Uniform decay for a plate equation partially dissipative

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Abstract

In this article we consider a plate made of by two types of materials: a part is conservative and the other dissipative. We show that the dissipation produced for the second part is strong enough to obtain the uniform decay of the energy. moreover, we found that the rate of decay of the solutions depend on behavior of the dissipative term in a neighborhood of zero, that is, for a linear dissipation we obtain exponential decay while for a polynomial dissipation we obtain polynomial decay.

Keywords: Uniform decay, transmission problem, plate equation, partial damping.

1 Introduction

Stability of some problems with partial damping in elasticity were studied by many authors as for example [4, 11, 12, 14] among others. Those works deal about strong and uniform decay of the energy provided the damping is effective in a localized strategic part of his domain. The goal of this paper is to study the stability of a plate made of two types of materials where only a part of him is dissipative, thus, the corresponding plate equation can be seen as a problem with partial damping and discontinuous coefficients due to the different nature of their components.

*Supported by CNPq, Brazil
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To introduce this model, let us denote by $\Omega$ and $\Omega_1$ two bounded open sets in $\mathbb{R}^2$ with smooth boundaries $\Gamma$ and $\Gamma_1$ respectively such that $\bar{\Omega}_1 \subset \Omega$. We assume that a plate made of two materials, in equilibrium, occupy the region $\Omega$ which is conservative in $\Omega_1$ and dissipative in $\Omega_2 := \Omega \setminus \bar{\Omega}_1$ as shows the figure 1.

Figure 1. The set $\Omega$.

If we denote by $u(x, t)$ and $v(x, t)$ the transversal displacements in $\Omega_1$ and $\Omega_2$ at the time $t$, the equations that model this problem are given by

$$
\rho_1 u_{tt} + \alpha_1 \Delta^2 u = 0 \quad \text{in} \quad \Omega_1 \times \mathbb{R}^+,
$$

(1.1)

$$
\rho_2 v_{tt} + \alpha_2 \Delta^2 v + g(v_t) = 0 \quad \text{in} \quad \Omega_2 \times \mathbb{R}^+,
$$

(1.2)

satisfying the boundary condition

$$
v = 0, \quad \frac{\partial v}{\partial \nu} = 0 \quad \text{on} \quad \Gamma \times \mathbb{R}^+,
$$

(1.3)

$$
u = v, \quad \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu}, \quad \alpha_1 \mathcal{B}_1 u = \alpha_2 \mathcal{B}_1 v, \quad \alpha_1 \mathcal{B}_2 u = \alpha_2 \mathcal{B}_2 v \quad \text{on} \quad \Gamma_1 \times \mathbb{R}^+,
$$

and initial data

$$
u(x, 0) = u^0(x), \quad u_t(x, 0) = u^1(x) \quad \text{in} \quad \Omega_1,
$$

$$
u(x, 0) = v^0(x), \quad v_t(x, 0) = v^1(x) \quad \text{in} \quad \Omega_2.
$$

(1.4)

Here the mass densities $\rho_i$ and the elasticity coefficients $\alpha_i$ are positive. The nonlinearity $g$ is a non-decreasing continuous function satisfying

$$
g(w)w \geq 0 \quad \text{and} \quad |g(w)| \leq 1 + C|w|,
$$

(1.5)

and the operators $\mathcal{B}_1, \mathcal{B}_2$ are given by

$$
\mathcal{B}_1 u = \Delta u + (1 - \mu) \mathcal{B}_1 u,
$$

$$
\mathcal{B}_2 u = \frac{\partial \Delta u}{\partial \nu} + (1 - \mu) \frac{\partial \mathcal{B}_2 u}{\partial \nu},
$$

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where \( \nu = (\nu_1, \nu_2) \) denotes the unitary normal vector outside of \( \Omega_1 \), \( \tau = (-\nu_2, \nu_1) \) is the tangent vector and
\[
B_1 u := 2\nu_1\nu_2 \frac{\partial^2 u}{\partial x \partial y} - \nu_1^2 \frac{\partial^2 u}{\partial y^2} - \nu_2^2 \frac{\partial^2 u}{\partial x^2},
\]
\[
B_2 u := (\nu_1^2 - \nu_2^2) \frac{\partial^2 u}{\partial x \partial y} + \nu_1 \nu_2 \left( \frac{\partial^2 u}{\partial y^2} - \frac{\partial^2 u}{\partial x^2} \right).
\]

The boundary conditions given by the second part of (1.3) are the transmission conditions by means of which the dissipative properties in \( \Omega_2 \) are transmitted for \( \Omega_1 \).

