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Setor de Ciências Exatas
Departamento de Matemática
CM044 - Cálculo IV - 2025/1
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Lista 7

✩ Séries de potências

1. Determine as séries de Maclaurin das funções abaixo:

(a) $f(z) = e^z$

(b) $f(z) = e^{z^2}$

(c) $f(z) = \operatorname{sen} z$

(d) $f(z) = \dfrac{\operatorname{sen} z}{z}$

(e) $f(z) = \cos z$

(f) $f(z) = \dfrac{1-\cos z}{z^2}$

(g) $f(z) = \sinh z$

(h) $f(z) = \cosh z$

(i) $f(z) = z\cosh(z^2)$

(j) $f(z) = \operatorname{sen}^2 z$

(k) $f(z) = \operatorname{cosech} z = \dfrac{1}{\sinh z}$

(l) $f(z) = \dfrac{1}{1-z^2}$

(m) $f(z) = \dfrac{\sinh z}{z}$

(n) $f(z) = \sec z = \dfrac{1}{\cos z}$

2. Obtenha a série de Laurent de cada f a seguir no domínio D indicado:

(a) $f(z) = \dfrac{z+1}{z-1}$, $D: |z| > 1$

(b) $f(z) = \dfrac{\sinh z}{z^2}$, $D: |z| > 0$

(c) $f(z) = \dfrac{1}{z^2(1-z)}$, $D: |z| > 1$

(d) $f(z) = \dfrac{e^z - 1}{z^3}$, $D: |z| > 0$

(e) $f(z) = \dfrac{e^{1/z}}{z^2}$, $D: |z| > 0$

(f) $f(z) = \dfrac{1}{z(z-1)}$, $D: 0 < |z| < 1$

(g) $f(z) = \dfrac{1}{z(z-1)}$, $D: |z| > 1$

(h) $f(z) = \dfrac{1}{z-1} - \dfrac{1}{z-2}$, $D: 1 < |z| < 2$

(i) $f(z) = \dfrac{1}{z-1} - \dfrac{1}{z-2}$, $D: |z| > 2$

(j) $f(z) = \dfrac{1}{z(z^2+1)}$, $D: 0 < |z| < 1$

(k) $f(z) = \dfrac{1}{z(z^2+1)}$, $D: |z| > 1$

3. Considere $f(z) = \sinh(z^2)$, $z \in \mathbb{C}$.

① Obtenha a série de Maclaurin de f.

② Mostre que $f^{(n)}(0) = 0$ para todo n *ímpar* e $f^{(m)}(0) = 0$ para todo m múltiplo de 4.

4. Obtenha os primeiros termos das séries de Taylor/Laurent das funções abaixo (em torno da origem):

(a) $\operatorname{cosec} z = \frac{1}{z} + \frac{1}{3!}z + \left(\frac{1}{5!} - \frac{1}{(3!)^2}\right)z^3 + \ldots$, $0 < |z| < \pi$

(b) $\frac{1}{e^z - 1} = \frac{1}{z} - \frac{1}{2} + \frac{1}{12}z - \frac{1}{720}z^3 + \ldots$, $0 < |z| < 2\pi$

(c) $\operatorname{tg} z = z + \frac{1}{3}z^3 - \frac{2}{15}z^5 + \frac{17}{315}z^7 + \ldots$, $|z| < \pi/2$

(d) $\operatorname{cosec} z = \frac{1}{z} + \frac{1}{6}z + \frac{7}{360}z^3 + \frac{31}{15120}z^5 + \ldots$, $0 < |z| < \pi$

(e) $\sec z = 1 + \frac{1}{2}z^2 + \frac{5}{24}z^4 + \frac{61}{720}z^6 + \ldots$, $|z| < \pi/2$

(f) $\operatorname{cotg} z = \frac{1}{z}z - \frac{1}{3}z - \frac{1}{45}z^3 - \frac{2}{945}z^5 + \ldots$, $0 < |z| < \pi$

☆ Singularidades e Resíduos

5. Para cada função a seguir, determine o tipo de singularidade que a mesma apresenta em z_0 (polo de ordem m ou singularidade essencial) e calcule o resíduo de f em $z = z_0$:

