TURBULENT BUOYANT JETS AND PLUMES

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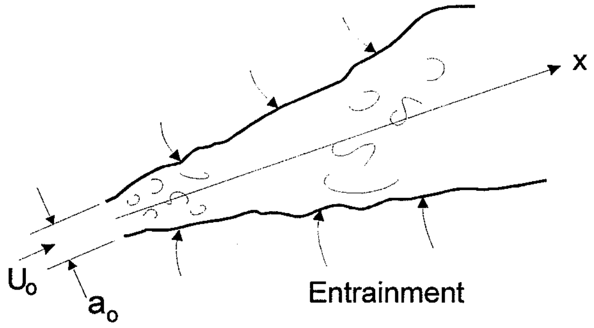
Active dispersal through induced turbulence

Pure jet



Momentum flux

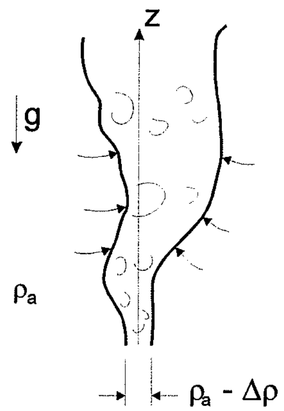
$$M_o = U_o^2 a_o$$

$$a_o = D^2 \pi / 4$$

Fully turbulent if

$$Re = \frac{U_o D}{\nu} \geq 2000$$

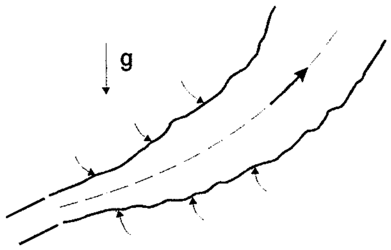
Pure plume



Buoyancy flux

$$J_o = \left(\frac{\Delta\rho}{\rho_a} g\right) U_o a_o$$

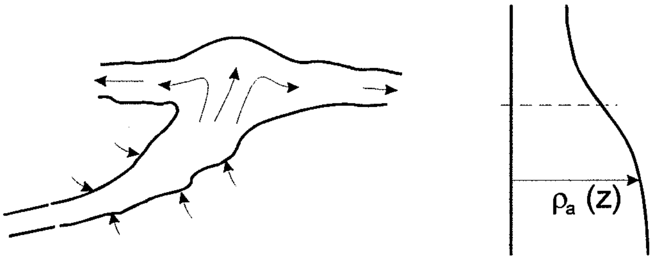
Combination:
buoyant jet or
forced plume



Homogeneous ambient

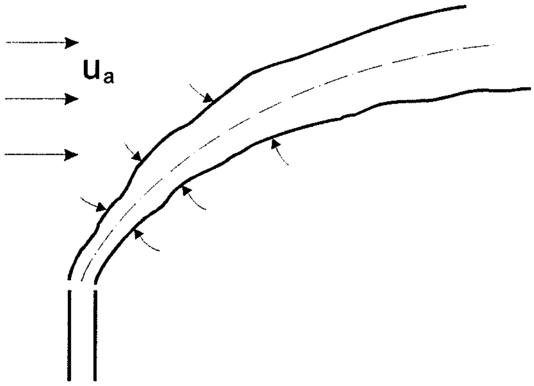
ρ_a = const.

Ambient density
stratification



$$\frac{\partial \rho_a}{\partial z} < 0$$

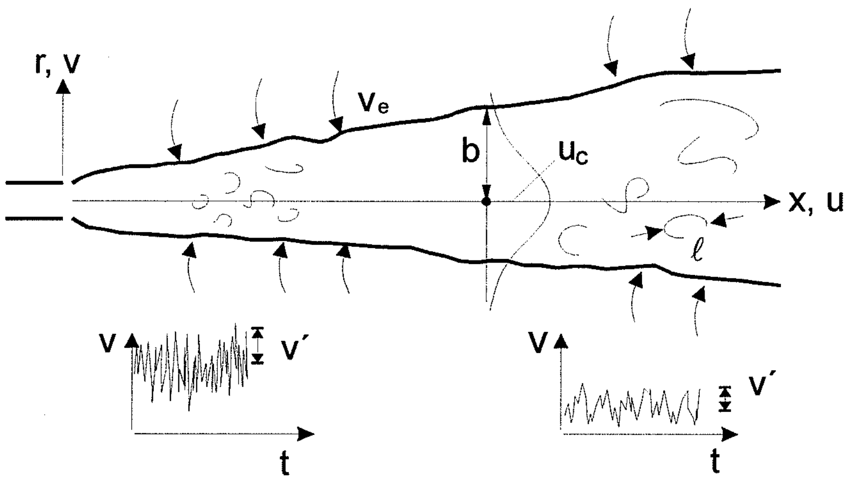
Ambient crossflow



Geometry: 3-D round jet
multiport diffuser (many 3-D round jets)
2-D plane jet

Fundamentals: **"FREE TURBULENCE" (Prandtl, Tollmien, Taylor ...)**

Jet:



Mean:

b = jet width

u_c = centerline velocity

Turbulence:

ℓ = integral length scale

v´ = integral velocity scale (r.m.s.)

1. Invariance $\quad \frac{\ell}{b} = \text{const.} \quad (\approx 0.3)$

2. Erosion = growth rate $\quad \frac{Db}{Dt} \cong u_c \frac{db}{dx} \sim v'$

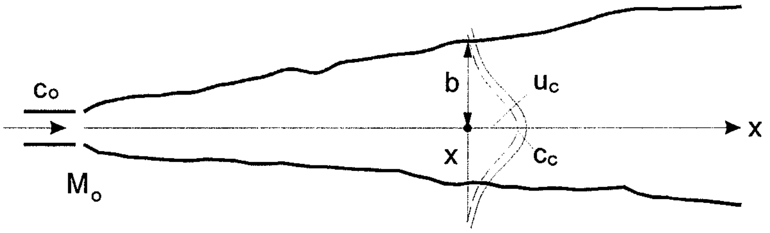
3. Prandtl´s approach (mixing length) $\quad v' = \text{const.}\ \ell \frac{\partial u}{\partial r} \sim \ell \frac{u_c}{b}$

$$u_c \frac{db}{dx} \sim \ell \frac{u_c}{b} \sim \text{const. } u_c$$

→	$\frac{db}{dx} = k$	linear spreading	"Jet diffusion"	spreading of zone containing momentum (and mass)
	$\frac{v'}{u_c} = \text{const.}$	constant turbulent intensity		
→	$\frac{v_e}{u_c} = \alpha$	constant entrainment rate	"Jet entrainment"	suction of outside fluid into zone (dilution)

Dimensional analysis:

Simple Jet



$M_o = U_o^2 a_o = U_o^2 \frac{\pi D^2}{4}$ Momentum flux conserved

$Q_{c_0} = c_o U_o a_o$ Scalar flux

Dynamic jet properties:

$b = f(x, M_o, a_o, \nu \ldots)$ $x = [L]$

$u_c = g(x, M_o, a_o, \nu \ldots)$ $M_o = \left[\frac{L^4}{T^2}\right]$

$$\frac{b}{x} = \text{const.}, \quad \frac{u_c\, x}{\sqrt{M_o}} = \text{const.}, \quad \rightarrow \quad b = kx, \quad \frac{u_c}{U_o} = K\frac{\sqrt{a_o}}{x}$$

Experiment: $b = 0.10\,x$, $u_c / U_o = 7.1 / (x / D)$

Passive properties:

$Q_{c_o} = c_o U_o a_o \cong c_c u_c b^2$ Mass flux conserved

$$\frac{c_c}{c_o} \sim \frac{U_o a_o}{u_c\, b^2} = \frac{1}{\bar{K}} \frac{\sqrt{a_o}}{x}$$

Dilution: $S = \frac{c_o}{c_c} = \bar{K}\frac{x}{\sqrt{a_o}}$!

