Introduction to Julia Sets of Rational Functions

André Pedroso Kowacs
andrekowacs@gmail.com

Advisor: Elizabeth W. Karas
Department of Mathematics
Federal University of Paraná

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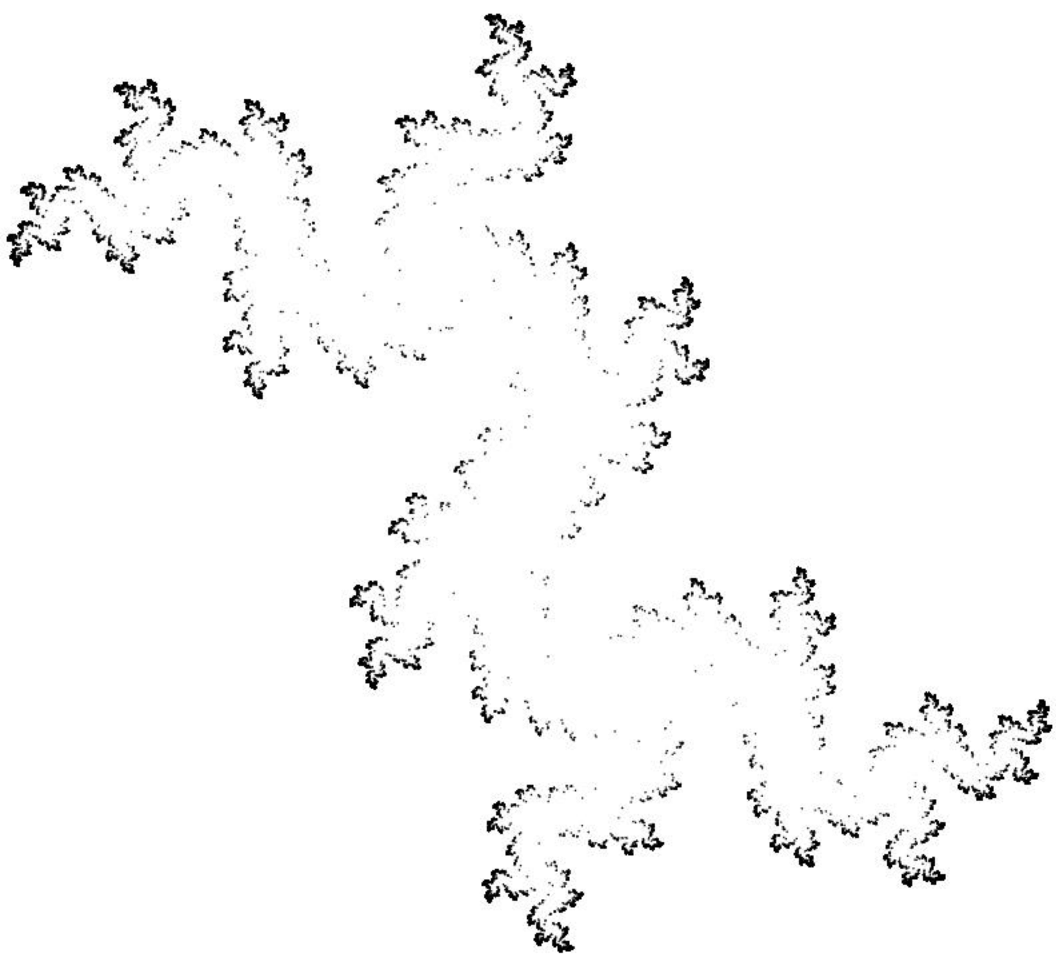


Abstract

The main goal of these notes is to introduce and present some properties of Julia Sets for rational functions, based primarily in [2] and [5]. Inspired to follow the work of [7], the notes are aimed at the undergraduate mathematics students, who are expected to have some basic knowledge in Complex Analysis and Topology. If any difficulty regarding these concepts may present, the reader may check [1] and [6]. These notes are structured so that some basic concepts and tools are presented in the first section, "Complex Analysis". The reader who is familiar with these concepts may wish to skip to the second section, "Iteration of Rational functions", in which we define the Julia sets for rational functions. We then define another family of sets related to the concept of normality of the family of iterates of rational functions. We prove many interesting properties about these sets and conclude by showing that they coincide with the Julia sets. We also, we mention some techniques for "graphing" Julia sets and present some examples.

1 Complex Analysis

This first section is dedicated to some important results in Complex Analysis that will be needed later on.

1.1 Extended Complex Plane

Definition 1.1. Taking an abstract point ∞ called *infinity*, we define the *extended complex plane* as:

$$\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$$

Definition 1.2. We say $f : D \subset \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ is *meromorphic* in D if for all $z_0 \in D$ there exists neighbourhood $V_{z_0} \subset D$ such that f or $1/f$ is holomorphic at V_{z_0}. The *poles* of f holomorphic are points such that $1/f$ is analytic around w, with $1/f(w) = 0$.

Note that given any meromorphic function defined in a subset of the complex plane, there's a natural unique extension to the extended complex plane setting $f(w) = \infty$ where w is a pole of f.

Definition 1.3. A function is said to be defined in some neighbourhood of ∞ if it is defined on some set $\{|z| > r\} \cup \{\infty\}$. In this case, it is holomorphic at ∞ if $1/f$ is meromorphic at 0.

Again, if a complex function f is defined such that $\lim_{z\to\infty} f(z)$ makes sense and exists, we may extend it uniquely by setting $f(\infty) = \lim_{z\to\infty} f(z)$.

Example 1.4. Let $P(z) = a_n z^n + ... + a_0$, $a_n \neq 0$, $n > 0$ be a complex polynomial. Then P is meromorphic in $\overline{\mathbb{C}}$ and in fact, $P(\infty) = \infty$. Clearly it is holomorphic at ∞ since $P(1/z) = a_n/z^n + ... + a_0$ has a pole at the origin, because:

$$\frac{1}{P(1/z)} = \frac{z^n}{a_n + ... + a_0 z^n}$$

is holomorphic at 0, with value 0 there.

Definition 1.5. We say $U \subset \overline{\mathbb{C}}$ is *bounded* if there exists C such that $|z| < C$, for all $z \in U$ and $\infty \notin U$.

Note: A more rigorous approach to dealing with the extended complex plane is achieved by visualizing it as a Riemann surface, specifically the complex sphere, in which taking another metric called the chordal metric, we may treat ∞ as any other point. However, since all calculus is done with local charts in the complex plane, we have no real necessity to deal with this metric. For that reason, we chose to omit these definitions in these notes. The interested reader may check [5].

1.2 Roots of Analytic Functions

Definition 1.6. We say a meromorphic function f has n roots at $z_0 \in \mathbb{C}$, or that z_0 is a root of multiplicity n of f if, in a neighbourhood of z_0:

$$f(z) = (z - z_0)^n g(z)$$

where g is analytic with $g(z_0) \neq 0$. We say $z_0 \in \mathbb{C}$ is a solution of multiplicity n for $f(z) = w$, $w \in \mathbb{C}$ if z_0 is a root of multiplicity n for the function $f(z) - w$.

Notice these definitions are motivated by the power series definition of f.

Proposition 1.7. The number $z_0 \in \mathbb{C}$ is a solution of multiplicity n for the equation $f(z) = w$, $w \in \mathbb{C}$ if and only if:

$$\lim_{z \to z_0} \frac{f(z) - w}{(z - z_0)^n}$$

exists and is different from 0 and ∞.

Proof. ($\Longleftarrow$) Writing $f(z) - w$ as a power series at z_0, we have:

$$f(z) - w = \sum_{k=0}^{\infty} a_k (z - z_0)^k = (z - z_0)^n \sum_{k=0}^{\infty} a_n (z - z_0)^{k-n}.$$

But then the above limit implies:

$$a_i = \lim_{z \to z_0} \frac{f(z) - w}{[z - z_0]^n} [z - z_0]^{n-i} = 0,\ i = 0, 1, ..., n-1 \implies a_0, ..., a_{n-1} = 0$$

so that

$$f(z) = (z - z_0)^k \sum_{k=n} a_{k-n} (z - z_0)^{k-n} = (z - z_0)^n g(z)$$

where $g(z_0) \neq 0$. The converse is immediate. □

Corollary 1.8. If $z_0 \in \mathbb{C}$ is a solution of multiplicity n for $f(z) - w$, $w \in \mathbb{C}$, then $f^{(i)}(z_0) = 0$ for $i = 1, ..., n-1$, $f^{(n)}(z_0) \neq 0$.

Example 1.9. Let $f(z) = z^2 - 2z$. Then $z_0 = 1$ is a solution of multiplicity 2 for the equation $f(z) = -1$, since

$$f(z) - (-1) = 0 \iff z^2 - 2z + 1 = 0 \iff (z-1)^2 = 0.$$

Note also that

$$\lim_{z \to 1} \frac{f(z) - (-1)}{z - 1} = \lim_{z \to 1} \frac{(z-1)^2}{z-1} = 0$$

while

$$\lim_{z \to 1} \frac{f(z) - (-1)}{(z-1)^2} = \lim_{z \to 1} \frac{(z-1)^2}{(z-1)^2} = 1.$$

Finally, note that $f'(z) = 2z - 2$, so that $f'(1) = 0$, $f''(1) = 2 \neq 0$.

Definition 1.10. Let f be analytic, $z_0 \in \overline{\mathbb{C}}$ such that $f'(z_0) = 0$, then we say z_0 is a critical point of f

Proposition 1.11. Let f be analytic. Then f is injective at $z_0 \in \overline{\mathbb{C}}$ iff $f'(z_0) \neq 0$.

Proof. If $z_0 = \infty$, simply compose f with an injective map g that sends ∞ to a point $z_0' \in \mathbb{C}$ such that $g'(\infty) \neq 0$. So, in the following we will assume $z_0 \in \mathbb{C}$. Let f, z_0 be such that $f'(z_0) \neq 0$. Viewing $\mathbb{C}$ as $\mathbb{R}^2$ we can apply the Inverse Function Theorem. Alternatively, we provide the following proof: Claim: There exists $\delta' > 0$ such that $f(z) \neq f(z_0)$, $\forall z_0 \neq z \in B(z_0, \delta')$.
Suppose not. Then $\forall \delta' = 1/n$, there would exist $z_0 \neq z_n \in B(z_0, 1/n)$, $f(z_n) - f(z_0) = 0$. But then:

$$0 = \lim_{n\to\infty} \frac{f(z_n) - f(z_0)}{z_n - z_0} = f'(z_0)$$

which is a contradiction.
If necessary, take δ' smaller so that f has no poles at $B(z_0, \delta')$. Denote $C = S(z_0, \delta')$ and $\Gamma = f(C)$. Now, by the argument principle we have that:

$$\frac{1}{2\pi i} \int_\gamma \frac{f'}{f} = \mathbb{Z} - \mathbb{P}$$

Where $\mathbb{Z} =\#$ zeros of f in λ, $\mathbb{P} =\#$ poles of f inside γ. We have:

$$1 = \frac{1}{2\pi i} \int_{S(z_0,\delta')} \frac{f'(z)}{f(z) - f(z_0)} dz = \frac{1}{2\pi i} \int_{S(z_0,\delta')} \frac{dw}{w - \alpha} = \frac{1}{2\pi i} \int_{S(z_0,\delta')} \frac{dw}{w - \beta}.$$

For all β sufficiently close to α, i.e: $\beta \in B(\alpha, \epsilon)$, since we claim the winding number

$$n(\gamma, a) := \frac{1}{2\pi i} \int_\gamma \frac{dz}{z - a}$$

is locally constant.
We will show that it is an integer valued continuous function, which implies the former. Let γ be given by $z(t)$, $0 \leq t \leq 1$ and

$$F(s) = \int_0^s \frac{z'(t)}{z(t) - a} dt,\ 0 \leq s \leq 1.$$

Then $F(s)$ is given piecewise as branches of log, so that:

$$(z(s) - a)e^{-F(s)} = \frac{(z(s) - a)^2}{z(0) - a}. \tag{1}$$

Therefore

$$\begin{aligned}\left[(z(s) - a)e^{-F(s)}\right]' &= z'(S)e^{-F(S)} - (z(s) - a)e^{-F(s)}F'(s) \\ &= \frac{z'(s)(z(s) - a)}{z(0) - a} - \frac{(z(s) - a)^2 z'(s)}{(z(s) - a)(z(s) - a)} = 0.\end{aligned}$$

So that (1) is constant. But at $s = 0$ we have $(z(s) - a)e^{-F(s)} = z(0) - a$. Thus

$$e^{F(s)} = \frac{z(s) - a}{z(0) - a}.$$

Therefore

$$e^{F(1)} = \frac{z(1) - a}{z(0) - a} = 1,$$

since $z(1) = z(0)$. Hence, $F(1) = 2\pi k i$, $k \in \mathbb{Z}$ and $n(\gamma, a) = \frac{1}{2\pi i} F(1) = k$. Notice that $F(s, a)$ varies continuously with a inside γ, so that n does too.
Therefore, $\exists \delta < \delta'$ such that $z \in B(z_0, \delta) \implies f(z) = f(z_0)$, it follows, for $z_1, z_2 \in B(z_0, \delta)$, $C = S(z_0, \delta)$, $\Gamma = f(C)$:

$$\begin{aligned} 1 &= \frac{1}{2\pi i} \int_\Gamma \frac{dw}{w - f(z_1)} = \frac{1}{2\pi i} \int_\Gamma \frac{dw}{w - f(z_2)} \\ &= \frac{1}{2\pi i} \int_C \frac{f'(z)dz}{f(z) - f(z_1)} = \frac{1}{2\pi i} \int_C \frac{f'(z)dz}{f(z) - f(z_2)}. \end{aligned}$$

Such that $f(z_1)$ and $f(z_2)$ are both assumed only once inside $f(C)$, that is, $f(z_1) \neq f(z_2)$ if $z_1 \neq z_2$. And f is injective.

