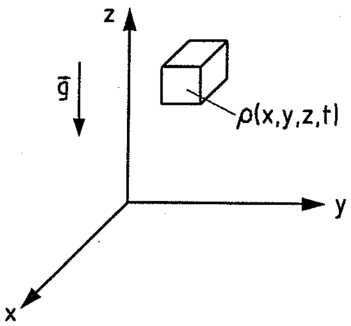
STRATIFIED FLOW

Gerhard H. Jirka, University of Karlsruhe

1. Definitions, Equations of Motion, Parameters

Stratified fluid: fluid involving density variations ρ (x, y, z, t) in gravitational field $\vec{g}$



$$\rho = \left[\frac{\text{mass}}{\text{volume}}\right] = \left[\frac{M}{L^3}\right]$$

Sources of density change:
- temperature (heat)
- dissolved phases (solids, fluids, vapors)
- suspended solids (sediments)
- pressure

Ex. Seawater ρ (S, T, p)

↑ ↑ ↑_ pressure
| L temperature
L salinity

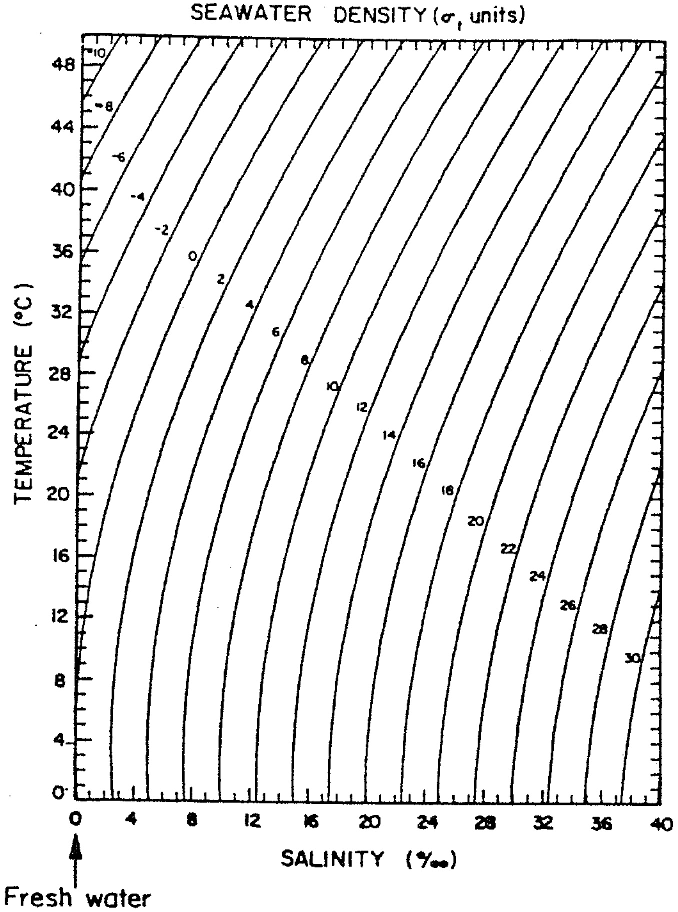
$$S = \left[\frac{\text{mass of salt}}{\text{mass of mixture}}\right] ‰$$

UNESCO (1980) equation of state

$$\sigma_t = \rho\left[\frac{kg}{m^3}\right] - 1000 \qquad \text{“sigma-t” units}$$

Freshwater	4°C	$\sigma_t = 0$	Reference value
	20°C	$\sigma_t = -1.77$	0.18% lighter
Seawater	0°C	$\sigma_t = 27.3$	2.73% heavier
S = 34‰	20°C	$\sigma_t = 24.0$	2.4 % heavier

DENSITY OF SEA AND FRESH WATER



Density $[kg/m^3] = 1000.0 + \sigma_t$

Diagram for density of seawater as a function of temperature and salinity

The One Atmosphere International Equation of State of Seawater, 1980

Definition

The density (ρ, kg m^{-3}) of seawater at one standard atmosphere ($p = o$) is to be computed from the practical salinity (S) and the temperature (t, °C) with the following equation : S (‰)

$$\rho(S,t,o) = \rho_w + (8.244\,93 \times 10^{-1} - 4.0899 \times 10^{-3}\, t$$

$$+ 7.6438 \times 10^{-5}\, t^2 - 8.2467 \times 10^{-7}\, t^3 + 5.3875 \times 10^{-9}\, t^4) S$$

$$+(-5.724\,66 \times 10^{-3} + 1.0227 \times 10^{-4}\, t - 1.6546 \times 10^{-6}\, t^2) S^{3/2}$$

$$+ 4.8314 \times 10^{-4}\, S^2$$

where ρ_w, the density of the Standard Mean Ocean Water (SMOW) taken as pure water reference, is given by·

$$\rho_w = 999.842\,594 + 6.793\,952 \times 10^{-2}\, t - 9.095\,290 \times 10^{-3}\, t^2$$

$$+ 1.001\,685 \times 10^{-4}\, t^3 - 1.120\,083 \times 10^{-6}\, t^4$$

$$+ 6.536\,332 \times 10^{-9}\, t^5$$

The one atmosphere International Equation of State of Seawater, 1980 is valid for practical salinity from 0 to 42 and temperature from −2 to 40°C.

Reprint, Unesco [1981b]

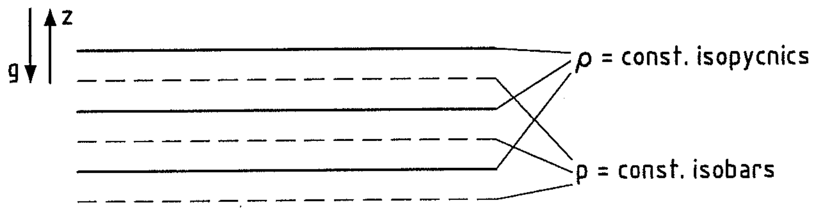
Hydrostatics: Equilibrium of stratified fluids $\rho\,(z)$

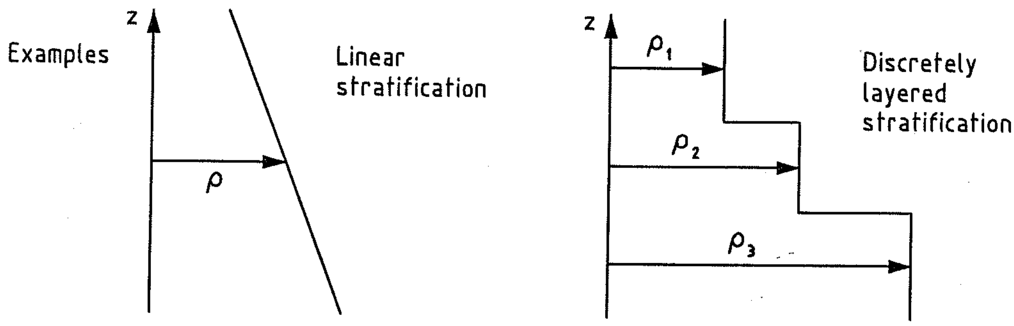
$$\frac{\partial p}{\partial z} = -\rho g \qquad \vec{g} = (0, 0, -g)$$

Incompressible fluid: $\rho \neq f\,(\text{pressure})$

$$p = p(0) - g\int_0^z \rho(z)\,dz$$

p (0) = reference pressure





Equilibrium states:

Stable	→	ρ decreases with z
Neutral	→	ρ = const.
Unstable	→	ρ increases with z

Deviations from equilibrium → ρ (x, y, z) → horizontal pressure forces induce flow!

Compressible fluids: Atmosphere, deep ocean
ρ = f (pressure) Gas law

Stability conditions given by entropy variations
(i.e. adiabatic movements of fluid parcels)

Concept of potential density (potential temperature) (see Turner)

Governing equations of stratified flow: $\vec{u} = (u, v, w)$, p

Ref. Turner (1973)

- Mass conservation:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{u}) = 0 = \underbrace{\frac{\partial \rho}{\partial t} + \vec{u} \cdot \nabla \rho}_{\frac{D\rho}{Dt}} + \rho \nabla \cdot \vec{u}$$

$\frac{D\rho}{Dt} = 0$ incompressibility (constant density following fluid motion)

$\therefore \quad \nabla \cdot \vec{u} = 0$ continuity eq.

- Momentum conservation (Navier-Stokes Eq.):

$$\rho \left(\frac{\partial \vec{u}}{\partial t} + \vec{u} \cdot \nabla \vec{u} \right) = \rho \frac{D\vec{u}}{Dt} = -\nabla p + \rho \vec{g} + \mu \nabla^2 \vec{u}$$

μ = molecular viscosity

Neglecting viscous forces:

$\rho \frac{D\vec{u}}{Dt} = -\nabla p + \rho \vec{g}$ Euler Eq. ρ = variable!

Reference state: Hydrostatic condition (no motion)

$$0 = -\nabla p_o + \rho_o \vec{g}$$

Deviations: $p = p_o(z) + p'$
$\rho = \rho_o(z) + \rho'$

$$\rho \frac{D\vec{u}}{Dt} = -\nabla(p - p_o) + (\rho - \rho_o)\vec{g} = -\nabla p' + \rho' \vec{g}$$

$\therefore$ Stratified flow motions are result of density and pressure variations from equlibirium!

- Vorticity conservation: $\vec{\zeta} = \nabla \times \vec{u}$

$$\frac{D\vec{\zeta}}{Dt} = \nu\nabla^2\vec{\zeta} + \vec{\zeta}\cdot\nabla\vec{u} - \frac{1}{\rho^2}(\nabla p \times \nabla\rho)$$

diffusion from solid boundaries	stretching of vortex lines	generation due to non-parallel isopycnics and isobars

1) Stratified flow is always rotational

2) In layered stratified flow vorticity is generated at density interfaces, but flow may be irrotational within layers.

Boussinesq approximation:

Small density changes relative to total density
ρ = const. = ρ_o reference density in acceleration term

$$\frac{D\vec{u}}{Dt} = -\frac{1}{\rho_o}\nabla p' + \frac{\rho'}{\rho_o}\vec{g} = -\frac{1}{\rho_o}\nabla p' + \vec{g}'$$

$$\vec{g}' = \frac{\rho'}{\rho_o}\vec{g} = \text{buoyant acceleration}$$

Small amplitude approximation:

$$\frac{D\vec{u}}{Dt} = \frac{\partial\vec{u}}{\partial t} + \underbrace{\vec{u}\cdot\nabla\vec{u}}_{\approx 0} = -\frac{1}{\rho_o}\nabla p' + \vec{g}'$$

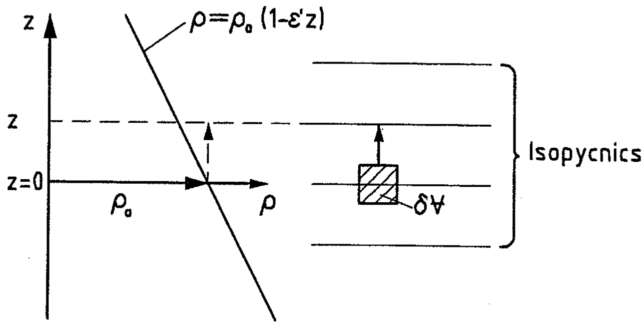
useful for linear wave theory

≈ 0 small spatial gradients

Parameters of stratified flow:

- Displacement of fluid parcel in local stratification:

$$\varepsilon' = -\frac{1}{\rho_o}\frac{d\rho}{dz}$$ density gradient



$$\rho_o \delta\forall \frac{d^2z}{dt^2} = -(\rho_o\,\varepsilon' z) g \delta\forall$$

$$\frac{d^2z}{dt^2} + \varepsilon' g z = \frac{d^2z}{dt^2} + N^2 z = 0$$ harmonic eq.

$$z = A\cos Nt + B\sin Nt$$

$$N = \left(-\frac{g}{\rho_o}\frac{d\rho}{dz}\right)^{1/2}$$ Brunt-Vaisälä (buoyancy) frequency

period $T = \frac{2\pi}{N} = 0(\mathrm{min})$ atmosphere, lakes, upper ocean

- Shear flow $\frac{du}{dz}$

$$Ri = \frac{N^2}{\left(\frac{du}{dz}\right)^2} = \frac{-\frac{g}{\rho_o}\frac{d\rho}{dz}}{\left(\frac{du}{dz}\right)^2}$$ Gradient Richardson Number

- Layered shear flow $-\frac{d\rho}{dz} \approx \frac{\Delta\rho}{L}$; $\frac{du}{dz} \approx \frac{u}{L}$

$$R_{i_o} = \frac{\frac{\Delta\rho}{\rho_o} gL}{U^2} = \frac{g' L}{U^2}$$ Bulk Richardson Number

$$F = \frac{U}{(g'L)^{1/2}} = \frac{1}{R_{i_o}^{\;1/2}}$$ Densimetric Froude Number

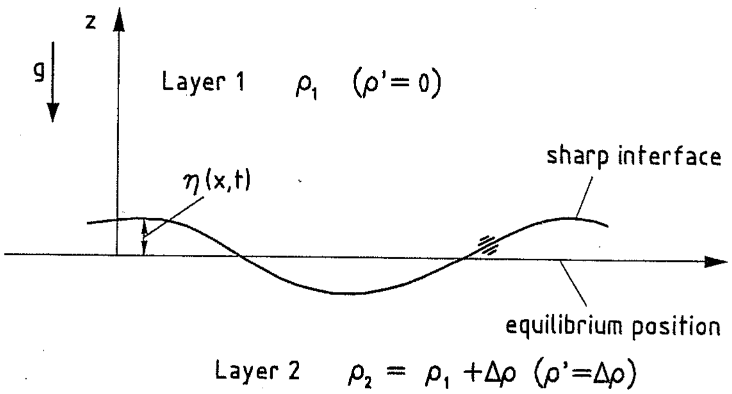
- Viscous, diffusive effects ν, D, K

Reynolds number $Re = \frac{UL}{\nu}$

Schmidt (Prandtl) number $Sc = \frac{\nu}{D}$, $Pr = \frac{\nu}{K}$

2. Internal Waves: Small Amplitude (Linear) Theory

A) Discretely Layered Sytems , 2-D (x, z), $\vec{u} = (u, w)$



For each layer: $\vec{u}_1$, $\vec{u}_2$

- flow is irrotational $\vec{\zeta} = \nabla \times \vec{u} = 0$
- satisfied if $\vec{u} = -\nabla\phi$ $\left[u = -\frac{\partial\phi}{\partial x},\ w = \frac{\partial\phi}{\partial z}\right]$
- continuity $\nabla \cdot \vec{u} = 0$

Thus: $\nabla \cdot \nabla\phi = \nabla^2\phi = 0$ Laplace Eq., Potential Flow

Interface conditions: $\eta = f(x, t)$

1) Kinematic: $w = \frac{\partial\eta}{\partial t} = -\left.\frac{\partial\phi_1}{\partial z}\right|_{z=0} = -\left.\frac{\partial\phi_2}{\partial z}\right|_{z=0}$

2) Pressure continuous: $p_1 = p_2$ at $z = \eta$

z-momentum eq.: $$\rho_1\frac{\partial w_1}{\partial t} = \rho_1\frac{\partial}{\partial t}\left(-\frac{\partial\phi_1}{\partial z}\right) = -\rho_1\frac{\partial}{\partial z}\left(\frac{\partial\phi_1}{\partial t}\right) = -\frac{\partial p_1}{\partial z}$$

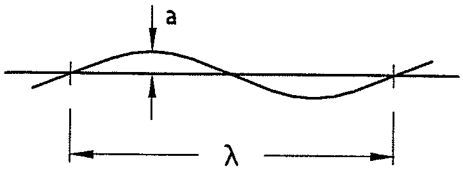
$$\rho_1\frac{\partial\phi_1}{\partial t} = p_1 + \text{Const.}$$

$$\rho_2\frac{\partial\phi_2}{\partial t} = p_2 + \Delta\rho\, g z + \text{Const.}$$

at $z = \eta$

$$\rho_1\left.\frac{\partial\phi_1}{\partial t}\right|_{z=0} = \rho_2\left.\frac{\partial\phi_2}{\partial t}\right|_{z=0} - \Delta\rho\, g\,\eta$$

<u>Natural modes:</u> $\boxed{\eta = a\cos(kx + \omega t)}$



a = amplitude

k = wave number = $\dfrac{2\pi}{\lambda}$ λ = wave length

ω = frequency = $\dfrac{2\pi}{T}$ T = period

Note: $\eta = \mathrm{Re}\left[a e^{ikx} e^{\omega}\right] = \mathrm{Re}\left[a\cos(kx + \omega t) + i a \sin(kx + \omega t)\right]$

(i) <u>Two infinitely deep layers:</u> Progressive Waves

$$\left.\begin{aligned} \phi_1 &= A_1 e^{-kz} e^{ikx + i\omega t} \\ \phi_2 &= A_2 e^{+kz} e^{ikx + i\omega t} \end{aligned}\right\}$$

Satisfies $\nabla^2 \phi = 0$!

$e^{\mp kz} \rightarrow$ decays away from interface!