Experiment: $\frac{c_c}{c_o} = \frac{7.1}{\frac{x}{D}}$, $S_c = 0.14\frac{x}{D}$ centerline dilution

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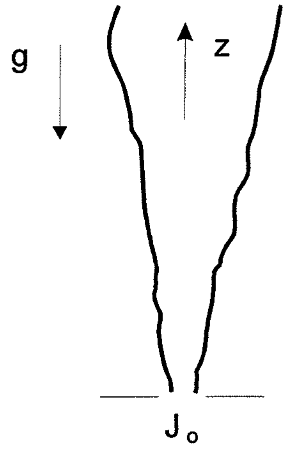
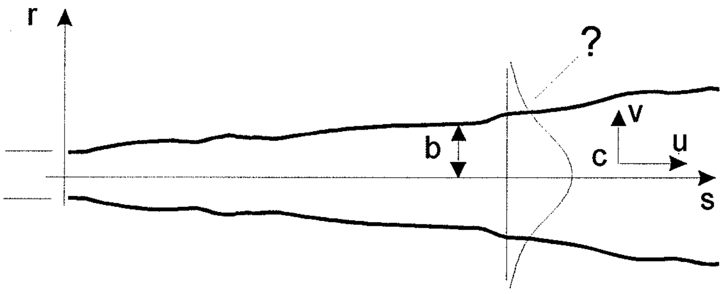
Simple Plume (J_o, z) $b \sim z$, $u_c \sim z^{-1/2}$, $S_c \sim z^{5/3}$

$J_o = \left[\frac{L^4}{T^3}\right]$



Formal solution methods:



Unknowns (u,v,p,c)

p_∞ = const.

Boundary layer approximations: $\frac{b}{s} << 1\,,\ \frac{\partial}{\partial r} >> \frac{\partial}{\partial s}$

$$\frac{\partial p}{\partial r} = 0,\quad \frac{\partial p}{\partial s} = 0$$

$$\frac{\partial u}{\partial s} + \frac{1}{r}\frac{\partial rv}{\partial r} = 0 \qquad \text{Continuity}$$

$$u\frac{\partial u}{\partial s} + v\frac{\partial u}{\partial r} = -\frac{1}{r}\frac{\partial}{\partial r}(r\,\overline{u'v'}) \qquad \text{Momentum}$$

$$\left[u\frac{\partial c}{\partial s} + v\frac{\partial c}{\partial r} = -\frac{1}{r}\frac{\partial}{\partial r}(r\,\overline{v'c'})\right] \qquad \text{Scalar transport (uncoupled)}$$

} Jet equation
u,v,c

I.C.: $s = 0: \quad u = U_o\,,\quad c = C_o\,,\quad v = 0$

B.C.: $r \rightarrow "\infty": \ u \rightarrow 0,\ c \rightarrow 0,$

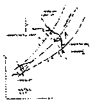
$\overline{u'v'} \rightarrow 0,\ \overline{v'c'} \rightarrow 0$

Solutions:

- Similarity methods with simple turbulence closure ⇒ classical, simple geometries
  $\overline{u'v'} = \varepsilon\frac{\partial u}{\partial r}$
- Numerical integration (P.D.E.) with advanced turbulence closure e.g. k-ε ⇒ more general geometries
  LES
- Integral methods (conversion to O.D.E.)

INTEGRAL METHOD FOR BUOYANT JET ANALYSIS

BUOYANT JET (Round jet, 2-D trajectory, stagnant unstratified ambient)



$$g' = \frac{\Delta\rho}{\rho_a} g$$

1. Profile Specification:

$$\frac{u}{u_c} = e^{-r^2/b^2} \qquad b = \text{"}\frac{1}{e}-\text{Width" (37 \%)}$$

$$\frac{g'}{g'_c} = \frac{c}{c_c} = e^{-\frac{r^2}{(\lambda b)^2}} \qquad \lambda > 1 \text{ "Dispersion ratio " (turbulent Schmidt number)}$$

2. Definition of integral quantities:

Volume flux

$$Q = 2\pi \int_0^\infty ur\,dr = 2\pi u_c\, b^2 \underbrace{\int_0^\infty \left(\frac{r}{b}\right) e^{-\left(\frac{r}{b}\right)^2} d\left(\frac{r}{b}\right)}_{1/2} = \pi u_c\, b^2$$

Momentum flux

$$M = \frac{\pi}{2} u_c^2 b^2$$

Buoyancy flux

$$J = \frac{\lambda^2}{1+\lambda^2} \pi\, u_c g'_c\, b^2$$

Scalar flux

$$Q_c = \frac{\lambda^2}{1+\lambda^2} \pi\, u_c c_c b^2$$

(„tracer, pollutant“)

3. Conservation equations: Local element ds

$$(1)\quad \frac{dQ}{ds} = \text{“entrainment“} = 2\pi b\, v_e$$

$$\text{"turbulence closure"} \qquad v_e = \alpha u_c \qquad \alpha = \overset{\text{jet}}{\alpha_1} + \overset{\text{plume contribution}}{\frac{\alpha_2 \sin\theta}{F_\ell^{\,2}}}$$

$$F_\ell^{\,2} = \frac{u_c^{\,2}}{g_c' b} \qquad \text{local Froude number}$$

$$(2)\quad \frac{dM}{ds} = \pi\lambda^2 g_c' b^2 \sin\theta \qquad \text{(buoyancy force)}$$

$$(3)\quad \frac{d(M\cos\theta)}{ds} = 0$$

$$(4)\quad \frac{dJ}{ds} = 0 \qquad (5)\quad \frac{dQ_c}{ds} = 0$$

$$(6)\quad \frac{dx}{ds} = \cos\theta \qquad (7)\quad \frac{dz}{ds} = \sin\theta$$

Unknowns: $u_c, b, g_c', c_c,$ θ, x, z

B.C.: $U_o,\ b_o \cong \frac{D}{2},\ g_o',\ c_o,$ θ_0, $x_0 = 0$, $z_0 = 0$

Dynamic parameter:

$$F_o = \frac{U_o}{\sqrt{g_o' D}} \quad \text{Densimetric Froude number}$$

Empirical Constants for the Axisymmetric Jet and Plume with Gaussian Profile Specification