Connected with transmission problems we have the following works: the controllability for the transmission problem of the plate equation was studied by Aassila [1] and Liu et al [10]. They proved simultaneously the exact controllability at the boundary applying the Hilbert Uniqueness Method (HUM). Concerning to the controllability for hyperbolic systems we can mention the works of Aassila [2], Lagnese [7] and Lions [8]. They proved the exact controllability by using HUM. Stability for the transmission problem for the wave equation was studied by Liu and Williams [9]. They considered a linear frictional damping at the boundary and showed exponential decay of the energy. For the stationary case, some results about existence and regularity of solutions were obtained by Athanasiadis et al [3] and Schechter [13].

The existence and uniqueness of solutions for system (1.1)-(1.4) satisfying condition (1.5) may be established by using the standard semigroup theory (see Komornik [6, section 7] for a similar study of the wave equation). We summarize the existence result in the following theorem.

**Theorem 1.1** Let \((u^0, v^0)\) be in \(H^2(\Omega_1) \times H^2(\Omega_2)\) and \((u^1, v^1)\) in \(L^2(\Omega_1) \times L^2(\Omega_2)\) satisfying
\[
v^0 = 0, \quad \frac{\partial v^0}{\partial \nu} = 0 \quad \text{on} \quad \Gamma,
\]
\[
u^0 = v^0, \quad \frac{\partial u^0}{\partial \nu} = \frac{\partial v^0}{\partial \nu} \quad \text{on} \quad \Gamma_1,
\]
(1.6)

Then there exists a unique weak solution of (1.1)-(1.4) satisfying
\[
(u, v) \in \bigcap_{\mu=0}^1 C^\mu(\mathbb{R}_0^+; H^{2-2\mu}(\Omega_1) \times H^{2-2\mu}(\Omega_2)).
\]

Moreover, if \((u^0, v^0) \in H^4(\Omega_1) \times H^4(\Omega_2), (u^1, v^1) \in H^2(\Omega_1) \times H^2(\Omega_2)\) and satisfy
\[
v^1 = 0, \quad \frac{\partial v^1}{\partial \nu} = 0 \quad \text{on} \quad \Gamma,
\]
\[
u^1 = v^1, \quad \frac{\partial u^1}{\partial \nu} = \frac{\partial v^1}{\partial \nu}, \quad \alpha_1 B_1 u^0 = \alpha_2 B_1 v^0, \quad \alpha_1 B_2 u^0 = \alpha_2 B_2 v^0 \quad \text{on} \quad \Gamma_1,
\]
(1.7)
then the solution has the following regularity property

\[(u, v) \in \bigcap_{\mu=0}^{2} W^{\mu,\infty}(\mathbb{R}^+; H^{4-2\mu}(\Omega_1) \times H^{4-2\mu}(\Omega_2)).\]

To finish this section we enunciate a Green’s formula which be often used in the next section. Let us consider the following bilinear operator

\[\mathcal{D}(u, v) = \frac{\partial^2 u \partial^2 v}{\partial x^2 \partial y^2} + \frac{\partial^2 u \partial^2 v}{\partial y^2 \partial x^2} + \mu \left( \frac{\partial^2 u \partial^2 v}{\partial x^2 \partial y^2} + \frac{\partial^2 u \partial^2 v}{\partial y^2 \partial x^2} \right) + 2(1 - \mu) \frac{\partial^2 u}{\partial x \partial y} \frac{\partial^2 v}{\partial x \partial y}.\]

\[\text{Lemma 1.2} \quad \text{Let us suppose that } w \text{ and } z \text{ are functions of } H^4(\Omega), \text{ then we have}\]

\[\int_{\Omega} (\Delta^2 w) z \, dx = \int_{\Omega} \mathcal{D}(w, z) \, dx + \int_{\Gamma} \left\{ (B_2 w) z - (B_1 w) \frac{\partial z}{\partial \nu} \right\} \, d\Gamma\]

\[\text{Proof:} \quad \text{See [5].}\]

The remainder part of this paper is organized as follows. In the next section we find some estimates for strong solutions of (1.1)-(1.4) and show the exponential decay and in section 3 we show the polynomial decay.