Now, let f be injective at z_0 and without loss of generality, $f(z_0) = 0$. Then:

$$\begin{aligned} f(z) &= a_k(z - z_0)^k + a_{k+1}(z - z_0)^{k+1} + \dots \\ &= (z - z_0)^k [a_k + a_{k+1}(z - z_0) + \dots] \\ &:= (z - z_0)^k g(z). \end{aligned}$$

With $a_k = f^{(k)}(z_0)/k! \neq 0$ the least coefficient corresponding to the non-null derivative (notice that $f(z_0) = 0 \implies a_0 = 0$ so that $k \geq 1$). Then $(g(z_0) \neq 0$, so that there exists an analytic k-th root branch for g in a disk $B(z_0, \delta)$, $g^{1/k}$. Then

$$f(z) = [h(z)]^k$$

where h is the analytic function defined by:

$$h(z) = (z - z_0) g^{1/k}(z)$$

Notice that $h(z_0) = 0$ and $h'(z_0) = g^{1/k}(z_0) \neq 0$, so that by the previous part h is injective. But $f = l \circ h$, where $l(z) = z^k$. So that for $\epsilon < 1$, $B(0, \epsilon \subset h(B(z_0, \delta)$, then $l(B(0, \epsilon)) \subset B(0, \epsilon)$ implies $f(h^{-1}(B(0, \epsilon))) \subset B(0, \epsilon)$. But since $deg(l) = k$ and h injective this implies $f(z) = w$, $w \in B(0, \epsilon)$ has k solutions counting multiplicities in $B(0, \epsilon)$ (with the only solution with multiplicity greater than 1 being 0). So that f is "k to 1" in $B(0, \epsilon) \backslash \{0\}$. But since f is injective, we must have $k = 1$, so that $f'(z_0) \neq 0$. □

Definition 1.12. Let $z_0 \in \mathbb{C}$ be a fixed point of an analytic function f . We say that f has k-fixed points at z_0 if z_0 is a root of multiplicity k for the function $f(z) - z$.

Lemma 1.13. Let $z_0 \in \mathbb{C}$ be a fixed point of an analytic function f, ϕ be analytic, injective and finite in a neighbourhood of z_0. Then $\phi \circ f \circ \phi^{-1}$ has the same number of fixed points at $\phi(z_0)$ as f has at z_0.

Proof. Suppose f has k fixed points at z_0. Since

$$\frac{\phi \circ f \circ \phi^{-1}(z) - z}{[z - \phi(z_0)]^k} = \left(\frac{\phi \circ f \circ \phi^{-1}(z) - \phi \circ \phi^{-1}(z)}{f \circ \phi^{-1}(z) - \phi(z)} \right) \left(\frac{f \circ \phi^{-1}(z) - \phi^{-1}(z)}{[\phi^{-1}(z) - z_0]^k} \right) \cdot \left(\frac{[\phi^{-1}(z) - \phi \circ \phi^{-1}(z)]^k}{[z - \phi(z_0)]^k} \right). \tag{2}$$

Now, $\phi(z_0)$ is zero of multiplicity k of $\phi \circ f \circ \phi^{-1} - z$ iff the right hand side of (2) tends to a finite non-zero number, as $z \to \phi(z_0)$. Let $f \circ \phi^{-1}(z) = u$ and $\phi^{-1}(z) = v$. Then the first term in the right hand side becomes:

$$\frac{\phi(u) - \phi(v)}{u - v}$$

And as $z \to \phi(z_0)$, $u \to z_0$, $v \to z_0$ and since ϕ is injective, $u \neq v$ for $z \neq z_0$, so that this term is always finite and tends to $\phi'(z_0) \neq 0$ since ϕ is injective, by Proposition 1.11.
Next, setting, notice that the second term in (2) is:

$$\frac{f(v) - v}{[v - z_0]^k}$$

and as $z \to \phi(z_0)$, $v \to z_0$. Since z_0 is fixed point of order k for f, the above limit is finite and non-zero. Again, by injectivity of ϕ, the denominator never vanishes.
Finally, let $\phi(z_0) = w$. Then the last term of (2) becomes:

$$\left[\frac{\phi^{-1}(z) - \phi^{-1}(w)}{z - w} \right]^k$$

and as $z \to \phi(z_0) = w$, the above expression tends to

$$[(\phi^{-1})'(w)]^k = [(\phi^{-1})'(\phi(z_0)]^k = [\phi'(z_0)]^{-k} \neq 0$$

again by Proposition 1.11. □

Motivated by this Lemma, we define:

Definition 1.14. A meromorphic map f has n fixed points at ∞ if $\phi \circ f \circ \phi^{-1}$ has n fixed points at $\phi(\infty)$, where ϕ is any injective meromorphic function such that $\phi(\infty) \in \mathbb{C}$.

Corollary 1.15. If z_0 is a fixed point of f with multiplicity greater than 1, then $f'(z_0) = 1$.

Proof. Follows from Corollary 1.8 that $g(z) = f(z) - z$ is such that $0 = g'(z_0) = f'(z_0) - 1$, so that $f'(z_0) = 1$. □

Example 1.16. The map $f(z) = \frac{z^2+1}{z}$ has a fixed point at ∞. Taking $\phi(z) = 1/z$, so that $\phi(\infty) = 0$, we have that $h(z) = \phi \circ f \circ \phi^{-1}(z) = \frac{z}{z^2+1}$. Then,

$$h(z) = z \iff \frac{z}{z^2+1} - z = 0 \iff z^3 = 0.$$

It follows that 0 is a fixed point of multiplicity 3 for h, thus ∞ is a fixed point of multiplicity 3 for f.

Theorem 1.17. Let $f : U \to \overline{\mathbb{C}}$ be meromorphic, $f(0) = 0$, $f'(0) = \lambda \neq 0, 1$. Then $\exists r > 0, \phi : f(B(0,r)) \to \overline{\mathbb{C}}$, such that $\phi \circ f(z) = \lambda \phi(z)$.

Proof. Suppose, without loss of generality, that $|\lambda| < 1$ (The case $|\lambda| > 1$ follows by applying the same theorem to the the local inverse of f, whose existence is guaranteed since $f'(0) \neq 0$). Take $c \in \mathbb{R}$ such that $c^2 < |\lambda| < c < 1$. Since $f(0) = 0$, $|f'(0)| = |\lambda| < c$ and f' is continuous, there exists $r > 0$ such that $|f'(z)| < c, \forall z \in B(0,r)$. Hence:

$$\begin{aligned} |f(z) - f(0)| &< c|z - 0| \\ |f(z)| &< c|z|, \forall z \in B(0,r). \end{aligned}$$

Thus, given $z_0 \in B(0,r)$, it follows that:

$$\begin{aligned} |f^2(z)| &= |f(f(z_0))| \\ &< c|f(z_0)| \\ &< c^2|z_0|. \end{aligned}$$

By induction, it follows that

$$|f^n(z_0)| < c^n|z_0| < c^n r < r \tag{3}$$

Which implies $f^n(z_0) \in B(0,r)$, $\forall n, z, \forall z_0 \in B(0,r)$, that is,

$$f^n(B(0,r)) \subset B(0,r) \tag{4}$$

By Taylor's Theorem, we also have that:

$$f(z) = \lambda \cdot z + \frac{f''(\xi)}{2!} z^2$$

with ξ between z and 0. Therefore, since f'' is bounded in $B(0,r)$, it follows that

$$|f(z) - \lambda z| \leq C|z|^2, \forall z \in B(0,r)$$

where $C = \sup_{\xi \in B(0,r)} |f''(\xi)|$. Then, by (3) and (4) we have that:

$$|f^{n+1}(z) - \lambda f^n(z)| \leq C|f^n(z)|^2 \leq Cr^2c^{2n}. \tag{5}$$

Now, consider the sequence of analytic functions $\phi : B(0,r) \to \overline{\mathbb{C}}$:

$$\phi_n(z) = \frac{f^n(z)}{\lambda^n}.$$

Then, from (5), we have that for all $z \in B(0,r)$:

$$\begin{aligned} |\phi_{n+1}(z) - \phi_n(z)| &= \left| \frac{f^{n+1}(z)}{\lambda^{n+1}} - \frac{f^n(z)}{\lambda^n} \right| \\ &= \frac{1}{|\lambda|^{n+1}} \left| f^{n+1}(z) - \lambda f^n(z) \right| \\ &\leq \frac{1}{|\lambda|^{n+1}} C r^2 c^{2n} = \frac{Cr^2}{|\lambda|} \left(\frac{c^2}{|\lambda|} \right)^n . \end{aligned}$$

Since $c^2/|\lambda| < 1$, it follows that $|\phi_{n+1}(z) - \phi_n(z)| \to 0$, as $n \to \infty$, uniformly on $B(0,r)$. So ϕ_n is Cauchy, therefore it converges to an analytic function ϕ. But then, note that:

$$\phi_n(0) = \frac{f^n(0)}{\lambda^n} = 0, \, \forall n.$$

So that $\phi(0) = 0$ and

$$\phi_n \circ f(z) - \lambda \phi_{n+1}(z) = \frac{f^{n+1}(z)}{\lambda^n} - \frac{\lambda f^{n+1}(z)}{\lambda^{n+1}} = 0$$

for all $z \in B(0,r)$, $n \in \mathbb{N}$. Taking the limit as $n \to \infty$, it follows that:

$$\begin{aligned} \phi \circ f(z) - \lambda \phi(z) &= 0 \\ \phi \circ f(z) &= \lambda \phi(z). \end{aligned}$$

□

1.3 Rational Maps

Definition 1.18. A *rational map* is a function $R : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ of the form

$$R(z) = \frac{P(z)}{Q(z)}$$

where P and Q are both polynomials, not both being the zero polynomial. If $P = 0$, then $R = 0$ and if $Q = 0$ then $R = \infty$. We also assume that P and Q have no common zeros, that is, they are *coprime*. Thus R determines P and Q uniquely up to a scalar multiple, and the *degree* of R is

$$deg(R) = \max\{deg(P), deg(Q)\}$$

except in the degenerate cases where R is constant. In that case, $deg(R) = 0$.

Example 1.19. Take $R(z) = \frac{z^3+2z+3}{z-1}$. Then $deg(R) = 3$, $R(1) = \infty$, $R(\infty) = \infty$.

Proposition 1.20. If R is a rational map, $deg(R) = d > 0$, then $\forall w \in \overline{\mathbb{C}}$ the equation $R(z) = w$ has d solutions, counting multiplicities.

Proof. Let $R = P/Q$, $deg(P) = n$, $deg(Q) = m$, so that:

$$R(z) = \frac{a_n z^n + ... + a_0}{b_m z^m + ... + b_0}.$$

If $n = m$, then $R(\infty) \in \mathbb{C}$, so that all roots and poles of R are in $\mathbb{C}$. Notice that there are exactly $n = m = d$ of each, being those the roots of P ad Q respectively. If $n > m$, then R has $n = d$ roots and m poles at the respective roots in $\mathbb{C}$. Notice that there is also a pole at ∞, whose multiplicity is by definition the multiplicity of 0 in $1/R(1/z)$. That is in $Q(1/z)/P(1/z)$ or:

$$\frac{b_m z^{n-m} + ... + b_0 z^n}{a_n + ... + a_0 z^n}$$

so that 0 has multiplicity $n - m$. There are, therefore, $m + n - m = n = d$ poles in total. Similarly, we show that the same holds for the case $m > n$ (in this case there is a zero at ∞, with multiplicity $m - n$).
Finally, if $w \neq \infty$the number of solutions of the equation $R(z) = w$ is, by definition, the number of roots of the equation $R(z) - w = 0$, that is:

$$R(z) - w = \frac{P(z) - wQ(z)}{Q(z)}.$$

Notice that since P and Q have no common zeros, $P - wQ$ and Q have no common zeros. Then $R(z) - w$ and $R(z)$ have the same degree, so that as we saw above they have the same number of roots, that is, d. If $w = \infty$, then the number of solutions of $R(z) = w$ is the number of poles of R, which, as we saw, is also d. So that it holds for any $w \in \overline{\mathbb{C}}$. □

Corollary 1.21. If R is a rational map and $\forall w \in \overline{\mathbb{C}}$ the equation $R(z) = w$ has exactly d solutions, counting multiplicities, then $deg(R) = d$.

Proposition 1.22. If R, S are a rational maps, then:

$$deg(R \circ S) = deg(R) \cdot deg(S).$$

Proof. Notice that the composition of rational maps is a rational map. Now, let $w \in \overline{\mathbb{C}}$. Then the equation $R(z) = w$ has exactly $deg(R) = d$ solutions, counting multiplicities. Let $\xi_1, ..., \xi_d$, not necessarily distinct be all those solutions. Then for each $i = 1, ..., d$, the equation $S(z) = \xi_i$ has exactly $deg(S)$ solutions, counting multiplicity. So, the equation $R(S(z)) = w$ has, in total, exactly $deg(R) \cdot deg(S)$ solutions, counting multiplicity, so that $R \circ S$ has degree $deg(R) \cdot deg(S)$ by the above result. □

Corollary 1.23. If R is a rational map, then $deg(R^n) = [deg(R)]^n$, where $R^n = R \circ ... \circ R$, n times.

Definition 1.24. A rational map of the form:

$$g(z) = \frac{az + b}{cz + d}, \; ad - bc \neq 0$$

is said to be a *Mobius map*. We say two meromorphic functions f, h are *conjugate* if there exists a Mobius map g such that $h = g \circ f \circ g^{-1}$.