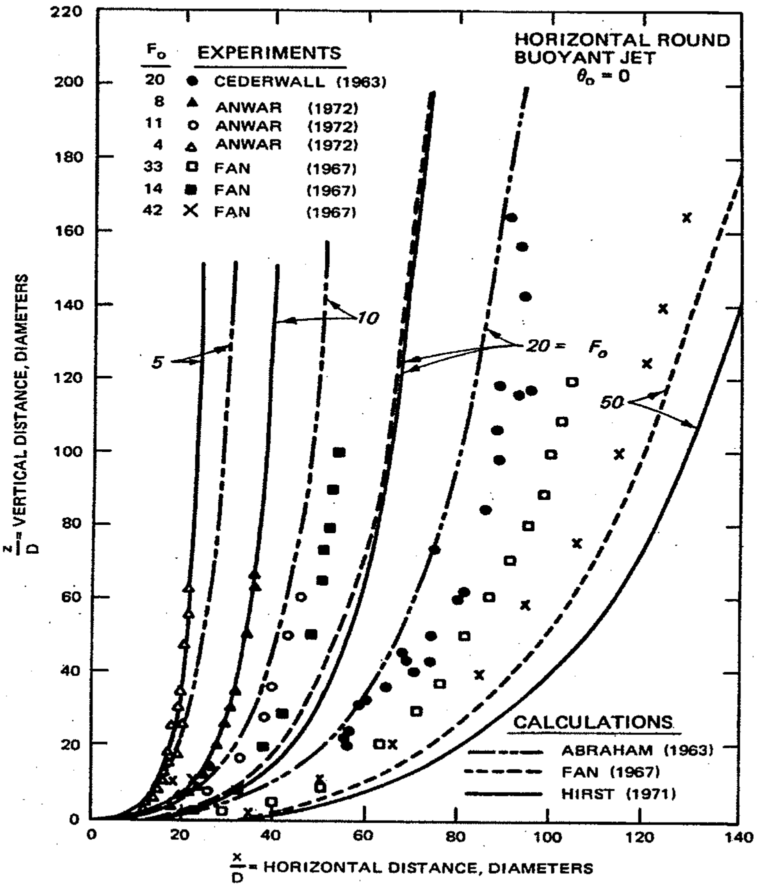
	Round Jet	**Round Plume**	
Spreading coefficient	$k_j = 0.110$	$k_p = 0.100$	
Ratio α/k	$\alpha_j/k_j = 0.5$	$\alpha_p/k_p = 5/6$	
Entrainment coefficient	$\alpha_j = 0.055$	$\alpha_p = 0.083$	**$\alpha = v_e / U_c$**
Scalar dispersion coefficient	$\lambda_j = 1.20$	$\lambda_p = 1.20$	
Equilibrium Froude number $F_\ell = \frac{u_c}{(g_c' b)^{0.5}}$	$F_{\ell j} = \infty$	$F_{\ell p} = 4.66$	

Notes: 1) Round jet integral coefficients for Gaussian profiles are:

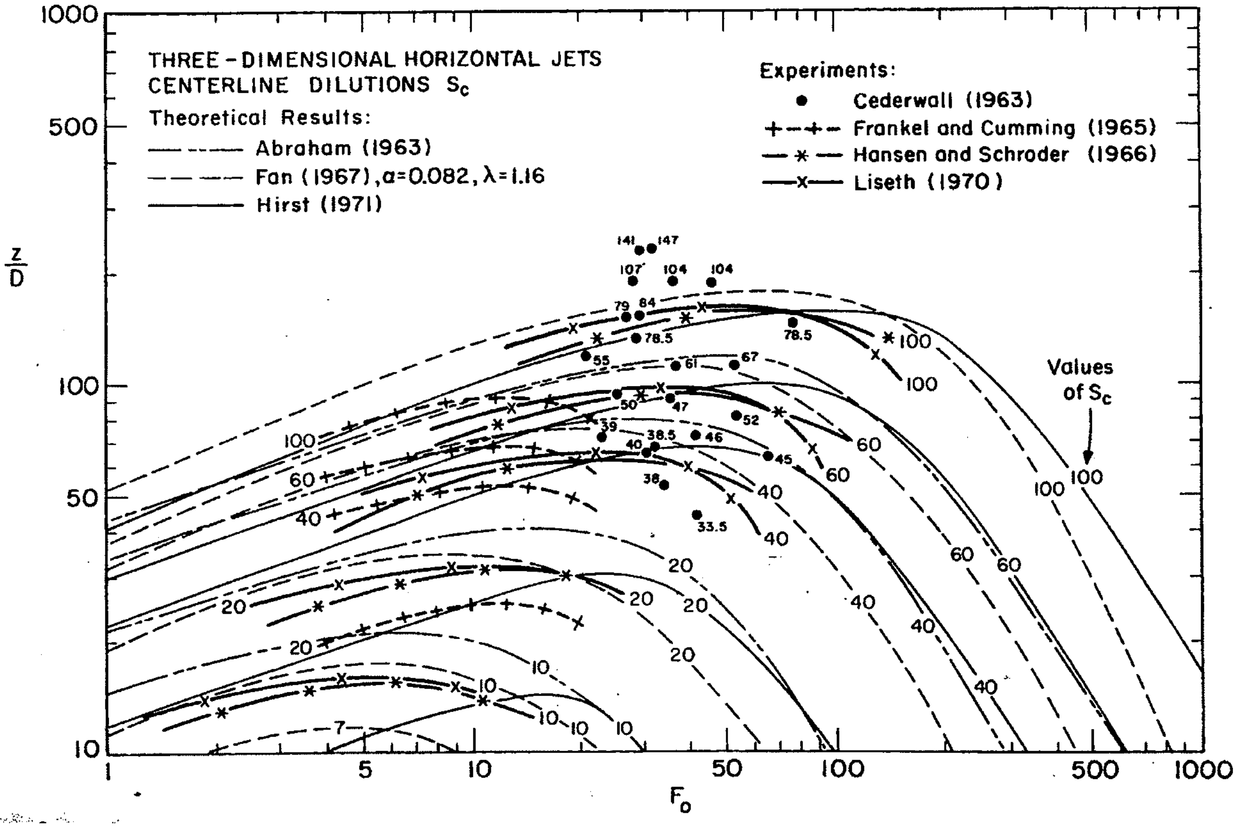
$I_1 = 0.5$, $I_2 = 0.25$, $I_3 = 0.5\ \lambda^2/(1+\lambda^2)$, $I_4 = \lambda^2/2$

2) The above constants are based on the data surveys of Jirka et al. (1975), Chen and Rodi (1980) and List (1982)

STAGNANT HOMOGENEOUS AMBIENT



Three-Dimensional Horizontal Jet Trajectories.
Comparison Between Theoretical Pedictions and Experimental Data



Three-Dimensional Horizontal Jets Centerline Dilutions S_c.
Comparison Between Theoretical Pedictions and Experimental Data

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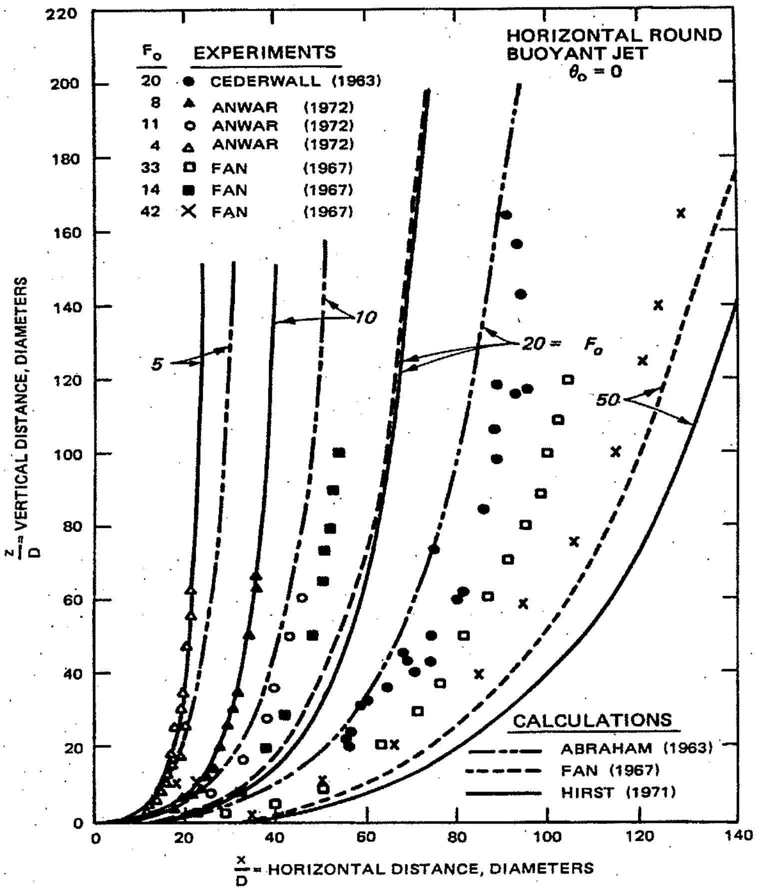

