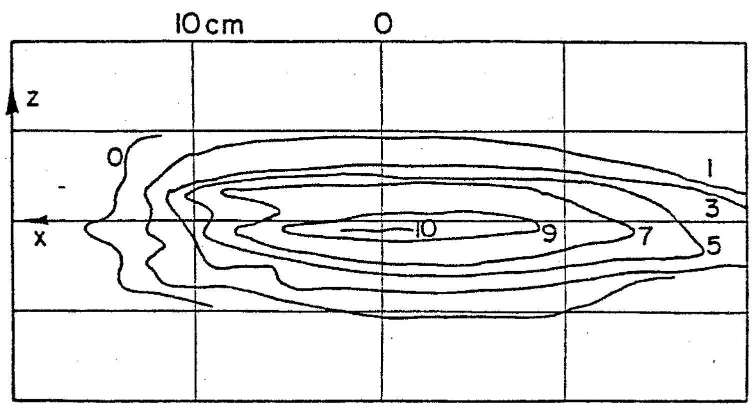
$E_L = 5.9\, u_* h$ Elder (1959)

$E_Z \sim$ parabolic



Compare: $E_y \approx E_x = 0.2\, u_* h$

$E_L >> E_x$



Plan view of a drop of dye diffusing in the turbulent flow in an open channel. Distribution of concentration C, normalized to have a maximum of 10. The flow is to the left; h = 1.43 cm, x/h = 90 (Elder, 1959)

2. Turbulent pipe flow:

$E_L = 10.1\, u_* r_o$ Taylor (1954)

r_o = pipe radius

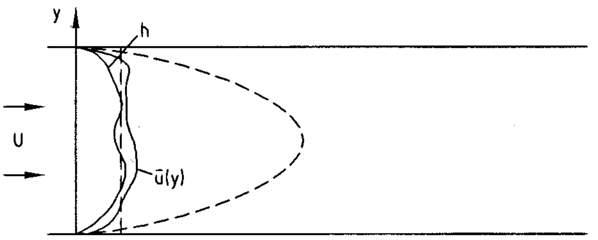
3. Laminar pipe flow:

$$E_L^{mol} = \frac{r^2{}_o U^2}{48 D}$$ Taylor (1953)

General rule for applicability: $t > t_M$ = transverse mixing time (vertical)

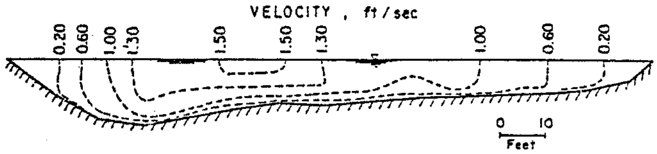
$$= 0.4 \frac{h^2}{E_t}$$ E_t = transverse diffusivity

Natural Rivers Data: $\frac{E_L}{u_* h} = 100 \text{ to } 1000$ 3-D non-uniformities



Lateral velocity deviations dominate

$\rightarrow$ Stretching
+
Lateral (transverse) diffusion



Typical cross section of a natural stream (Fischer 1968)

Fischer (1965) $\bar{u}(y), E_y(y)$, A = cross-section

$$E_L = -\frac{1}{A}\int_0^B (\bar{u}-U)h\int_0^y \frac{1}{E_y h}\int_0^y (\bar{u}-U)h \, dy dy dy$$

$\downarrow$ $\downarrow$

velocity deviation in lateral direction y

u(y): measurements or numerical model

- approximate formula (typical conditions):

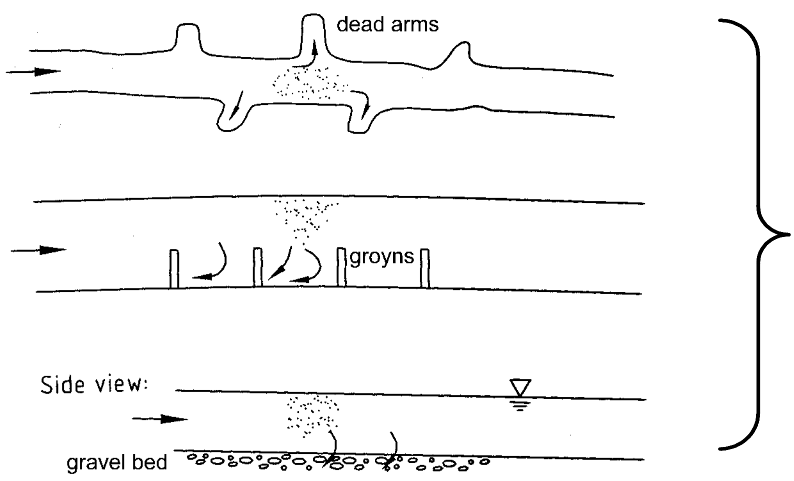
$$E_L = 0.011\frac{U^2 B^2}{u_* \bar{h}}$$

good within factor 4 ($\sigma_L \sim$ within 2)

