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Lista 1

✩ Sequências e séries de números reais

1. Verifique se cada uma das sequências abaixo é convergente ou divergente e determine o limite das que forem convergentes:

(1) $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}, \frac{6}{7}, \ldots$

(2) $1, \frac{1}{2}, 1, \frac{1}{4}, 1, \frac{1}{8}, 1, \frac{1}{16}, \ldots$

(3) $\frac{1}{2}, -\frac{1}{2}, \frac{1}{4}, -\frac{3}{4}, \frac{1}{8}, -\frac{7}{8}, \ldots$

(4) $a_n = \left(4 + \frac{1}{n}\right)^{\frac{1}{2}}$

(5) $a_n = \frac{\sqrt{n}+1}{n-1}$, $n \geq 2$

(6) $a_n = \frac{n^3+3n+1}{4n^3+2}$

(7) $a_n = \sqrt{n+1} - \sqrt{n}$

(8) $a_n = \frac{n+(-1)^n}{n-(-1)^n}$

(9) $a_n = \frac{2n}{n+1} - \frac{n+1}{2n}$

(10) $a_n = n(\sqrt{n^2+1} - n)$

(11) $a_n = \frac{\operatorname{sen} n}{n}$

(12) $a_n = \operatorname{sen} n$

(13) $a_n = \frac{2n + \operatorname{sen} n}{5n+1}$

(14) $a_n = \frac{(n+3)! - n!}{(n+4)!}$

(15) $a_n = \sqrt[n]{n^2+n}$

(16) $a_n = \frac{n \operatorname{sen}(n!)}{n^2+1}$

(17) $a_n = \frac{3^n}{2^n + 10^n}$

(18) $a_n = \left(\frac{n+2}{n+1}\right)^n$

(19) $a_n = \frac{(n+1)^n}{n^{n+1}}$

(20) $a_n = na^n$, $a \in \mathbb{R}$

(21) $a_n = \frac{n!}{n^n}$

(22) $a_n = n - n^2 \operatorname{sen} \frac{1}{n}$

(23) $a_n = \sqrt[n]{a^n + b^n}$, $0 < a < b$

(24) $a_n = (-1)^n + \frac{(-1)^n}{n}$

(25) $a_n = \left(1 - \frac{1}{2}\right)\left(1 - \frac{1}{3}\right) \ldots \left(1 - \frac{1}{n}\right)$