Satisfying interface conditions:

Solutions: $A_1 = -\dfrac{i\omega}{k} a$, $A_2 = +\dfrac{i\omega}{k} a$

$$\boxed{\omega^2 = gk \frac{\Delta\rho}{\rho_1 + \rho_2}}$$

Dispersion relation

Interface movement: $\eta(x,t)$

$$d\eta = \frac{\partial \eta}{\partial x} dx + \frac{\partial \eta}{\partial t} dt = 0 \qquad \text{for } \eta = \text{const.}$$

$$-ak\sin(kx + \omega t)dx - a\omega \sin(kx + \omega t)dt = 0$$

$$\frac{dx}{dt} = -\frac{\omega}{k} = c \qquad \text{Phase speed}$$

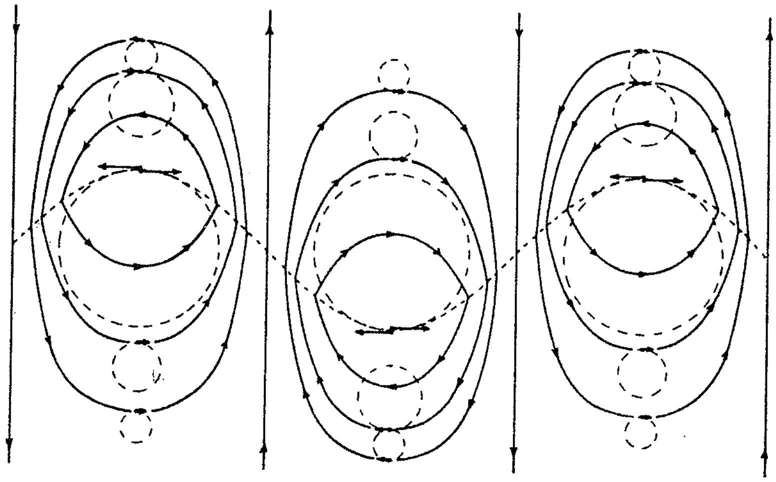
$$c = \frac{\omega}{k}\left(\frac{g}{k}\frac{\Delta\rho}{\rho_1 + \rho_2}\right)^{1/2}$$

Long waves ($k \rightarrow 0$) $\Rightarrow$ c fast

Short waves ($k \rightarrow \infty$) $\Rightarrow$ c slow

Progressive internal wave:

____ instantaneous streamlines
- - - - particle orbits



Note: Shear at interface (vorticity generation)

Surface wave: $\rho_1 = 0$ (air) , $\Delta\rho = \rho_2$

$$\omega^2 = gk \quad , \qquad c = \left(\frac{g}{k}\right)^{1/2}$$

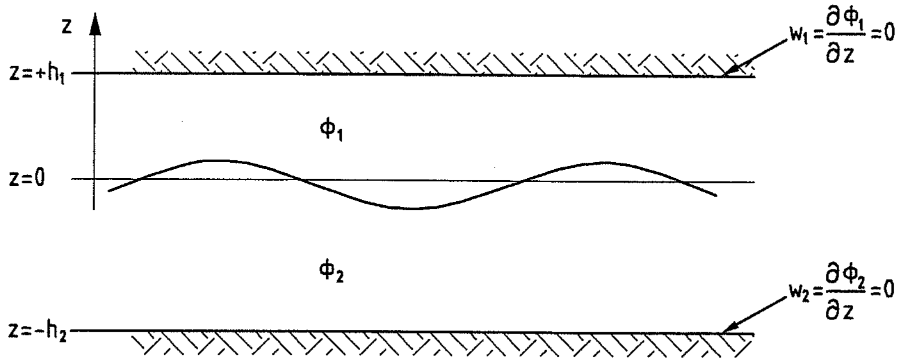
Boussinesq case: $\rho_1 \approx \rho_2 = \rho$

$$\omega^2 = \underbrace{\frac{1}{2}\frac{\Delta\rho}{\rho}}_{\text{lower}} gk \quad , \quad c = \underbrace{\left(\frac{1}{2}\frac{\Delta\rho}{\rho}\right)^{1/2}}_{\text{slower}} \left(\frac{g}{k}\right)^{1/2} = \left(\frac{1}{2}\right)^{1/2} \left(\frac{g'}{k}\right)^{1/2}$$

lower
longer period T

slower

(ii) Two finite layers:



$$\phi_1 = A_1 \cosh(z-h_1)e^{ikx+i\omega t}$$

$$\phi_2 = A_2 \cosh(z+h_2)e^{ikx+i\omega t}$$

$$\omega^2 = gk\frac{\Delta\rho}{\rho_1 \coth kh_1 + \rho_2 \coth kh_2} \quad ; \quad c^2 = \frac{\omega^2}{k^2}$$

1) Deep case (short waves):

$kh \to \infty$, $\coth kh \to 1$ $\to$ as before

In practice: $kh \geq 2$ or $\lambda \leq \pi h$ (10% error)

2) Shallow case (long waves):

$kh \to 0$, $\coth kh \to 1/kh$

In practice: $kh \leq 0.5$ or $\lambda \geq 4\pi h$ (10%)

$$c^2 = \frac{g}{k}\frac{\Delta\rho}{\rho_1\frac{1}{kh_1}+\rho_2\frac{1}{kh_2}} = g\frac{\Delta\rho h_1 h_2}{\rho_1 h_2 + \rho_2 h_1}$$ non-dispersive!

Boussinesq case: $c^2 = g'\frac{h_1 h_2}{h_1 + h_2}$

3) One shallow layer (long waves):

$kh_2 \to 0$ (shallow), $kh_1 \to \infty$ (deep)

$$c^2 = \frac{g}{k} \frac{\Delta\rho}{\rho_1 + \rho_2 \frac{1}{kh_2}} = \frac{g}{k} \Delta\rho \frac{kh_2}{\rho_1 kh_2 + \rho_2} = \frac{\Delta\rho}{\rho_2} gh_2 = g'h_2$$

non-dispersive, shallow layer controls!

Free surface case: $\Delta\rho = \rho_2$ $\quad c^2 = gh_2$

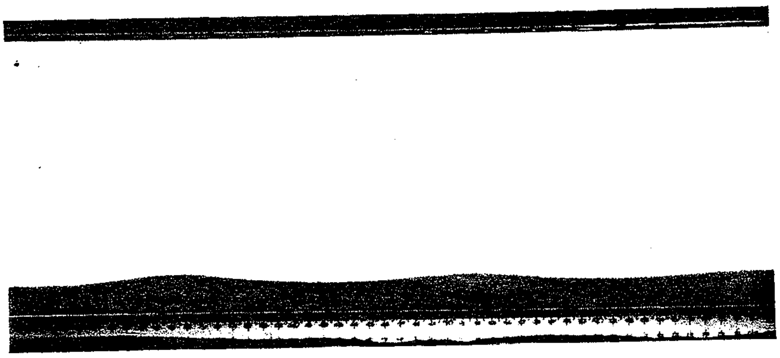
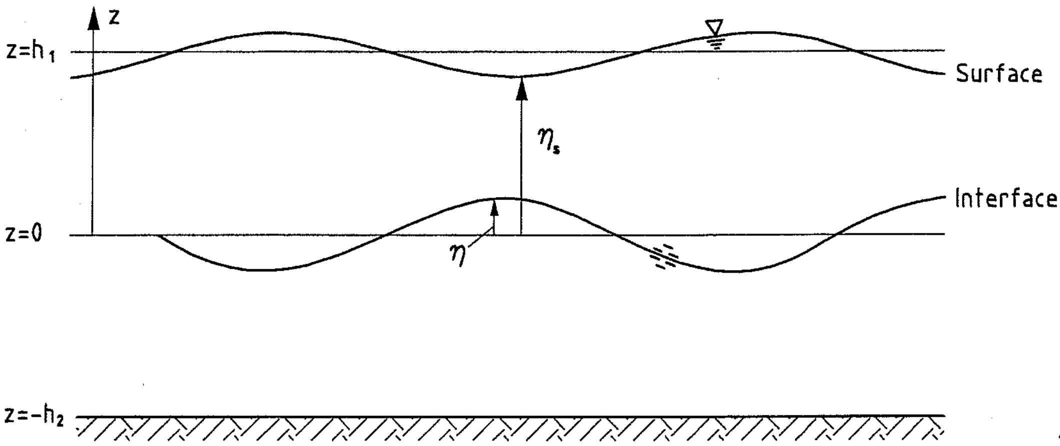


Fig. 2.1. A wave on the interface between two homogeneous fluid layers with different densities and depths. The lower layer is dyed.

(iii) <u>Two-layer system with free surface:</u>



Dispersion relation $\quad \omega^4 = f\left(k, h_1, h_2, \Delta\rho/\rho_2\right)$

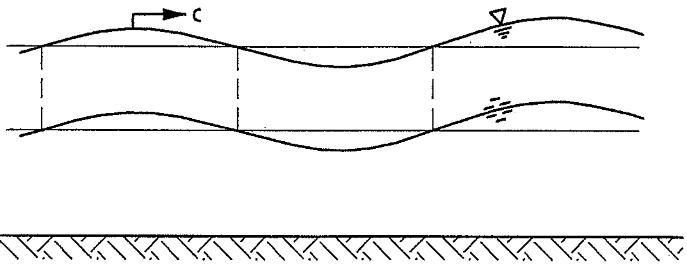
$$c^4 = f\,(.......................)$$

Surface / interface interaction! $\to$ 2 roots

Solution assuming: Long waves and Boussinesq case!

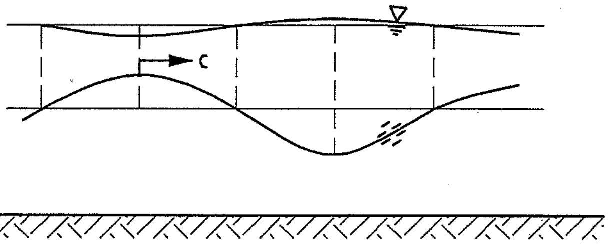
$$c^2 = g(h_1 + h_2) = gH$$

FAST MODE
(external, barotropic, sinuous)
density effects unimportant



$$c^2 = g' \frac{h_1 h_2}{H}$$

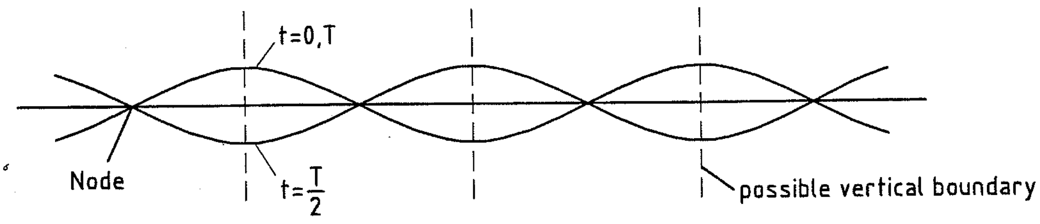
SLOW MODE $g' << g$
(internal, baroclinic, varicose)
Free surface almost flat, yields hydrostatically



In shallow waves phase speed is equal to energy transport velocity ("group velocity").

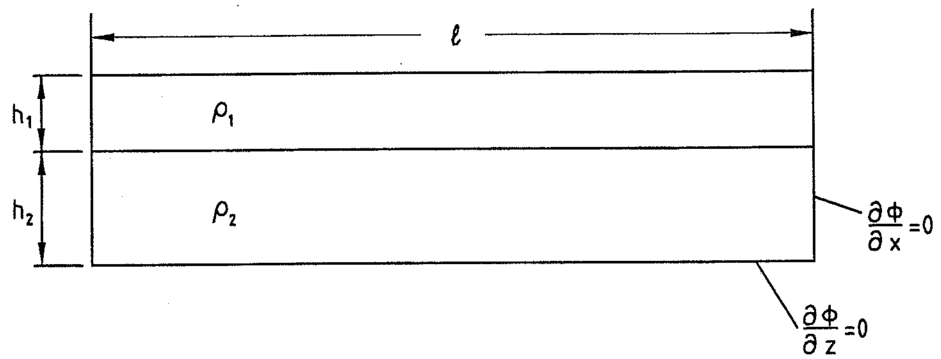
Standing waves:

Superposition of two opposing progressive wave trains



Dispersion relations are the same!

Assume: Shallow basin of finite extent ℓ



Possible wave numbers: $k = \frac{2\pi}{\lambda} = n\frac{\pi}{\ell}$ $\quad$ n = 1, 2, 3 .....

mode number

Thus: $\lambda = \frac{2\ell}{n}$

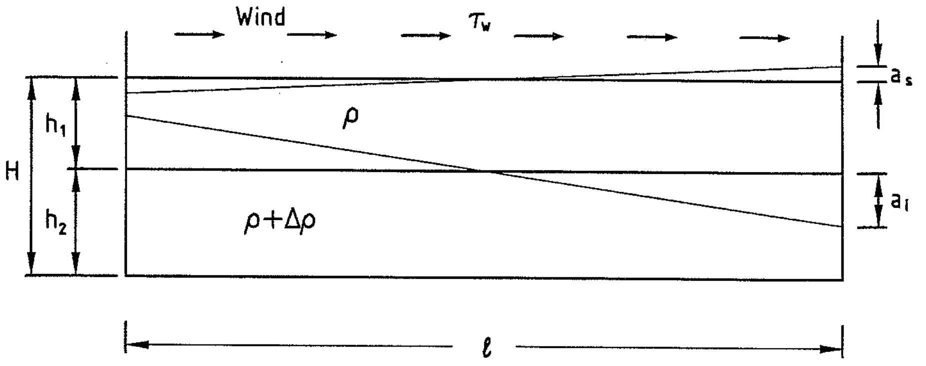
Note: $c = \frac{\omega}{k} = \frac{\frac{2\pi}{T}}{\frac{2\pi}{\lambda}} = \frac{\lambda}{T} = \frac{2\ell}{nT}$

Associated periods: $T = \frac{2\ell}{nc}$

"Seiche" motions:

- External seiches: $T_e = \frac{2\ell}{n\sqrt{gH}}$
- Internal seiches: $T_i = \frac{2\ell}{n\sqrt{g'\frac{h_1 h_2}{H}}}$

Generation of seiche motions: Wind set-up



Steady wind τ_W

$\tau_W = c_D \rho_{air} U^2_W$

$c_D \approx 10^{-3}$ typically

Surface set-up a_s: $\rho g 2 a_s H = \tau_W \ell$ Force balance

$$a_s = \frac{\tau_W}{\rho g} \frac{\ell}{2H}$$ "barotropic forcing"

Interface set-down a_i:

Pressure at basin bottom must be constant

$p_{left} = \rho g\,(h_1 - a_s - a_i) + (\rho + \Delta\rho)\, g\,(h_2 + a_i)$

$p_{right} = \rho g\,(h_1 + a_s + a_i) + (\rho + \Delta\rho)\, g\,(h - a_i)$

Set equal: $a_i = \frac{\rho}{\Delta\rho} a_s$ "Baroclinic adjustment"

Wind stops $\rightarrow$ a_s gives initial condition for external seiche

a_i for internal seiche

E. B. HENSON

LIMNOLOGY AND OCEANOGRAPHY, 1959. Vo

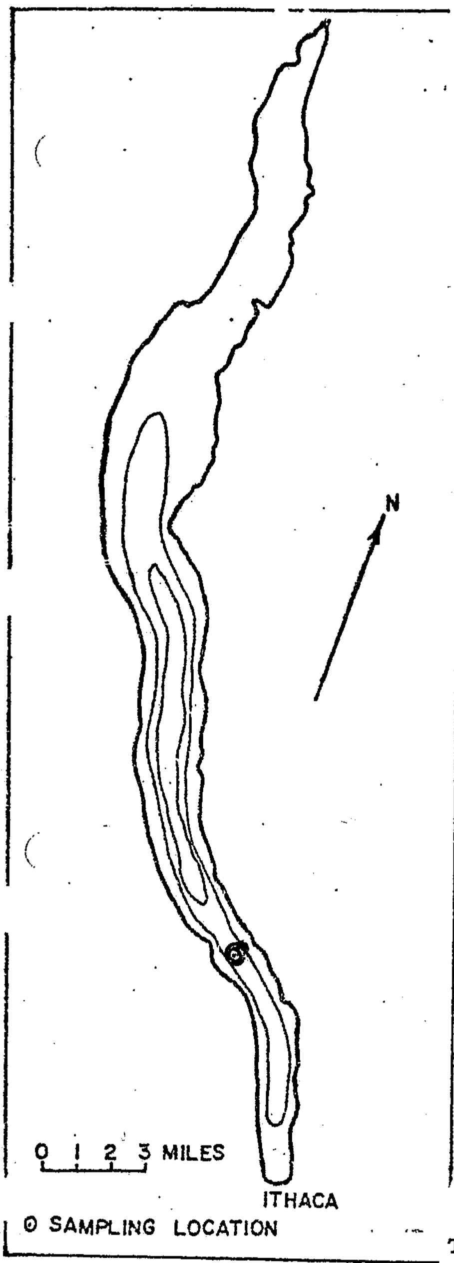


FIG. 1. Map of Cayuga Lake, New York showing the location of the Taughannock sampling station. Contour intervals are 200 feet.

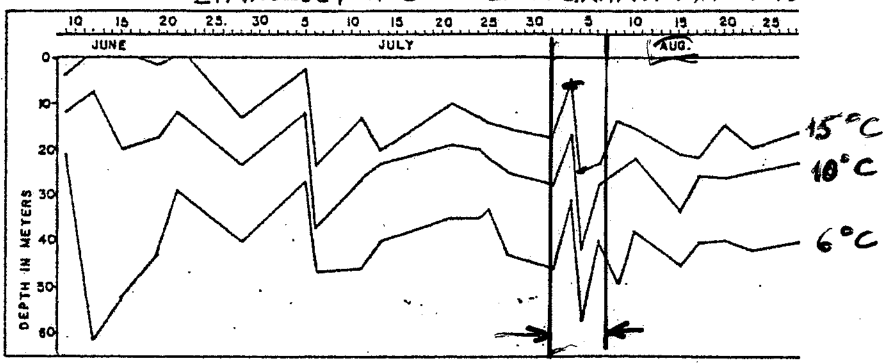


FIG. 2. Fluctuations of the 6°, 10°, and 15°C isotherms as observed from the Taughai
Cayuga Lake, from June 9 through September 7, 1951.

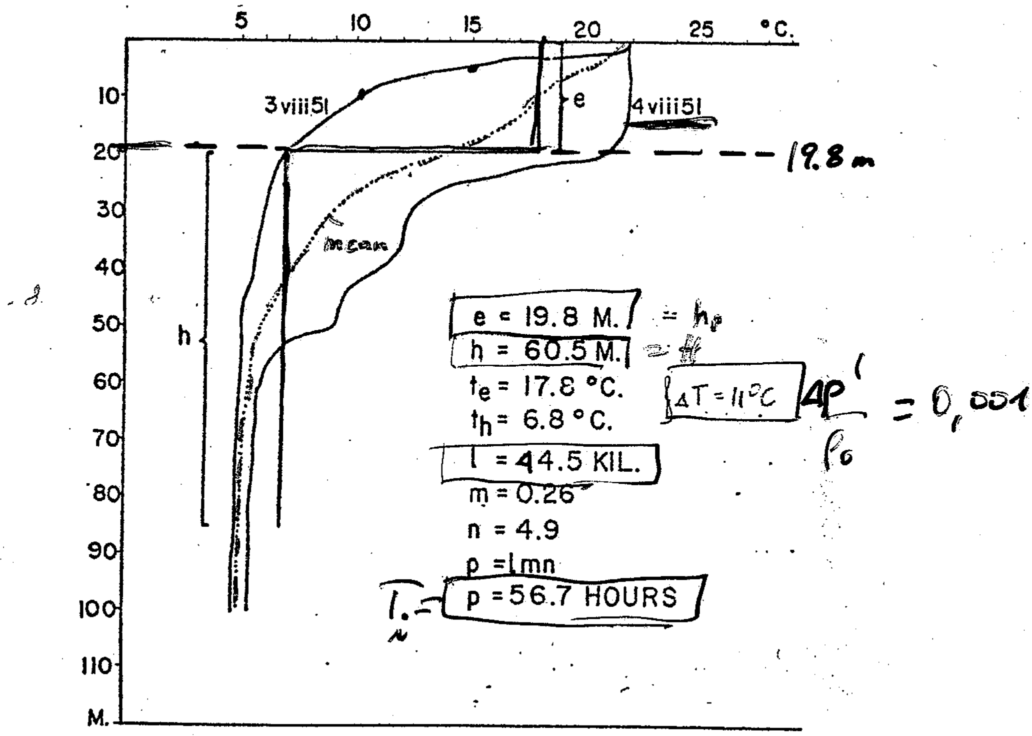


FIG. 4. Temperature profiles for August 3 and 4, 1951, from the Taughannock station, C
The mean temperature conditions of August 1 through August 8 are represented by the dotte
various factors listed are used in calculating the theoretical period of oscillation of a uninoc
ture seiche.