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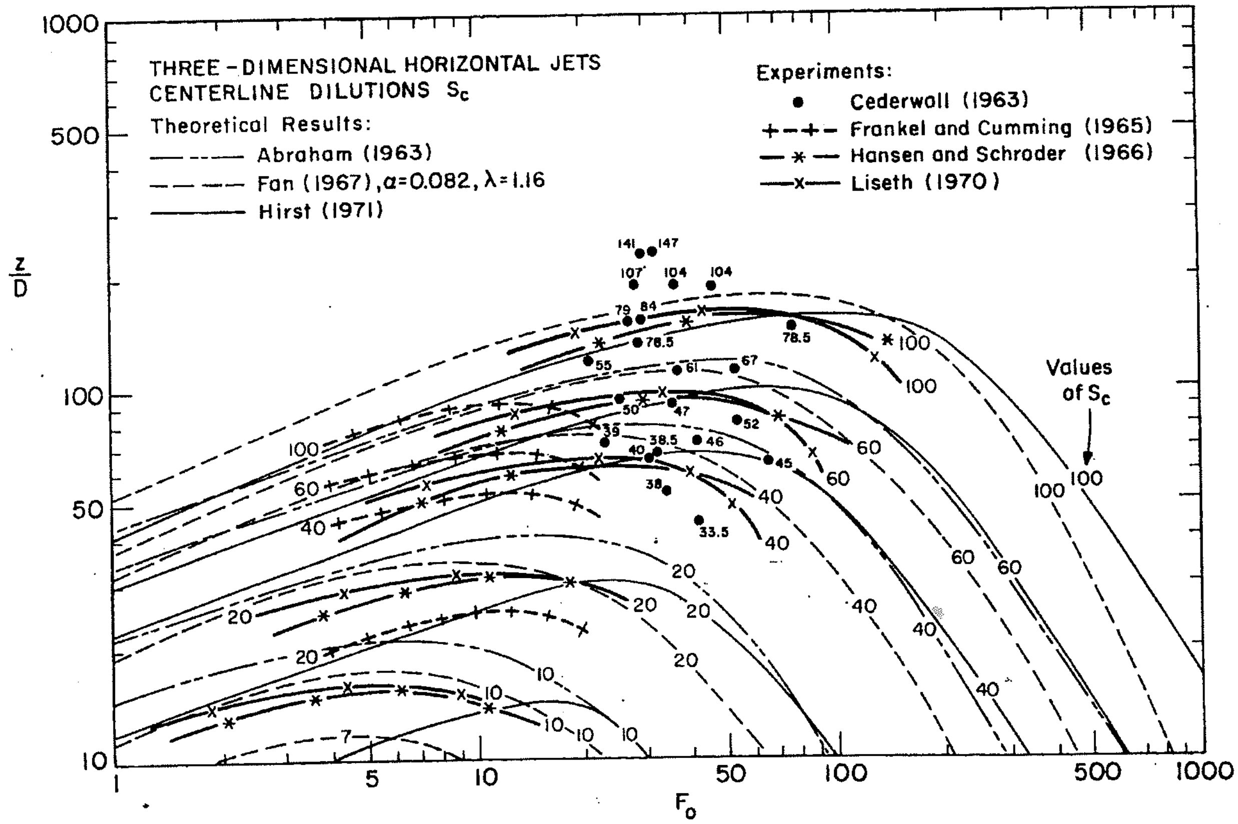


Three-Dimensional Horizontal Jets Centerline Dilutions S_c.
Comparison Between Theoretical Pedictions and Experimental Data

Dimensional analysis:

Any buoyant jet property e.g. $x = f(z, M_o, J_o, Q_o, D\ldots)$

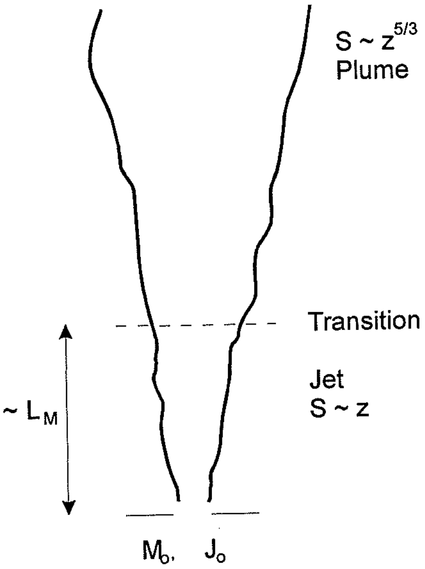
Normalized property $= f\left(\frac{z}{L_M}\right)$

$$L_M = \frac{M_o^{3/4}}{J_o^{1/2}} \quad = \text{momentum length scale}$$

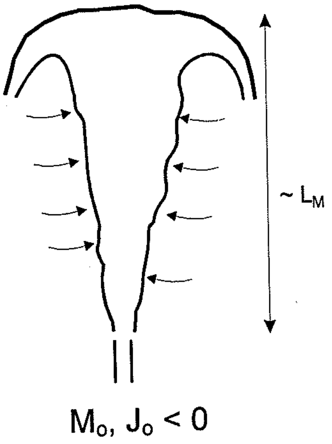
Thus: $DF_o = \left(\frac{4}{\pi}\right)^{1/4} L_M$ correct scaling! $L_M \sim D\,F_o$

Examples:

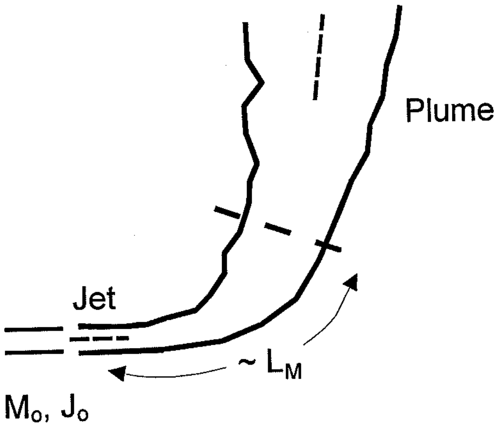
Vertical buoyant jet



Negative buoyant jet

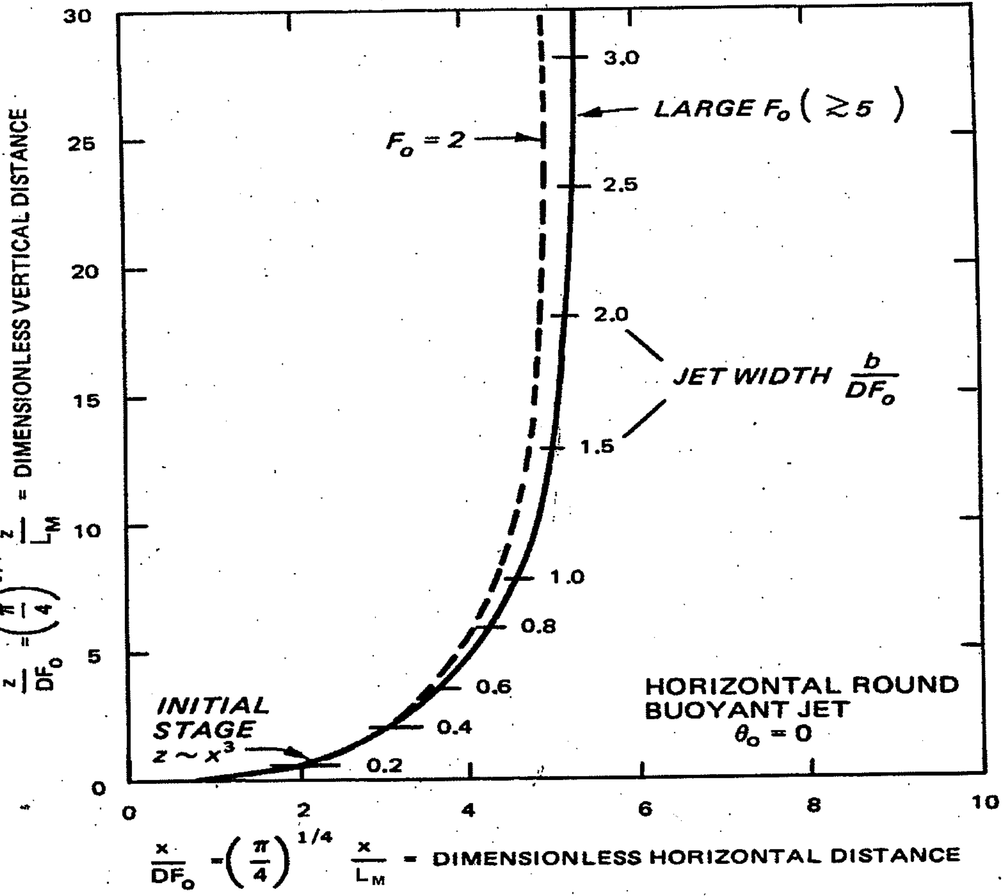


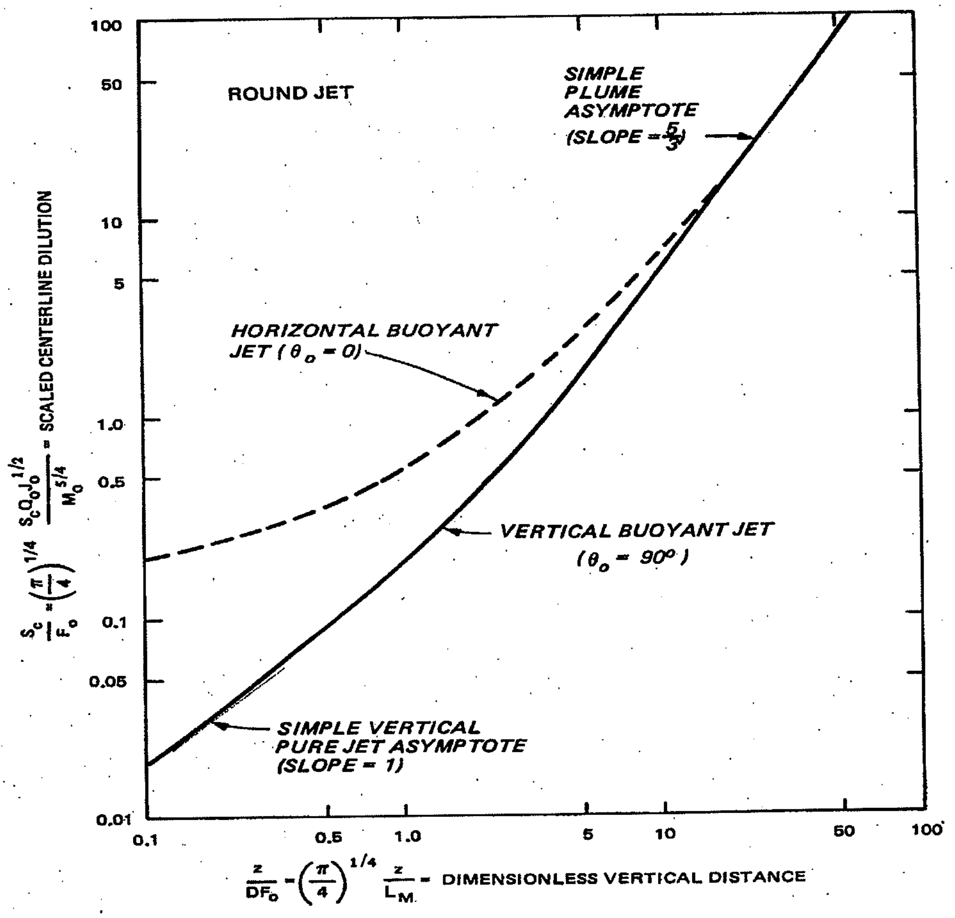
Horizontal buoyant jet



BUOYANT JETS

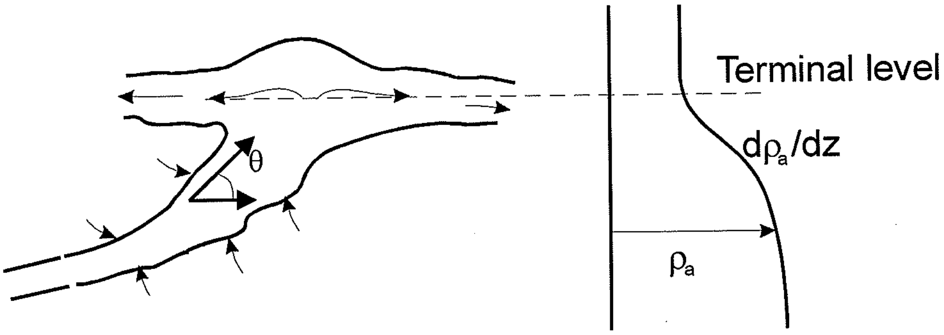
Stagnant, Unstratified Ambient





Amplifications of the Integral Method

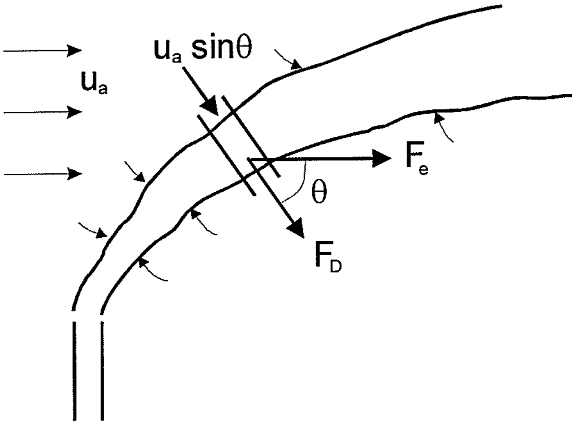
Ambient stratification:



(4) $$\frac{dJ}{ds} = \pi u_c b^2 \frac{d}{dz}\left(\frac{\rho_a g}{\rho_o}\right) \sin\theta$$ = change of buoyancy flux

$$T = \frac{\Delta\rho_o}{D\left(-\frac{d\rho_a}{dz}\right)}$$ Stratification parameter