(26) $a_n = \frac{\sqrt{n} + \operatorname{sen}(2n! - 7)}{n + 3\sqrt{n}}$

(27) $a_n = \frac{1}{n} \cdot \frac{1.3.5\ldots(2n-1)}{2.4.6\ldots(2n)}$

(28) $a_n = \sqrt[n]{n}$

(29) $a_n = \frac{n^\alpha}{e^n}$, $\alpha \in \mathbb{R}$

(30) $a_n = \frac{\ln n}{n^\alpha}$, $\alpha > 0$

(31) $a_n = \sqrt[n]{n!}$

(32) $a_n = \sqrt[n]{a}$, $a > 0$

(33) $a_n = \left(\frac{n-1}{n}\right)^n$

(34) $a_n = \left(\frac{n+1}{n}\right)^{n^2}$

(35) $a_n = \left(\frac{n+1}{n}\right)^{\sqrt{n}}$

(36) $a_n = \left(\dfrac{3n+5}{5n+11}\right)^n$

(37) $a_n = \left(\dfrac{3n+5}{5n+1}\right)^n \left(\dfrac{5}{3}\right)^n$

(38) $a_n = \left(1+\dfrac{1}{n^2}\right)^n$

(39) $a_n = \operatorname{sen}(\dfrac{n\pi}{2})$

(40) $a_n = \dfrac{n}{\sqrt[n]{n!}}$

(41) $a_n = \dfrac{1}{n}\sqrt[n]{(n+1)(n+2)\cdot\ldots\cdot(2n)}$

(42) $a_n = \sqrt[n]{\dfrac{(2n)!}{n!^2}}$

(43) $a_n = \dfrac{n^2-1}{n^5+(-1)^n n^2}$

(44) $a_n = \sqrt[n]{n^4+2012n^3-5}$

(45) $a_n = \left(1+\dfrac{1}{n}\right)^{1/n}$

(46) $a_n = \dfrac{n!^2}{n^{2n}}$

(47) $a_n = \dfrac{5^n}{2^n+3^n+4^n}$

(48) $a_n = \dfrac{n+\sqrt{2n+3}}{\sqrt[4]{n}+\sqrt[7]{17n-8}}$

(49) $a_n = \dfrac{3n^3-n^2+11n}{n^4-2n^3}$

(50) $a_n = \left(\dfrac{5n+7}{3n+8}\right)^{2n-4}$

(51) $a_n = \dfrac{(2n)!}{(n!)^2}$

2. Considere a sequência $a_1 = \sqrt{2}$, $a_2 = \sqrt{2\sqrt{2}}$, $a_3 = \sqrt{2\sqrt{2\sqrt{2}}}, \ldots$

   (a) Verifique que a sequência é crescente e limitada superiormente por 2.

   (b) Calcule $\lim_{n\to\infty} a_n$.

3. Mostre que a sequência $\sqrt{2}, \sqrt{2+\sqrt{2}}, \sqrt{2+\sqrt{2+\sqrt{2}}}, \ldots$ converge para 2.

4. Calcule $\sqrt{2}^{\sqrt{2}^{\sqrt{2}^{\cdots}}}$.

5. Decida se cada uma das séries abaixo é convergente e calcule sua soma:

(1) $\displaystyle\sum_{n=0}^{\infty}\left(\frac{1}{10^n}+2^n\right)$

(2) $\displaystyle\sum_{n=0}^{\infty}(-1)^k x^{\frac{k}{2}}$, $0<x<1$

(3) $\displaystyle\sum_{n=0}^{\infty}u^n(1+u^n)$, $|u|<1$

(4) $\displaystyle\sum_{n=0}^{\infty}x^n\cos\left(\frac{n\pi}{2}\right)$, $|x|<1$

(5) $\displaystyle\sum_{n=0}^{\infty}\operatorname{sen}^{2n}x$, $|x|<\frac{\pi}{2}$

(6) $\displaystyle\sum_{n=0}^{\infty}\frac{1}{e^{n/2}}$

(7) $\displaystyle\sum_{n=0}^{\infty}\cos\left(\frac{n\pi}{2}\right)\frac{\pi^{n/2}}{e^n}$

