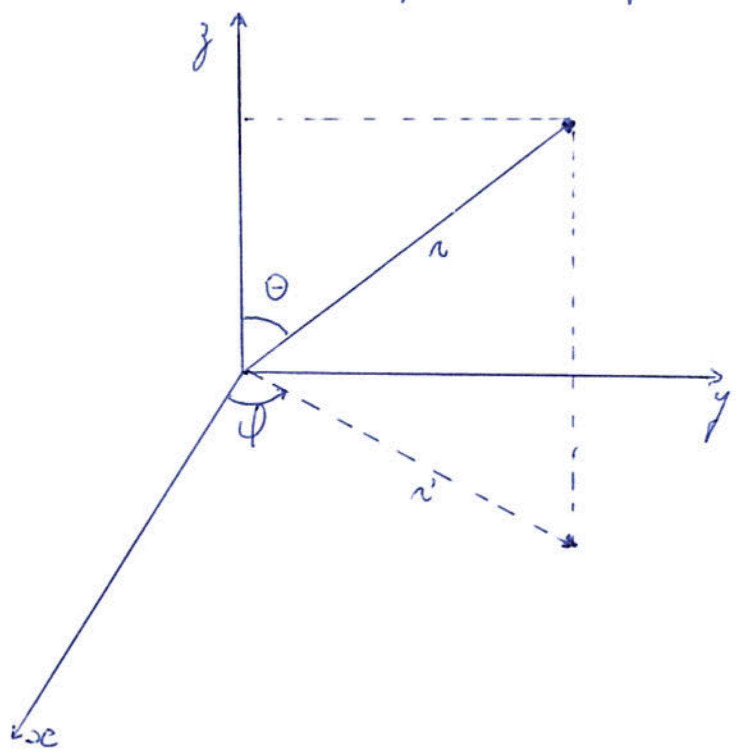
Operador Laplaciano em Coordenadas Esféricas

Eduardo Lemos de Sá



$$r^2 = x^2 + y^2 + z^2$$

$$\cos\Theta = \frac{z}{r}$$

$$tg\,\varphi = \frac{y}{x}$$

$$x = r\,sen\Theta\cos\varphi$$
$$y = r\,sen\Theta\,sen\varphi$$
$$z = r\cos\Theta$$

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial}{\partial \theta}\frac{\partial \theta}{\partial x} + \frac{\partial}{\partial \varphi}\frac{\partial \varphi}{\partial x}$$ e analogamente para $\frac{\partial}{\partial y}$ e $\frac{\partial}{\partial z}$

$$\hat{L}_x = \frac{\hbar}{i}\left\{ y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y} \right\} \qquad \hat{L}_y = \frac{\hbar}{i}\left\{ z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z} \right\}$$ e

$$\hat{L}_z = \frac{\hbar}{i}\left\{ x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x} \right\}$$

Calculando os termos $\frac{\partial\, esférico}{\partial\, cartesiano}$:

$$r^2 = x^2 + y^2 + z^2 \Rightarrow 2r\,dr = 2x\,dx \Rightarrow \frac{\partial r}{\partial x} = \frac{x}{r} = \frac{r\,sen\theta\cos\varphi}{r} = sen\,\theta\cos\varphi$$

$$\frac{\partial r}{\partial y} = \frac{y}{r} = \frac{r\,sen\theta\,sen\varphi}{r} = sen\theta\,sen\varphi$$

$$\frac{\partial r}{\partial z} = \frac{z}{r} = \frac{r\cos\theta}{r} = \cos\theta$$

$$\cos\theta = \frac{z}{r} = \frac{z}{(x^2+y^2+z^2)^{1/2}} \Rightarrow -\operatorname{sen}\theta\, d\theta = z \times \left(-\frac{1}{\cancel{2}}\right)(x^2+y^2+z^2)^{-3/2} \times \cancel{2}x\, dx$$

$$+\operatorname{sen}\theta\, d\theta = +z\, r^{-3}\, x\, dx \Rightarrow \frac{\partial\theta}{\partial x} = \frac{x z}{r^3 \operatorname{sen}\theta} = \frac{r\operatorname{sen}\theta\cos\varphi\; r\cos\theta}{r^3\operatorname{sen}\theta} \Rightarrow$$