Applicability: $t > t_m$ = transverse (lateral mixing time

$$= 0.4\frac{(B/2)^2}{E_y}$$

Other physical non-uniformities:



Overall effects:

- stretching of "cloud"
- increased longitudinal dispersion $E_L \uparrow$

Practical longitudinal dispersion equation

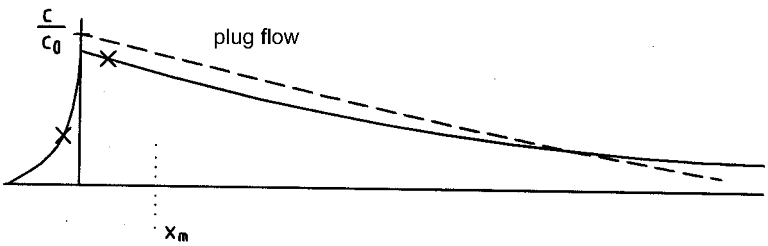
$$\frac{\partial C}{\partial t} + U\frac{\partial C}{\partial x} = E_L \frac{\partial^2 C}{\partial x^2} - kC$$

$t > t_M$

(often neglected; but careful!)

Instantaneous Area Source: $C = \frac{m''}{\sqrt{4\pi E_L t}} \exp - \left[\frac{(x-Ut)^2}{4E_L t}\right] \exp[-kt]$

Continuous Area source: $C = \frac{q''}{u\sqrt{1+\frac{4kE_L}{u^2}}} \exp\left[\frac{xU}{2E_L}\left(1 \mp \sqrt{1+\frac{4kE_L}{U^2}}\right)\right]$



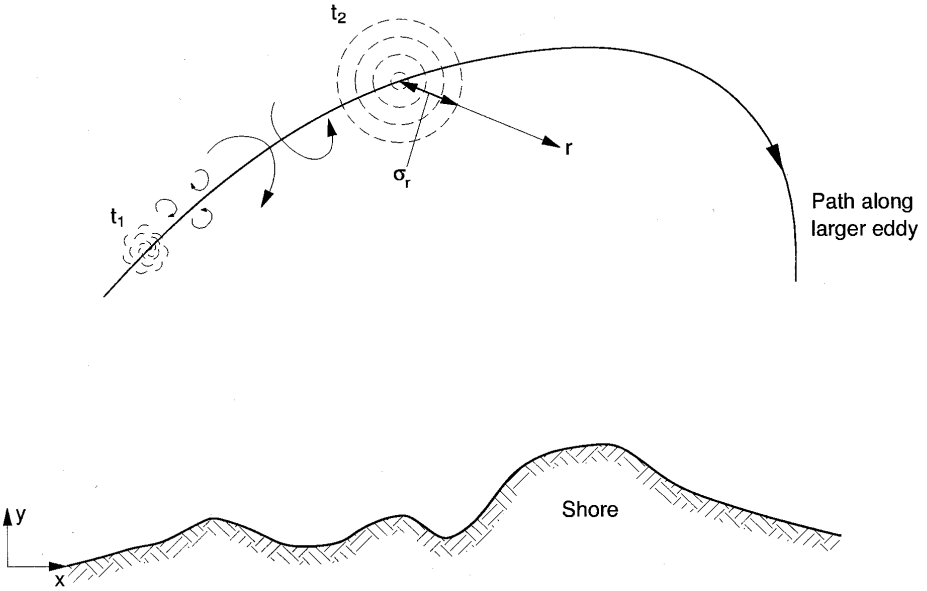
In continuous problems
→ long. Dispersion usually negligible

Plug flow: $\frac{4kE_L}{U^2} \to 0$

$$C = \frac{q''}{U} \exp\left(-\frac{kx}{U}\right)$$

OCEANIC DIFFUSION (coastal zone, estuaries, large lakes)

- eddy structure



RELATIVE DIFFUSION"

diffusion relative to center of mass, independent of actual location in fixed space (x, y)

- mass will follow larger eddies, but will be diffused by action of smaller eddies (of increasing size)

→ accelerating growth rate

$E_r = C\,\varepsilon^{1/3}\,\sigma_r^{4/3}$ „Richardson´s (1921) 4/3 Law“

↑ cloud size

$$\frac{d\sigma_r^2}{dt} = 4E_r$$ cylindrical coordinate r, standard deviation σ_r