6. Verifique se cada uma das séries abaixo é convergente ou divergente:

(1) $\displaystyle\sum_{n=3}^{\infty}\frac{1}{\sqrt{n^2-4}}$

(2) $\displaystyle\sum_{n=2}^{\infty}\frac{\arctan n}{n^2}$

(3) $\displaystyle\sum_{n=1}^{\infty}\frac{e^{\frac{1}{n}}}{n^2}$

(4) $\displaystyle\sum_{n=1}^{\infty}\frac{2^n}{(n!)^\lambda}$, $\lambda>0$

(5) $\displaystyle\sum_{n=1}^{\infty}\frac{(2n)!}{n!^2}$

(6) $\displaystyle\sum_{n=2}^{\infty}\frac{\ln n}{n}$

(7) $\sum_{n=1}^{\infty} \frac{\sqrt[3]{n+2}}{\sqrt[4]{n^3+3}\sqrt[5]{n^3+5}}$

(8) $\sum_{n=2}^{\infty} \frac{1}{n^{\ln n}}$

(9) $\sum_{n=1}^{\infty} \left(1-\cos\frac{1}{n}\right)$

(10) $\sum_{n=2}^{\infty} \frac{1}{(\ln\ n)^n}$

(11) $\sum_{n=2}^{\infty} \frac{\ln\ n}{n^2}$

(12) $\sum_{n=2}^{\infty} \frac{\ln n}{n^p}$, $p>0$

(13) $\sum_{n=2}^{\infty} \ln\left(1+\frac{1}{n^p}\right)$, $p>0$

(14) $\sum_{n=2}^{\infty} \sqrt{n}\ln\left(\frac{n+1}{n}\right)$

(15) $\sum_{n=1}^{\infty} \frac{n!3^n}{n^n}$

(16) $\sum_{n=1}^{\infty} \frac{n!e^n}{n^n}$

(17) $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n+1}+\sqrt{n}}$

(18) $\sum_{n=1}^{\infty} \frac{n}{\operatorname{sen} n}$

(19) $\sum_{n=1}^{\infty} \frac{n+2}{\sqrt{n+2012}}$

(20) $\sum_{n=1}^{\infty} \frac{2+\cos n}{n}$

(21) $\sum_{n=1}^{\infty} \operatorname{tg}\left(\frac{1}{n}\right)$

(22) $\sum_{n=1}^{\infty} \frac{e^{\frac{1}{n}}}{n^n}$

(23) $\sum_{n=1}^{\infty} 3^n\left(\frac{n}{n+1}\right)^{n^2}$

(24) $\sum_{n=1}^{\infty} \frac{n^3}{(\ln 2)^n}$

(25) $\sum_{n=1}^{\infty} \frac{1}{(\arctan n)^n}$

(26) $\sum_{n=0}^{\infty} \frac{n+2}{(n+1)^3}$

(27) $\sum_{n=1}^{\infty} \left(\sqrt[n]{2}-1\right)$

(28) $\sum_{n=1}^{\infty} \operatorname{sen}\left(\frac{1}{n}\right)$

(29) $\sum_{n=0}^{\infty} \frac{1+2^n}{1+3^n}$

(30) $\sum_{n=0}^{\infty} \frac{n}{(1+n^2)^p}$, $p>0$

(31) $\sum_{n=1}^{\infty} \frac{1}{n+\sqrt[17]{n}}$

(32) $\sum_{n=0}^{\infty} \left(\frac{2n+1}{3n+4}\right)^n$

(33) $\sum_{n=0}^{\infty} \frac{\operatorname{sen} 4n}{4^n}$

(34) $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^p}$, $p>0$

(35) $\sum_{n=1}^{\infty} \ln(\cos(1/n))$

(36) $\sum_{n=0}^{\infty} \left(\frac{n^2+1}{2n^2+1}\right)^n$

(37) $\sum_{n=1}^{\infty} \frac{n!}{n^n}$

(38) $\sum_{n=2}^{\infty} \frac{1}{(n\ln n)^p}$, $p>0$

(39) $\sum_{n=1}^{\infty} (\sqrt{1+n^2}-n)$

(40) $\sum_{n=1}^{\infty} \frac{\ln n}{n^p e^n}$, $p>0$

(41) $\sum_{n=0}^{\infty} e^{-n}n!$

(42) $\sum_{n=2}^{\infty} \frac{1}{(\ln n)^p}$, $p>0$

(43) $\sum_{n=1}^{\infty} \frac{\sqrt[8]{n^7+3n^3-2}}{\sqrt[6]{n^9+7n^2}}$

(44) $\sum_{n=1}^{\infty} \operatorname{sen}\left(\frac{1}{n\sqrt[4]{n^3+6}}\right)$