$$\boxed{\frac{\partial\theta}{\partial x} = \frac{\cos\varphi\cos\theta}{r}}$$

Por analogia: $\boxed{\dfrac{\partial\theta}{\partial y} = \dfrac{\operatorname{sen}\varphi\cos\theta}{r}}$ e $\boxed{\dfrac{\partial\theta}{\partial z} = -\dfrac{\operatorname{sen}\theta}{r}}$

$$\operatorname{tg}\varphi = \frac{y}{x} \Rightarrow d(\operatorname{tg}\varphi) = d(\operatorname{sen}\varphi/\cos\varphi) = \frac{\cos\varphi\cos\varphi - \operatorname{sen}\varphi \times -\operatorname{sen}\varphi}{\cos^2\varphi}\, d\varphi = \frac{1}{\cos^2\varphi}\, d\varphi$$

$$d\operatorname{tg}\varphi = \frac{1}{\cos^2\varphi}\, d\varphi = -\frac{y}{x^2}\, dx \Rightarrow \frac{\partial\varphi}{\partial x} = \frac{-y\cos^2\varphi}{x^2} = \frac{-r\operatorname{sen}\theta\operatorname{sen}\varphi\cancel{\cos^2\varphi}}{r^2\operatorname{sen}^2\theta\cancel{\cos^2\varphi}} = -\frac{\operatorname{sen}\varphi}{r\operatorname{sen}\theta}$$

$\boxed{\dfrac{\partial\varphi}{\partial x} = -\dfrac{\operatorname{sen}\varphi}{r\operatorname{sen}\theta}}$ Por analogia, $\dfrac{\partial\varphi}{\partial y} = \dfrac{\cos\varphi}{r\operatorname{sen}\theta}$ e $\dfrac{\partial\varphi}{\partial z} = \emptyset$ pois z não depende de φ

Assim:

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial}{\partial\theta}\frac{\partial\theta}{\partial x} + \frac{\partial}{\partial\varphi}\frac{\partial\varphi}{\partial x} = \operatorname{sen}\theta\cos\varphi\frac{\partial}{\partial r} + \frac{\cos\varphi\cos\theta}{r}\frac{\partial}{\partial\theta} - \frac{\operatorname{sen}\varphi}{r\operatorname{sen}\theta}\frac{\partial}{\partial\varphi}$$

$$\frac{\partial}{\partial y} = \operatorname{sen}\theta\operatorname{sen}\varphi\frac{\partial}{\partial r} + \frac{\operatorname{sen}\varphi\cos\theta}{r}\frac{\partial}{\partial\theta} + \frac{\cos\varphi}{r\operatorname{sen}\theta}\frac{\partial}{\partial\varphi}$$

$$\frac{\partial}{\partial z} = \cos\theta\frac{\partial}{\partial r} - \frac{\operatorname{sen}\theta}{r}\frac{\partial}{\partial\theta}$$

Assim, $\hat{L}_x = \dfrac{\hbar}{i}\left(y\dfrac{\partial}{\partial z} - z\dfrac{\partial}{\partial y}\right) \Rightarrow$

$$\hat{L}_x = \frac{\hbar}{i}\left(r\operatorname{sen}\theta\operatorname{sen}\varphi\left(\cancel{\cos\theta\frac{\partial}{\partial r}} - \frac{\operatorname{sen}\theta}{\cancel{r}}\frac{\partial}{\partial\theta}\right) - r\cos\theta\left(\cancel{\operatorname{sen}\theta\operatorname{sen}\varphi\frac{\partial}{\partial r}} + \frac{\operatorname{sen}\varphi\cos\theta}{\cancel{r}}\frac{\partial}{\partial\theta} + \frac{\cos\varphi}{r\operatorname{sen}\theta}\frac{\partial}{\partial\varphi}\right)\right)$$

$$\hat{L}_x = \frac{\hbar}{i}\left(\operatorname{sen}\varphi\underbrace{\left(-\operatorname{sen}^2\theta - \cos^2\theta\right)}_{-1}\frac{\partial}{\partial\theta} - \cos\varphi\operatorname{cotg}\theta\frac{\partial}{\partial\varphi}\right) = i\hbar\left(\operatorname{sen}\varphi\frac{\partial}{\partial\theta} + \cos\varphi\operatorname{cotg}\theta\frac{\partial}{\partial\varphi}\right)$$

$$\hat{L}_y = \frac{\hbar}{i}\left\{ z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}\right\} = -i\hbar\left\{ z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}\right\}$$