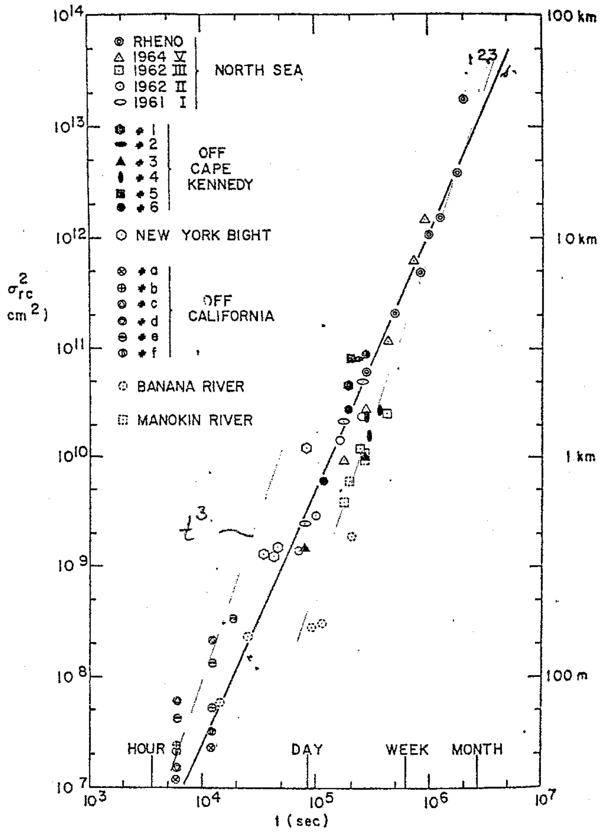
$\Rightarrow \sigma_r^2 \sim \varepsilon t^3$ rapid growth

$E_r \sim \varepsilon t^2$ ≠ const.!, diffusivity increases with time

<u>Data:</u> $\sigma^2_r \sim t^{2.3}$

$E_r \sim \sigma^{1.1}_r$

Okubo (1971) → Oceanic Diffusion Diagrams



ocean, coastal zone

large lake, coastal zone

mid-size

small lake

A. Okubo, "Oceanic Diffusion Diagrams", Deep Sea Research, Vol. 18, 1971

Typical growth rates:

1 hr	~	σ_r = 50 m
1 day	~	1 km
1 week	~	10 km
1 month	~	100 km

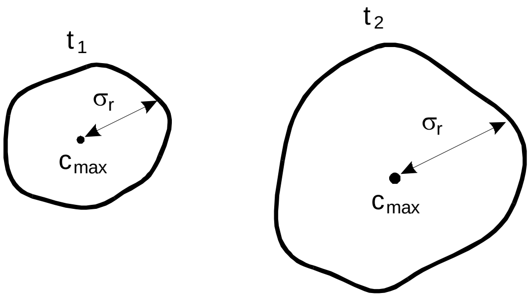
<u>Applications:</u>

Instantaneous Point Source (Accident):

$$c_{max} \sim \frac{1}{\sigma^2_r}$$

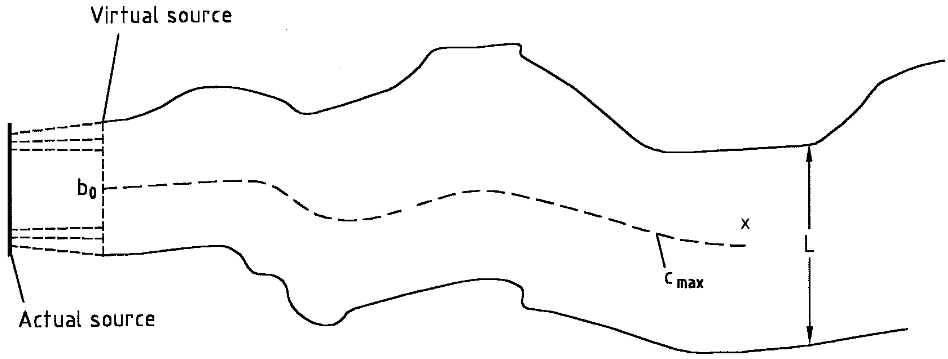
2-D growth

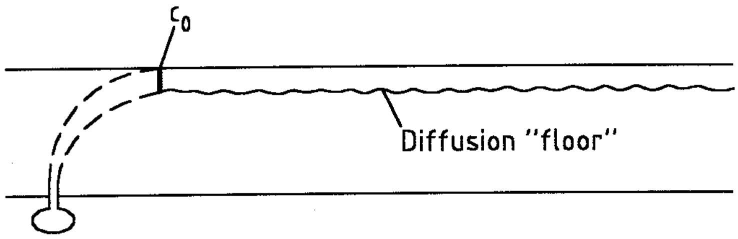
(vertically mixed)



$$c = \frac{M}{h\,2\pi\,\sigma_r^2}\, e^{-\left(\frac{r^2}{2\sigma_r^2}\right)}$$

Continuous release (Routine problems):





2-D

$$U\frac{\partial c}{\partial x}=\frac{\partial}{\partial y}\left(E_y\frac{\partial c}{\partial y}\right)-k\,c$$

Brooks (1960)

$$E_y = E_{y_0}\left(\frac{L}{b_0}\right)^{4/3}$$

4/3 Law

$$L=2\sqrt{3}\,\sigma_y$$

22% width

$$\frac{L}{b}=\left[1+\frac{8E_{y_0}\,x}{Ub^2{}_0}\right]^{3/2}\sim x^{3/2}\ !$$

$$\frac{c_{max}}{c_0}=\text{erf}\sqrt{\frac{3/2}{\left(1+\frac{8E_{y_0}\,x}{Ub^2{}_0}\right)^3-1}}$$

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