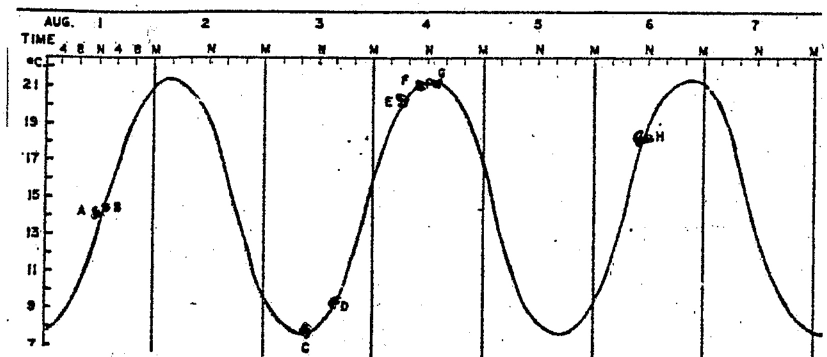


FIG. 5. The observed temperatures of the metalimnion (19.8 meters) at the Taughannock
Cayuga Lake for August 1 through August 8, 1951 are represented by points *A* through *K*. Th
represents the theoretical temperature conditions at 19.8 meters.

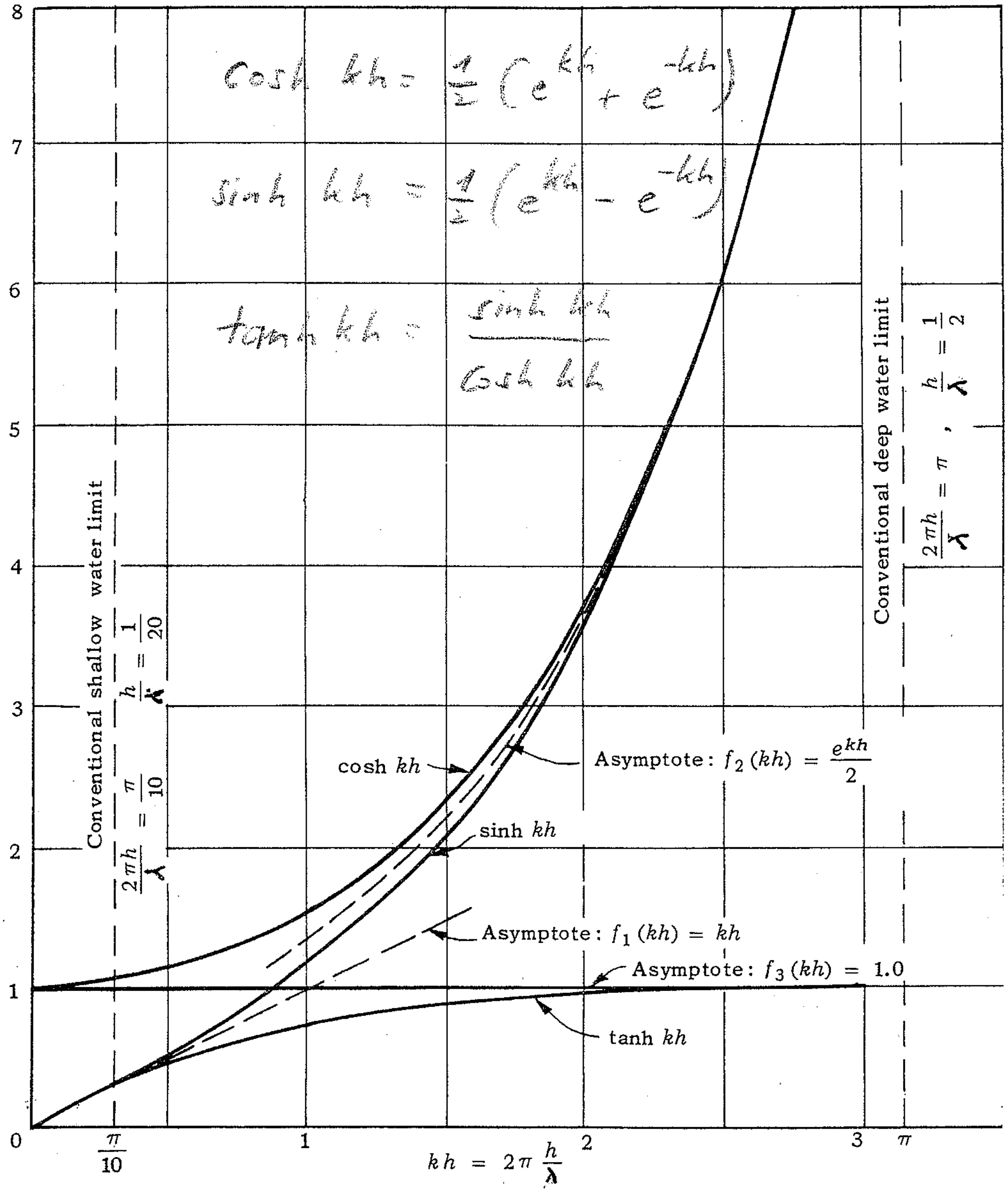


FIG. 1.9. Hyperbolic functions and asymptotes.

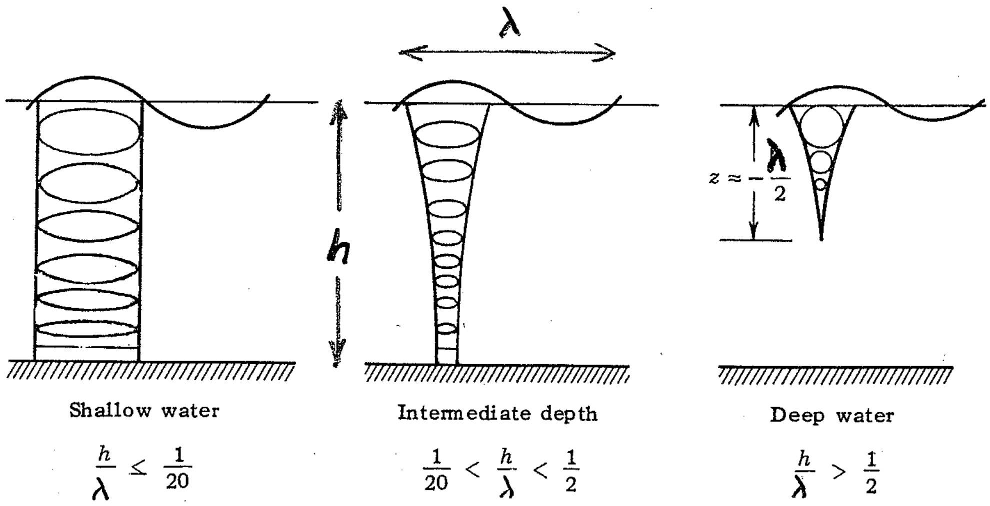
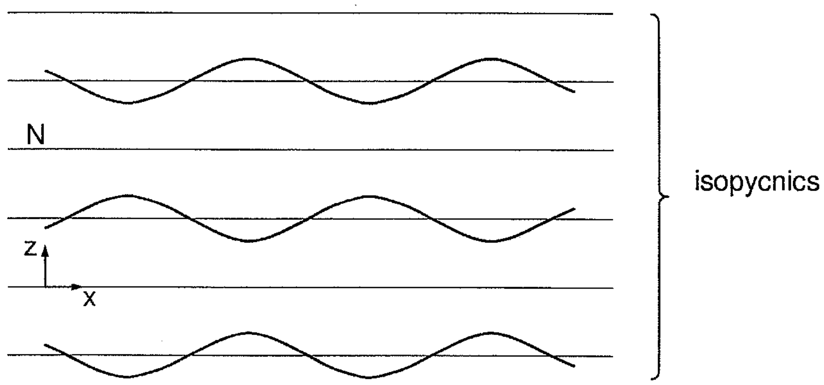


FIG. 1.11. Schematic representation of water particle trajectories.

B) Continuously Stratified Systems

(x, y), $\vec{u} = (u, w)$



$$-\frac{g}{\rho_o}\frac{\partial \rho_o}{\partial z} = N^2$$

ρ_o = equilibrium density

wave motions

→ rotational flow!

(1) $\dfrac{\partial u}{\partial x} + \dfrac{\partial w}{\partial z} = 0$ Continuity

(2) $\dfrac{\partial u}{\partial t} + \dfrac{1}{\rho}\dfrac{\partial p'}{\partial x} = 0$ x-momentum

(3) $\dfrac{\partial w}{\partial t} + \dfrac{1}{\rho}\dfrac{\partial p'}{\partial z} + g' = 0$ z-momentum

(4) $\dfrac{\partial g'}{\partial t} - N^2 w = 0$ Mass conservation $\dfrac{\partial \rho}{Dt} = 0 = \dfrac{\partial \rho'}{\partial t} + w\dfrac{\partial \rho_o}{\partial z}$

$u, w, p', g' = \dfrac{\rho - \rho_o}{\rho_o} g$

Eq. (4) → density variation 90° out of phase with velocity w

2-D wave structure:

$\lambda_h = \dfrac{2\pi}{k}$ = horizontal wave length

$\lambda_v = \dfrac{2\pi}{m}$ = vertical wave length

Dispersion relation $\omega = N\left(\dfrac{k^2}{k^2+m^2}\right)^{1/2}$

Since $\left(\dfrac{k^2}{k^2+m^2}\right)^{1/2} \leq 1$: $\omega \leq N$

N is maximum frequency of excitation!

Applications:

- <u>Standing wave patterns</u> in rectangular basin

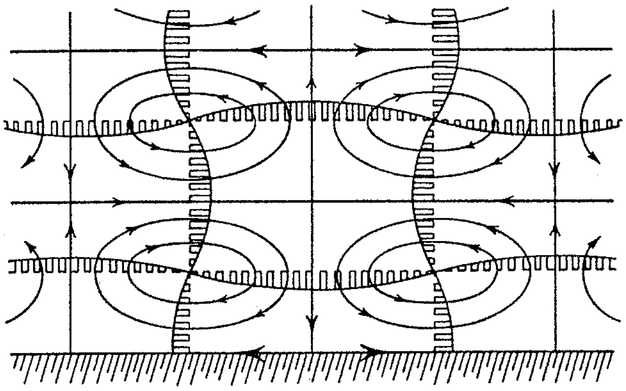


Fig. Displacements and streamlines in a cellular standing internal gravity wave. (From Prandtl 1952.)

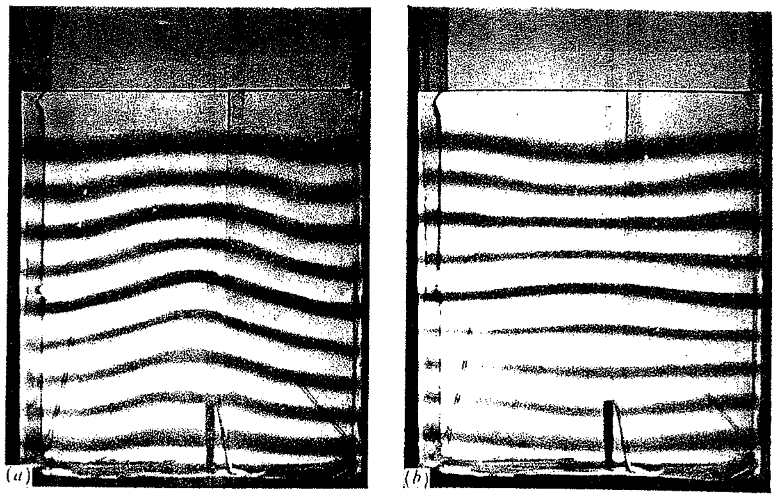
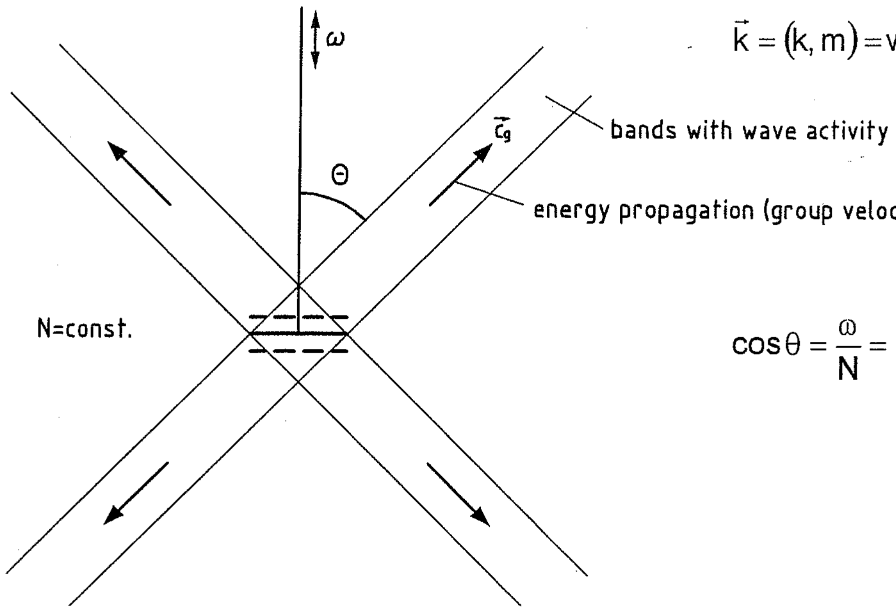


Fig. Laboratory experiments on standing internal waves in a continuously stratified fluid (*a*) mode (2, 1), (*b*) mode (2, 3). The dyed layers marking surfaces of constant density were inserted during the filling of the tank. (From Thorpe 1968*a*.)

- <u>Wave radiation from oscillating source</u>

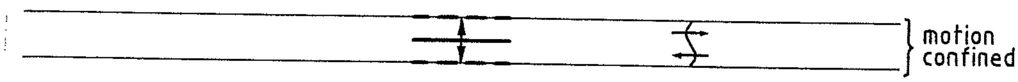
$\vec{k} = (k, m) =$ wave number vector



$$\cos\theta = \frac{\omega}{N} = \left(\frac{k^2}{k^2 + m^2}\right)^{1/2}$$

Slow excitation $\omega \ll N$; $\theta = 90°$

$\rightarrow$ horizontal motion only



Eigenfrequency $\omega = N$ $\theta = 0°$: vertical motions

Fast excitation $\omega = N$ No waves! Local dissipation!

WAVE RADIATION IN STRATIFIED FLUID (LINEAR STRATIFICATION)