Example 1.25. If f is a quadratic polynomial, then f is conjugate to $h_c(z) = z^2 + c$, for some $c \in \mathbb{C}$. In fact, taking $g(z) = az + b$, $a \neq 0$, then $g^{-1}(z) = \frac{z-b}{a}$ so that:

$$(g \circ h_c \circ g^{-1})(z) = a\left(\frac{z^2 - 2zb + b^2}{a^2} + c\right) + b = \frac{1}{a}z^2 + \frac{-2b}{a}z + \frac{b^2}{a} + c + b$$

So that if $f(z) = \alpha z^2 + \beta z + \gamma$, $\alpha \neq 0$, then:

$$a = \frac{1}{\alpha},\ b = \frac{-\beta}{2\alpha} \text{ and } c = \gamma + \frac{\beta}{2\alpha} - \frac{\beta^2}{4\alpha}$$

Proposition 1.26. If R, S are conjugate rational maps, $S = g \circ R \circ g^{-1}$ then:

1. $deg(R) = deg(S)$.
2. $S^n = g \circ R^n \circ g^{-1}$ so that S^n and G^n are conjugate.
3. If R has n fixed points at z_0, then, S also has n fixed points at $g(z_0)$, so that R and S have the same number of fixed points, counting multiplicity.
4. R and S have the same number of critical points.

Proof. 1. Follows from Proposition 1.22.

2. Note that

$$\begin{aligned} S^2 &= g \circ R \circ g^{-1} \circ g \circ R \circ g^{-1} \\ &= g \circ R^2 \circ g^{-1} \end{aligned}$$

so that it follows from induction

3. Follows from Lemma 1.13.
4. If z_0 is critical point of R, $R'(z_0) = 0$. But then

$$S'(g(z_0)) = g'(R(z_0))R'(z_0)(g^{-1})'(g(z_0)) = 0$$

so that S has a critical point at $g(z_0)$.

□

Proposition 1.27. Let R be a rational map, $deg(R) = d$. Then R has at most $2d-2$ critical points.

Proof. Choose $\xi \in \mathbb{C}$ such that $f'(\xi) \neq 0$, $f(\xi) \neq \xi$ and $f(z) = \xi$ has $d = deg(R)$ distinct solutions (that is, for $z_1, ..., z_d$, $f(z_i) = \xi$, $i = 1, ..., d$ it holds that $f'(z_i) \neq 0$, $i = 1, ..., d$). Let g be the Mobius map:

$$g(z) = \frac{R(\xi) - \xi}{z - \xi}.$$

Then, if $S = g \circ f \circ g^{-1}$, then $S(\infty) = 1$, $S'(\infty) \neq 0$ and $S(z) = \infty$ has d distinct solutions, $g^{-1}(z_1) = y_1, ..., g^{-1}(z_d) = y_d \in \mathbb{C}$, which must then be simple poles.

Follows that all critical points of S must lie in $\mathbb{C}$, so they must be the zeroes of $S'(z)$. Writing $S = P/Q$, we have that

$$S'(z) = \frac{P'(z)Q(z) - P(z)Q'(Z)}{Q(z)^2}$$

where the denominator and nominator are coprime, for suppose not: Then the common zero must be a y_i, since $Q(z)^2 = 0 \implies Q(z) = 0 \implies S(z) = \infty$. But then $Q'(y_i) \neq 0$, since they are simple poles, which implies $P(y_i) = 0$ which is a contradiction since P, Q are coprime. Since $S(\infty) \neq \infty, 0$, $S'(\infty) \neq \infty, 0$, so that $d = deg(Q) = deg(P)$ and $deg(P'Q - PQ') = \deg(Q^2)$ and then $deg(S'(z)) = deg(Q(z)^2 S'(z))$, which is a polynomial, so it's degree can be computed by it's growth rate as z tends to ∞. First note that $Q(z)^2/z^{2d}$ tends to a finite non-zero value as z tends to ∞. Next, since $S'(\infty) \neq 0$, it is injective in a neighbourhood of infinity, so that $S(1/z)$ is injective in a neighbourhood of the origin and then

$$S(1/z) = 1 + a_1 z + \dots$$

with $a_1 \neq 0$. Differentiating both sides yields:

$$-S'(1/z)\frac{1}{z^2} = a_1 + \cdots + o(z^2)$$

for z near 0, implies $S'(z)z^2 = -a_1 + \cdots + o(1/z^2)$ for z near infinity, which means

$$\lim_{z \to \infty} S'(z)z^2 = -a_1 \neq 0.$$

It follows that

$$\frac{Q(z)^2 S'(z) z^2}{z^{2d}} = \frac{Q(z)^2 S'(z)}{z^{2d-2}}$$

tends to a finite non-zero value as z tends to infinity and, therefore, $deg(S') = 2d - 2$ so the result follows. $\square$

Example 1.28. Let

$$R(z) = \frac{z^2 + 1}{z}.$$

Then $deg(R) = 2$, so by the previous proposition, R has at most $2 \cdot 2 - 2 = 2$ critical points. We see that is the case, since:

$$R'(z) = 0 \iff \frac{z^2 - 1}{z^2} = 0 \iff z = 1 \text{ or } z = -1.$$

Corollary 1.29. If R is a rational map, $deg(R) \geq 2$, then by Proposition 1.8, for all $w \in \overline{\mathbb{C}}$, but at most $2 \cdot deg(R) - 2$ exceptions, $R^{-1}(w)$ has d distinct elements, so that R is not (globally) injective.

Proposition 1.30. If R is a rational map, then R has exactly $d+1$ fixed points, counting multiplicity.

Proof. Since conjugation preserves the number of fixed points, we may assume ∞ is not a fixed point of R. Then, the fixed points of R are given by:

$$R(z) = z \overset{z \neq \infty}{\iff} R(z) - z = 0$$

Expressing $R = P/Q$, with $deg(P) \leq deg(Q) = d$ by hypothesis, that is equivalent to:

$$\frac{P(z)}{Q(z)} - z = 0 \iff P(z) - zQ(z) = 0$$

But since $deg(P(z)) < deg(zQ(z))$, follows that $deg(P(z) - zQ(z)) = d+1$, so the result follows. $\square$

2 Julia Sets of Rational functions

When not stated otherwise, all functions from now on will be considered $f : U \subset \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ analytic, where U is open. Also $f^k := f \circ f \circ \cdots \circ f$ (k-times). We begin with a series of definitions:

Definition 2.1. We say $w \in \overline{\mathbb{C}}$ is a *fixed point of* f if $f(w) = w$ and a *periodic point of period* p if it is a fixed point for f^p, $p \geq 1$.

Definition 2.2. Given a function f, $z \in U$, we denote the forward, backwards and total orbits of z, respectively, by:

$$\mathbb{O}^+(z) = \bigcup_{k=1}^{\infty} f^k\left(\{z\}\right), \ \mathbb{O}^-(z) = \bigcup_{k=1}^{\infty} f^{-k}\left(\{z\}\right) \ , \mathbb{O}(z) = \mathbb{O}^+(z) \cup \mathbb{O}^-(z).$$

Definition 2.3. If $w_0 \in \mathbb{C}$ is a periodic point of period $p \geq 1$, we call the forward orbit of w_0: $\mathbb{O}^+(w) = \{w_1, ..., w_{p-1}, w_0\}$ a periodic cycle. Note that any two points in a periodic cycle have the same periodic cycle.
The *multiplier* of a periodic cycle of period p is

$$\lambda = (f^p)'(w)$$

where w is any point in the cycle. If $p = 1$, so that the cycle consists of a single fixed point, we may interchange the words cycle and fixed point. We say a periodic cycle is:

a) *super attractive* if $\lambda = 0$;

b) *attractive* if $0 \leq |\lambda| < 1$;

c) *indifferent* if $|\lambda| = 1$;

d) *repelling* if $|\lambda| > 1$.

Notice that conjugation preserves multipliers: if $f^n(w) = w$, $(f^n)'(w) = \lambda$, then, for $g^{-1}(z) = w$ we have:

$$(g \circ f^n \circ g^{-1})'(z) = g'(f^n(g^{-1}(z))) \cdot (f^n)'(g^{-1}(z)) \cdot (g^{-1})'(z) = g'(g^{-1}(z)) \cdot \lambda \cdot (g^{-1})'(z) = \lambda.$$

So we may define the multiplier of a periodic cycle at infinity as follows:

Definition 2.4. If $w_0 = \infty$ is a periodic point of period $p \geq 1$, we define the multiplier of its cycle as $(g \circ f^p \circ g^{-1})(0)$, where $g = 1/z$. That is:

$$\lambda = \frac{1}{f'(\infty)}.$$

It is worthy to mention that, by Corollary 1.15, it follows that if w is a fixed point with multiplicity greater than 1, then it is indifferent.
We justify these names with the following proposition:

Proposition 2.5. Let z_0 be a fixed point of f. There exists a neighbourhood U of z_0 such that for all $z \in U$:

1. $f^n(z) \to z_0$, if z_0 is attractive or super attractive.

2. If $z \neq z_0$, $\exists N$ such that $f^n(z) \notin U$, if z_0 is repelling.

Proof. Since conjugation preserves iteration and multipliers, we may assume $z_0 \neq \infty$ We first prove 1. Let $|f'(z_0)| < c < 1$. Then, by definition, there exists $r > 0$ such that $\forall z \in B(z_0, r)\backslash\{z_0\}$:

$$\frac{|f(z) - f(z_0)|}{|z - z_0|} < c \implies |f(z) - z_0| < c|z - z_0| < cr.$$

So that $f(z) \in B(0, r)$, thus inductively it follows that:

$$|f(z) - z_0| < c^n r \to 0$$

as claimed. Now, for 2, let $1 < c < |f'(z_0)|$. Then, by definition, there exists $r > 0$ such that $\forall z \in B(z_0, r)\backslash\{z_0\}$:

$$\frac{|f(z) - f(z_0)|}{|z - z_0|} > c \implies |f(z) - z_0| > c|z - z_0|$$

Now, let $z \in B(0, r)\backslash\{z_0\}$ and suppose said N does not exist. Then, $\forall n \in \mathbb{N}$, $f^n(z) \in B(z_0, r)$. That is, $|f^n(z) - z_0| < r$. But then, we may apply the above inequality inductively so that:

$$r > |f^n(z) - z_0| > c^n|z - z_0|, \forall n$$

an absurd since the $c^n|z - z_0| \to +\infty$, so that said N does exist. □

Definition 2.6. Given $f : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ an analytic function, we call the *Julia Set of f* the closure of the set of repelling periodic points of f, denoted by $J(f)$. The *Fatou Set of f* is denoted by $F(f)$ and given by $\overline{\mathbb{C}}\backslash J(f)$.

Example 2.7. Let $f(z) = z^2$. Then w is a periodic point if and only if:

$$f^p(w) = w \iff z^{2p} = z \overset{z\neq\infty}{\iff} z(z^{2p-1} - 1) = 0.$$

Thus w is periodic if and only if $w = 0$ or $w = \infty$ or $w = e^{\frac{2\pi i}{2p-1}q}$, $q = 0, 1, ..., 2p - 2$. Clearly,

$$(f^p)'(0) = 0 \qquad \text{and} \qquad \frac{1}{(f^p)'(\infty)} = 0$$

so $0, \infty$ are super attractive, whereas otherwise

$$|((f^p)'(w)| = |2p(w)^{2p-1}| = 2p|w|^{2p-1} = 2p > 1$$

so that all other periodic cycles are repelling. We conclude that

$$\begin{aligned} J(f) &= \overline{\left\{e^{\frac{2\pi i}{2p-1}q} | q = 0, ..., 2p - 2\,; p \geq 1\right\}} \\ &= \overline{\{\text{the “odd” roots of unit}\}} \\ &= S^1. \end{aligned}$$

Definition 2.8. If $w \in \overline{\mathbb{C}}$ is an attractive fixed point of f, we call the *basin of attraction of* w the set:

$$\mathcal{A}(w) = \{z \in \mathbb{C} : f^k(z) \to w,\, k \to \infty\}.$$

And the *immediate basin of attraction of* w, $\mathcal{A}_0(w)$ to be the connected component of $\mathcal{A}$ which contains w. If $\mathcal{O} = \{z_0, ..., z_{p-1}\}$ is an attracting periodic orbit of period p (that is, $f(z_i) = z_{i+1 \mod p}$ and $|(f^m)'(z_i)| < 1$, the basin of attraction of the orbit $\mathcal{O}$, $\mathcal{A}(\mathcal{O})$ is the union of the basins of attraction for each z_i as fixed points of f^m, and it's immediate basin of attraction $\mathcal{A}_0(\mathcal{O})$ is the union of the immediate basins of attraction.

Example 2.9. In the case $f(z) = z^2$, Then $A(0) = B(0,1)$ and $A(\infty) = \mathbb{C} \backslash \overline{B}(0,1)$.

Lemma 2.10. Let $w \in \overline{\mathbb{C}}$ be an attractive fixed point of f analytic. Then $A(w)$ is open.

Proof. Since w is attractive, there exists V neighbourhood of w, $V \subset A(w)$. So, given $z \in A(w) \implies f^k(z) \in V$ for some k. But then $z \in f^{-k}(V) \subset A(w)$ open. ☐

Corollary 2.11. Let $\mathcal{A}$ be the basin of attraction of a periodic point z_0 of an analytic function f, $f'(z_0) = \lambda$. Then $\exists \phi : \mathcal{A} \to \overline{\mathbb{C}}$ analytic such that $\phi \circ f(z) = \lambda\phi(z), \forall z \in \mathcal{A}$ and ϕ maps a neighbourhood of z_0 into a neighbourhood of 0.