(a) $f(z) = \frac{z^2+1}{z^2(z-1)}$, $z_0 = 0$

(b) $f(z) = \frac{z^2+1}{z^2(z-1)}$, $z_0 = 1$

(c) $f(z) = \frac{\operatorname{sen} z}{z^2}$, $z_0 = 0$

(d) $f(z) = \frac{z^2 + (1-i)z - i}{z^2+1}$, $z_0 = i$

(e) $f(z) = \frac{\sinh z}{z^4}$, $z_0 = 0$

(f) $f(z) = \frac{e^z}{z^2(z+1)}$, $z_0 = 0$

(g) $f(z) = \frac{e^z}{z^2(z+1)}$, $z_0 = -1$

(h) $f(z) = z\operatorname{sen}\left(\frac{1}{z}\right)$, $z_0 = 0$

(i) $f(z) = \frac{\cos z}{z}$, $z_0 = 0$

(j) $f(z) = e^{1/z}$, $z_0 = 0$

(k) $f(z) = \frac{\operatorname{Log}(1+z)}{z^2}$, $z_0 = 0$

(l) $f(z) = \frac{1}{e^z - 1}$, $z_0 = 0$

(m) $f(z) = \cosh\left(\frac{1}{z}\right)$, $z_0 = 0$

(n) $f(z) = \frac{\operatorname{cosec}(z^2)}{z^3}$, $z_0 = 0$

(o) $f(z) = z\cos\left(\frac{1}{z}\right)$, $z_0 = 0$

6. Use o teorema dos resíduos para calcular as seguintes integrais impróprias:

(a) $\int_{-\infty}^{\infty} \frac{dx}{x^2+1} = \pi$

(b) $\int_{-\infty}^{\infty} \frac{dx}{x^2+2x+2} = \pi$

(c) $\int_{-\infty}^{\infty} \frac{x^2}{x^4+1} = \frac{\pi}{\sqrt{2}}$

(d) $\int_0^{\infty} \frac{x^2}{(x^2+1)(x^2+4)}dx = \frac{\pi}{6}$

(e) $\int_0^{\infty} \frac{dx}{1+x^4} = \frac{\pi\sqrt{2}}{4}$

(f) $\int_0^{\infty} \frac{x^2}{1+x^6}dx = \frac{\pi}{6}$

(g) $\int_0^{\infty} \frac{dx}{(1+x^2)^2} = \frac{\pi}{4}$

(h) $\int_0^{\infty} \frac{\operatorname{sen} x}{x^2+4x+5} = -\frac{\pi \operatorname{sen} 2}{e}$

(i) $\int_0^{\infty} \frac{\cos(ax)}{1+x^2} = \frac{\pi e^{-a}}{2}$, $a \geq 0$

(j) $\int_0^{\infty} \frac{\cos x}{(1+x^2)^2} = \frac{\pi}{2e}$

(k) $\displaystyle\int_0^\infty \frac{\cos(ax)}{1+x^2} = \frac{\pi e^{-a}}{2}$, $a \geq 0$

(l) $\displaystyle\int_{-\infty}^\infty \frac{dx}{(x^2+1)(x^2+2x+2)} = -\frac{\pi}{5}$

(m) $\displaystyle\int_{-\infty}^\infty \frac{x^2}{x^4+x^2+1}dx$

(n) $\displaystyle\int_{-\infty}^\infty \frac{\cos x}{(x+a)^2+b^2} = \frac{e^{-b}\cos a}{b}$, $a, b > 0$

7. Use o teorema dos resíduos para calcular as seguintes integrais definidas:

(a) $\displaystyle\int_0^\pi \frac{\cos\theta}{5+4\cos\theta}d\theta = -\frac{\pi}{3}$

(b) $\displaystyle\int_0^\pi \frac{d\theta}{a^2+\cos^2\theta} = \frac{\pi}{a\sqrt{a^2+1}}$, $a > 0$

(c) $\displaystyle\int_{-\pi}^\pi \frac{d\theta}{1+\operatorname{sen}^2\theta} = \pi\sqrt{2}$

(d) $\displaystyle\int_0^\pi \frac{d\theta}{a+\cos\theta}d\theta = \frac{\pi}{\sqrt{a^2-1}}$, $a > 1$

(e) $\displaystyle\int_0^\pi \frac{d\theta}{1+a\operatorname{sen}\theta}d\theta = \frac{2\pi}{\sqrt{1-a^2}}$, $|a| < 1$

(f) $\displaystyle\int_0^{2\pi} \frac{d\theta}{(a+\cos\theta)^2}d\theta = \pi a(1+a^2)^{-3/2}$, $a > 1$

(g) $\displaystyle\int_0^\pi \operatorname{sen}^{2n}\theta d\theta = \pi\frac{(2n)!}{(2^n n!)^2}$, $n \in \mathbb{N}$

8. Neste exercício, vamos utilizar a conhecida fórmula

$$\int_0^\infty e^{-x^2}dx = \frac{\sqrt{\pi}}{2}$$

para determinar o valor das *integrais de Fresnel*. Integrando a função $f(z) = e^{-z^2}$ sobre a fronteira do setor circular $S_{R,\pi/4} : 0 \leq \theta \leq \pi/4,\ 0 \leq r \leq R$ e fazendo $R \to \infty$, mostre que

$$\int_0^\infty \operatorname{sen}(x^2)dx = \int_0^\infty \cos(x^2)dx = \frac{\sqrt{2\pi}}{4}.$$