No motion

ω/N = 0.32

ω/N = 0.37

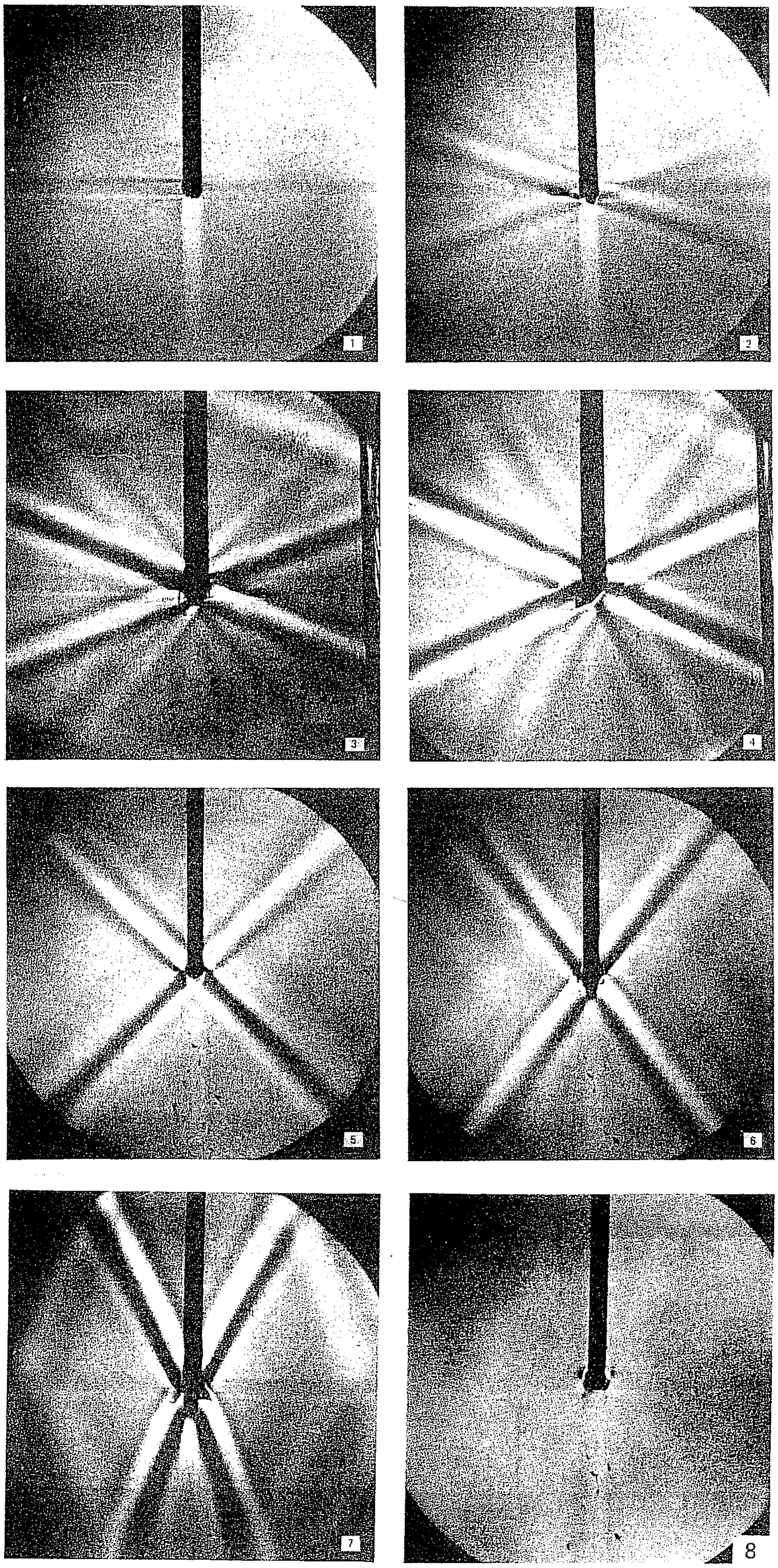
0.42

0.62

0.70

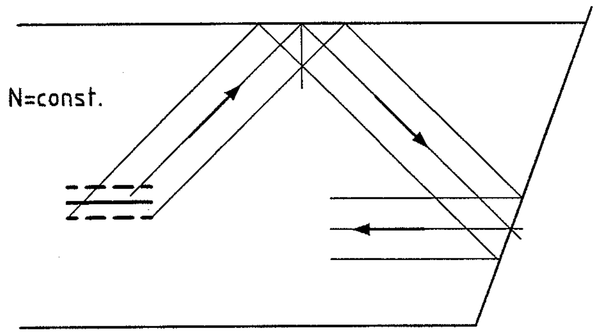
0.90

1.11



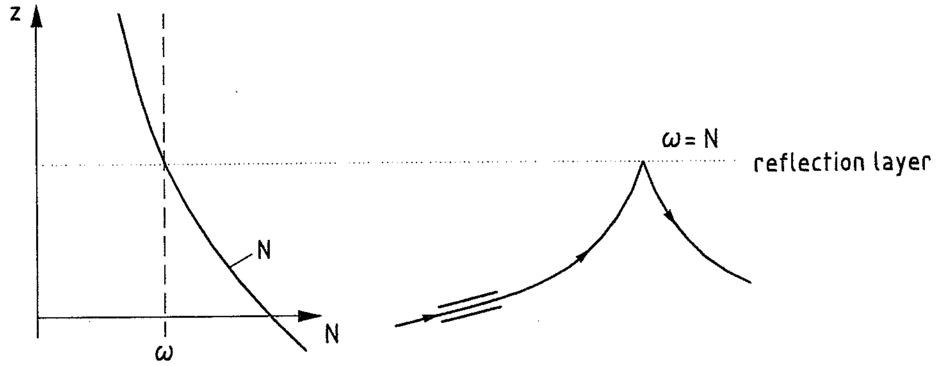
Mowbray and Rarity, 1967, JFM

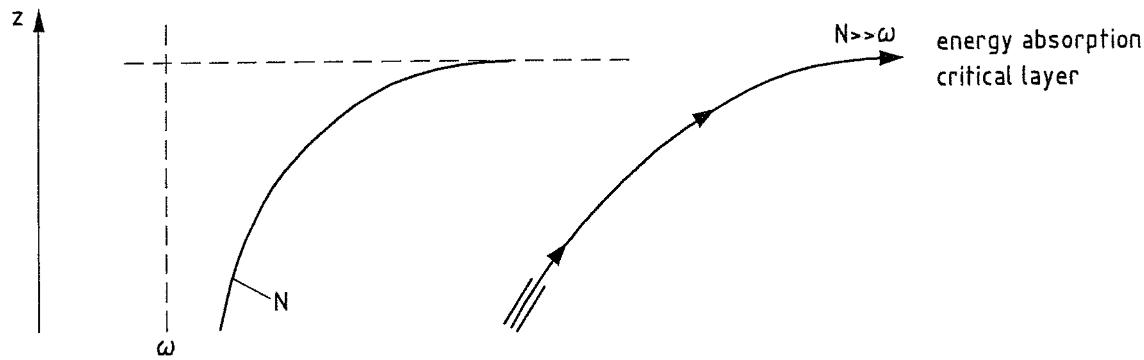
- boundary reflections



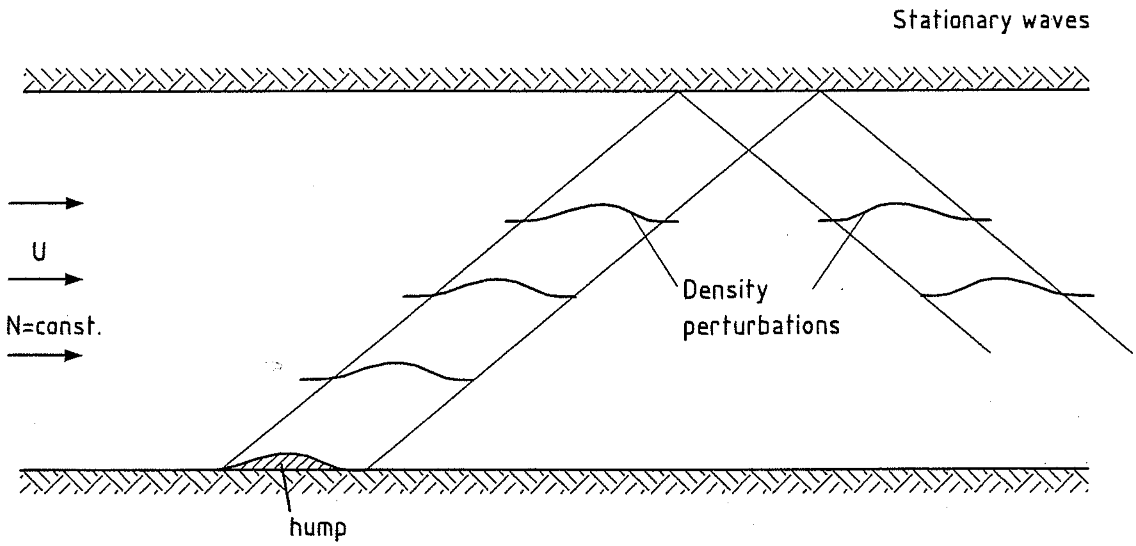
- basin shape!
- silent regions
- focusing regions

- nonlinear density variation $N \neq$ const.





- Lee waves $\Rightarrow$ superimposed mean flow U

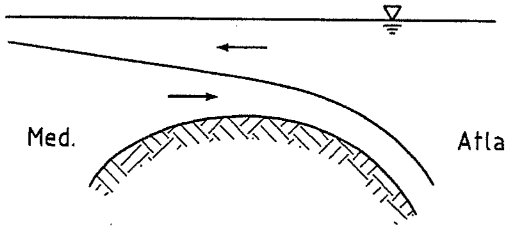


3. Two-layer stratified flow

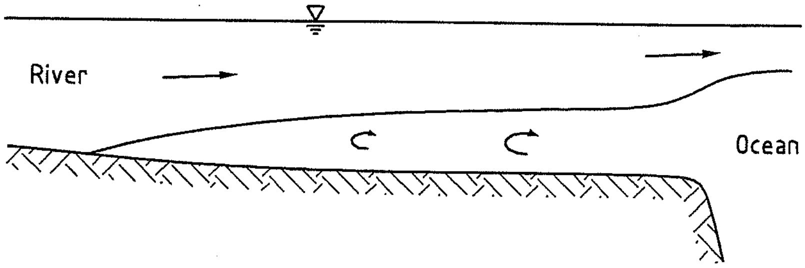
(Hydraulics, finite amplitude motions)

- predominantly horizontal flows driven by density differences
  (baroclinic effects) 1-D theory

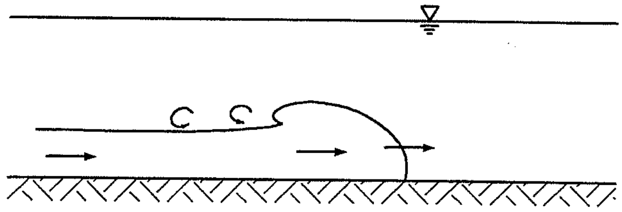
Examples:



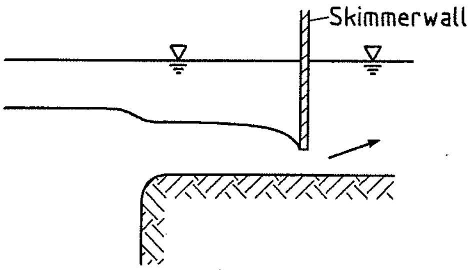
exchange flows over sills (Gibraltar)



salt wedge
in tideless estuaries



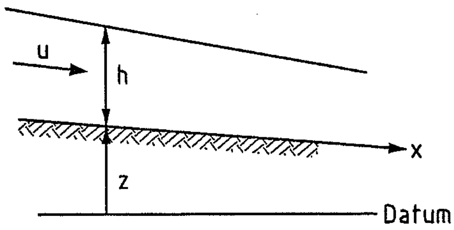
bottom density current
(e.g. turbidity current)



selective withdrawal
(cooling water intake)

- analogy to single-layer hydraulics
  - concepts of local control
  - effect of friction

Single-layer hydraulics:



1-D hydrostatic assumption

Total energy $E = \rho g(z+h) + \rho \frac{u^2}{2}$

$= \rho g z + p + \rho \frac{u^2}{2}$

- flow rate/width $q = uh$ = const.

- stationary
- constant width

Frictionless flow $\frac{dE}{dx} = 0$ (short distances)

$$\rho g\left(\frac{dz}{dx} + \frac{dh}{dx}\right) - \rho \frac{q^2}{h^3}\frac{dh}{dx} = 0$$

$$\frac{dz}{dx} = -\left(1 - \frac{q^2}{gh^3}\right)\frac{dh}{dx} = -\left(1 - F^2\right)\frac{dh}{dx}$$

$F = \frac{u}{\sqrt{gh}} =$ (free surface) Froude number

$$\frac{dh}{dx} = \frac{-\frac{dz}{dx}}{1 - F^2}$$

$F^2 = 1$: critical flow:

$\frac{dh}{dx} \to \infty$ unless $\frac{dz}{dx} = 0$

- flow over a hump
- flow through constriction

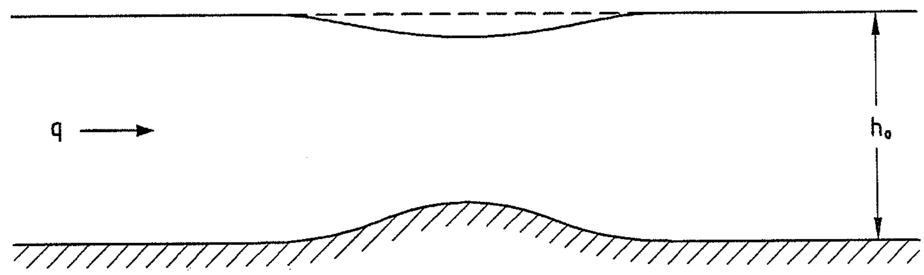
} concept of local control

Flow with friction: $\frac{dE}{dx} = -S_f =$ friction slope

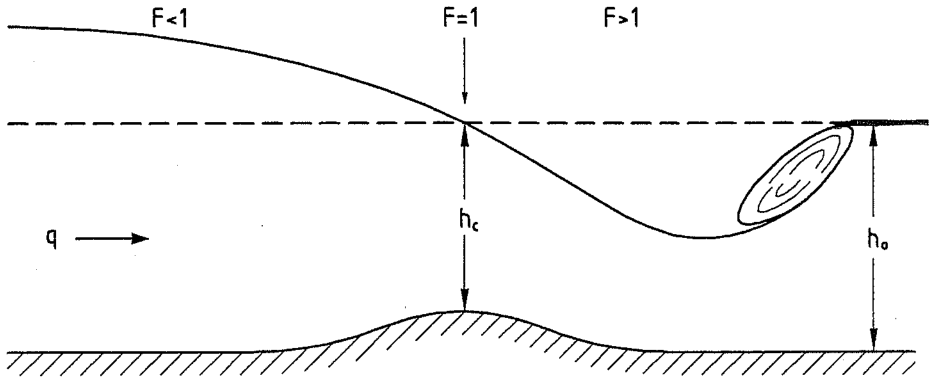
e.g. $S_f = \frac{f}{4R_h}\rho\frac{u^2}{2}$ Darcy-Weisbach

$R_h \approx h$

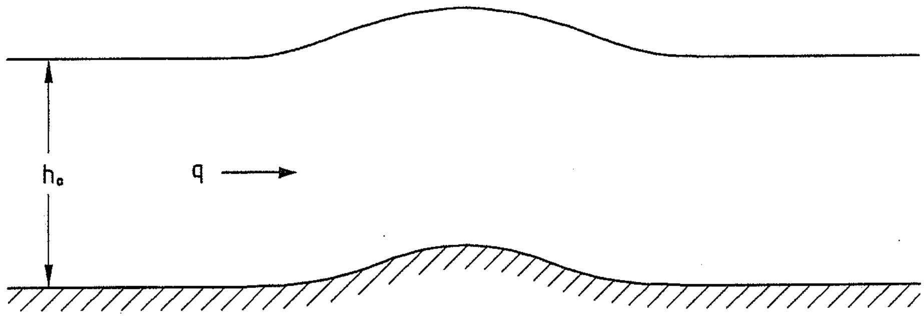
Frictionless flow over hump: h_o = depth without hump



Subcritical flow
F < 1



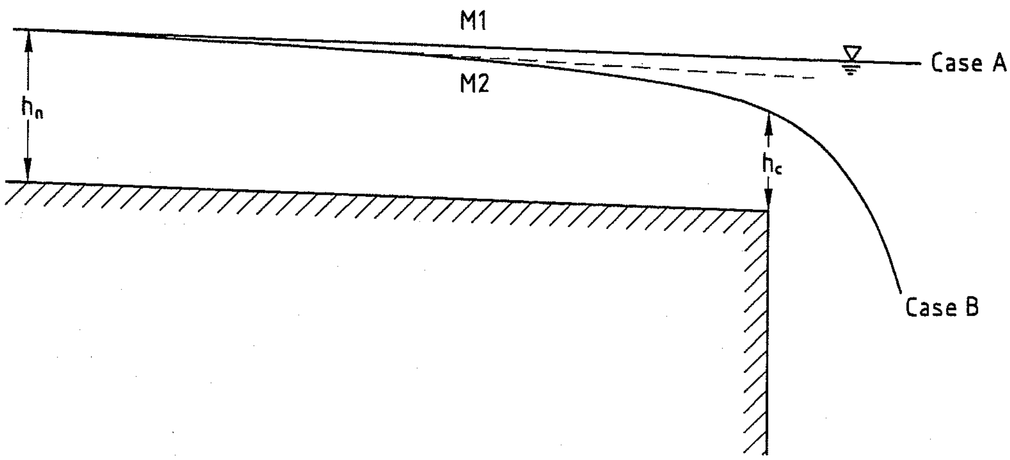
Crest-controlled flow
F = 1 at crest



Supercritical flow
F > 1

Flow with friction: Backwater curves
e.g. subcritical flow

h_n = normal depth



Two-layer hydraulics:

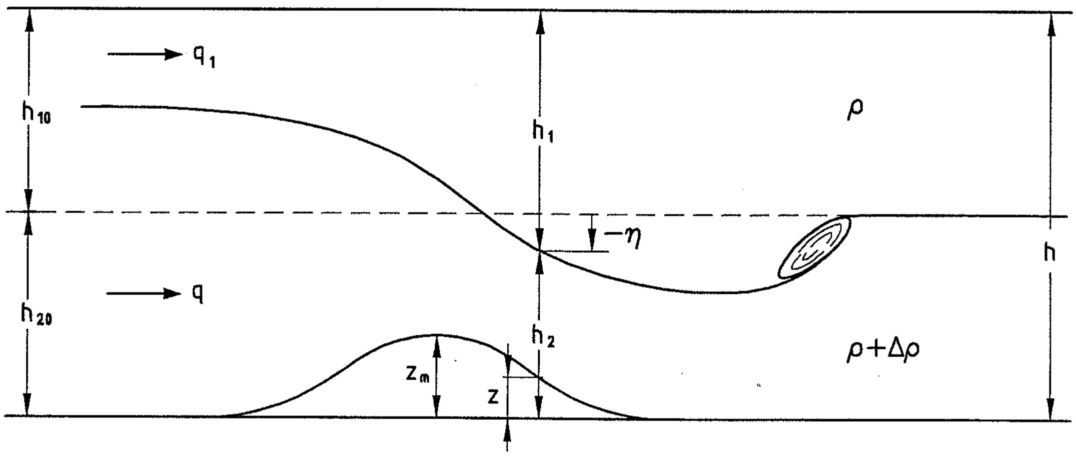
- similar principles
- more degrees of freedom

  q_1, q_2 (co- or counterflow)

Ex.: Frictionless Co-flow with barotropic forcing

i.e. constant velocities in both layers without obstacle (Lawrence, 1993)

Boussinesq case → free surface ≅ constant



Parameters: ρ, $\Delta\rho$, g, q_1, q_2, h, z_m

Four dimensionless groups: $r = \frac{q_2}{q}$, $\beta_m = \frac{z_m}{h}$, $G_o = \frac{q}{\sqrt{g' r(1-r)h^3}}$, $\frac{\Delta\rho}{\rho}$

where $q = q_1 + q_2$, $h_{20} = rh$, $g' = \frac{\Delta\rho}{\rho} g$

Define: $\eta = z + h_2 - h_{20}$ Disturbance of interface

$$\frac{d\eta}{dx} = \frac{-F_2^2 \frac{dz}{dx}}{1-G^2} \qquad F_1^2 = \frac{q_1^2}{g' h_1^3}, \quad F_2^2 = \frac{q_2^2}{g' h_2^3}$$

$$G^2 = F_1^2 + F_2^2$$

Composite local Froude number

Classification diagram
(Lawrence, 1993)

r = 0.5

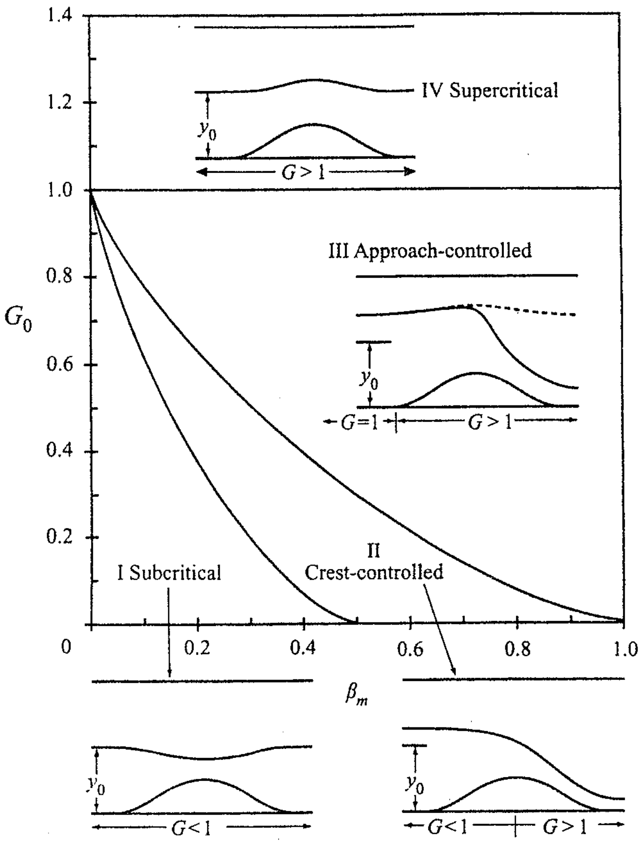
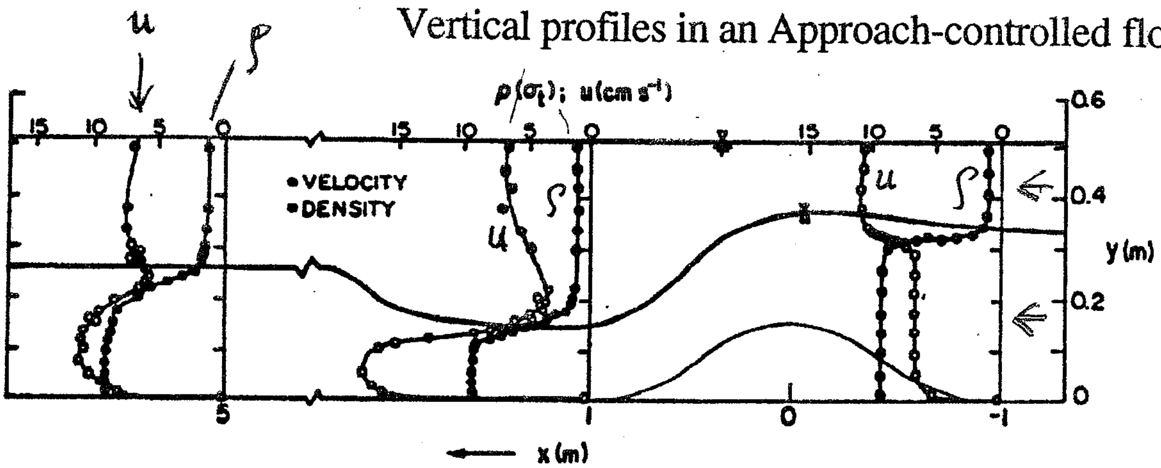


FIGURE Classification diagram for $r = 0.5$ showing the regions of the (β_m, G_0)-plane corresponding to each of the flow regimes (adapted from Lawrence 1993). The dashed line in the sketch of Approach-controlled flow represents the hydrostatic solution.