Proof. First, notice that taking $g(z) = z + z_0$, then:

$$g^{-1} \circ f \circ g(0) = 0,\ (g^{-1} \circ f \circ g(0))'(0) = f'(z_0).$$

Since conjugation also preserves iteration, there's no loss in generality in assuming $z_0 = 0$. By the Theorem 1.17, we know $\exists r > 0$ and $\phi : B(0,r) \to \overline{\mathbb{C}}$ such that $\phi \circ f(z) = \lambda\phi(z)$, $\forall z \in B(0,r)$. Now, we will extend ϕ by defining it as the limit of ϕ_n, but now $\phi_n : \mathcal{A} \to \overline{\mathbb{C}}$. It remains to prove the functions converge to an analytic function. Initially, notice that $\forall z \in \mathcal{A}$, $\exists N_z$ such that $f^{N_z}(z) \in B(0,r)$, since $f^n(z) \to 0$. But then $\phi_n(f^{N_z}(z))$ converges to $\phi(f^{N_z}(z))$. So, given $\epsilon > 0$, $\exists N$, $n, m \geq N$ implies:

$$\begin{gathered}
\left|\phi_n(f^{N_z}(z)) - \phi_m(f^{N_z}(z))\right| < \epsilon \cdot \left|\lambda^{N_z}\right| \\
\left|\frac{f^{N_z+n}(z)}{\lambda^n} - \frac{f^{N_z+m}(z)}{\lambda^m}\right| < \epsilon \cdot \left|\lambda^{N_z}\right| \\
\left|\frac{f^{N_z+n}(z)}{\lambda^{N_z+n}} - \frac{f^{N_z+m}(z)}{\lambda^{N_z+m}}\right| < \epsilon \\
|\phi_{N_z+n}(z) - \phi_{N_z+m}(z)| < \epsilon
\end{gathered}$$

hence $\phi_n(z)$ is Cauchy and therefore converges. Now, let:

$$\mathcal{A}_n = \{z \in \mathcal{A} | f^n(z) \in B(0,r)\}$$

which are all open. Notice that $B(0,r) \subset \mathcal{A}_1$, $\mathcal{A}_n \subset \mathcal{A}_{n+1}$, $\forall n$. Also,

$$\bigcup_{n=1}^{\infty} \mathcal{A}_n = \mathcal{A}.$$

Clearly, ϕ_n converges uniformly on each $\mathcal{A}_m$. I then claim that ϕ_n converges uniformly in every compact subset of $\mathcal{A}$. In fact, given $K \subset \mathcal{A}$, $\exists N$, $K \subset \mathcal{A}_N$. We prove by contradiction: suppose that is not the case. Then $\forall n$ there would exist $z_n \in K$ such that $z_n \notin A_n$, that is $f^n(z_n) \notin B(0,r)$. But then since K is compact, there would exist a subsequence z_{n_k} such that $z_{n_k} \to \bar{z} \in K$. But now $\bar{z} \in \mathcal{A}$, so there exists N such that $f^N(\bar{z}) \in B(0,r)$. Then, by continuity, $\exists \delta$, $\forall z \in B(\bar{z},\delta)$, $f^N(z) \in B(0,r)$. But $z_{n_k} \to \bar{z} \implies \exists K$, $z_{n_K} \in B(\bar{z},\delta)$. But then $f^N(z_{n_K}) \in B(0,r)$, $\forall k \geq K$ which leads to a contradiction, since eventually we would have $n_k \geq N$ and then $f^{n_k}(z_{n_k}) \in B(0,r)$.
Since $\mathcal{A}$ is open, every point admits a compact neighbourhood contained in $\mathcal{A}$, so every points admits a neighbourhood where ϕ is analytic. Follows that ϕ is analytic in $\mathcal{A}$. □

Theorem 2.12. Let $f : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ be a rational map, $\hat{z}$ attractive fixed point of f (not super attractive) with basin of attraction $\mathcal{A}$. Let $\phi : \mathcal{A} \to \overline{\mathbb{C}}$ be such as in the previous corollary. Then, $\exists \psi_\epsilon : B(0,\epsilon) \to \mathcal{A}_0$ analytic local inverse of ϕ, which extends to ψ_r defined in a maximal ball $B(0,r)$. Also, ψ_r extends to $\partial B(0,r)$ and $\psi_r(\partial B(0,r)) \subset \mathcal{A}_0$ contains a critical point of f.

Proof. Again, we may assume $\hat{z} = 0$. Notice that $f'(0) = \lambda \neq 0 \implies \phi_n'(0) = \lambda$, $\forall n \implies \phi(0) = \lambda \neq 0$. Then, by the inverse function theorem, $\exists \psi_\epsilon$ local inverse defined in a ball of radius ϵ around 0. Now, suppose it was possible to extend ψ to an arbitrarily large ball, we would then have $\psi : \mathbb{C} \to \mathbb{C}$ analytic. But then we could extend ϕ to each $z \in \mathbb{C}$, by the equality $\phi(\psi(z)) = z$. But then $\psi(\mathbb{C}) = \mathbb{C}$. But note that $f|_{\psi(\mathbb{C})}$ is bijective over it's image, since if $\psi(\xi_1) \neq \psi(\xi_2) \implies \xi_1 \neq \xi_2$, thus, since ψ is injective:

$$\phi(f(\psi(\xi_1)) = \lambda\xi_1$$
$$f(\psi(\xi_1)) = \psi(\lambda\xi_1) \neq \psi(\lambda\xi_2) = f(\psi(\xi_2)).$$

But then that would imply $f : \mathbb{C} \to \mathbb{C}$ is bijective, a contradiction by Corollary 1.29. So there exists $r > 0$ maximal such that ψ_ϵ extends analytically to $\psi_r = \psi : B(0,r) \to U = \psi(B(0,r)) \subset \mathcal{A}_0 \subset \mathcal{A}$, since $B(0,r)$ is connected and ψ is continuous.

$$\begin{tikzcd} U \arrow[r, "f"] \arrow[d, bend left, "\phi"] & f(U) \arrow[d, bend left, "\phi"] \\ B(0,r) \arrow[u, bend left, "\psi"] \arrow[r, "\cdot\lambda"] & B(0,\lambda r) \arrow[u, bend left, "\psi"] \end{tikzcd}$$

Now, note that $\overline{U}$ must be contained in $\mathcal{A}_0$, since $\lambda \cdot B(0,r) = B(0,\lambda r)$ is contained in the compact $K = \overline{B(0,\lambda r)} \subset B(0,r)$, it follows by continuity that:

$$f(\overline{U}) \subset \overline{f(U)}$$

$$\phi(\overline{f(U)}) \subset \overline{\phi(f(U))} = K$$

hence

$$\phi(f(\overline{U})) = \lambda\overline{U} \subset K = \lambda\overline{B(0,r)}.$$

So that finally

$$\overline{U} \subset \overline{B(0,r)} \subset \mathcal{A}_0.$$

In particular, ϕ is defined in a neighbourhood of $\overline{U}$. We will show that ∂U contains a critical point of f, since otherwise we would be able to extend ψ analytically to a larger ball, contradicting the maximality of r.
Given $w_0 \in \partial B(0,r)$, let $z_0 \in \partial U$ be the accumulation point of the curve $\gamma : [0,1) \to \overline{\mathbb{C}}$, $\gamma(t) = \psi(tw_0)$, such that $\gamma(t) \to z_0$ for $t \to 1$. Notice that it is well defined:
If $z_0 \in \partial U$, $\exists (z_n)_n \subset U$, $z_n \to z_0$. But then $\exists (y_n)_n \subset B(0,r)$, $\psi(y_n) = z_n \therefore \psi(y_n) \to z_0$. But then $(y_n)_n \subset \overline{B(0,r)}$ is a sequence in a compact, therefore admits convergent subsequence $y_{n_k} \to y_0 \in \overline{B(0,r)}$. Now, since $z_0 \in \partial U$ and U is open since ψ is a diffeomorphism, $z_0 \notin U$ By continuity we have that $\psi(y_0) = z_0 \implies y_0 \notin B(0,r)$. But then $y_0 \in \partial B(0,r)$, so that taking $w_0 = y_0$ in the previous curve, we do in fact have that $\gamma(t) \to z_0$ as $t \to 1$.
Suppose z_0 is not a critical point for f. Then, by the inverse function theorem, there are neighborhoods of U_{z_0} of z_0 and V_{z_0} of $f(z_0)$ and $g : V_{z_0} \to U_{z_0}$ analytic such that $f(g(z)) = z$, for $z \in V_{z_0}$. Let $\epsilon > 0$ be such that $\lambda \cdot |w_0 + \epsilon| < r$, so that for $B(\lambda w_0, \lambda\epsilon) \subset B(0,r)$ and $\psi(B(\lambda w_0, \lambda\epsilon)) \subset V_{z_0}$ (which exists since $\psi(\lambda w) = f(\psi(w))$. Then, by setting:

$$\psi_{z_0}(w) = g(\psi(\lambda(w))),\ w \in B(w_0, \epsilon)$$

follows that ψ_{z_0} is analytic and coincides with ψ in $B(0,r) \cap B(w_0,\epsilon)$, since, by definition:

$$\begin{aligned} (\phi \circ f)(\psi(w)) &= \lambda w \\ f(\psi(w)) &= \psi(\lambda w) \\ \psi(w) &= g(\psi(\lambda w)) \end{aligned}$$

it follows that ψ_{z_0} is analytic extension of ψ. Then, if there are no critical points in ∂U, we may apply the same process and obtain an extension of ψ over the whole boundary into a larger ball, contradicting the maximality of r.
It remains to show ψ extends to the boundary of $B(0,r)$. But this is clear, since if $w \in \partial B(0,r)$, then $\lambda w \in B(0, \lambda r) \subset B(0,r)$. But then $\exists y \in f(U)$ such that $\phi(y) = \lambda w$. But $f(U) = f(\psi(B(0,r))$, so that, as before, f is locally bijective and we may set

$$\psi(w) = g(\psi(\lambda w))$$

as before and extend ψ. So finally, we may conclude that $\exists z_0 \in \psi(\partial B(0,r)) \subset \mathcal{A}$ such that z_0 is a critical point of f. □

Example 2.13. Taking $f(z) = z^2 + 0.7iz$, we have that f has an attractive fixed point at 0. The Julia Set for f, as will be proved, is the boundary for the basin of attraction of 0. Therefore, in Figure 2, the region inside the Julia Set is the immediate basin of attraction of 0, so that there we may calculate ϕ as in the theorem. Now, note that $c = 0 - .35i$ is the closest critical point of f in the immediate basin of attraction of 0, so that the inverse ψ may only extend to a maximal radius r such that there is a point z_0, $|z_0| = r$ and $\psi(z_0) = c \iff \phi(c) = z_0 \implies |\phi(c)| = |z_0| = r$. Therefore, the subset U that ϕ maps bijectively onto $B(0,r)$ is such that $\partial U = \{z || \phi(z)| = |\phi(c)|\}$, as marked in Figure 2.

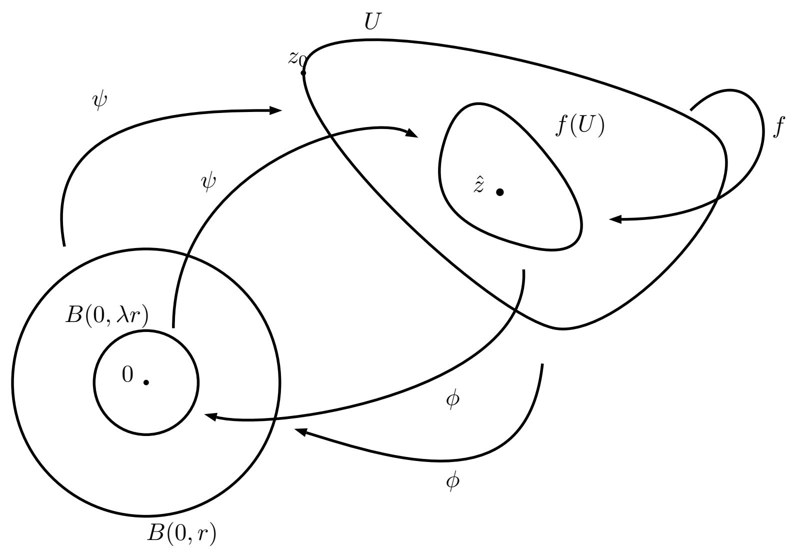


Figure 1: Drawing illustrating the result

Corollary 2.14. If f is a rational map of degree at least 2, then for each periodic cycle, there is at least one critical point of R in it's immediate basin of attraction.

Proof. Let $\{z_1, ..., z_p\}$ be a periodic cycle. If it is super attractive, then $(f^p)'(z_i) = 0 \implies f'(z_j) = 0$, for some $j = 1, 2, ..., p$, hence one of the points in the cycle is itself critical and is obviously in the immediate basin. If it is not super attractive, then applying the previous theorem to z_1 as fixed point of f^p, we have that there exists a critical point c of f^p in the immediate basin of attraction z_1. But this implies $(f^p)'(c) = 0 \implies f'(f^m(c)) = 0$, for some $m = 0, 1, ..., p-1$, that is, $f^m(c)$ is critical point of f. But since c was in the immediate basin of attraction of z_1, this implies $f^m(c)$ is in the immediate basin of z_{1+m}, so that it is the immediate basin of the cycle. $\square$

Corollary 2.15. If f is a rational map of degree $d \geq 2$, there are at most $2d - 2$ attractive periodic cycles.

Proof. By the previous result, for each attractive periodic cycle there is a critical point in it's immediate basin of attraction. Since those are open and disjoint, these must all be different. Since R has at most $2d - 2$ critical points, the result follows. $\square$

Theorem 2.16. Let f be rational map of degree $d \geq 2$. Then f has at most $4d-4$ indifferent cycles.