Crossflow:



Entrainment force =

$$F_e = u_a \frac{dQ}{ds}$$

Drag force =

$$F_D = C_D \frac{(u_a \sin\theta)^2}{2} (2b)$$

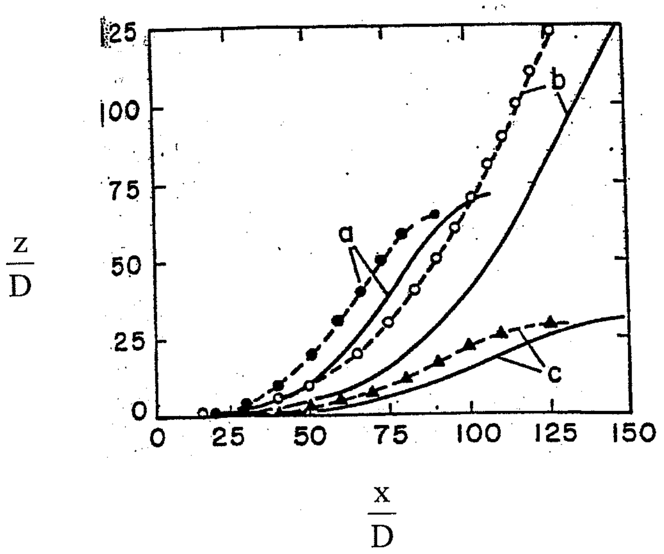
(2) $$\frac{dM}{ds} = \pi\lambda^2 g_c' b \sin\theta + F_e \cos\theta$$

(3) $$\frac{dM \cos\theta}{ds} = F_D \sin\theta + F_e$$

Entrainment → additional components: $E = 2\pi b u_c\left(\alpha_1 + \frac{\alpha_2 \sin\theta}{F_\ell^{\,2}}\right) + 2\pi b u_a \alpha_3 \sin\theta\cos\theta$

$$R = \frac{u_a}{U_o}$$ Crossflow parameter

Buoyant jet in ambient stratification



● a: $F_o = 26$, $T = 1200$, $\theta_o = 0°$

o b: $F_o = 40$, $T = 5000$, $\theta_o = 0°$

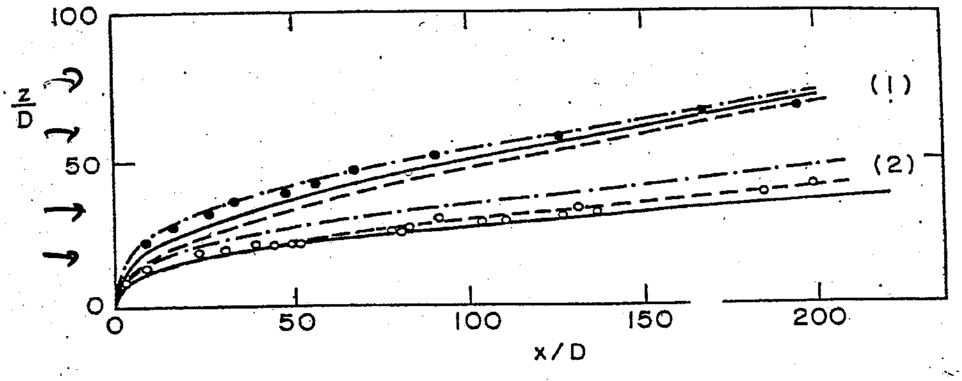
▲c: $F_o = 60$, $T = 510$, $\theta_o = 0°$

- - - Fan's model $\alpha = 0.082$, $\lambda = 1.16$

—— Hirst's model α Eq. (4.4), $\lambda = 1.16$

Experiments by Fan (1967)

Buoyant jet in crossflow



a) Jet Trajectories

● (1) : $F_o = 40$, $R = 0.063$, $\theta_o = 90°$

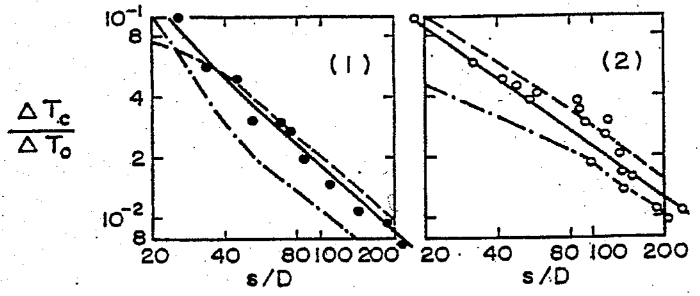
o (2) : $F_o = 40$, $R = 0.125$, $\theta_o = 90°$

- - - Fan's model, $\alpha = 0.4$, $C_D = 0.3$ (1), 0.7 (2) (best fit)

—— Abraham's model, α Eq. (4.13), $C_D = 0.4$ (best fit)

—●— Hirst's model , no drag

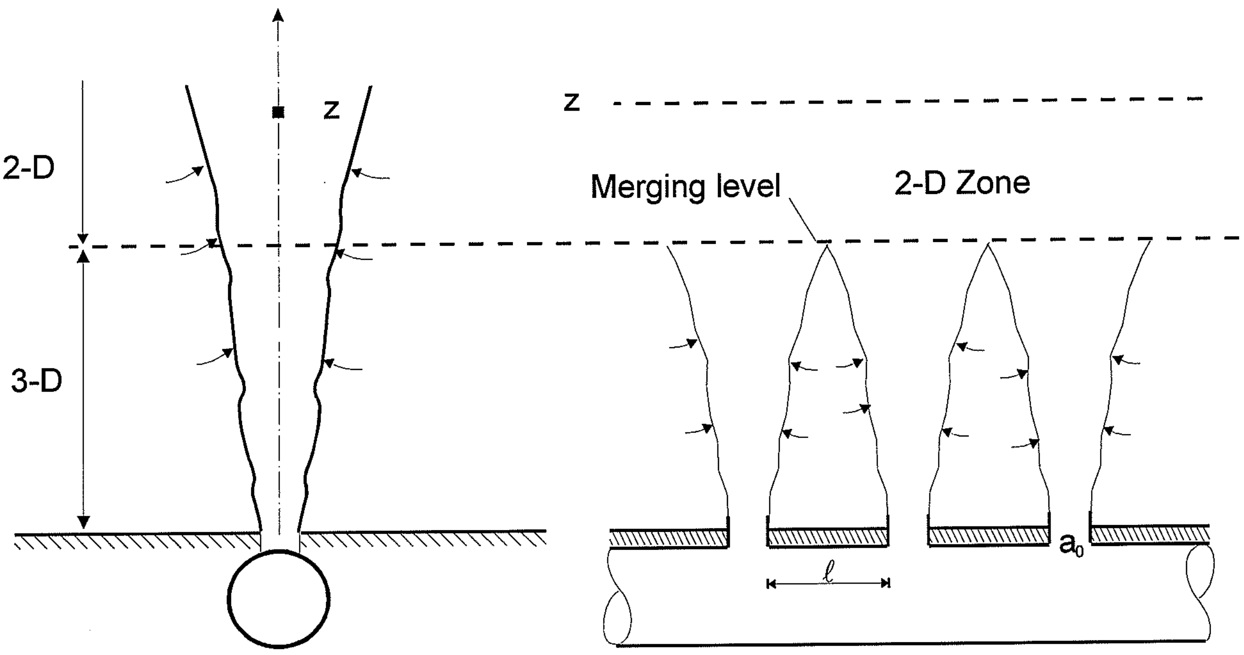
Experiments by Fan (1967)



b) Centerline temperature (Concentration) Decay
Buoyant Jets in Crossflow. Comparison of Theoretical
Prediction with Experimental Data