Vertical profiles in an Approach-controlled flow



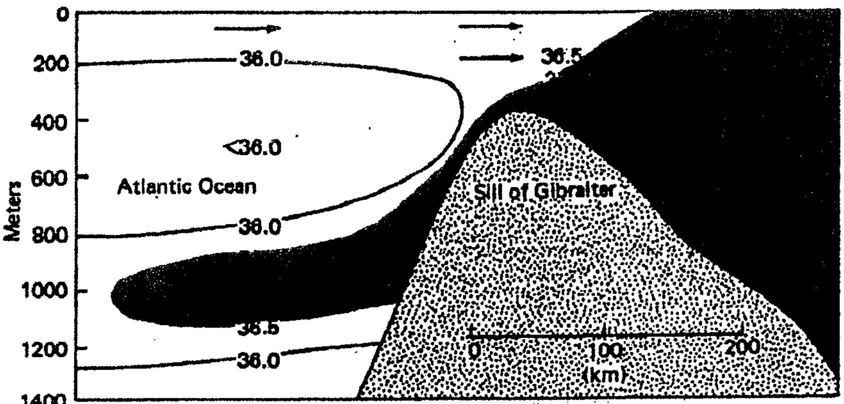
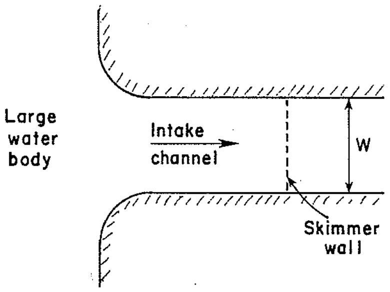
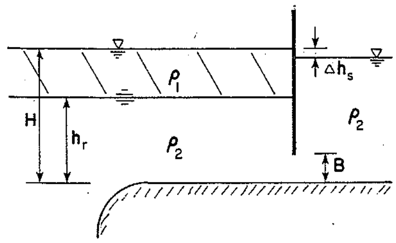


Figure . Dense, highly saline water of the Mediterranean flows into the North Atlantic over the sill at the Straits of Gibraltar. As it sinks it mixes with the surrounding water and reaches a density equilibrium at about 1000 m. Spreading of the high-salinity Mediterranean water can be traced across the entire Atlantic (from Knauss, 1978).

Ex: <u>Selective Withdrawal with skimmer wall</u>
(Cooling water intakes) (Jirka, 1979)

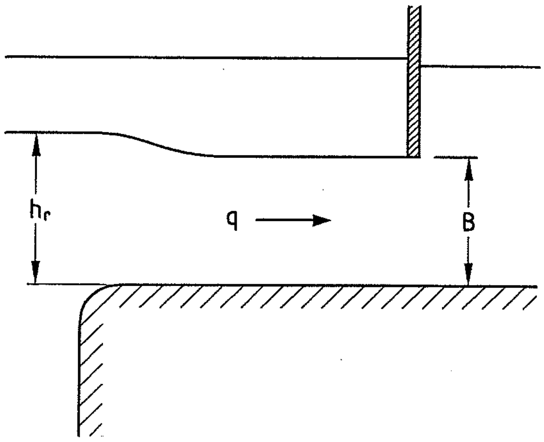


a) Plan view



b) Hydrostatic conditions

Operating condition: Lower layer flow only

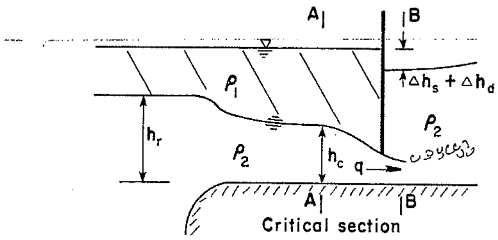


$$h_r = \frac{q^2}{2gB^2} + B$$

$$q = B\sqrt{2g'(h_r - B)}$$

permissible withdrawing rate

$$\frac{dq}{dB} = 0 \quad \Rightarrow \quad q_{max} = \sqrt{g'\left(\frac{2}{3}h_r\right)^3} \quad ; \quad B = \frac{2}{3}h_r$$



c) Incipient

Two-dimensional skimmer wall in two-layer flow with design condition $B < 2/3\ h_r$.

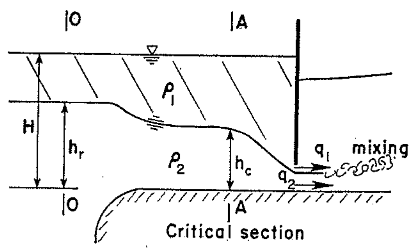
Critical flow at gate

if $B < \frac{2}{3}h_r$ critical section shifts forward

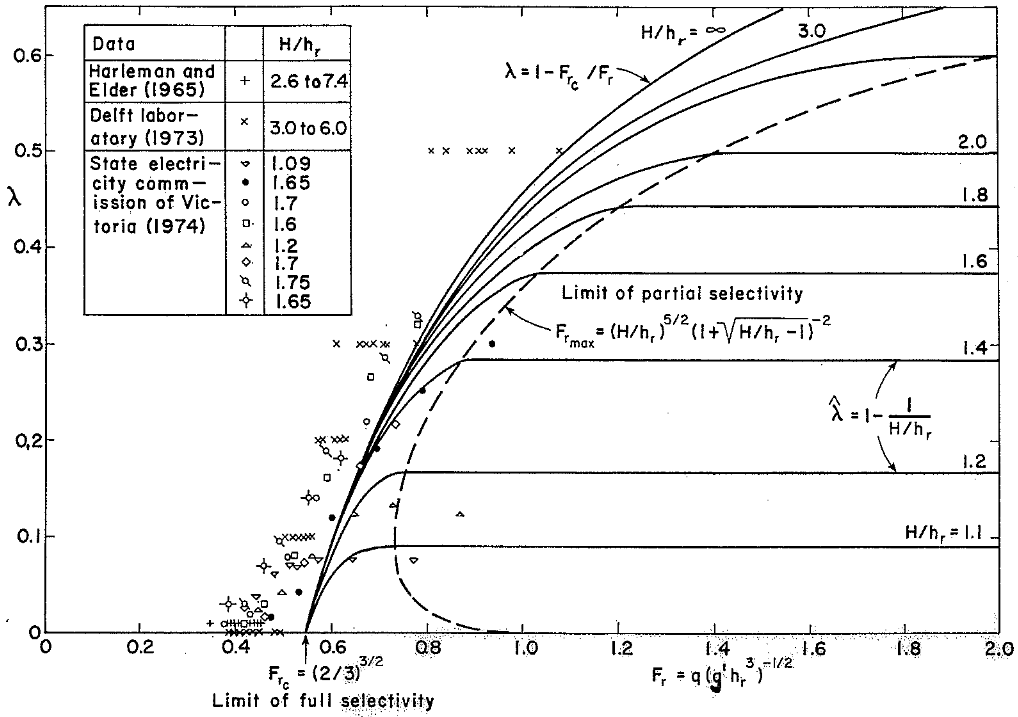
if $q > q_{max}$:

Both layers flow

- but selectivity is maintained

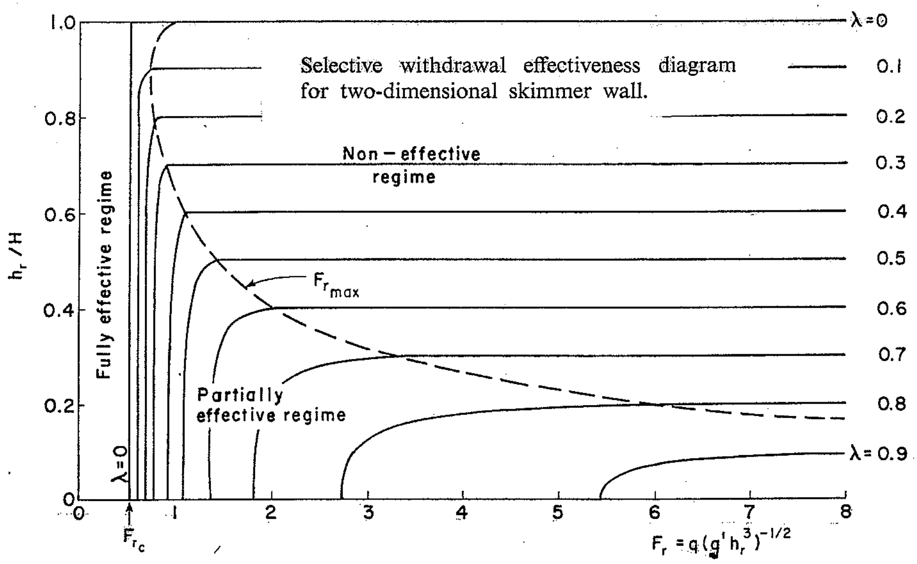


d) Supercritical withdrawal



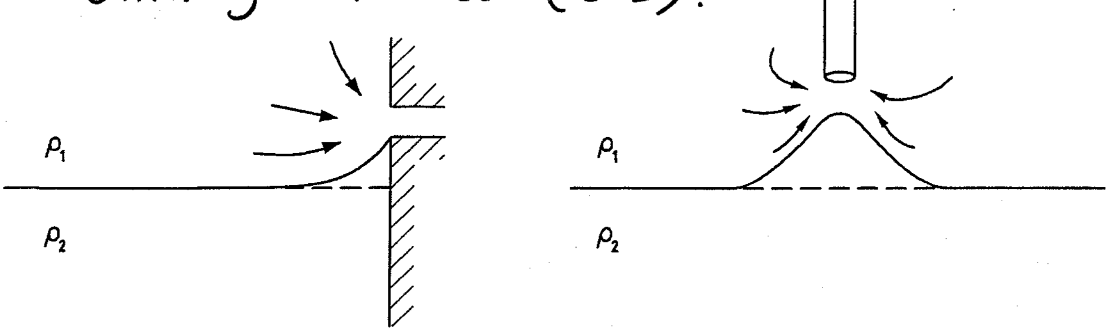
Comparison of complete skimmer wall theory with experimental data for $B < 2/3h_r$.

$$F_r = \frac{q}{\left(g' h_r^{\,3}\right)^{1/2}}$$ withdrawal Froude No.



Selective withdrawal effectiveness diagram for two-dimensional skimmer wall.

Other geometries (3-D):

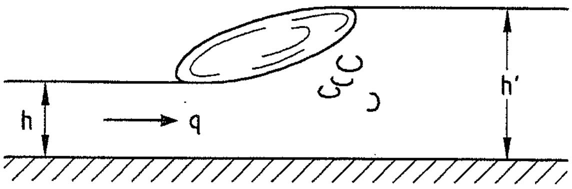


Jirka and Katavola (1979)

Ex: Internal hydraulic jump

→ sudden flow transition from strongly supercritical condition to less supercritical (shocks) or subcritical conditions; with energy dissipation

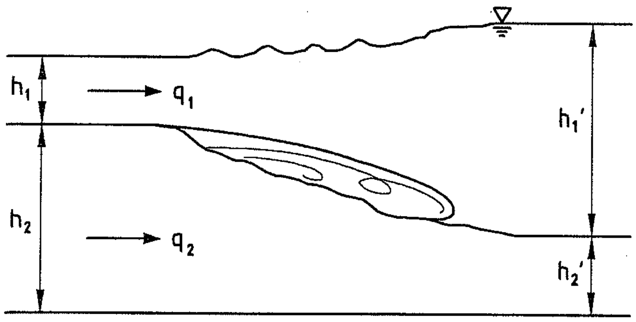
Compare: free surface flow



conjugate depth h'

"Flow force" = $\frac{1}{2}\rho g h^2 + \rho q \frac{q}{h} = \text{const.}$

$$\frac{h'}{h} = \frac{1}{2}\left(\sqrt{1+8F^2} - 1\right)$$



Extension: Yih and Guha (1953)
4 possible conjugate states

- neglect entrainment

Boussinesq case: Jirka and Harleman (1979)

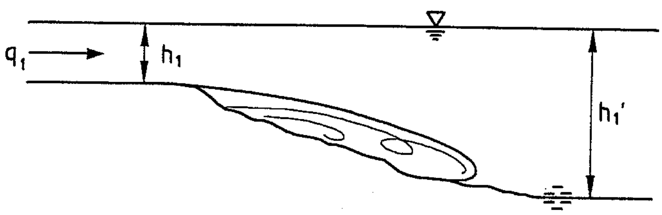
$$h_1 + h_2 = h_1' + h_2' = H$$

$$f\left(h_1'\right) = \left[\left(\frac{h_1'}{h_1} - 1\right)\frac{h_1}{h_2} - \frac{3}{2}\right]^2 = \frac{1}{4} + \frac{2F_2^{\,2}}{1 - \dfrac{2F_1^{\,2}}{\dfrac{h_1'}{h_1}\left(\dfrac{h_1'}{h_1} + 1\right)}}$$

$$F_1^{\,2} = \frac{q_1^{\,2}}{g' h_1^{\,3}}, \quad F_2^{\,2} = \frac{q_2^{\,2}}{g' h_2^{\,3}}$$

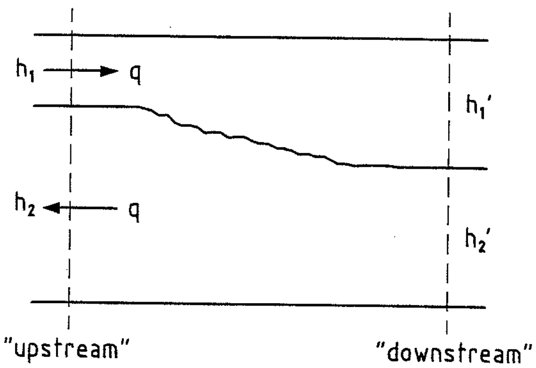
Special case:
Inverted jump for surface layer

$$h_2 \to \infty, \quad F_2 \to 0$$



$$\frac{h_1'}{h_1} = \frac{1}{2}\left(\sqrt{1+8F_1^{\,2}} - 1\right)$$

Special case: Equal counterflow



Upstream supercriticality

$$S_u = F_1^2 + F_2^2$$

Downstream

$$S_d = F_1'^2 + F_2'^2$$

For jumps $S_d < S_u$

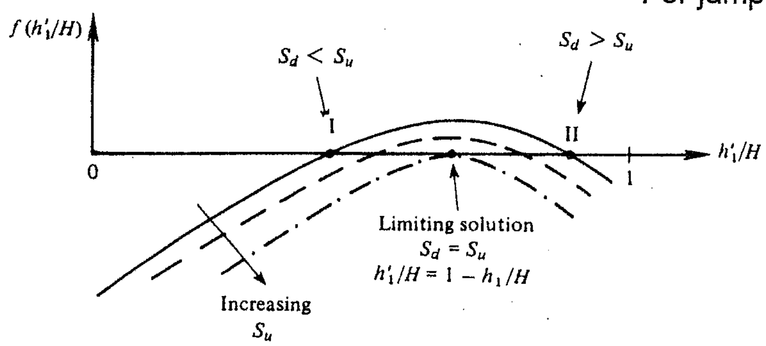


Figure 7. Solution properties of internal hydraulic jump equation (32) as a function of upstream supercriticality S_u. Solution I is the general admissible solution with lower specific energy.

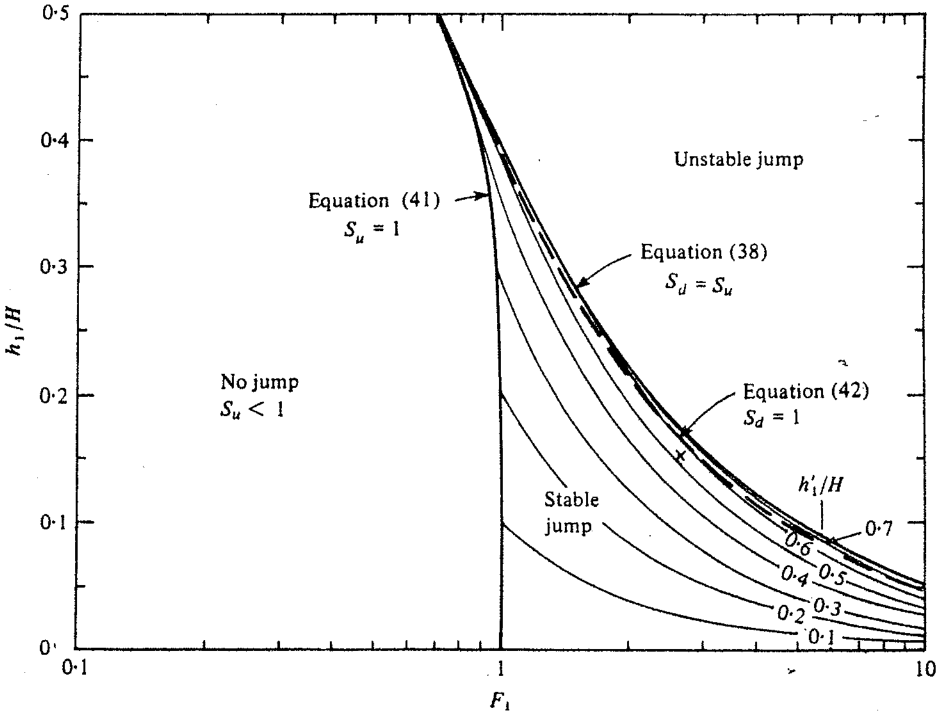


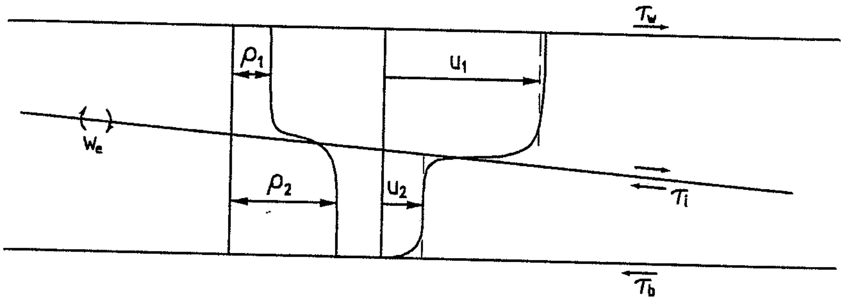
Figure 8. Stability plot for internal hydraulic jumps with equal flow in both layers as a function of upstream conditions F_1 and h_1/H (× indicates conditions for pure plume discharge).