Proof. If $f(z) = z^d$, for some $d \in \mathbb{Z}$, then f has no indifferent cycles. Now, if $f(z) \neq z^d$, let $f = p/q$, with p, q coprime and define:

$$f_t(z) = \frac{p(z) - tz^d}{q(z) - t},\ f_0(z) = f(z),\ f_\infty(z) = z^d$$

for $t \in \overline{\mathbb{C}}$. Notice that f_t is a smooth function on t, except when at least one root of $p(z) - tz^d$ coincides with a root of $q(z) - t$ in a point $\hat{T}$, such that $deg(f_t) = d$ for t close to $\hat{t}$, but

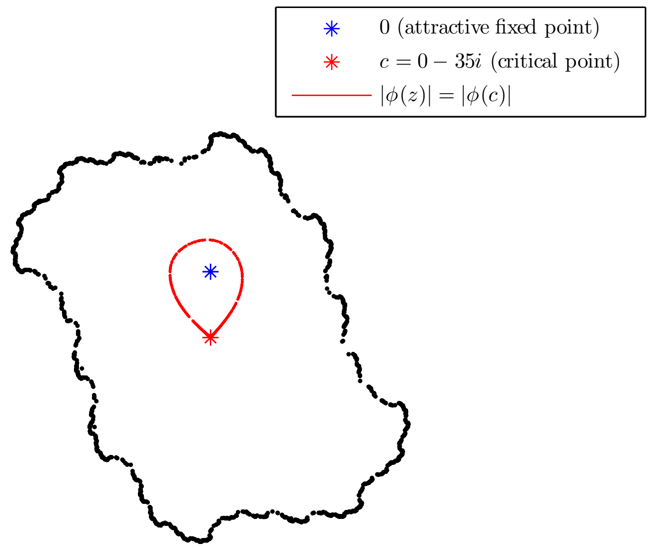


Figure 2: Julia Set for $f(z) = z^2 + 0.7iz$ illustrating the result

$deg(f_{\hat{t}}) < d$. But when this happens, we have that:

$$\begin{aligned} p(z) - \hat{t}z^d = 0 = q(z) - \hat{t} \\ \implies p(z) = \hat{t}z^d,\ q(z) = \hat{t} \\ \implies f(z) = \frac{p(z)}{q(z)} = \frac{\hat{t}z^d}{\hat{t}} = z^d. \end{aligned}$$

But since $f(z) \neq z^d$, $f(z) = z^d \iff f(z) - z^d = 0$ has a finite number of solutions. Hence, there exists a finite number of $\hat{t}$, since $\hat{t} = q(z)$.
Now, since $f = f_0$, there exists a neighbourhood U of 0 such that $t \mapsto f_t$ is a smooth function on t. Now, suppose f has at least $k = 4d - 3$ indifferent periodic orbits, $\mathcal{O}(0)_j$, $j = 1, 2, ..., k$. Let $z_j(0) \in \mathcal{O}(0)_j$ and let l_j be the period of $z_j(0)$, $\lambda_j(0) = (f^{l_j})'(z_j(0))$.
Note that $|\lambda_j(0)| = 1 \implies (f_0^{l_j})'(z_j(0)) \neq 0$, $f^{l_j}(z_j(0)) = z_j(0)$ and since $t \mapsto f_t$ is smooth in U, by the Implicit Function Theorem, there exists $U' \subset U$ neighbourhood of 0 such that $\forall t \in U'$, $\exists z_j(t)$ such that

$$f_t^{l_j}(z_j(t)) = z_j(t)$$

for $j = 1, 2, ..., k$. Similarly, for each j, define $\lambda : U' \to \mathbb{C}$

$$\lambda_j(t) = (f_t^{l_j})'(z_j(t)).$$

Again, by the Implicit Function Theorem, we have that $z_j(t)$ and $\lambda_j(t)$ are analytic, for $j = 1, ..., k$.
Claim: None of the functions λ_j is constant in a neighbourhood around 0.
Suppose at least one of the functions $\lambda_{j_0} = \lambda_0$ is constant in a neighbourhood around 0. Choose $\theta \in [0, 2\pi)$ such that the ray $R([0, \infty])$, given by $R : [0, \infty] \to \overline{\mathbb{C}}$,

$$R(r) = re^{i\theta}$$

does not intercept any one of the finitely many $\hat{t}$.
We now claim it is possible to extend $z_{j_0}(R(r)) = z_0(R(r))$ analytically along the ray up to $r = \infty$. Let

$$A = \{\hat{r} \mid z_0(R(r)) \text{ can be extended analytically up to } r = \hat{r}\}.$$

Then by hypothesis $A \neq \emptyset$, so that we can define:

$$\alpha = \sup A.$$

Now, clearly A is closed, since if $(r_n)_n \subset A$, $r_n \to \bar{r}$, we can define $z_0(R(\bar{r})) = \lim z_0(R(r_n))$. Then clearly this extends z_0, since by continuity it still holds that $f^{l_0}_{R(\bar{r})}(z_0(R(\bar{r})) = z_0(R(\bar{r}))$. So that $\alpha \in A$. But then suppose $\alpha \neq \infty$. Since $\alpha \in A$, we have that $z_0(R(\alpha))$ is a periodic point of period l_0. Note also that we may extend $\lambda_0(R(r))$ up to α by the same formula. But being constant in a neighbourhood of 0 and analytic implies $\lambda_0(R(r)) = \lambda_0(R(0))$, $\forall r \in A$, hence $\lambda_0(R(\alpha)) = \lambda_0(R(0)) \neq 0$. But $\lambda_0(R(\alpha)) = (f^{l_0}_{R(\alpha)})'(z_0(R(\alpha))$, so that by the Implicit Function Theorem there exists a neighbourhood $(\alpha - \delta, \alpha + \delta)$ in which we can define $r \mapsto \tilde{z}_0(R(r))$ periodic point of $f^{l_0}_{R(r)}$. But by uniqueness of the Implicit Function, we have that $\tilde{z}_0$ and z_0 coincide in the open set $(\alpha - \delta, \alpha)$, so that we may extend z_0 to $[0, \alpha + \delta)$ contradicting the fact that $\alpha = \sup A$. Hence $\alpha = \infty$. But then it is possible to extend z_0 and λ_0 up to $t = \infty$, and by connectivity $\lambda_0(\infty) = \lambda_0(0)$. But for $t = \infty$, $f_t(z) = z^d$, which has no indifferent points, a contradiction. Thus we may conclude all λ_j are not constant in a neighbourhood of 0.
Now, for each j, since $\lambda_j(0) \neq 0$ and λ_j is not constant, we may write it as a power series around 0:

$$\begin{aligned}\frac{\lambda_j(t)}{\lambda_j(0)} &= 1 + a_j t^{n_j} + \sum_{k=n_j+1}^{\infty} b_k z^k \\ &= 1 + a_j z^{n_j} + z^{n_j+1} g(z)\end{aligned}$$

where $a_j \neq 0$, $n_j \geq 1$ and $g(z) = \sum_{i=0}^{\infty} c_i z^i$ is bounded and analytic.
Then:

$$\begin{aligned}|\lambda_j(t)|^2 &= \lambda_j(t)\overline{\lambda_j(t)} \\ &= (1 + a_j t^{n_j} + t^{n_j+1} g(t))(\overline{1 + a_j t^{n_j} + t^{n_j+1} g(t)}) \\ &= 1 + 2\Re(a_j t^{n_j}) + |a_j|^2 |t|^{2n_j} + 2\Re(t^{n_j+1} g(t)) + a_j |t|^{2n_j} \overline{t g(t)} + \\ &+ \overline{a_j} |t|^{2n_j} t g(t) + |t|^{2n_j+2} |g(t)|^2 \\ &= 1 + 2\Re(a_j t^{n_j}) + |a_j|^2 |t|^{2n_j} + t^{n_j+1} G(t)\end{aligned}$$

where

$$G(t) = \frac{2\Re(t^{n_j+1} g(t)) + a_j |t|^{2n_j} \overline{t g(t)} + \overline{a_j} |t|^{2n_j} t g(t) + |t|^{2n_j+2} |g(t)|^2}{t^{n_j+1}}$$

is bounded in a neighbourhood of 0. Now, since the power series for $\sqrt{1+x}$ (x real) in a neighbourhood of 0, is given by:

$$\sqrt{1+x} = 1 + 1/2x + O(x^2).$$

We have that

$$\begin{aligned}|\lambda_j(t)| &= \sqrt{1 + 2\Re(a_j t^{n_j}) + |a_j|^2|t|^{2n_j} + t^{n_j+1}G(t)} \\ &= 1 + \Re(a_j t^{n_j}) + 1/2(|a_j|^2|t|^{2n_j} + t^{n_j+1}G(t)) + O(|t|^{2n_j}) \\ &= 1 + \Re(a_j t^{n_j}) + O(|t|^{n_j+1}).\end{aligned}$$

Now, note that we may divide the complex plane into n_j sectors where $\Re(a_j t^{n_2} > 0$ and n_j sectors where $\Re(a_j t^{n_j} < 0$ (follows from the polar form of complex numbers) such that we may define:

$$\sigma_j(\theta) = sign(\Re(a_j e^{i\theta n_j})).$$

Notice that, for each j, σ_j is a stair function with values ± 1 except at $2n_j$ discontinuity points where it is 0. Also, it's average value is 0 in $[0, 2pi)$.
Claim: $\exists \delta > 0$ such that:

$$\begin{aligned}\sigma_j(\theta) = 1 &\iff |\lambda_j(re^{i\theta})| > 1 \\ \sigma_j(\theta) = -1 &\iff |\lambda_j(re^{i\theta})| < 1\end{aligned}$$

for all $0 < r < \delta$, for all $j = 1, .., k$.
It suffices to show the first implication, the proof of the second is analogous. Let $|\lambda_j(t)| = 1 + \Re(a_j t^{n_j}) + h(t)$, where $h(t) = O(|t|^{n_j})$. First, assume $\sigma_j(\theta) = 1$, that is, $\Re(a_j e^{i\theta n_j}) > 0$, and suppose the implication is false. Then, for each $\delta = 1/m$, there would exist $r_m \in (0, \delta)$ such that:

$$g(r_m e^{i\theta}) < -\Re(a_j r_m^{n_j} e^{i\theta n_j}),\ \forall m.$$

But since:

$$\lim_{m\to\infty} -\frac{\Re(a_j r_m^{n_j} e^{i\theta n_j})}{r_m^{n_j+1}} = -\infty$$

it follows that

$$\lim_{m\to\infty} \frac{g(r_m e^{i\theta})}{r_m^{n_j+1}} = \infty$$

so that g is not $O(|t|^{n_j+1})$, a contradiction. Since for each case we can find such a $\delta_j^{\pm}$, we may take δ to be the minimum of these and we're done.
Let $\sigma = \sigma + \sigma_2 + ... + \sigma_k$. Then, since σ is the sum of stair function with average 0, it's average too is 0. We claim $\exists\theta$ such that $\sigma_j(\theta) = -1$ for at least $(k+1)/2$ of the σ_j's. Suppose not, then, $\forall\theta$, at most $(k+1)/2$ of the σ_j's have the value -1. But then this would imply that the average of δ would be greater then 0 in $[0, 2\pi)$., a contradiction. Let $\theta_0 \in [0, 2\pi)$ be this value. Then, taking $r_0 \in (0, \delta)$, such that $t_0 = r_0 e^{i\theta_0})$ is not one of the problem points $\hat{t}$, we have that $|\lambda_j(t_0)| < 1$, for $(k+1)/2$ values of j, so that f_{t_0} has $(k+1)/2$ attractive periodic cycles. But

$$k = 4d - 3 \implies (k+1)/2 = 2d - 1.$$

So that f_{t_0} is a rational map of degree d and $2d - 1$ attractive cycles, a contradiction with Corollary 2.15. So we may conclude f has at most $4d - 4$ indifferent periodic cycles.

□

Corollary 2.17. If f is a rational map of degree $d \geq 2$, then the number of non-repelling periodic cycles of R is at most $6d-6$, so that the number of non-repelling periodic points of R is finite.

Proof. Follows from Corollary 2.15 and Theorem 2.16 that the number of non-repelling cycles is at most $2d-2+4d-4 = 6d-6$. Then, let $p \in \mathbb{N}$ be the greatest period of all these cycles. Then the number of non-repelling periodic points is at most $(6d-6)p$. □

Lemma 2.18. Let f be a rational map with degree at least 2, z_0 be a fixed point of f with multiplicity m and that it's multiplier is either 1 or not a root of unity. Then, for all $n \in \mathbb{N}$, z_0 is also a fixed point of f^n with multiplicity k.

Proof. Since conjugation preserves fixed points, iterates and multiplicity, we may assume $z_0 = 0$, so that we may write f as a power series around 0:

$$f(z) = az + bz^k + ...$$

with $a^n \neq 1$, $n \geq 2$, $b \neq 0$ and $k > 1$, since $degree(f) = d \geq 2$.
If $a \neq 1$, then $z_0 = 0$ has multiplicity 1. Then:

$$f^n(z) = a^n z + nbz^k + ...$$

$$f^n(z) - z = (a^n - 1)z + nbz^k + ...$$

So that $z_0 = 0$ is a simple root and therefore also has multiplicity one.
If $a = 1$, then $k = m$ and:

$$f^n(z) = z + nbz^m + ...$$

$$f^n(z) - z = nbz^m + ...$$

Since $nb \neq 0$, follows that $z_0 = 0$ is also a fixed point of multiplicity m.

□

Theorem 2.19. A rational map f of degree $d \geq 2$ has infinitely many periodic points.