Multiple Jets: Multiport diffusor

Analogy to 2-D plume



$\frac{J_o}{L_D} = j_o = \frac{\Delta\rho_o}{\rho_a} gU_oB$ buoyancy flux / length $\quad B = \frac{a_o}{\ell}$ equivalent slot width

$q_{c_o} = c_o U_o B$

Dynamic properties:

$F_1 (b, z, j_o, \ldots) = 0 \qquad b = k z$

$F_2 (u_c, z, j_o, \ldots) = 0 \qquad u_c = K j_o^{1/3} = \text{const !}$

$F_3 (q, z, j_o, \ldots) = 0 \qquad q = \overline{K} j_o^{2/3} z$

Scalar: $q_{co} = c_o U_o B \cong c_c u_c b$

$S = \frac{c_o}{c_c} \sim z$ Dilution

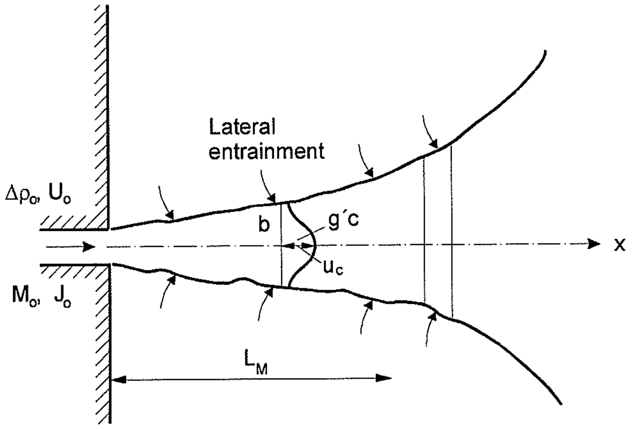
Compare: 3-D plume $S \sim z^{5/3}$

Appropriate length scale for 2-D buoyant jet

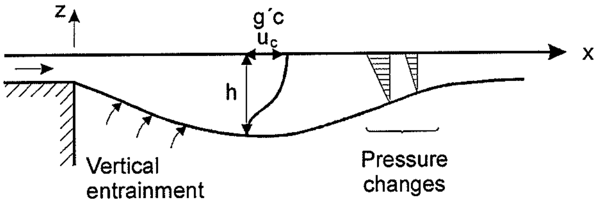
$\ell_M = \frac{m_o}{j_o^{2/3}}$ = jet/plume transition length scale

$m_o = \frac{M_o}{L_D}, \qquad j_o = \frac{J_o}{L_D}$ $\qquad L_D$ = diffusor length

Surface Buoyant Jet



stagnant
ambient



$$Q = \int_A u_c \, f\left(\frac{y}{b}, \frac{z}{h}\right) dA$$

(1) $$\frac{dQ}{dx} = \overbrace{\underbrace{c_1 \alpha_h u_c h}_{\text{lateral } \alpha_h = \alpha} + \underbrace{c_2 \alpha_v u_c b}_{\text{vertical } \alpha_v = f(F_\ell)}}^{\text{entrainment rate}}$$

$F_\ell = u_c / (g_c' h)^{1/2}$

e.g. $\alpha = \alpha \, e^{-5 \cdot F_\ell}$ local buoyancy effect

(2) $$\frac{dM}{dx} = \frac{dP}{dx}$$

$$P = \int_A \left(\int_{-\infty}^{z/h} \frac{\Delta\rho}{\rho} g \, d\left(\frac{z}{h}\right) \right) dA$$ buoyant pressure force

(3) $$\frac{db}{dx} = \underset{\text{jet spreading} \approx 0{,}1}{k} + \underset{\text{buoyant spreading}}{\left(\frac{db}{dx}\right)_B}$$

$$u_f = \left(\frac{db}{dt}\right)_B \cong \left(\frac{\Delta\rho_c}{\rho} gh\right)^{\frac{1}{2}} = (g'_c \, h)^{0,5}$$ spreading velocity

$$\frac{dx}{dt} \cong u_c$$

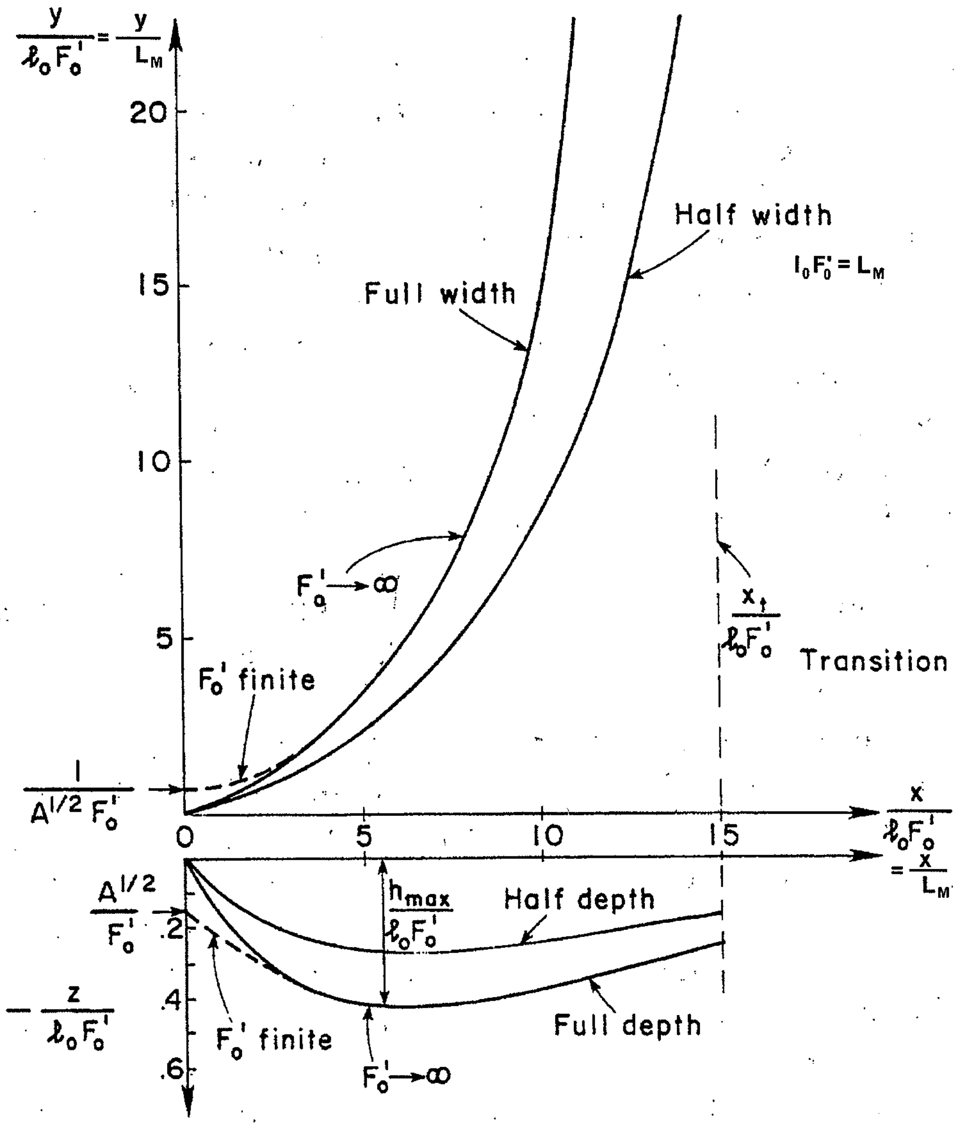
$$\left(\frac{db}{dx}\right)_B = \left(\frac{c_3}{F_\ell}\right)$$ global buoyancy effect