(Jirka et al., 1979)

- Also: Radial internal jumps

Ex: <u>Flow with Friction, Horizontal Bottom</u>

Boussinesq case



$$\tau_b = \frac{f_o}{8}\rho u_2 |u_2| \quad \text{bottom shear}$$

$$\tau_i = \frac{f_i}{8}\rho (u_1 - u_2)|u_1 - u_2| \quad \text{interfacial shear}$$

$$\tau_w = c_D \rho_{air} U_a^2 \quad \text{wind shear (neglected)}$$

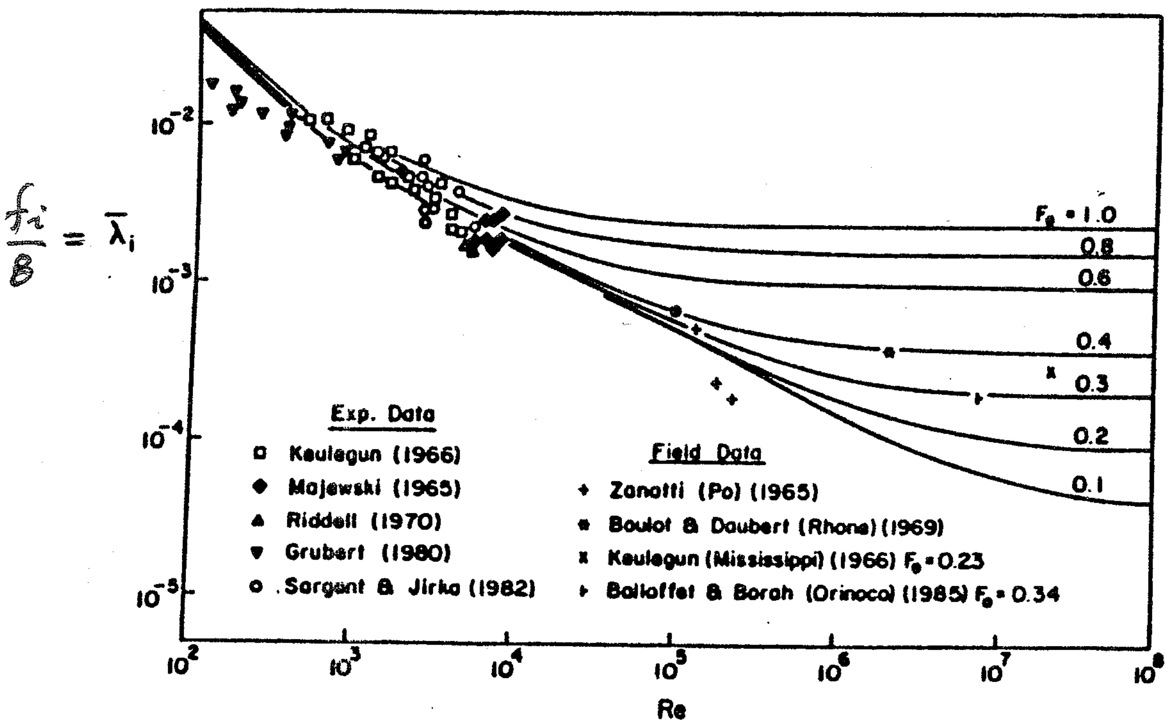


Fig. . : Interfacial Shear Stress Coefficient λ_i as a Function of Reynolds Number and Froude Number (Arita and Jirka, 1987)

Schijf and Schönfeld (1953)

$$\frac{q_1^2}{gh_1^2}\frac{dh_1}{dx} = h_1\left[\frac{dh_1}{dx} + \frac{dh_2}{dx}\right] + \frac{\tau_i}{\rho_1 g}$$

$$\frac{q_2^2}{gh_2^2}\frac{dh_2}{dx} = \frac{1}{\rho_2}h_2\left[\rho_1\frac{dh_2}{dx} + \rho_2\frac{dh_2}{dx}\right] + \frac{\tau_b - \tau_i}{\rho_2 g}$$

Boussinesq case; quadratic friction law

$$\frac{dh_2}{dx} = \frac{-\frac{f_o}{8}F_2^2\,\mathrm{sign}(q_2) - \frac{f_i}{8}F_1^2\frac{H}{h_2}\left(1 - \frac{q_2}{q_1}\left(\frac{H}{h_2} - 1\right)\right)^2 \mathrm{sign}(q_1 - q_2)}{1 - F_1^2 - F_2^2}$$

where F_1 , F_2 are densimetric Froude numbers defined as

$$F_1^2 = \frac{q_1^2}{g'h_1^3}, \quad F_2^2 = \frac{q_2^2}{g'h_2^3} \qquad \text{local}$$

and sign (a) = (+1) if a is positive and (-1) if a is negative.

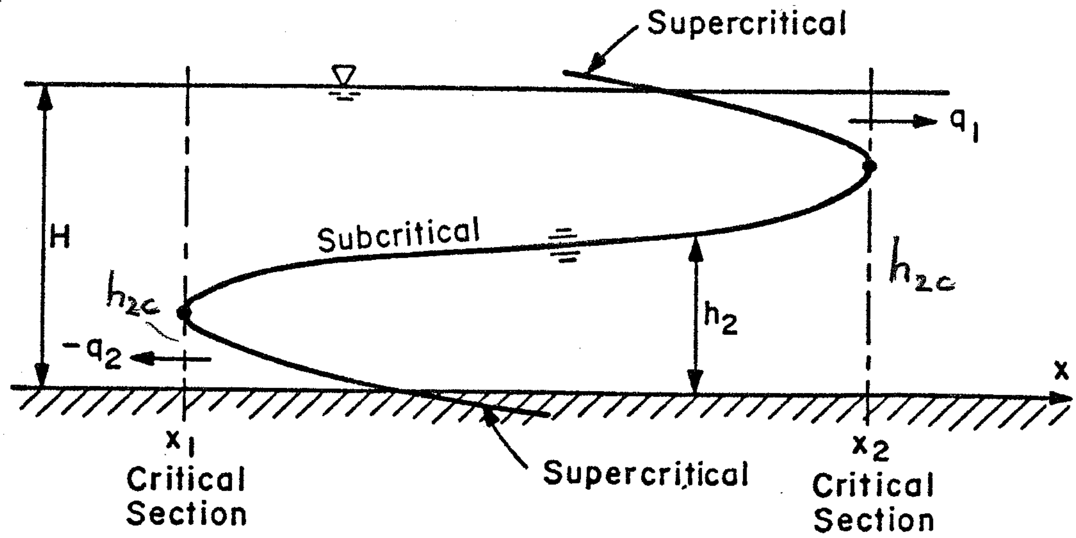


Fig. : Interface Profiles: General Solution for One-Dimensional Stratified Flow Equation

Parameters: q_1 , q_2 , g', H , f_o , f_i

$$F_{2H} = \frac{q_2}{\sqrt{g'H^3}}, \quad Q = \frac{q_1}{q_2}, \quad f_o, \quad f_i$$

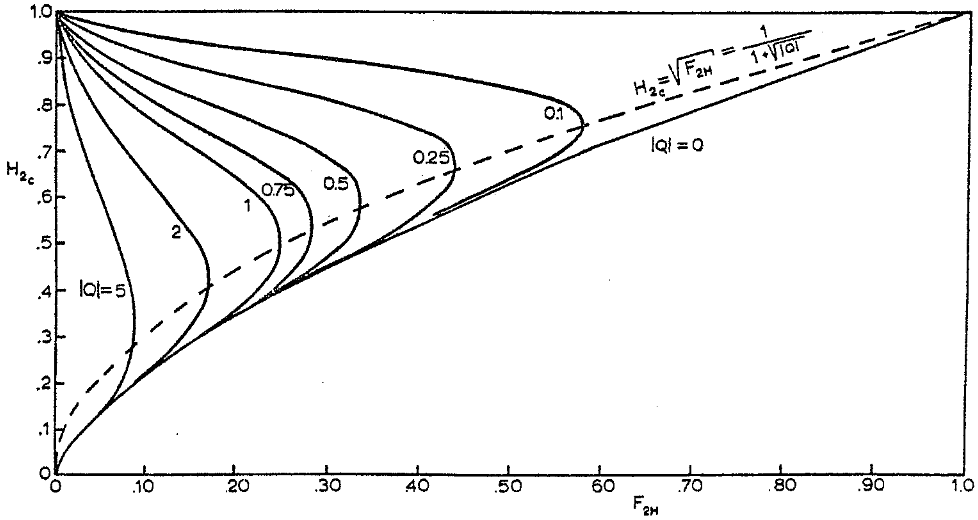


Fig. : Critical Depth as a Function of Densimetric Froude Number and Flow Ratio

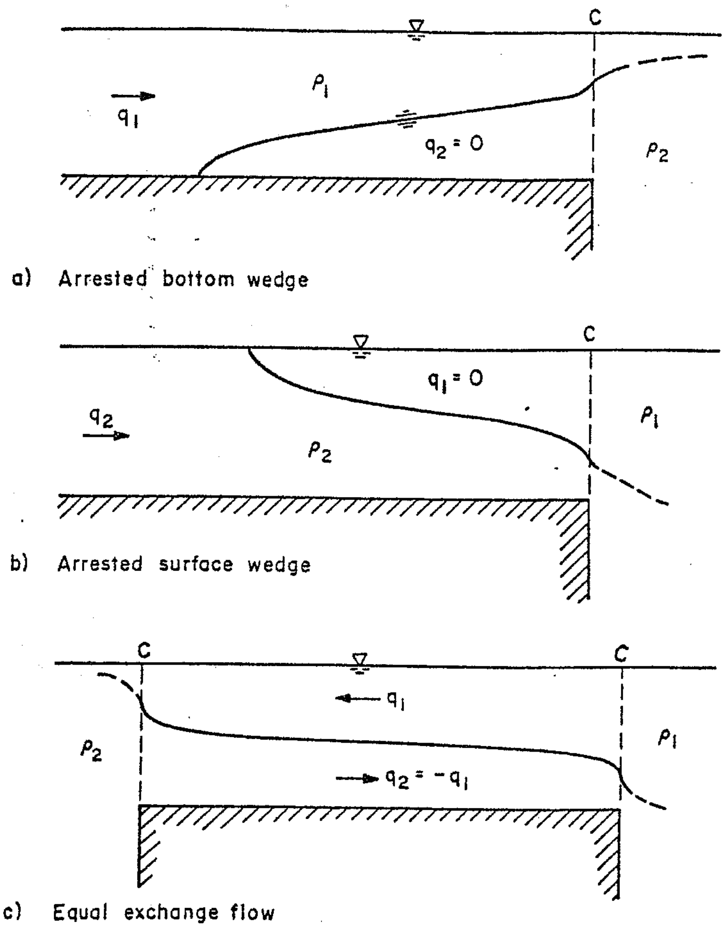


Fig. : Three Examples of One-Dimensional Stratified Flow ("C" denotes a critical section)

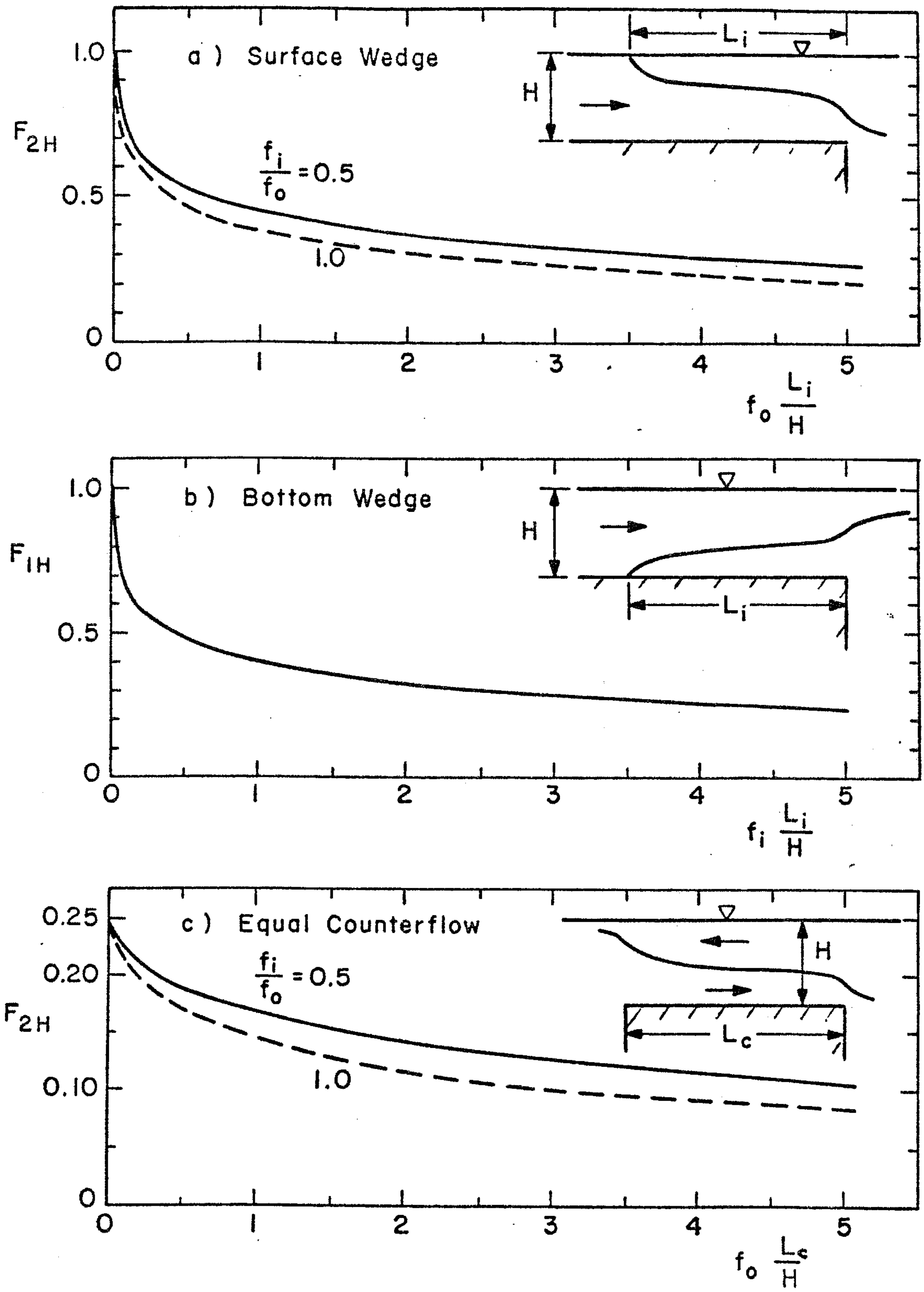


Fig. : Non-Dimensional Solutions of Wedge or Channel Length as a Function of Densimetric Froude Number

Salt Wedge

Intrusion length $$\frac{L_i}{H} = \frac{2}{f_i}\left[\frac{1}{5F_{1+4}} - 2 + 3F_{1H}^{\;2/3} - \frac{6}{5}F_{1H}^{\;4/3}\right]$$

Sargent and Jirka, J. Hydr. Eng., (113), 1987

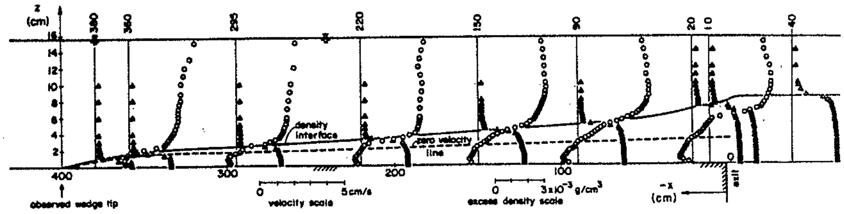


Fig. . Velocity and Density Distribution for Wedge Experiment 4/17 (F_0 = 0.39)

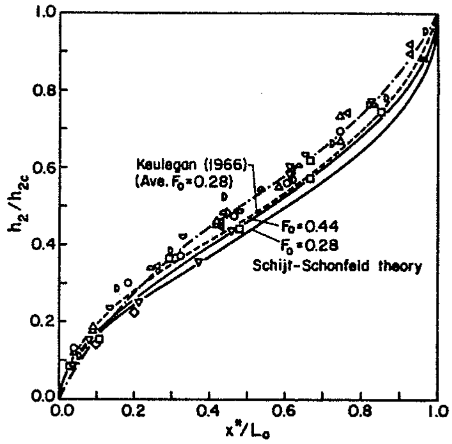


Fig. .. Normalized Interface Shape for Salt Wedge. Comparison of Experiments and Schijf-Schonfeld Theory

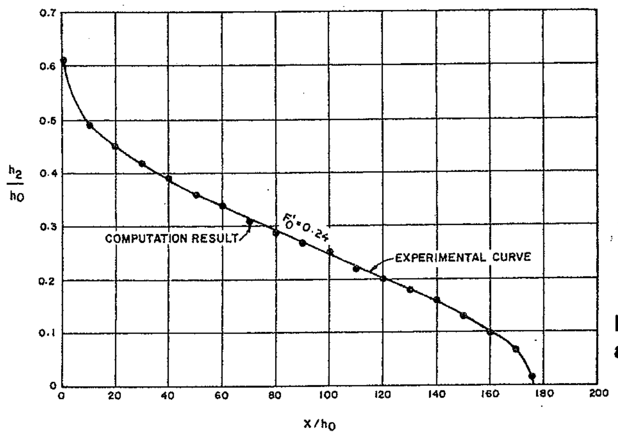


FIG. —Comparison of Experimental and Computed Salt Wedge Profile in Horizontal and Rectangular Canal