Proof. Suppose all fixed points of f have multipliers that are either 1 or not a root of unity. Let $p > 2$ be a prime number. We claim that there f^p has a fixed point different from those of f. Suppose not, and let ξ be a fixed point of f. Since f can have at most $d+1$ fixed points, counting multiplicity, ξ has multiplicity at most $d+1$. but by the previous lemma, ξ is also a fixed point of f^p with multiplicity at most $d+1$. Since f can have at most $d+1$ fixed points in total, each with multiplicity at most $d+1$, f^p can have at most $(d+1)(d+1) = (d+1)^2$ fixed points, counting multiplicity. But since f^p has exactly $d^p + 1 > (d+1)^2$ (for $d, p > 2$), we have a contradiction. Hence f^p must have a fixed point different from those of f, for all $p > 2$ prime. That means f has at least one periodic cycle of every prime period greater than 2. But since period cycles of different prime periods cannot have terms in common, this means they must all be different and, therefore, f must have infinitely many periodic points.
Now, assume f does have fixed points that are root of unity. Suppose, by contradiction that f has finitely many periodic points. Let $\xi_1, ..., \xi_m$ be all the periodic points of f such that

their respective multipliers $\lambda_1, ...\lambda_m$ are all $n_1, ..., n_m$ roots of unity, with period $p_1, ..., p_n$ (note that these may not all be distinct). Let

$$p = \prod_{i=1}^{n} p_i n_i$$

It follows that ξ_i is a fixed point of f^p, for all $i = 1, 2, ..., n$. Moreover, by the chain rule:

$$(f^p)'(\xi_i) = \prod_{j=1}^{p} f'(\xi_{i_j \mod p_i}) = (\lambda_i)^{p/p_i} = 1^{p/(p_i n_i)} = 1$$

where $\{\xi_{i_j} | \, j = 0, ..., p_i - 1\} \subset \{\xi_i, | \, i = 1, ..., m\}$ are the points in the same cycle. So that all fixed points of f^p have multiplier either 1 or not a root of unity. It follows we may apply the previous result to f^p and obtain infinitely many periodic points for f^p, which are also periodic points of f contradicting the assumption f has only finitely many periodic points. Either way, f must have infinitely many periodic points. □

Definition 2.20. Let $U \subset \overline{\mathbb{C}}$ be an open set, $\{g_k\}_{k \in K}$ be a family of holomorphic functions, $g_k : U \to \overline{\mathbb{C}}$. Then $\{g_k\}_{k \in K}$ is *normal* if, for every sequence in $(g_{k_n})_n \subset \{g_k\}_{k \in K}$, there exists an uniformly converging subsequence in every compact subset of U, to either a finite holomorphic function, or to $f = \infty$. The family is said to be *normal in z* if there is exists a neighbourhood of z where $\{g_k\}_{k \in K}$ is normal.

Example 2.21. Every finite family $F = \{f_1, f_2, ..., f_n\}$ of complex functions is normal. In fact, every sequence in F repeats at least one term infinitely often, so it has a constant (thus uniformly converging) subsequence.

Example 2.22. By Arzelà-Ascoli Theorem (see ref. [6] p.206), if the family $\{g_k\}_{k \in K}$ is equicontinuous and uniformly bounded, then its closure is compact, hence every sequence has converging subsequence. Therefore it is normal.

Example 2.23. If f is a contraction, then there exists $\overline{z}$, such that for every $z \in U$, $f^k(z) \to \overline{z}$ (see ref. [6] p.215), so any sequence (hence any subsequence) of the family $\{f^k\}_k$ converges to the constant function $h(z) = \overline{z}$. So $\{f^k\}_k$ is normal.

Theorem 2.24. (Montel)
Let $\{g_k\}_{k \in K}$ be a family of complex analytic functions defined in $U \subset \overline{\mathbb{C}}$ an open set. If $\{g_k\}_k$ is not normal, then $\forall w \in \overline{\mathbb{C}}$, with at most 2 exceptions, $\exists z \in U$, $k \in K$ such that: $g_k(z) = w$

Proof. The proof of this theorem is beyond the scope of this book, hence it shall not be covered. It can be found in [4] □

Note that the converse is false: the family f, where $f(z) = z$ is normal, but, $\forall z \in \overline{\mathbb{C}}$, $\exists z = z \in \overline{\mathbb{C}}, f(z) = z$.

Definition 2.25. Let $f : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ be a complex analytic function. Then

$$J_0(f) := \{z \in \overline{\mathbb{C}} : \{f^k\}_k \text{ is not normal at } z\}$$

$$F_0(f) = \overline{\mathbb{C}} \backslash J_0(f).$$

Note: it follows that $F_0(f)$ is open, since:

$$F_0(f) = \bigcup_{z \in F_0(f)} V_z, \text{ where } V_z \text{ is a neighborhood of } z \text{ on which } f \text{ is normal.}$$

So that $J_0(f)$ is always closed.

Example 2.26. If $f(z) = z^2$, then $J_0(f) = S^1$:
Given $z_0 \in B(0,1)$, there exists U neighbourhood of z_0, $U \subset B(0,1)$. If $z \in U, z = 0$, then $f^k(0) = 0$. If $z \in U, z \neq 0$, then:

$$|f^k(z)| = |z|^{2k} = |z|^{2(k-1)}||z|^2 < |z|^{2(k-1)} = |f^{k-1}(z)|$$

so that $f^k(z) \to 0$. So $\{f^k\}_k$ (and any subsequence) converges to the constant function $h(z) = 0$ in U. Therefore, $(f^k)_k$ is normal for every $z \in B(0,1)$. By a completely analogous argument, for $z_0 \in \overline{\mathbb{C}} \backslash \overline{B}(0,1)$, there exists U neighbourhood of z_0 contained in $\overline{\mathbb{C}} \backslash \overline{B}(0,1)$, and in such $(f^k)_k$ (and any subsequence) converges to ∞, so that it is normal. Meanwhile, for any $z_0 \in S^1$, in every neighbourhood U of z_0 there exists $z_1 \in B(0,1) \cap U$, $z_2 \in \overline{\mathbb{C}} \backslash \overline{B}(0,1) \cap U$ so that $f^k(z_1) \to 0$ and $f^k(z_2) \to \infty$, so the same holds for any subsequence, hence there is no convergent subsequence and $\{f^k\}_k$ is not normal.

We see that for $f(z) = z^2$, $J(f) = J_0(f)$. A natural question to ask is if this is true or not for any f. We shall prove it is true in case f is a rational map, but first, we need a few propositions.

Proposition 2.27. If f is a polynomial of degree at least 2, then $J_0(f)$ is bounded.

Proof. Let

$$f(z) = a_n z^n + \cdots + a_2 z^2 + a_1 z + a_0, \; a_n \neq 0.$$

Define $A := \max_{i=0,\ldots,n-1}\{|a_i|\}$ Then, for $z \neq 0$:

$$\begin{aligned}
|f(z)| &= |a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0| \\
&= |z^n| \left| a_n + \frac{a_{n-1}}{z} + \cdots + \frac{a_1}{z^{n-1}} + \frac{a_0}{z^n} \right| \\
&\geq |z^n| \left(|a_n| - \left| \frac{a_{n-1}}{z} + \cdots + \frac{a_1}{z^{n-1}} + \frac{a_0}{z^n} \right| \right) \\
&\geq |z^n| \left(|a_n| - \left(\frac{|a_{n-1}|}{|z|} + \cdots + \frac{|a_1|}{|z|^{n-1}} + \frac{|a_0|}{|z|^n} \right) \right) \\
&\geq |z^n| \left(|a_n| - \left(\frac{A}{|z|} + \cdots + \frac{A}{|z|^{n-1}} + \frac{A}{|z|^n} \right) \right).
\end{aligned}$$

Consider $M := \max\{\frac{2(n-1)A}{|a_n|}, 1\}$. So, for $|z| > M$:

$$\begin{aligned}
|f(z)| &\geq |z|^n \left(|a_n| - \left(\frac{|a_n|}{2(n-1)} + \cdots + \frac{|a_n|}{2(n-1)} \right) \right) \\
&\geq \frac{|a_n|}{2} |z|^n.
\end{aligned}$$

So that for $|z| > \max\{M, 4/|a_n|\} = R$, we have: $|f(z)| > 2|z|$, hence:

$$|f^k(z)| > 2^k R.$$

And $f^k(z) \to \infty$, so f is normal for every z, $|z| > R$, hence $J_0(f)$ is bounded. $\square$

To prove that $J_0(f)$ isn't empty for rational functions, is surprisingly more difficult than the proof for polynomials. Reference [3] aided in that regard by presenting Theorem 2.19, from which follows:

Proposition 2.28. If f is a rational map, then $J(f) \subset J_0(f) \neq \emptyset$.

Proof. By Theorem 2.19, f has infinitely many periodic points. But by Corollary 2.17, f has a finite number of non-repelling periodic points, thus there exists a periodic repelling point and $J(f)$ is not empty. Let $w \in \overline{\mathbb{C}}$ be a repelling periodic point of f with period p, such that $g(w) := f^p(w) = w$. Without loss of generality we may assume $w \neq \infty$, since conjugation preserves periodic points and multipliers. Suppose $\{g^k\}_k$ is normal at w. Then $\exists V$ neighbourhood of w in which there is a subsequence $\{g^{k_i}\}$ which converges to a finite analytic function g_0, since $g^k(w) = w, \forall k$. Therefore it's derivatives also converge in the compact subsets of V to finite analytic functions, that is, $(g^{k_i})' \to g_0'$. However, since w is repulsive, $|g'(w)| > 1$, and since it is a fixed point of g, $(g^{k_i})'(w) = g'(w)^{k_i}$, so that:

$$|(g^{k_i})'(w)| = |g'(w)^{k_i}| = |g'(w)|^{k_i} \to \infty$$

hence g_0' cannot be finite, a contradiction. So $\{g^k\}_k$ cannot be normal at w. Therefore, $w \in J_0(g) = J_0(f)$. So all repulsive periodic points belong to $J_0(f)$. Since $J_0(f)$ is closed, it follows it's closure, $J(f)$ is also contained in it, that is: $J(f) \subset J_0(f)$.

$\square$

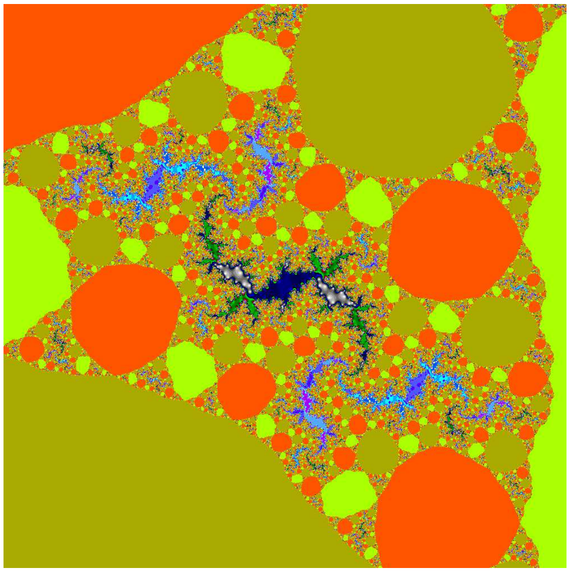
Proposition 2.29. If f is a rational map, w attractive fixed point of f, then

$$\partial A(w) = J_0(f).$$

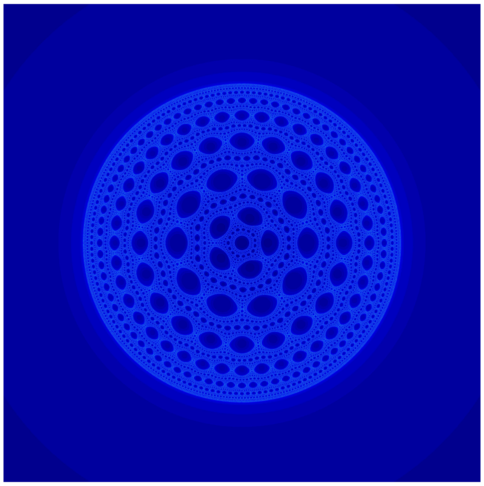
Proof. Let $z \in J_0(f)$. Then $f^k(z) \in J_0(f), \forall k$, so that $z \notin A(w)$. Now let U be a neighbourhood of z. Then $f^k(U) \cap A(w) \neq \emptyset$, by Proposition 2.35. But then $\exists z_0 \in U \cap A(w) \implies z \in \overline{A(w)} \implies z \in \partial A(w)$ so $J_0(f) \subset \partial A(w)$.
Now take $z \in \partial A(w)$, and suppose $z \notin J_0(f)$. Then z has connected neighbourhood V in which $\{f^k\}_k$ converges to an analytic function or ∞. Taking a smaller neighbourhood if necessary, we can assume the convergence is uniform. But then, $\forall z \in A(w) \cap V$, $f^k(z) \to w$ ($A(w) \cap V \neq \emptyset$ since $z \in \partial A(w)$). But $A(w) \cap V \subset V$ is open. So we conclude $f^k \to g$, where g is analytic and constant in an open subset, which implies $g(z) = w, \forall z \in V$. But this implies $f^k(z) \to w, \forall z \in V \implies V \subset A(w)$ contradiction with the fact that $z \in \partial A(w)$. So $\partial A(w) \subset J_0(f)$. $\square$

From Proposition 2.29 the author derived two methods for illustrating Julia Sets. If one considers a rational map with ∞ as an attractive point, one may select a mesh grid in the complex plane, iterate all points there and color them by how many iterates it takes for their absolute value to get above a certain threshold, that is, how fast they converge to infinity.