(45) $\sum_{n=0}^{\infty} \frac{n^2 2^n}{n!}$

(46) $\sum_{n=1}^{\infty} \frac{n^p}{e^{-an}}$, $a,p>0$

(47) $\sum_{n=1}^{\infty} a^n n^p$, $a,p>0$

(48) $\sum_{n=1}^{\infty} \frac{(2n)^n}{n^{2n}}$

(49) $\sum_{n=2}^{\infty} \frac{1}{(\ln n)^{\ln n}}$

(50) $\sum_{n=1}^{\infty} \frac{1}{n^{1+1/n}}$

(51) $\sum_{n=1}^{\infty} \left(1+\frac{1}{n}\right)^{2012} e^{-n/3}$

(52) $\sum_{n=1}^{\infty} \frac{1+n+n^2}{\sqrt{1+n^2+n^6}}$

(53) $\sum_{n=1}^{\infty} \frac{n^n}{(n+1)^{n+1}}$

7. Classifique as séries abaixo absolutamente convergentes, condicionalmente convergentes ou divergentes:

(1) $\sum_{n=1}^{\infty} (-1)^n\frac{1}{\sqrt{n}}$

(2) $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^{\frac{3}{2}}}$

(3) $\sum_{n=1}^{\infty} (-1)^n\frac{2n^2+1}{n^3+3}$

(4) $\sum_{n=2}^{\infty} (-1)^n\frac{1}{\ln\ n}$

(5) $\sum_{n=2}^{\infty} (-1)^n\frac{\ln n}{n}$

(6) $\sum_{n=2}^{\infty} \frac{(-1)^n}{n(\ln n)^2}$

(7) $\sum_{n=2}^{\infty}(-1)^n \frac{\sqrt{n}}{\ln n}$

(8) $\sum_{n=1}^{\infty}(-1)^{2n+1} \frac{1}{\sqrt{n}}$

(9) $\sum_{n=1}^{\infty}(-1)^n \operatorname{sen}\frac{1}{n^p}$, $p>0$

(10) $\sum_{n=2}^{\infty}(-1)^n \frac{\ln n}{n^2}$

(11) $\sum_{n=1}^{\infty}(-1)^n \frac{\ln n}{\sqrt{n}}$

(12) $\sum_{n=1}^{\infty}\frac{(-1)^n n^n}{(n+1)^{n+1}}$

(13) $\sum_{n=1}^{\infty}\frac{(-1)^n}{(1+n^2)^p}$, $p>0$

(14) $\sum_{n=1}^{\infty}\frac{(-1)^n}{(\ln n)^p}$, $p>0$

(15) $\sum_{n=1}^{\infty}(-1)^n \frac{\sqrt{n}}{n+1}$

(16) $\sum_{n=1}^{\infty}(-1)^n n! e^{-n}$

(17) $\sum_{n=1}^{\infty}(-1)^n \frac{\ln n}{n^p}$, $p>0$

(18) $\sum_{n=1}^{\infty}(-1)^n \frac{2\cdot 4\cdot\ldots(2n)}{3\cdot 5\cdot(2n+1)}$

(19) $\sum_{n=1}^{\infty}\frac{(-1)^n}{n+\sqrt[3]{n^2+3n}}$

(20) $\sum_{n=1}^{\infty}(-1)^n \operatorname{sen}\left(\frac{1}{n}\right)$

(21) $\sum_{n=1}^{\infty}\frac{(-1)^n}{10^n n!}$

(22) $\sum_{n=1}^{\infty}(-1)^n \frac{\sqrt{n}}{1+7\sqrt{n+2}}$

(23) $\sum_{n=1}^{\infty}\frac{\cos(n\pi)}{n^{3/4}}$

(24) $\sum_{n=1}^{\infty}\frac{(-1)^n n!}{2\cdot 5\cdot\ldots\cdot(3n+2)}$

(25) $\sum_{n=1}^{\infty}(-1)^n n \operatorname{tg}(1/n)$

(26) $\sum_{n=1}^{\infty}\frac{(-2)^n n!}{e^{n^2}}$

(27) $\sum_{n=0}^{\infty}(-1)^n \frac{\operatorname{sen}(7n)}{9+5^n}$

(28) $\sum_{n=0}^{\infty}\frac{5^n}{3^n+4^n}$

(29) $\sum_{n=0}^{\infty}\frac{3^n}{4^n+5^n}$

(30) $\sum_{n=1}^{\infty}\frac{(n!)^n}{n^{4n}}$

(31) $\sum_{n=1}^{\infty}(-1)^n \frac{\sqrt{n}}{n+4}$

(32) $\sum_{n=1}^{\infty}(-1)^n \frac{\cos(n!)}{\sqrt[3]{n^4+\operatorname{sen} n}}$