Ballofet and Borah, J. Hydr. Eng. (111), 1985

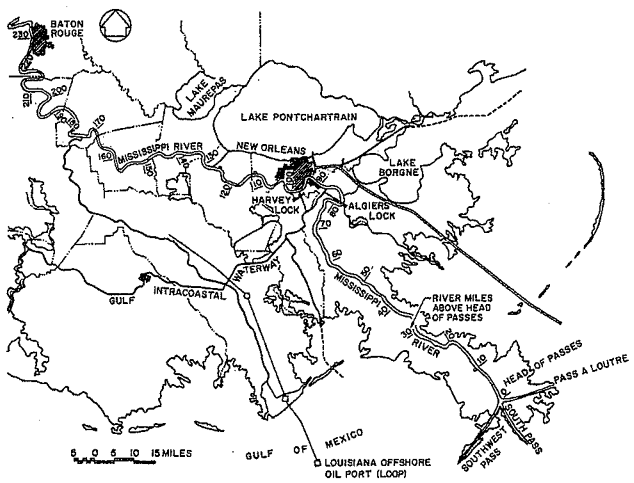


FIG. —Map of Lower Mississippi

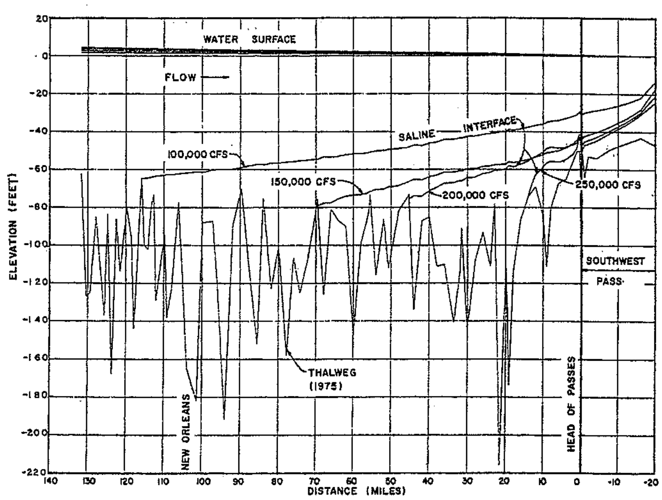


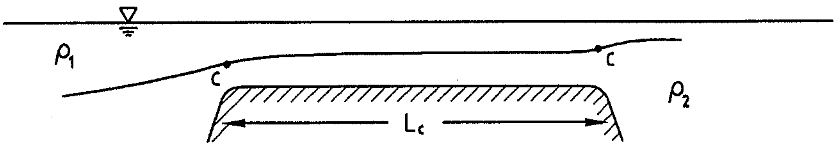
FIG. —Steady-State Saline Interfaces in Lower Mississippi: Existing Channel (1 ft = 0.305 m, 1 mile = 1.61 km, 1 cfs = 0.028 m³/s)

Extended Salt Wedge Model: Arita and Jirka (1987)

- with entrainment and salt circulation

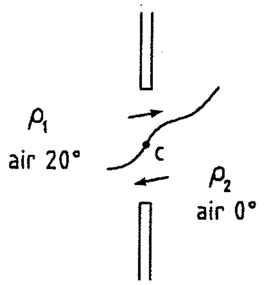
<u>Exchange Flow Between Basins</u>

- e.g. warm/cold lakes
  fresh/brackish/salt water: e.g. Baltic Sea



limitations on mixing

- open door/window



$L_c \Rightarrow 0$

$$F_{2H} = \frac{q_2}{\left(g' H^3\right)^{1/2}} = \frac{1}{4}$$

$$g' = \left(\frac{20}{273+20}\right) g = 0.067 \times 10 = 0.67 \frac{m}{s^2}$$

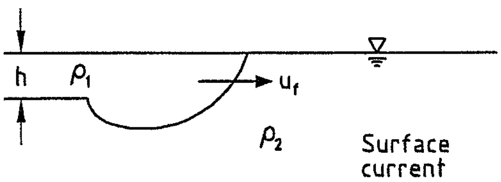
Opening H = 1m, B = 1m

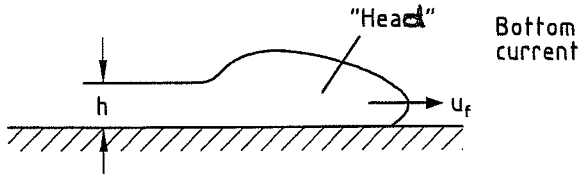
$$q_c = \frac{1}{4}\sqrt{0.67 \times 1^3} = 0.20 m^3 / s, m$$

$$Q_c = q_c B = 12.3 m^3 / min$$

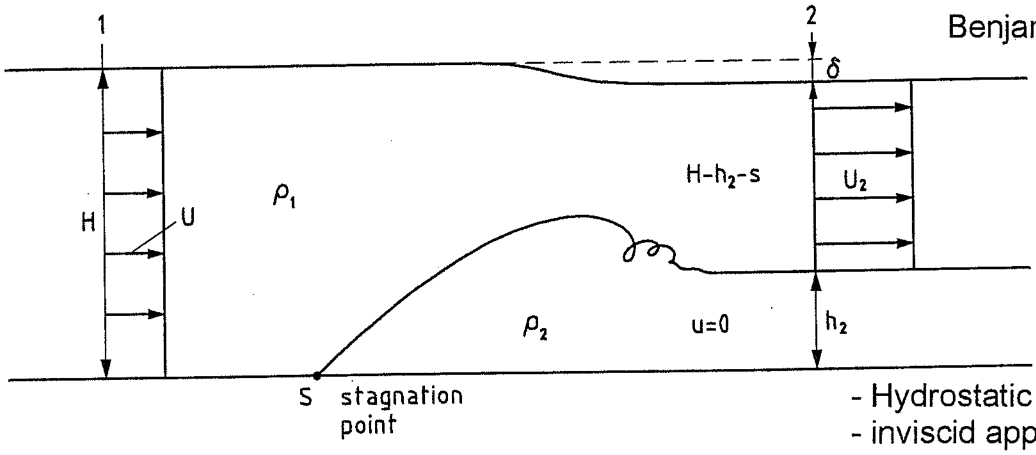
- tunnel ventilation, flushing mechanisms
  fire control

Ex. Density (Gravity) Currents (Fronts)





Benjamin (1968)



- Hydrostatic
- inviscid approach
- no mixing

$$UH = q = U_2(H - h_2 - \delta)$$

$$\rho_1 qU + \frac{1}{2}\rho g H^2 = \rho_2 q U_2 + \frac{1}{2}\rho_1 g (H-\delta)^2 + \frac{1}{2}(\rho_2 - \rho_1) g h_2^{\,2}$$

$$\rho_1 \frac{U^2}{2} + \rho_1 g H = P_s \qquad \text{Fluid 1}$$

$$\rho_1 g (H - h_2 - \delta) + \rho_2 g h_2 = p_s \qquad \text{Fluid 2}$$

Unknowns: U_2, U, δ, p_s ; Given: $\Delta\rho$, g, h_2, H

$$F^2 = \frac{U^2}{g' h_2} = \frac{(2-n)(1-n)}{1+n} \qquad n = \frac{h_2}{H}$$

Froude number for density front

Energy eq. for upper layer:

$$\frac{U^2}{2g} + H = \frac{U_2^{\,2}}{2g} + (H - h_2 - \delta) + h_\ell$$

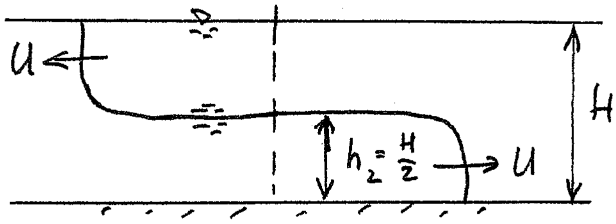
h_ℓ = energy loss per unit weight

$$\frac{h_\ell}{H} = \frac{n^2(1-2n)}{2(1+n)(1-n)}$$

normalized head loss

Special cases:

<u>n = 0.5</u> Lock exchange flow

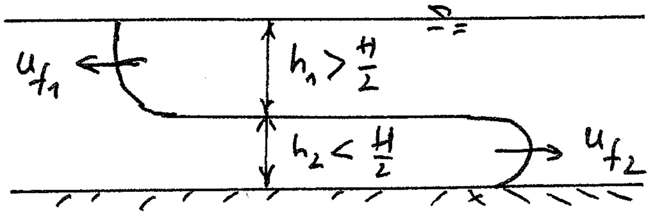


$$h_2 = \frac{1}{2}H$$

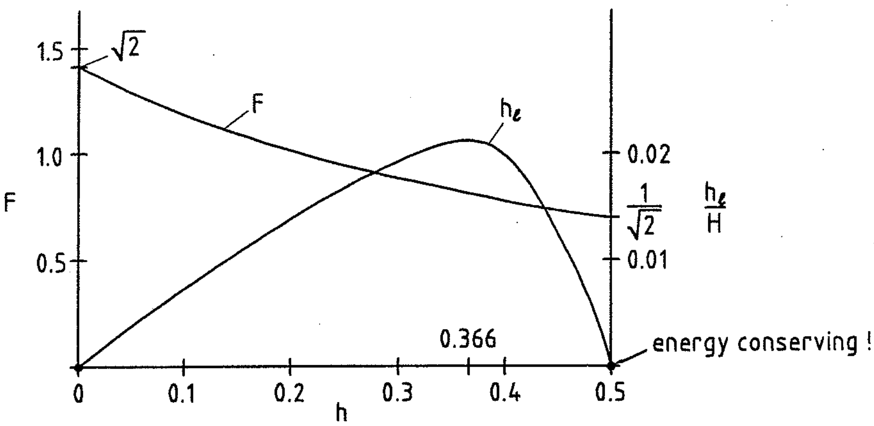
$$\frac{h_\ell}{H} = 0 \qquad \text{energy conserving!}$$

$$F^2 = \frac{1}{2} = \frac{U^2}{g'h_2} \;; \qquad F^2{}_H = \frac{U^2}{g'H} = \frac{1}{4} \;; \quad U = \frac{1}{2}\sqrt{g'H} = \frac{1}{\sqrt{2}}\sqrt{g'h_2}$$

In practice:



$$u_{f_1} = 0.59\sqrt{g'H} \qquad\qquad u_{f_2} = 0.47\sqrt{g'H}$$



Benjamin (1968)

<u>n = 0</u> $H \to \infty$ thin layer

$$F = \frac{U}{\sqrt{g'h_2}} = \sqrt{2}$$

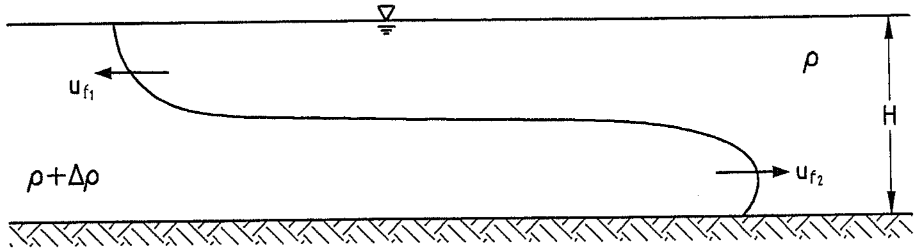
$$\frac{h_\ell}{H} \to 0, \quad \text{but not energy conserving!}$$

$$U = \sqrt{2}\,\sqrt{g'h_2}$$

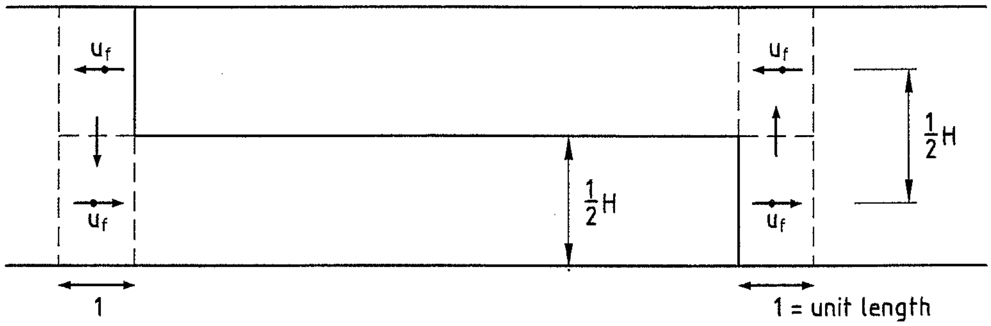
→ Dissipation in wake behind head
→ can be expressed as "drag force"

- effect of entrainment / mixing
- viscous effects at nose

LOCK EXCHANGE FLOW



approximate ⇓ as: $u_{f_1} \approx u_{f_2} = u_f$ Front velocity



Net change (release) in P.E. = Total gain in K.E.

$$\left[(\rho+\Delta\rho)\frac{1}{2}H\left(g\frac{1}{2}H\right) - \rho\frac{1}{2}H\left(g\frac{1}{2}H\right) = \rho\frac{1}{2}u_f^2 H + (\rho+\Delta\rho)\frac{1}{2}u_f^2 H\right]$$

Parcel on left P. on right Top layer Bottom layer

$$\Delta\rho\, g\frac{1}{4}H^2 = \rho u_f^2 H + \Delta\rho\frac{1}{2}u_f^2 H$$

neglect as $\Delta\rho << \rho$

$$u_f^2 = \frac{1}{4}\frac{\Delta\rho}{\rho}g\,H$$

Front velocity $$u_f = \frac{1}{2}\sqrt{\frac{\Delta\rho}{\rho}g\,H}$$

In non-dimensional form: $$\frac{u_f}{\sqrt{\frac{\Delta\rho}{\rho}g\,H}} = \frac{1}{2}$$

Densimetric Froude number

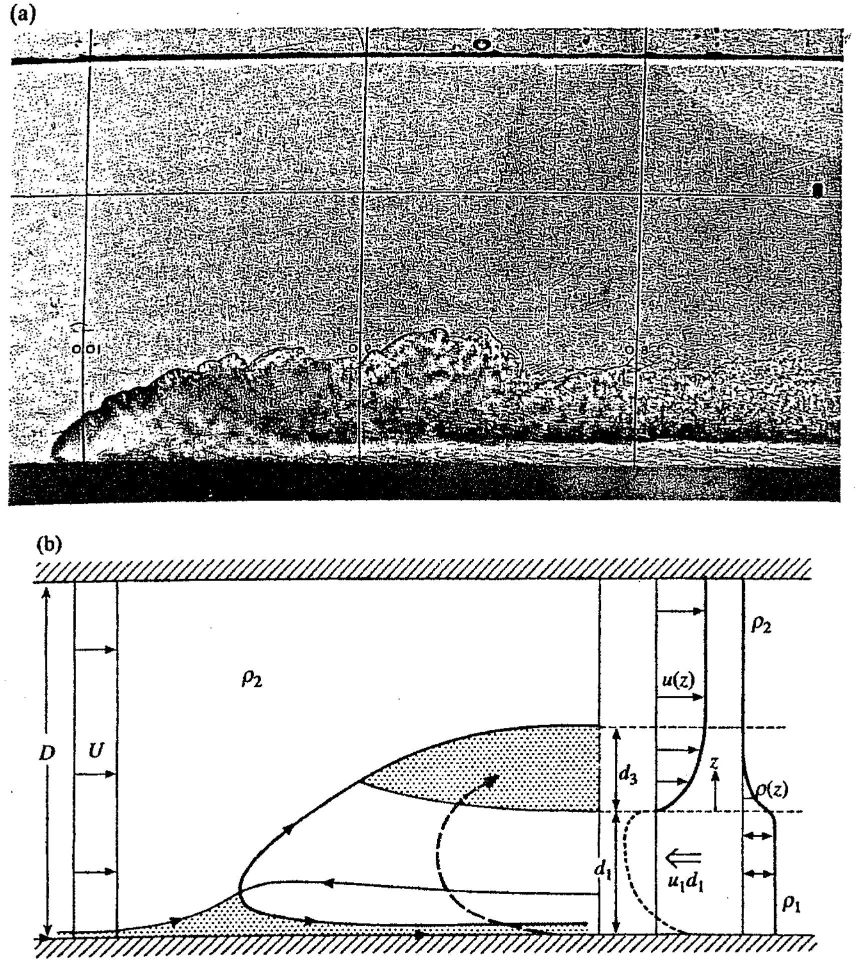


Fig. 3.3. (a) Shadowgraph of a gravity current head; $d_1/D = 0.06$, $\Delta\rho/\rho_1 = 0.008$, and head height is $4.5d_1$. (Photograph courtesy of J. Simpson.) (b). A two-dimensional model of the flow near a gravity current head, in axes moving with the head. O is a stagnation point at the nose. Representative velocity and density profiles behind the nose are shown on the right. (Modified from Simpson & Britter 1979.)

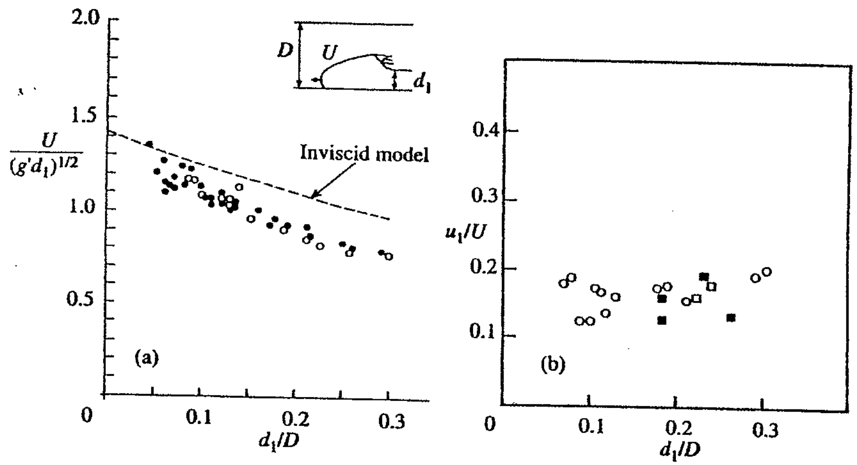


Fig. 3.4. (a) Observed speeds U of the head, scaled with $(g'd_1)^{1/2}$, as a function of d_1/D. (○): steady-state moving floor experiments; (●): lock exchange experiments. The dashed line is from the inviscid, non-mixing model of Benjamin (1968). (b) Observed fluid speed u_1 towards the head, relative to head speed U, as a function of d_1/D. (Modified from Simpson & Britter 1979 – different symbols refer to different measurement techniques.)

Advancing density (gravity) current

From:

Baines (1995)

"Topographic Effects in Stratified Flows"

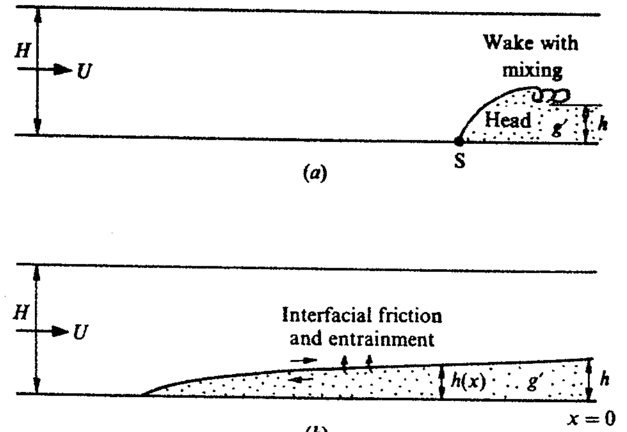


FIGURE 1. Steady-state shapes of intrusions of a dense fluid layer with buoyancy g' and initial thickness h into an ambient flow with relative velocity U (vertical average) and depth H. (a) Density current, (b) density wedge.