(4) $$\frac{dJ}{dx} = 0$$ (no buoyancy loss; e.g. heat loss)

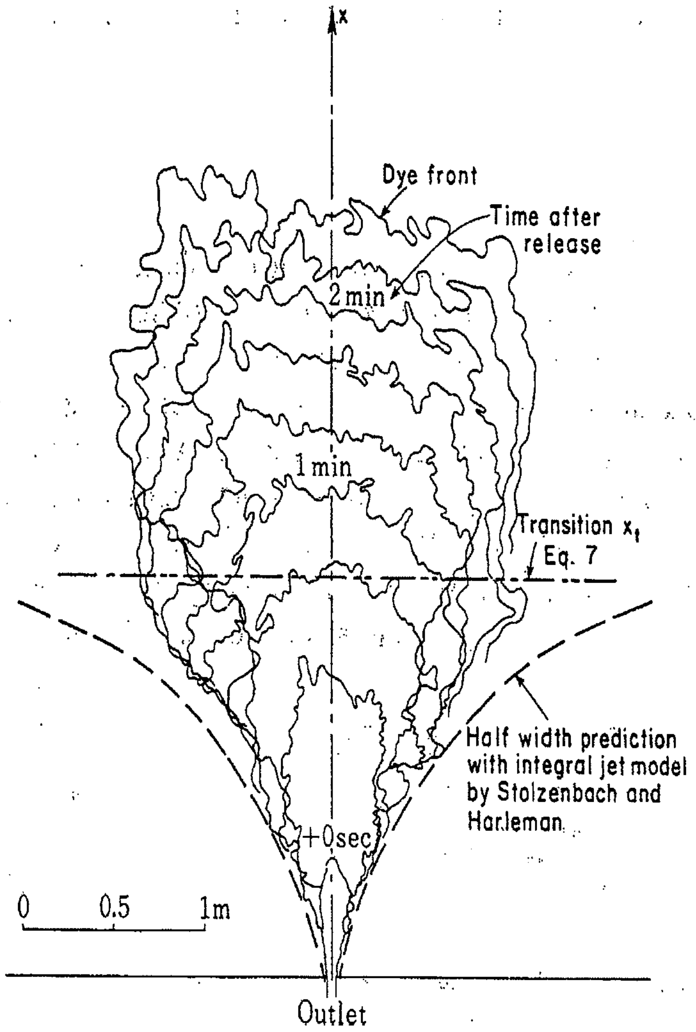
Parameter: F_o

Length Scale: $L_M = \dfrac{M_o^{3/4}}{J_o^{1/2}}$

Buoyant Surface Jets in Stagnant Ambient

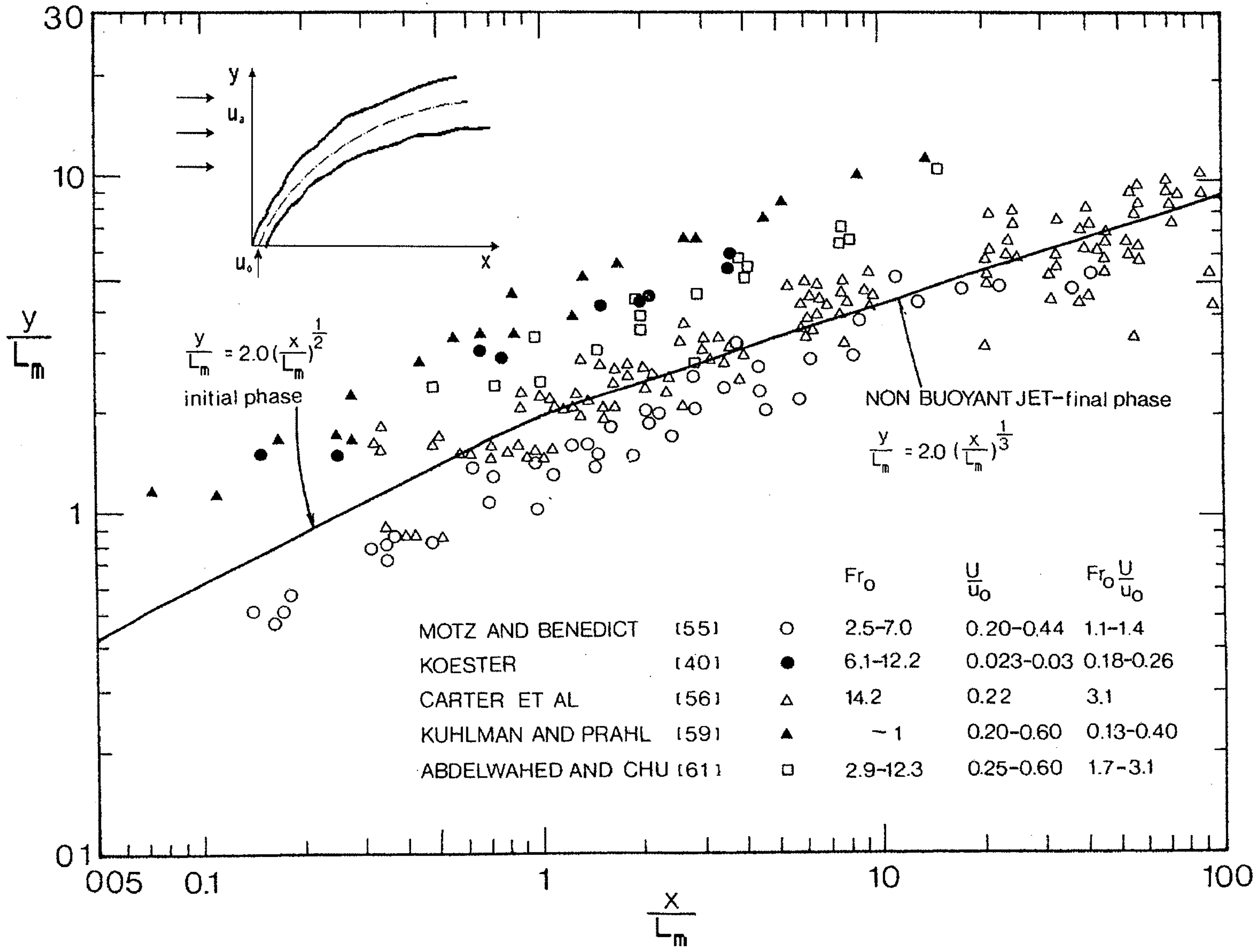


Self-similar Geometry of Buoyant Surface Jets in Deep Receiving Water Computed by Mathematical Model



Unsteady Dye Fronts within Steady Buoyant Surface Jet Showing Nonlinear Spread and Transition. Experiments by Hayashi and Shuto and Mathematical Model Predictions for Jet Half Width

Buoyant Surface Jets in Crossflow



Trajectories of free deflected buoyant jets in a crossflow in comparison to trajectory laws for non-buoyant jets

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