☆ Séries de potências

8. Determine o intervalo máximo de convergência de cada uma das séries de potências abaixo:

(1) $\sum_{n=1}^{\infty}\frac{n}{4^n}x^n$

(2) $\sum_{n=1}^{\infty} n!\, x^n$

(3) $\sum_{n=1}^{\infty}\frac{x^n}{n^3+1}$

(4) $\sum_{n=1}^{\infty}\frac{(3n)!}{(2n)!}x^n$

(5) $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{(x-5)^n}{n3^n}$

(6) $\sum_{n=1}^{\infty}\frac{(x+1)^n}{(n+1)\ln^2(n+1)}$

(7) $\sum_{n=1}^{\infty}\frac{10^n}{(2n)!}(x-7)^n$

(8) $\sum_{n=1}^{\infty}\frac{\ln n}{e^n}(x-e)^n$

(9) $\sum_{n=1}^{\infty}\frac{n!}{n^n}(x+3)^n$

(10) $\sum_{n=1}^{\infty}(-1)^n \frac{(x-3)^n}{(2n+1)\sqrt{n+1}}$

(11) $\sum_{n=0}^{\infty}\frac{n^2}{4^n}(x-4)^{2n}$

(12) $\sum_{n=2}^{\infty}\frac{(-1)^n}{n(\ln n)^2}\, x^n$

(13) $\sum_{n=1}^{\infty}2^n x^{n^2}$

(14) $\sum_{n=1}^{\infty}\frac{3^n}{n4^n}x^n$

(15) $\sum_{n=1}^{\infty}\frac{(2n)!}{(n!)^2}x^n$

(16) $\sum_{n=1}^{\infty}\frac{(2n)!}{(n!)^2}x^{2n}$

(17) $\sum_{n=1}^{\infty}\frac{(2n)!}{(n!)^2}x^{n^2}$

(18) $\sum_{n=1}^{\infty}\frac{n!}{n^n}x^n$

(19) $\sum_{n=1}^{\infty}\frac{n!}{n^n}x^{3n}$

(20) $\sum_{n=1}^{\infty}\frac{n!}{n^n}x^{n!}$

(21) $\sum_{n=1}^{\infty}\frac{x^n}{(2+(-1)^n)^n}$

(22) $\sum_{n=1}^{\infty}\frac{(x+1)^n}{a^n+b^n}$, $b>a>0$

(23) $\sum_{n=1}^{\infty}\left(\frac{2^n+3}{3^n+2}\right)x^n$

(24) $\sum_{n=1}^{\infty}\left(\frac{3n+2}{5n+7}\right)^n x^n$

(25) $\sum_{n=1}^{\infty}(\operatorname{sen} n)x^n$

(26) $\sum_{n=0}^{\infty}\frac{1}{(3+(-1)^n)^n}x^n$

(27) $\sum_{n=1}^{\infty}\frac{n!}{1\cdot 3\cdot\ldots\cdot(2n-1)}x^n$

(28) $\sum_{n=1}^{\infty} \frac{n^2}{2\cdot 4\cdot \ldots \cdot (2n)} x^n$

9. Suponhamos que $\sum_{n=0}^{\infty} a_n x^n$ converge quando $x = -4$ e diverge quando $x = 6$. O que pode ser dito sobre a convergência ou divergência das séries a seguir?