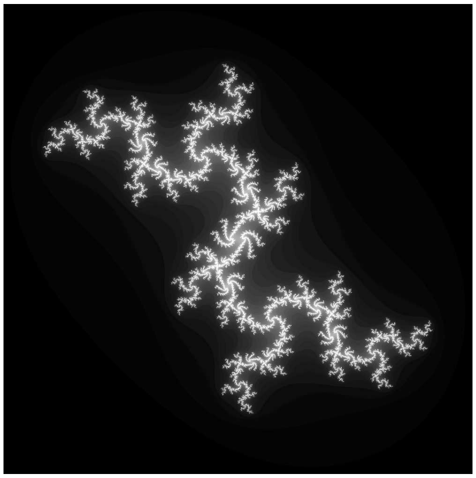
This will give a rough idea of the basin of attraction of ∞, and therefore the Julia Set will appear as it's boundary. We will call this method 1. Finally, given a rational map f, if one know that $\xi_1, ..., \xi_n$ are attractive periodic points, one may again take a mesh grid of the complex plane, iterate all points there and color differently depending on which periodic cycle they converge to. We shall call this method 2. Below are some examples of Julia Sets computed by these methods, in the software MATLAB. The codes can be found in the end of the paper.



(a) Julia Set around 0 for $f(z) = \dfrac{z^2 - 0.2 + 075i}{0.1z^3 + 1}$

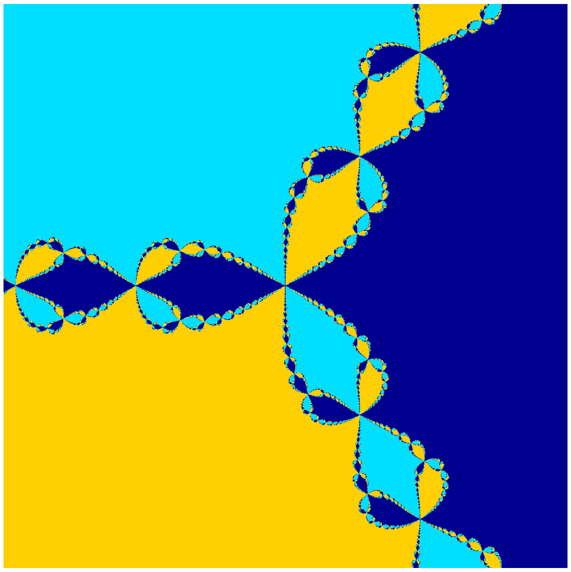


(b) Julia Set around 0 for $f(z) = \dfrac{z^5 - 0.0001}{z^2}$

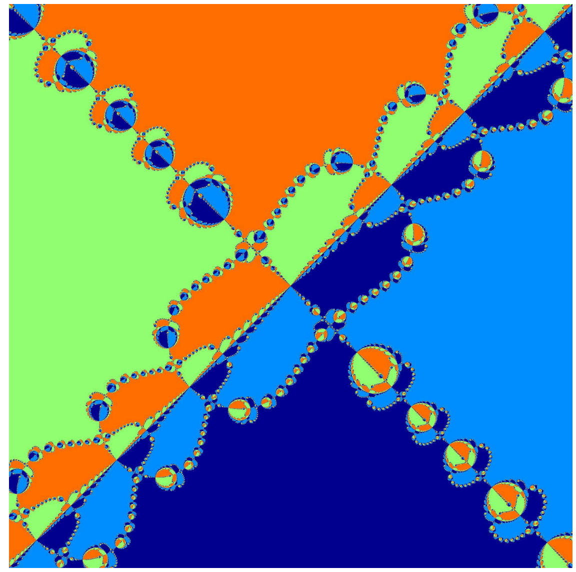


(c) Julia Set around 0 for $f(z) = z^2 + 0.7i$

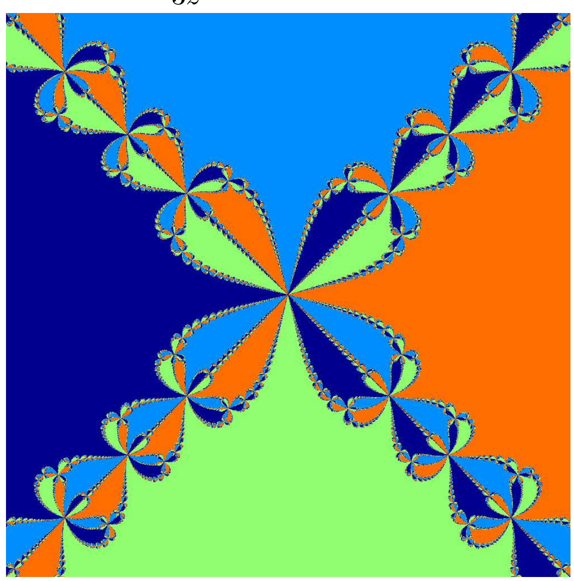
Figure 4: Method 1



(a) Julia Set around 0 for $f(z) = \dfrac{2z^3+1}{3z^2}$



(b) Julia Set around 0 for $f(z) = \dfrac{3z^4+1}{4z^3+iz}$



(c) Julia Set around 0 for $f(z) = \dfrac{3z^4+1}{4z^3}$

Figure 5: Method 2

Figure 5(a) and 5(c) deserve a special mention: the corresponding rational map corresponds to the application of the Newton Method [4, Cap.9] to obtain the cubic and quartic roots of unity, respectively. Therefore, the super attractive fixed points correspond tho these, and, in these cases, the Julia Sets, being the boundary of the basins of attraction, are the set of points for which the method fails.

Proposition 2.30. If f is a rational map, then $J_0(f)$ is invariant by f, that is, $f(J_0(f)) = J_0(f) = f^{-1}(J_0(f))$.

Proof. We show that $F_0(f)$ is invariant by f (which is equivalent). Let $V \subset \overline{\mathbb{C}}$ be open, $\{f^k\}_k$ normal at V. Since f is continuous, $f^{-1}(V)$ is open. Let $(f^{k_i})_i$ be sequence in $\{f^k\}_k$. Then $(f^{k_i-1})_i$ also is, so there is uniformly converging subsequence $(f^{k'_i-1})_i$ in compact subsets of V. Therefore, if $D \subset f^{-1}(V)$ is compact, then $(f^{k'_i-1})_i$ is uniformly convergent in $f(D) \subset V$ (also compact, since f is continuous). First, assume it converges to a bounded analytic function. Then the subsequence is Cauchy, so that given $\epsilon > 0$, there exists $I \in \mathbb{N}$ such that $\forall i, j \geq I$:

$$|f^{k'_i-1}(y) - f^{k'_j-1}(y)| < \epsilon, \forall y \in f(D)$$
$$|f^{k'_i-1}(f(x)) - f^{k'_j-1}(f(x))| < \epsilon, \forall x \in D$$
$$|f^{k'_i}(x) - f^{k'_j}(x)| < \epsilon, \forall x \in D.$$

So that $(f^{k'_i})_i$ is Cauchy, thus converges uniformly, in D and, therefore, is normal in $f^{-1}(V)$. If it converges uniformly to $f = \infty$, then, given $R > 0$, there exists $I \in \mathbb{N}$ such that $\forall i \geq I$:

$$|f^{k'_i-1}(y)| > R, \forall y \in f(D)$$
$$|f^{k'_i-1}(f(x))| > R, \forall x \in D$$
$$|f^{k'_i}(x)| > R, \forall x \in D.$$

So that $(f^{k'_i})_i$ converges uniformly to ∞, in D and, therefore, is normal in $f^{-1}(V)$. Since V was arbitrary, this implies $f^{-1}(F_0(f)) \subset F_0(f)$. Since f is surjective, applying f to both sides yields:$F_0(f) \subset f(F_0(f))$
Now taking an open V as above and $(f^{k_i+1})_i$ sequence,$(f^{k'_i+1})_i$ converging subsequence. Since f meromorphic, it is an open map, so that $f(V)$ is also open. Let $D \subset f(V)$ be compact. Then $f^{-1}(D) \subset V$ is compact since f is proper, hence $(f^{k'_i+1})_i$ converges uniformly. First, assume it converges to a bounded analytic function. Then the subsequence is Cauchy, in $f^{-1}(D)$, that is, given $\epsilon > 0$, $\exists I \in \mathbb{N}$, $\forall i, j \geq I$:

$$|f^{k'_i+1}(x) - f^{k'_j+1}(x)| < \epsilon, \forall x \in f^{-1}(D)$$
$$|f^{k'_i}(f(x)) - f^{k'_j}(f(x))| < \epsilon, \forall x \in f^{-1}(D)$$
$$|f^{k'_i}(y) - f^{k'_j}(y)| < \epsilon, \forall y \in D.$$

With the last inequality valid since f is surjective. So that $(f^{k'_i})_i$ is also Cauchy and converges uniformly in D and, therefore, is normal in $f^{-1}(V)$. If it converges uniformly to $f = \infty$, then, given $R > 0$, there exists $I \in \mathbb{N}$ such that $\forall i \geq I$:

$$|f^{k'_i+1}(x)| > R, \forall x \in f^{-1}(D)$$
$$|f^{k'_i}(f(x))| > R, \forall x \in f^{-1}(D)$$
$$|f^{k'_i}(y)| > R, \forall y \in D.$$

Again, with the last inequality valid since f is surjective. So that $(f^{k'_i})_i$ is also Cauchy and converges uniformly to ∞ in D and, therefore, is normal in $f^{-1}(V)$. This now implies $f(F_0(f)) \subset F_0(f)$, so that $F_0(f) \subset f^{-1}(F_0(f))$. Together with the previous inclusion, we conclude that $f^{-1}(F_0(f)) = F_0(f) = f(F_0(f))$ □

Example 2.31. In case $f(z) = z^2$, $J_(f) = S^1$, it is clear that $f(S^1) = S^1$:
Let $z \in S^1$, then $z = e^{\theta i}$, $\theta \in \mathbb{R}$. Then, $f(z) = e^{2\theta i} \in S^1$. Also, if $y \in S^1$, $y = e^{\theta i}$. Then clearly if $z = e^{\theta/2i} \in S^1$, then $f(z) = y$.

Proposition 2.32. $J_0(f^p) = J_0(f)$, $\forall p \in \mathbb{N}$.

Proof. We show that $F_0(f^p) = F_0(f)$. Clearly, if every subsequence of $\{f^{k_i}\}$ has converging subsequence $\{f^{k_i'}\}$ in an open subset of the plane, then the same is true for $\{(f^p)^{k_i}\}$, since $\{(f^p)^{k_i}\} = \{f^{pk_i}\}$ and $\{f^{pk_i'}\}$ is a subsequence of $\{f^{k_i'}\}$, so it is converging. Therefore $F_0(f^p) \subset F_0(f)$. Now, if D is compact, $\{g_k\}$ uniformly convergent in D, the same holds for $\{h \circ g_k\}$ for any continuous h (in fact, any continuous function). So, if $\{(f^p)^k\} = \{f^{pk}\}$ is normal in an open subset, then $\{f^{pk+r}\}$, $r = 0, 1, ..., p-1$ also is. But any subsequence of $\{f^k\}$ contains an infinite number of terms of the form $\{f^{pk+r}\}$ for at least some r, $0 \leq r \leq p-1$, which has an uniformly convergent subsequence. Therefore $\{f^k\}$ is normal and $F_0(f) \subset F_0(f^p)$. $\square$

Example 2.33. In case $f(z) = z^2$, $J_0(f) = S^1$, by analogous arguments, it is easy to see that $J_0(f^p) = J_0(z^{2p}) = S^1$

Proposition 2.34. If f is a rational map of degree $d \geq 2$, then there are at most 2 points with finite total orbits and none belong to $J_0(f)$.

Proof. Suppose z_0 has finite grand orbit $\mathbb{O}(z_0)$. Then f maps $\mathbb{O}(z_0)$ bijectively into itself, so that it must consist of a single periodic orbit, $\mathbb{O}(z_0) = z_0, z_1, ..., z_{n-1}$. Now, since for each z_i it's only pre-image is $z_{i-1 \mod n}$ they all must be a critical points of f by Proposition 1.8. So the number of finite total orbits must be finite. Since the points are periodic, it also means they are are super attracting periodic cycle, and so are in $F_0(f)$. Suppose now that there are at least 3 of those points. Then for any open set U such that none of these points belong to U, $\{f^n\}$ is normal at U, so that f is normal at $\overline{\mathbb{C}}$ and $J_0(f)$ is empty, a contradiction with Proposition 2.28. So there can only be 2 points with finite total orbit. $\square$

Proposition 2.35. If f is a rational map, $w \in J_0(f)$ and U neighborhood of w, then:

$$W = \bigcup_{k=1}^{\infty} f^k(U) = \overline{\mathbb{C}} \backslash V$$

where V has at most 2 points, not in $J_0(f)$, independent on w and U.