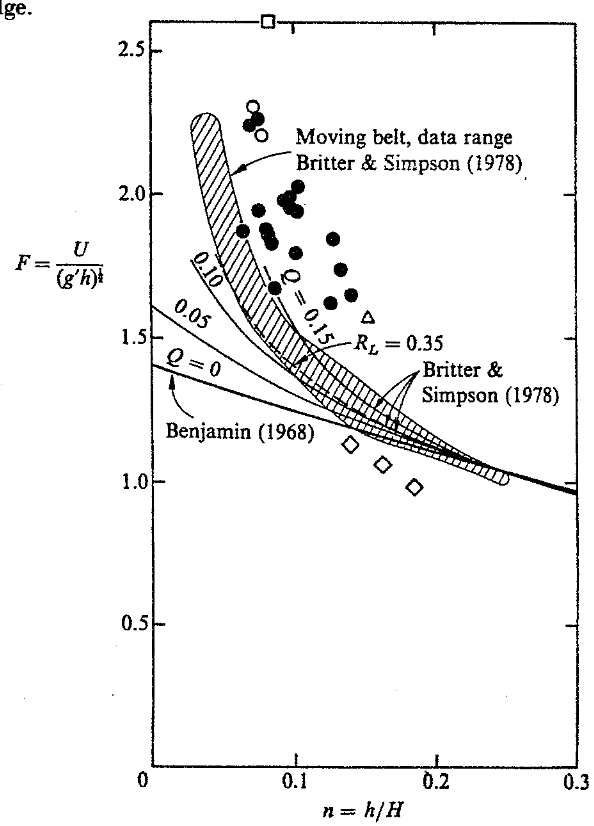


FIGURE 6. Comparison of steady-state density-current data with flow-force theories of Benjamin (1968) and Britter & Simpson (1978) neglecting second-order effects. Data: ●, step device (Sargent & Jirka, 1982); ○, step device; □, barrier device; ◇, suction control.

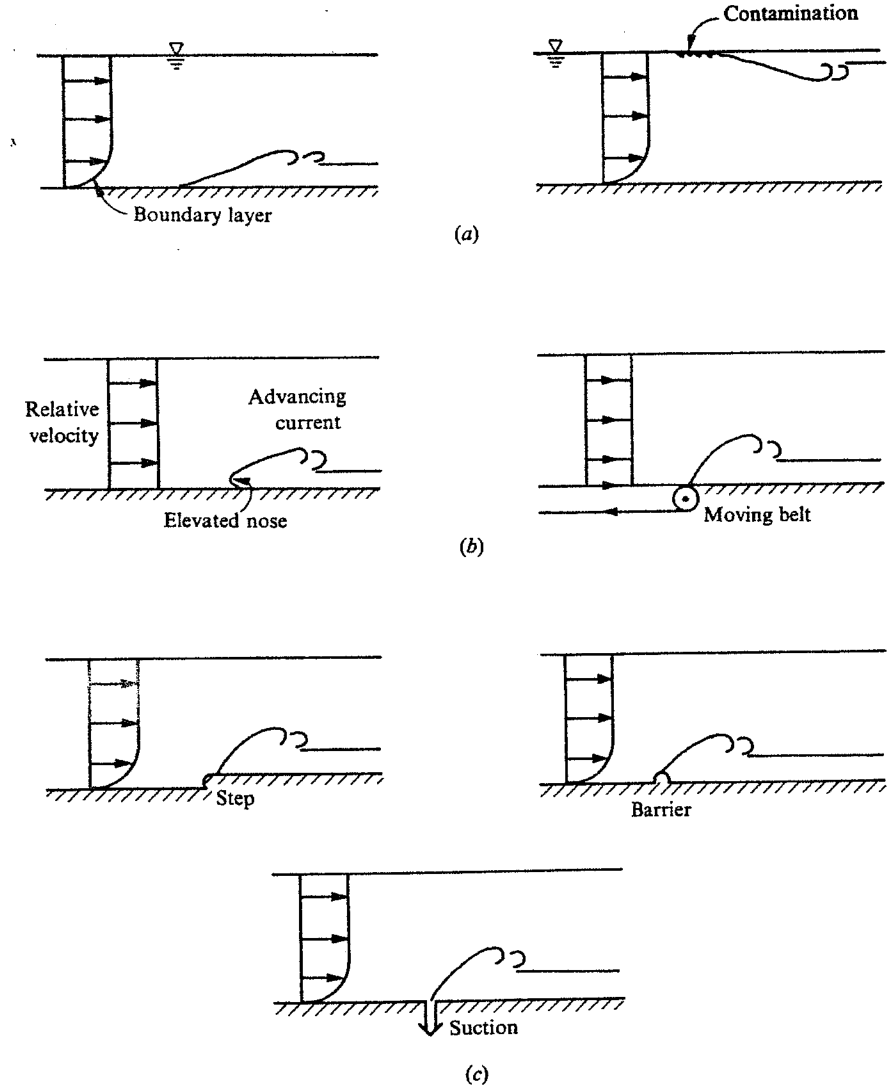
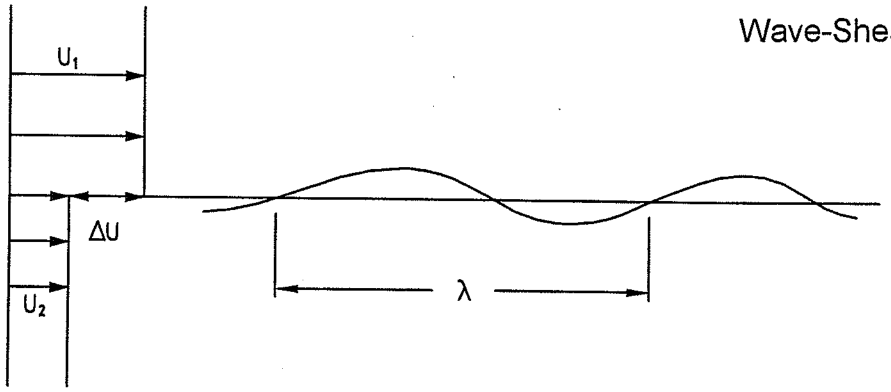


FIGURE 3. Effect of ambient flow and of boundary-layer control on the head shape of density currents. (a) Unstable density currents evolving into wedges, (b) stable density currents in a uniform channel, (c) stable density currents with local channel non-uniformities (control methods).

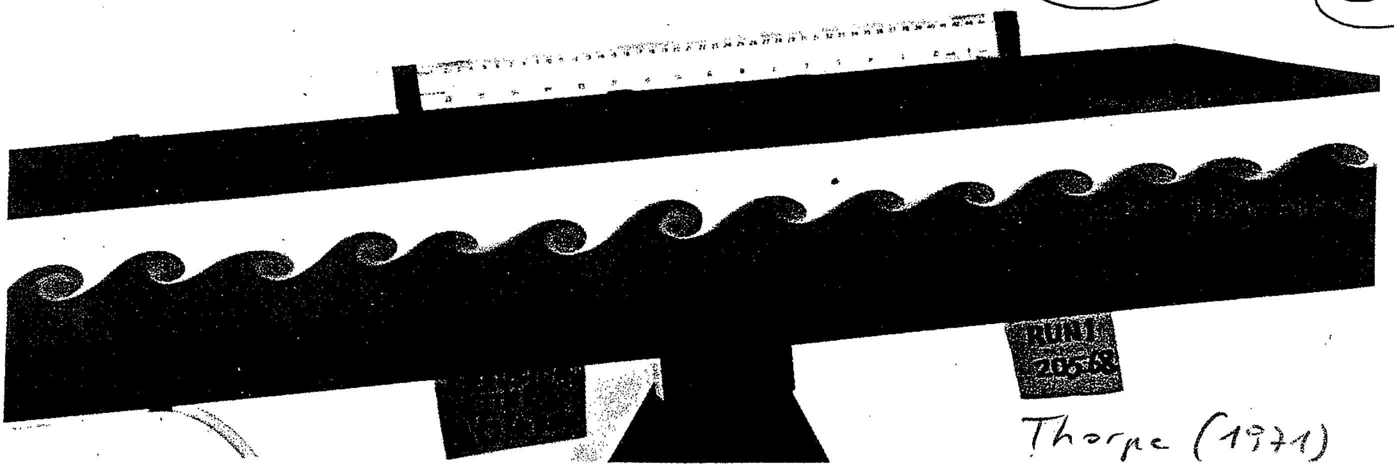
4. Interfacial Instabilities, Mixing, Entrainment

Kelvin-Helmholtz Instability Mechanism



Wave-Shear flow interaction

Instabilities → wave growth → roll-up



Thorpe (1971)

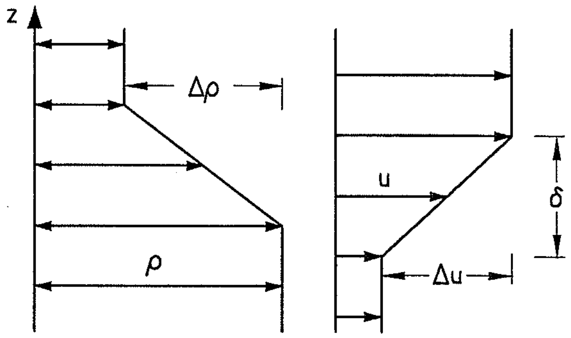
Short waves unstable for $\frac{(\Delta U)^2}{g'\lambda} > \frac{1}{\pi}$

i.e. initially sharp interface intrinsically unstable

Long waves $\frac{(\Delta U)^2}{g'(h_1 + h_2)} > 1$ i.e. supercritical flow is unstable!

Subcritical flow tends to be stable
(for long waves!)

Stratified Shear Flow (Taylor-Goldstein)



Buoyancy $-\frac{g}{\rho}\frac{\partial \rho}{\partial z} = N^2$ stabilizes

Shear $\frac{\partial u}{\partial z}$ destabilizes

$$R_i = -\frac{g}{\rho}\frac{\frac{\partial \rho}{\partial z}}{\left|\frac{\partial u}{\partial z}\right|^2} < \frac{1}{4} \quad \text{for instability}$$

For layer thickness δ: $R_{i_o} = \frac{g\frac{\Delta\rho}{\rho}\delta}{(\Delta U)^2} = \frac{g'\delta}{(\Delta U)^2} < \frac{1}{4}$

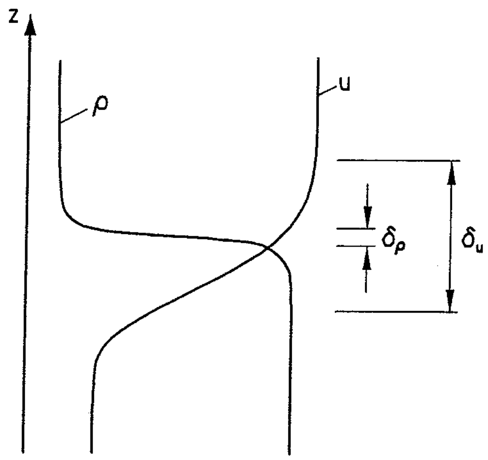
Bulk R_i

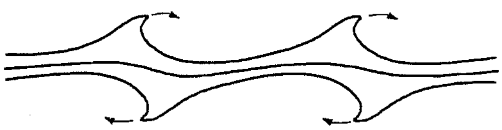
if unstable → wave breaking → mixing

until new layer $\delta^* > \delta$ forms for which $R_{i_o}^{crit} = \frac{g'\delta^*}{(\Delta U)^2} = \frac{1}{4}$ "Marginal stability"

Other wave types: Holmboe Waves

- sharp density gradients in gradual shear flow

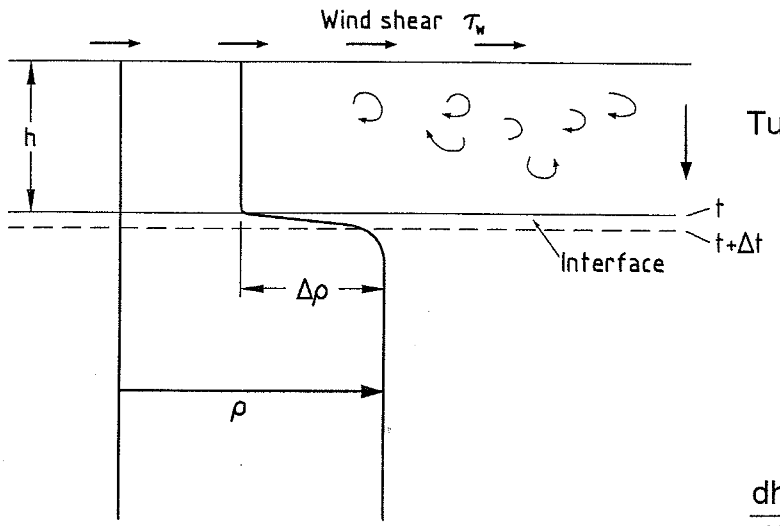




cusps; moderate mixing

Mixed layer entrainment

- effect of turbulent kinetic energy on stratified interface → instabilities and entrainment



Turbulence diffusing downward

$$\frac{dh}{dt} = w_e = \text{entrainment velocity}$$

(mean)

$$u_* = \sqrt{\frac{\tau_w}{\rho}} = \text{shear velocity}$$

ρ = density of water!

$$w_e = f\left(u_*, \frac{\Delta\rho}{\rho} g = g', h\right)$$

$$E = \frac{w_e}{U_*} = f\left(R_{i_*}\right)$$

$$R_{i_*} = \frac{g'h}{U_*^2} = \text{shear Richardson number}$$

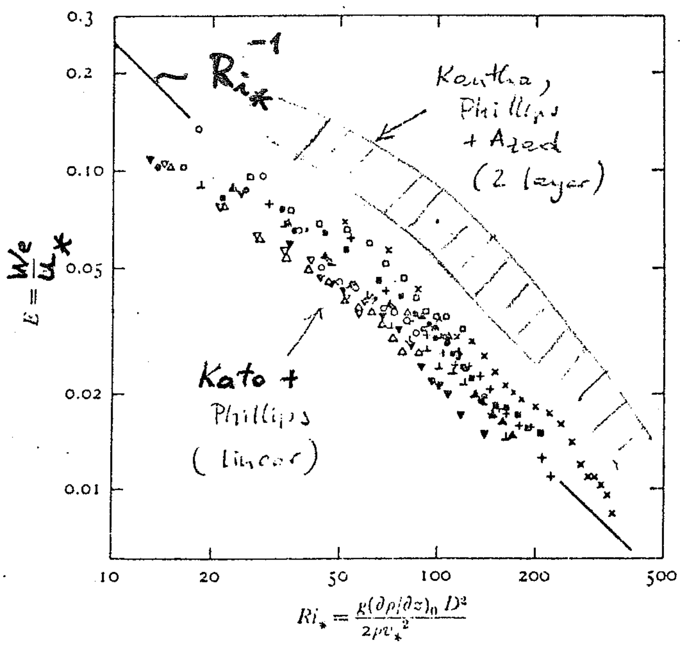


Fig. 9.4. Entrainment rates measured by Kato and Phillips (1969) i turbulent stratified flow produced by a surface stress. The overall Richa son number is defined using the friction velocity and the depth of mixed layer, and different symbols are used for experimental runs w various initial density gradients and surface stresses.

Laboratory

- moving belt
- grid stirring

$$\frac{w_e}{u_*} = 2.5\, R_{i_*}^{-1}$$

Turner (1973)

References: Stratified Flow

Akar, P.J. and Jirka, G.H., 1994, "Buoyant Spreading Processes in Pollutant Transport and Mixing. Part I: Lateral Spreading in Strong Ambient Current", J. Hydraulic Research, 32, 815-831; "Part II: Upstream Spreading in Weak Ambient Current", J. Hydraulic Research, 33, 87-100

Arita, M. and Jirka, G.H., 1987, "Two-Layer Model of the Saline Wedge. Part 1: Entrainment and Interfacial Friction; Part 2: Prediction of Mean Properties", J. Hydraulic Engineering, Vol. 113, No. 10, 1229-63

Arita, M., Jirka, G.H. and Tamai, N., 1986, "Classification and Mixing of Two-Dimensional Buoyant Surface Discharges", J. Hydraulic Research, Vol. 25, No. 5

Baines, P.G., 1995, "Topographic Effects in Stratified Flows", Cambridge Monographs on Mechanics

Barr, D.I.H., 1967, "Densimetric exchange flow in rectangular channels. III. Large scale experiments", La Houille Blanche, 22, 619-32

Benjamin, T.B., 1968, "Gravity currents and related phenomena" J. Fluid Mech., 31, 209-48

Fannelöp, T.K., 1994, "Fluid Mechanics for Industrial Safety and Environmental Protection", Elsevier, Amsterdam

Fischer, H.B., List, E.J., Koh, R.C.Y., Imberger, J. and Brooks, N.H., 1979, "Mixing in Inland and Coastal Waters", Academic Press

Hino, M., 1994, "Water Quality and its Control", Hydraulic Design Considerations, IAHR

Jirka, G.H.,1979, "Selective Withdrawal from Two-Layered Fluid Systems. Part 1: Two-Dimensional Skimmer Wall", J. of Hydraulic Research, Vol. 17, No. 1

Jirka, G.H., 1990, "Circulation in the Salt Wedge Estuary", in Physics of Shallow Bays and Estuaries, Cheng, R.T. (Ed.), Springer Verlag

Jirka, G.H. and Arita, M., 1987, "Density Currents or Density Wedges: Boundary Layer Influence and Control Methods", J. Fluid Mechanics, Vol. 177, 186-206

Jirka, G.H. and Harleman, D.R.F., 1979, "Stability and Mixing of Vertical Plane Buoyant Jet in Confined Depth", J. of Fluid Mechanics, Vol. 94, pp. 275-304

Jirka, G.H. and Katavola, D.S., 1979, "Selective Withdrawal from Two-Layered Fluid Systems. Part 2: Three-Dimensional Flow into Round Intake", J. of Hydraulic Research, Vol. 17, No. 1

Knauss, J.A., 1978, "Introduction to Physical Oceanography", Prentice Hall

Lawrence, G.A., 1993, "The hydraulics of steady two-layer flow over a fixed obstacle", J. Fluid Mech., 254, 605-633

List, E.J. and Jirka, G.H. (Eds.), 1990, Stratified Flows, American Society of Engineers, New York

Schijf, J.B. and Schonfeld, J.C., 1953, "Theoretical considerations on the motion of salt and fresh water", Proc. Minnesota Int. Hydr. Conv., ASCE and IAHR

Scorer, R.S., 1978, "Environmental Aerodynamics", Ellis Horwood, London

Thorpe, S.A., 1971, "Experiments on the instability of stratified shear flows: miscible fluids", J. Fluid Mech., 46, 299-319

Turner, J.S., 1973, "Buoyancy Effects in Fluids", Cambridge University Press

Wood, I.R., Bell, R.G. and Wilkinson, D.L., 1993, "Ocean Disposal of Wastewater", Advanced Series on Ocean Engineering – Vol. 8, World Scientific

Yih, C.S., 1980, "Stratified Flows", Academic Press