(1) $\sum_{n=0}^{\infty} a_n$

(2) $\sum_{n=0}^{\infty} a_n 8^n$

(3) $\sum_{n=0}^{\infty} a_n (-3)^n$

(4) $\sum_{n=0}^{\infty} (-1)^n a_n 9^n$

10. Usando derivação e integração termo a termo, calcule as somas das séries de potências:

(1) $\sum_{n=1}^{\infty} \frac{x^n}{n}$

(2) $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n}$

(3) $\sum_{n=1}^{\infty} \frac{x^{2n-1}}{2n-1}$

(4) $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^{2n-1}}{2n-1}$

(5) $\sum_{n=0}^{\infty} (n+1) x^n$

(6) $\sum_{n=1}^{\infty} n x^n$

(7) $\sum_{n=1}^{\infty} n x^{2n-1}$

(8) $\sum_{n=2}^{\infty} \frac{x^n}{n(n-1)}$

(9) $\sum_{n=1}^{\infty} n^2 x^{n-1}$

(10) $\sum_{n=1}^{\infty} n^3 x^n$

(11) $\sum_{n=0}^{\infty} (n+4) x^n$

(12) $\sum_{n=1}^{\infty} \frac{x^{4n}}{4n}$

11. Use as séries do exercício anterior para calcular:

(1) $\sum_{n=1}^{\infty} \frac{1}{n 2^n}$

(2) $\sum_{n=1}^{\infty} \frac{n^3}{2^n}$

(3) $\sum_{n=1}^{\infty} \frac{(-1)^n}{5^n (n-1)}$

12. Determine as expansões em séries de potências em torno de $x_0 = 0$ das seguintes funções e os valores de x para os quais essas expansões são válidas:

(1) $\frac{1}{(1+x)^2}$

(2) $\frac{x^3}{1+x^2}$

(3) $\frac{1}{(1+x)^3}$

(4) $\frac{2x}{1+x^4}$

(5) $\ln(1+x)$

(6) $\ln\left(\frac{1}{1+3x^2}\right)$

13. Verifique as identidades abaixo:

(1) $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$, $x \in \mathbb{R}$

(2) $\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n}$, $-1 < x \leq 1$

(3) $\operatorname{sen} x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$, $x \in \mathbb{R}$

(4) $\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$, $x \in \mathbb{R}$

(5) $\ln\left(\frac{1+x}{1-x}\right) = 2\sum_{n=0}^{\infty} \frac{x^{2n+1}}{2n+1}$, $|x| < 1$

(6) $\arctan\ x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$, $|x| \leq 1$

(7) $\sum_{n=0}^{\infty} \frac{(n+1)(n+2)(n+3)}{6} x^n = \frac{1}{(1-x)^4}$, $x \neq 1$

14. Utilizando as séries do exercício anterior, obtenha um valor aproximado de:

(1) e, com erro inferior a 10^{-5}

(2) $\operatorname{sen} 1$, com erro inferior a 10^{-5} e a 10^{-7}

(3) $\ln 2$ e $\ln 3$, com erro inferior a 10^{-5}

(4) $\arctan(1/2)$ e $\arctan(1/3)$, com erro inferior a 10^{-5}

(5) $\pi/4$, com erro inferior[1] a 10^{-5}

15. Seja $f(x) = \arctan x$, $x \in \mathbb{R}$. Determine

$$\frac{f^{2026}(0) + f^{2027}(0)}{f^{2025}(0) - f^{2024}(0)},$$

onde $f^{(n)}$ denota a derivada de ordem n de f.

16. Desenvolva em série de potências de x as seguintes funções, indicando os intervalos de convergência:

(1) $f(x) = x^2 e^x$

(2) $f(x) = \cos(\sqrt{x})$

(3) $f(x) = \operatorname{sen}(x^2)$

(4) $f(x) = \cos^2 x$

(5) $f(x) = \int_0^x \frac{\operatorname{sen} t}{t} dt$

(6) $f(x) = \int_0^x e^{-t^2} dt$

(7) $f(x) = \int_0^x \frac{\ln(1+t)}{t} dt$

(8) $f(x) = \int_0^x \operatorname{sen}(t^2) dt$

(9) $f(x) = \frac{e^{x^2} - 1}{x}$

[1] Usando a identidade $\operatorname{tg}(x+y) = \frac{\operatorname{tg}(x) + \operatorname{tg}(y)}{1 - \operatorname{tg}(x)\operatorname{tg}(y)}$, verifique que $\pi/4 = \arctan(1/2) + \arctan(1/3)$. Use esta última igualdade para estimar $\pi/4$ com a precisão indicada.