UFPR - Universidade Federal do Paraná
Setor de Ciências Exatas
Departamento de Matemática
CM044 - Cálculo IV - 2025/1
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Lista 3

✫ Funções elementares

1. Prove as seguintes afirmações:

① $e^{\pi i}+1=0$

② $|e^z| = e^{\operatorname{Re} z} > 0,\ z\in\mathbb{C}$

③ $e^z\in\mathbb{R}$ se e só se $\operatorname{Im} z = k\pi,\ k\in\mathbb{Z}$

④ $e^z\in i\mathbb{R}$ se e só se $\operatorname{Im} z = \left(k+\dfrac{1}{2}\right)\pi,\ k\in\mathbb{Z}$

⑤ $e^z=1$ se e só se $z=2k\pi i,\ k\in\mathbb{Z}$

⑥ $e^z=e^w$ se e só se $z=w+2k\pi$, para algum $k\in\mathbb{Z}$.

⑦ $\lim_{\operatorname{Re} z\to-\infty} e^z = 0$

2. Determine todos os valores de $z\in\mathbb{C}$ tais que:

① $e^z=-2$

② $e^z=i$

③ $e^z=1+i\sqrt{3}$

④ $e^{2z-1}=1$

⑤ $e^{z^2}=1$

⑥ $e^{z^2-2z}=i$

3. Mostre que:

1. $\operatorname{sen}^2 z+\cos^2 z=1$
2. $|\operatorname{sen} z|^2=\operatorname{sen}^2 x+\sinh^2 y,\ z=x+iy$
3. $|\cos z|^2=\cos^2 x+\sinh^2 y,\ z=x+iy$
4. $\operatorname{sen}(z+w)=\operatorname{sen} z\cos w+\operatorname{sen} w\cos z$
5. $\cos(z+w)=\cos z\cos w-\operatorname{sen} z\operatorname{sen} w$
6. $\operatorname{sen}(-z)=-\operatorname{sen} z$
7. $\cos(-z)=\cos z$
8. $\operatorname{sen}\left(\dfrac{\pi}{2}-z\right)=\cos z$
9. $\operatorname{sen} z=0$ se e só se $z=n\pi,\ n\in\mathbb{Z}$
10. $\cos z=0$ se e só se $z=n\pi+\dfrac{\pi}{2},\ n\in\mathbb{Z}$
11. $|\sinh y|\le|\operatorname{sen} z|\le\cosh y$, se $z=x+iy$
12. $|\sinh y|\le|\cos z|\le\cosh y$, se $z=x+iy$
13. $\cosh^2 z-\sinh^2 z=1$
14. $|\operatorname{sen} x|\le|\operatorname{sen} z|$, se $z=x+iy$
15. $|\cos x|\le|\cos z|$, se $z=x+iy$
16. $|\sinh x|\le|\cosh z|\le\cosh x$, se $z=x+iy$
17. $\operatorname{sen}(z+w)\operatorname{sen}(z-w)=\dfrac{1}{2}(\cos(2w)-\cos(2z))$
18. $\cos(z+w)\cos(z-w)=\dfrac{1}{2}(\operatorname{sen}(2w)-\operatorname{sen}(2z))$
19. $\cos z=\cos w$ se e só se $z\pm w=2k\pi,\ k\in\mathbb{Z}$
20. $\operatorname{sen} z=\operatorname{sen} w$ se e só se $z-w=2k\pi$ ou $z+w=(2k+1)\pi,\ k\in\mathbb{Z}$
21. $|\sinh z|^2=\sinh^2 x+\operatorname{sen}^2 y$
22. $|\cosh z|^2=\sinh^2 x+\cos^2 y$
23. $\sinh(iz)=i\operatorname{sen} z$
24. $\cosh(iz)=\cos z$
25. $\operatorname{sen}(iz)=i\sinh z$
26. $\cos(iz)=\cosh z$
27. $\sinh(z+\pi i)=-\sinh z$
28. $\cosh(z+\pi i)=-\cosh z$
29. $\tanh(z+\pi i)=\tanh z$

4. Determine todos os valores de $z \in \mathbb{C}$ que satisfazem as equações:

① $\cos z = 1$

② $\cos z = 3$

③ $\operatorname{sen} z = 4$

④ $\cos z = i$

⑤ $\cosh z = 1/2$

⑥ $\sinh z = i$

⑦ $\cosh z = -1$

⑧ $\sinh z = 0$

⑨ $\cosh z = 0$

⑩ $\tanh z = 0$

5. Prove as identidades abaixo, para $z, w \in \mathbb{C}$ e $x, y \in \mathbb{R}$:

1. $\sinh(z+w) = \sinh z \cosh w + \sinh w \cosh z$
2. $\cosh(z+w) = \cosh z \cosh w + \sinh w \sinh z$
3. $\sinh(-z) = -\sinh z$
4. $\cosh(-z) = \cosh z$
5. $\cosh(2z) = \cosh^2 z + \sinh^2 z$
6. $\sinh(2z) = 2 \sinh z \cosh z$
7. $\sinh(x+iy) = \sinh x \cos y + i \operatorname{sen} y \cosh x$
8. $\cosh(x+iy) = \cosh x \cos y + i \sinh x \operatorname{sen} y$
9. $\sinh z = 0$ se e só se $z = k\pi i$, $k \in \mathbb{Z}$
10. $\cosh z = 0$ se e só se $z = \left(k + \frac{1}{2}\right)\pi i$, $k \in \mathbb{Z}$
11. $\sinh(z+w)\sinh(z-w) = \frac{1}{2}(\cosh(2z) - \cosh(2w))$
12. $\sinh(z-w)\cosh(z+w) = \frac{1}{2}(\sinh(2z) - \sinh(2w))$
13. $\cosh z = \cosh w$ se e só se $z \pm w = 2k\pi i$, $k \in \mathbb{Z}$
14. $\sinh z = \sinh w$ se e só se $z - w = 2k\pi i$ ou $z + w = (2k+1)\pi i$, $k \in \mathbb{Z}$
15. $\cosh(2z) = \cosh^2 z + \sinh^2 z$
16. $\sinh(2z) = 2 \sinh z \cosh z$