$$\hat{L}_y = -i\hbar\left\{ r\cos\theta\left(\cancel{sen\theta\cos\varphi\frac{\partial}{\partial r}} + \cancel{\cos\theta\cos\varphi}\frac{\partial}{r\,\partial\theta} - \frac{sen\varphi}{r\,sen\theta}\right) - r\,sen\theta\cos\varphi\left(\cancel{\cos\theta\frac{\partial}{\partial r}} - \frac{sen\theta}{r}\frac{\partial}{\partial\theta}\right)\right)$$

$$L_y = -i\hbar\left\{ \cos\varphi\left(\underbrace{\cos^2\theta + sen^2\theta}_{1}\right)\frac{\partial}{\partial\theta} - cotg\,\theta\, sen\varphi\frac{\partial}{\partial\varphi}\right\}$$

$$\hat{L}_y = -i\hbar\left\{\cos\varphi\frac{\partial}{\partial\theta} - cotg\,\theta\, sen\varphi\frac{\partial}{\partial\varphi}\right\}$$

$$\hat{L}_z = \frac{\hbar}{i}\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial z}\right) = -i\hbar\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right)$$

$$\hat{L}_z = -i\hbar\left(r\,sen\theta\cos\varphi\left(\cancel{sen\theta\, sen\varphi\frac{\partial}{\partial r}} + \cancel{\cos\theta\, sen\varphi}\frac{\partial}{r\,\partial\theta} + \frac{\cos\varphi}{r\,sen\theta}\frac{\partial}{\partial\varphi}\right) - \right.$$

$$\left. - r\,sen\theta\, sen\varphi\left(\cancel{sen\theta\cos\varphi\frac{\partial}{\partial r}} + \cancel{\frac{\cos\theta\cos\varphi}{r}}\frac{\partial}{\partial\theta} - \frac{sen\varphi}{r\,sen\theta}\frac{\partial}{\partial\varphi}\right)\right)$$

$$\hat{L}_z = -i\hbar\left\{\left(\overbrace{\cos^2\varphi + sen^2\varphi}^{1}\right)\frac{\partial}{\partial\varphi}\right\} = -i\hbar\frac{\partial}{\partial\varphi}$$

$$\boxed{\hat{L}_z = -i\hbar\frac{\partial}{\partial\varphi}}$$

Agora, para se calcular $\hat{L}^2 = \hat{L}_x^2 + \hat{L}_y^2 + \hat{L}_z^2$

$$\hat{L}_x^2 = \hat{L}_x\hat{L}_x = i\hbar\left(sen\varphi\frac{\partial}{\partial\theta} + cotg\,\theta\cos\varphi\frac{\partial}{\partial\varphi}\right)\left(i\hbar\left(sen\varphi\frac{\partial}{\partial\theta} + cotg\,\theta\cos\varphi\frac{\partial}{\partial\varphi}\right)\right)$$

$$\hat{L}_x^2 = \underbrace{(i\hbar)^2}_{-\hbar^2}\left\{ sen\varphi\frac{\partial}{\partial\theta}\left(sen\varphi\frac{\partial}{\partial\theta} + cotg\,\theta\cos\varphi\frac{\partial}{\partial\varphi}\right) + cotg\,\theta\cos\varphi\frac{\partial}{\partial\varphi}\left(sen\varphi\frac{\partial}{\partial\theta} + cotg\,\theta\cos\varphi\frac{\partial}{\partial\varphi}\right)\right)$$

$$\hat{L}_x^2 = -\hbar^2 \left\{ \cancel{sen^2\varphi \frac{\partial^2}{\partial\theta^2}} + sen\varphi\cos\varphi \underbrace{\frac{\partial}{\partial\theta}\left(cotg\,\theta \frac{\partial}{\partial\varphi}\right)}_{-\cancel{\frac{1}{sen^2\theta}\frac{\partial}{\partial\varphi}} + \cancel{cotg\,\theta\frac{\partial^2}{\partial\theta\partial\varphi}}} + cotg\,\theta\cos\varphi \underbrace{\frac{\partial}{\partial\varphi}\left(sen\varphi\frac{\partial}{\partial\theta}\right)}_{\cancel{\cos\varphi\frac{\partial}{\partial\theta}} + \cancel{sen\varphi\frac{\partial^2}{\partial\varphi\partial\theta}}} + cotg^2\theta\cos\varphi \underbrace{\frac{\partial}{\partial\varphi}\left(\cos\varphi\frac{\partial}{\partial\varphi}\right)}_{-\cancel{sen\varphi\frac{\partial}{\partial\varphi}} + \cancel{\cos\varphi\frac{\partial^2}{\partial\varphi^2}}} \right\}$$