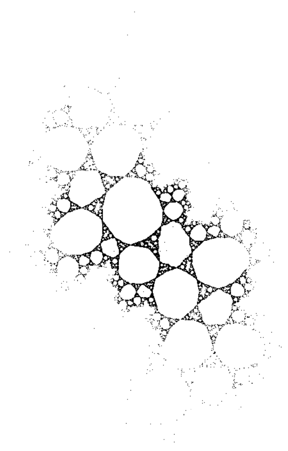
Proof. By definition, $\{f^k\}$ is not normal at w,implying it is not normal in U , thus, by Montel Theorem, $\forall z \in \overline{\mathbb{C}}$, $\exists w \in U$, $f(w) = z$, with at most 2 exceptions, call then $v, w \in \overline{\mathbb{C}}$. Suppose $f(z) = v$, for some $z \in \overline{\mathbb{C}}$. Since $f(W) \subset W$, this implies $z \notin W$, but then those points must be periodic and their grand orbit finite, so that they are in the $F_0(f)$ by the previous Proposition. Since v, w depend solely on f, they are independent of w and U as claimed. $\square$

Corollary 2.36. For all $z \in \overline{\mathbb{C}}$, with at most two exceptions, if $U \subset \overline{\mathbb{C}}$ is open, $U \cap J_0(f) \neq \emptyset$, then there exists an infinite sequence of $k_i \in \mathbb{N}$ such that $f^{-k_i}(z) \cap U \neq \emptyset$. Also, if $z \in J_0(f)$, then:

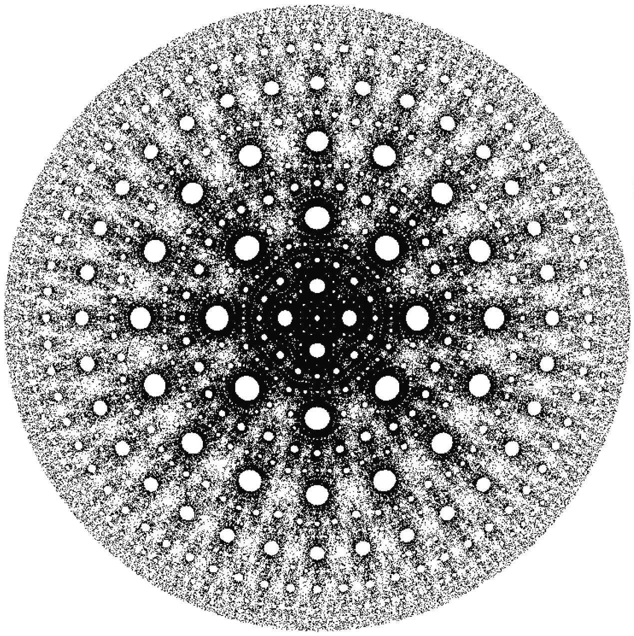
$$J_0(f) = \overline{\mathbb{O}^-(z)}.$$

Proof. Suppose $z \in \overline{\mathbb{C}}$ is not one of the exceptional points with finite total orbit. Then, $\exists k_1 > 0$ such that $z \in f^{k_1}(U)$ by Proposition 2.35. This implies that $f^{-k_1}(z) \cap U \neq \emptyset$. Take $z \neq z_1 \in f^{-k_1}(z)$ (it exists since otherwise it would imply z is the exceptional point, contradicting the choice of z). Notice that z_1 also is not the exceptional point, since that would imply z to also be. But then $\exists k_2' > 0$, $z_1 \in f^{k_2'}(U)$, which then implies $f^{-k_2'}(z_1) \subset f^{-k_2'}(f^{-k_1}(z)) = f^{-(k_1+k_2')}(z) = f^{-k_2}(z) \cap U \neq \emptyset$. Proceeding in a similar fashion, we obtain a sequence $(k_1, k_2, ...)$, with $k_i \neq k_j$, $i \neq j$, such that $f^{-k_i}(z) \cap U \neq \emptyset$.
Now, if $z \in J_0(f)$, then $f^{-k}(z) \subset J_0(f)$, by proposition 2.30, therefore $\overline{\mathbb{O}^-(z)} \subset J_0(f)$. On the other hand, if U is a neighborhood of $z \in J_0(f)$, then $f^{-k}(z) \cap U \neq \emptyset$, for some k by the above (z can not be the exceptional point since it is in $J_0(f)$). Thus, $\mathbb{O}^-(z)$ is dense in $J_0(f)$, that is $\overline{\mathbb{O}^-(z)} = J_0(f)$ □

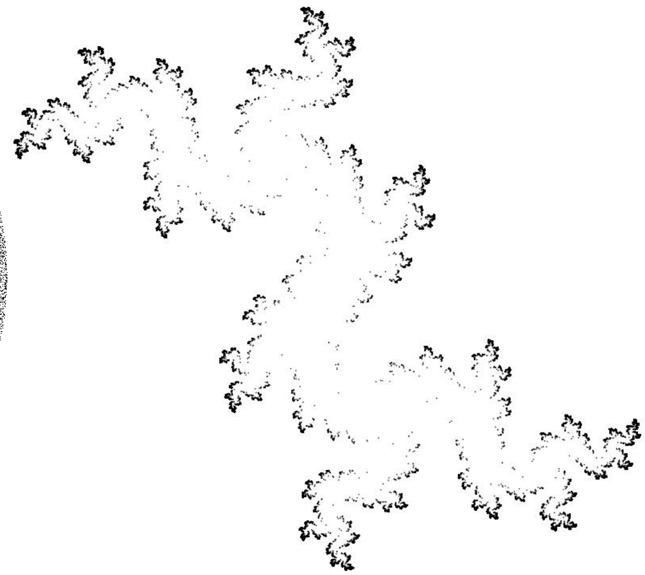
From Corollary 2.36 yet another method for illustrating the Julia Sets was derived. It consists in calculating the pre-images from repulsive periodic points. This yields a precise a list of points in fact in the Julia Set, though it may take many iterates to get a satisfactory result. We will call this method 3. Below are some examples of Julia Sets computed by this method, in the software MATLAB. The code can be found in the end of the paper.



(a) Julia Set around 0 for $\dfrac{z^2 - 0.2 + 075i}{0.1z^3 + 1}$



(b) Julia Set around 0 for $\dfrac{z^5 - 0.0001}{z^2}$



(c) Julia Set around 0 for $f(z) = z^2 + 0.7i$

Figure 6: Method 3

Example 2.37. Considering $f(z) = z^2$, take $1 \in J_0(f) = S^1$. Then clearly

$$\overline{\mathbb{O}^-(1)} = \overline{\{2n\text{-roots of unity}; n = 1, 2, ...\}} = S^1.$$

Corollary 2.38. If f is a polynomial, then $J_0(f)$ has empty interior. If f is a rational map and $J_0(f)$ has non-empty interior, then $J_0(f) = \overline{\mathbb{C}}$.

Proof. Suppose U is open, $U \subset J_0(f)$. Then $f^k(U) \subset J_0(f), \forall k$ by Proposition 2.30. If $U \neq \emptyset$, then by Proposition 2.35, $\bigcup_{k=1}^{\infty} f^k(U) = \mathbb{C} \subset J_0(f)$ or $\bigcup_{k=1}^{\infty} f^k(U) = \mathbb{C}\backslash\{0\} \subset J_0(f)$, contradicting the fact that it is bounded (Proposition 2.27). So $U = \emptyset$, that is $int(J_0(f)) = \emptyset$. Now suppose f is a rational map and $int(J_0(f)) \neq \emptyset$. Let $\emptyset \neq U \subset J_0(f)$ be open. Then by Proposition 2.35, $\overline{W} = \overline{\mathbb{C}}$, where $W = \bigcup_{k=1}^{\infty} f^k(U)$. But by Proposition 2.30 $W \subset J_0(f)$ $\therefore$ $\overline{W} \subset J_0(f)$ □

Proposition 2.39. If f is a rational map, then $J_0(f)$ is a perfect set (that is, closed and with not isolated points). Therefore it is uncountable.

Proof. Let $v \in J_0(f)$ and U neighborhood of v. We will show that $U \cap J_0(f)\backslash\{v\} \neq \emptyset$., that is, it is not an isolated point (since U was arbitrary). Consider the three possible cases:

1. If v is not a fixed nor periodic point, by Corollary 2.36, $U \cap \mathbb{O}^-(v) \neq \emptyset$. But $v \notin \mathbb{O}^-(v)$, since it is not periodic, so that $U \cap \mathbb{O}^-(v)\backslash\{v\} \neq \emptyset$. Since $v \in J_0(f)$, $\mathbb{O}^-(v) \subset J_0(f)$ by Proposition 2.30, so that the result follows.

2. If v is a fixed point, $f(v) = v$. Suppose $f(z) = v$ has no other solutions, then, just as in the proof of Proposition 2.35, $v \notin J_0(f)$, contradiction. So there exists $w \neq v$, $f(w) = v$. Note that this implies $w \in J_0(f)$, by Proposition 2.30. By Corollary 2.36 applied to U, $\exists k \geq 1, u \in \mathbb{C}$ such that $u \in f^{-k}(w) \cap U \neq \emptyset$. This also implies $u \in J_0(f)$, and $u \neq v$, since $f^k(u) = w \neq v = f^k(v)$, so the result follows.

3. If v is a periodic point of period greater than 1, that is $f^p(v) = v$, $p > 1$. By Proposition 2.32, $J_0(f) = J_0(f^p)$, so we may apply (2) to f^p.

In any case, $J_0(f)$ has no isolated points. Since it is closed, it is perfect. □

Theorem 2.40. If f is a rational map, then $J(f) = J_0(f)$.

Proof. We already have that $J(f) \subset J_0(f)$ by 2.28. Now let:

$$K := \{w \in J_0(f);, f(w) = w \wedge f'(z_0) = 0\}.$$

Notice that K is a finite proper subset of $J_0(f)$, thus discrete. Suppose $w \in J_0(f) \backslash K$. Then $\exists V$ neighbourhood of w such that $\exists g_1, g_2 : V \to \overline{\mathbb{C}}$ (g_1, g_2 two distinct branches of the local inverse of f) by the inverse function theorem, so that $g_1(z_0) = z_1$, $g_2(z_0) = z_2$. Define the family $\{h_k\}_k$ in V:

$$h_k(z) = \frac{(f^k(z) - g_1(z))(z - g_2(z))}{(f^k(z) - g_2(z))(z - g_1(z))}.$$

Let U be a neighbourhood of w, $U \subset V$. Since $w \in J_0(f)$ it follows that $\{f^k\}_k$ and therefore $\{h_k\}_k$ are not normal at U. By Montel's Theorem, $h_k(z)$ assumes at least 1 of the values $0, 1, \infty$ for some $k \geq 1$ and some $\bar{z} \in U$. But, if:

$$0 = h_k(\bar{z}) \implies f^k(\bar{z}) = g_1(\bar{z}) \text{ or } g_2(\bar{z}) = z$$

so that $f^{k+1}(\bar{z}) = \bar{z}$ or $f(\bar{z}) = \bar{z}$. And if:

$$\infty = h_k(\bar{z}) \implies f^k(\bar{z}) = g_2(\bar{z}) \text{ or } g_1(\bar{z}) = z$$

so that $f^{k+1}(\bar{z}) = \bar{z}$ or $f(\bar{z}) = \bar{z}$. And if:

$$1 = h_k(\bar{z}) \implies \frac{f^k(\bar{z}) - g_1(\bar{z})}{f^k(\bar{z}) - g_2(\bar{z})} = \frac{\bar{z} - g_1(\bar{z})}{\bar{z} - g_2(\bar{z})}$$

so that $f^{k+}(\bar{z}) = \bar{z}$.
Either way, U contains a periodic point. Since the number of non-repelling periodic points is finite, by Corollary 2.17, taking a smaller neighbourhood if necessary, we can guarantee it is repelling, so that $w \in J(f)$. Therefore $J_0(f)\backslash K \subset J(f)$. Since $J_0(F)$ has no isolated points, $\overline{J_0(f)\backslash K} = J_0(f) \subset J(f)$. □

We conclude listing the properties of Julia sets for rational map:
If f is a rational map, then it's Julia set is either the entire extended complex plane, or an uncountable closed set with empty interior and no isolated points. It is also the boundary of any basin of attraction of attractive periodic points.

MATLAB code used to generate Julia Sets

Code 1: Auxiliary function

```
function M= cmplxgrid(n)
M = zeros(n+1);
%Creates a matrix whose entries form a square grid centered at 0, with side
%length 3
for x = 0:n
    for y = 0:n
        M(x+1,y+1) = ((y-(n/2))/(n/3))+(((n/2)-x)*1i/(n/3));
    end
end
end
```

Code 2: Method 1

```
function [poli,M]=julia5(n,polin,polid)
%Finds the fixed points of the rational function, which are assumed to be
%repelling
poli=[polin;polid];
poli2 = [0 polin]-[polid, 0];
r=roots(poli2);
%Calculates the pre-images of the fixed points, one at a time, choosing
%randomly one of the pre-images and iterating
for w=1:length(r)
    M = zeros(n,2);
    z0=double(r(w));
    z0x = real(z0);
    z0y = imag(z0);
    M(1,1) = z0x;
    M(1,2)= z0y;
    for j=2:n
        poli0= polin-z0*polid;
        r2 = roots(poli0);
        k=randperm(length(r2));
        k=k(1);
        z0 = r2(k);
        z0x = real(z0);
        z0y = imag(z0);
        M(j,1) = z0x;
        M(j,2) = z0y;
    end
    %Plots the pre-images in the complex plane
    hold
    plot(M(:,1),M(:,2),'k.','MarkerSize',2);
    drawnow;
    hold
    axis equal
    axis off
end
```

Code 3: Method 2

```
function [polin,polid,M] = julia6(n,polin,polid)
M = cmplxgrid(n);
for e = 0:50
    M=polyval(polin,M)./polyval(polid,M);
end
M= (exp(-abs(M)));
imagesc(M)
end
```

Code 4: Method 3

```
function M = juliaroot2(n,polin,polid)
M=cmplxgrid(n);
%Finds the fixed points of the rational function, which are assumed to be
%attractive
poli2 = [0 polin]-[polid, 0];
r=roots(poli2);
len = length(r);
while polid(1)==0
    polid(1) =  [];
end
%Iterates the function
for e = 0:50
    M=polyval(polin,M)./polyval(polid,M);
end
%Colors the basins of attraction
for x=1:n+1
    for y=1:n+1
        l = M(x,y)*ones(len,1);
        diff = r-l;
        M(x,y)=len+1;
        for m=1:len
            if abs(diff(m))<10^-2
                M(x,y)=m;
                break
            end
        end
    end
end
imagesc(M)
end
```

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