$$\hat{L}_y^2 = \hat{L}_y\hat{L}_y = -i\hbar\left(\cos\varphi\frac{\partial}{\partial\theta} - cotg\,\theta\, sen\varphi\frac{\partial}{\partial\varphi}\right)(-i\hbar)\left(\cos\varphi\frac{\partial}{\partial\theta} - cotg\,\theta\, sen\varphi\frac{\partial}{\partial\varphi}\right)$$

$$\hat{L}_y^2 = \underbrace{(i\hbar)^2}_{-\hbar^2}\left\{\cos\varphi\frac{\partial}{\partial\theta}\left(\cos\varphi\frac{\partial}{\partial\theta} - cotg\,\theta\, sen\varphi\frac{\partial}{\partial\varphi}\right) - cotg\,\theta\, sen\varphi\frac{\partial}{\partial\varphi}\left(\cos\varphi\frac{\partial}{\partial\theta} - cotg\,\theta\, sen\varphi\frac{\partial}{\partial\varphi}\right)\right\}$$

$$\hat{L}_y^2 = -\hbar^2\left\{\cancel{\cos^2\varphi\frac{\partial^2}{\partial\theta^2}} - sen\varphi\cos\varphi\underbrace{\frac{\partial}{\partial\theta}\left(cotg\,\theta\frac{\partial}{\partial\varphi}\right)}_{-\cancel{\frac{1}{sen^2\theta}\frac{\partial}{\partial\varphi}} + \cancel{cotg\,\theta\frac{\partial^2}{\partial\theta\partial\varphi}}} - cotg\,\theta\, sen\varphi\underbrace{\frac{\partial}{\partial\varphi}\left(\cos\varphi\frac{\partial}{\partial\theta}\right)}_{-\cancel{sen\varphi\frac{\partial}{\partial\theta}} + \cancel{\cos\varphi\frac{\partial^2}{\partial\varphi\partial\theta}}} + cotg^2\theta\, sen\varphi\underbrace{\frac{\partial}{\partial\varphi}\left(sen\varphi\frac{\partial}{\partial\varphi}\right)}_{\cancel{\cos\varphi\frac{\partial}{\partial\varphi}} + \cancel{sen\varphi\frac{\partial^2}{\partial\varphi^2}}}\right\}$$

$$\hat{L}_z = -i\hbar\frac{\partial}{\partial\varphi} \Rightarrow \hat{L}_z^2 = \hat{L}_z\hat{L}_z = (-i\hbar)^2\frac{\partial}{\partial\varphi}\frac{\partial}{\partial\varphi} = i^2\hbar\frac{\partial^2}{\partial\varphi^2} \Rightarrow \hat{L}_z^2 = -\hbar^2\frac{\partial^2}{\partial\varphi^2}$$

Assim, $\hat{L}^2 = \hat{L}_x^2 + \hat{L}_y^2 + \hat{L}_z^2 = -\hbar^2\left\{\frac{\partial^2}{\partial\theta^2}\underbrace{\left(sen^2\varphi + \cos^2\varphi\right)}_{1} + \frac{\partial}{\partial\theta}\underbrace{\left(cotg\,\theta\cos^2\varphi + cotg\,\theta\, sen^2\varphi\right)}_{cotg\,\theta} + \underbrace{cotg^2\theta\underbrace{\left(\cos^2\varphi + sen^2\varphi\right)}_{1}\frac{\partial^2}{\partial\varphi^2} + \frac{\partial^2}{\partial\varphi^2}}_{\underbrace{\left(cotg^2\theta + 1\right)}_{\frac{1}{sen^2\theta}}\frac{\partial^2}{\partial\varphi^2}}\right\}$

$$\boxed{\text{Assim: } \hat{L}^2 = -\hbar^2\left\{\frac{\partial^2}{\partial\theta^2} + cotg\,\theta\frac{\partial}{\partial\theta} + \frac{1}{sen^2\theta}\frac{\partial^2}{\partial\varphi^2}\right\}}$$