\section{Exponential decay}

In this section we study the asymptotic behavior of the solutions of system (1.1)-(1.4) where the dissipative term given by \( g \) belong to a class de functions confined between two lines, as for example, the function \( g(w) = (\alpha + \sin(w))w \) for \( \alpha > 1 \). In the remainder of this paper we denote by \( C \) a positive constant independent of the initial data which takes different values in different places. Also, we will denote by \( C_\sigma \) a positive constant depending of a determined parameter \( \sigma \).

We shall assume that the material type in \( \Omega_2 \) is more dense and stiffer than that in \( \Omega_1 \), that is

\[\rho_1 \leq \rho_2, \quad \alpha_1 \geq \alpha_2.\]  

\[(2.1)\]

Also, we need to establish some geometric conditions for the common surface between \( \Omega_1 \) and \( \Omega_2 \), more precisely, we assume that there exists \( x_0 \in \mathbb{R}^2 \) such that

\[r(x) \cdot \nu(x) \geq 0 \quad \text{on} \quad \Gamma_1,\]

\[(2.2)\]
where \( r(x) := x - x_0 \). The first order energy associated to the system (1.1)-(1.3) is given by
\[
E(t) := \frac{1}{2} \int_{\Omega_1} (\rho_1 |u_t|^2 + \alpha_1 D(u, u)) \, dx + \frac{1}{2} \int_{\Omega_2} (\rho_2 |v_t|^2 + \alpha_2 D(v, v)) \, dx.
\]

The exponential asymptotic behavior which is the main result of this section is given by the following theorem.

**Theorem 2.1** Let us assume that (2.1)-(2.2) hold and there exist positive constants \( c_1 \) and \( c_2 \) such that
\[
c_1 |w| \leq |g(w)| \leq c_2 |w|, \quad \forall w \in \mathbb{R}.
\]
(2.3)

Then, for any \((u^0, v^0) \in H^2(\Omega_1) \times H^2(\Omega_2)\) and \((u^1, v^1) \in L^2(\Omega_1) \times L^2(\Omega_2)\) satisfying the compatibility conditions (1.6) there exists a positive constant \( \gamma \) such that
\[
E(t) \leq 4E(0)e^{-\gamma t}.
\]

We shall show this theorem for strong solutions, that is, for solutions with initial data \((u^0, v^0) \in H^4(\Omega_1) \times H^4(\Omega_2)\) and \((u^1, v^1) \in H^2(\Omega_1) \times H^2(\Omega_2)\) satisfying (1.6)-(1.7). Hence our conclusion follows by a density argument. We shall apply a piecewise multiplier method to obtain appropriate inequalities for the strong solutions of (1.1)-(1.4).

The dissipative properties of the solutions for the transmission problem is given by the following lemma

**Lemma 2.2** The first order energy satisfies the following identity
\[
\frac{d}{dt} E(t) = - \int_{\Omega_2} g(v_t)v_t \, dx.
\]

**Proof:** Multiplying equation (1.1) by \( u_t \), equation (1.2) by \( v_t \) and performing an integration by parts we easily arrive to this identity. \( \square \)

Let us denote by \( B_\delta(\Omega_1) = \{ x \in \mathbb{R}^2 : \inf_{y \in \Omega_1} |y - x| < \delta \} \) for \( \delta > 0 \) and let us consider the following decomposition: \( \Omega_2 = \Omega_2' \cup \Omega_2'' \) where \( \Omega_2' = \Omega_2 \cap B_\delta(\Omega_1) \) and \( \Omega_2'' = \Omega_2 \setminus B_\delta(\Omega_1) \) for some \( \delta \) small. Let \( \psi \in C_0^1(\Omega) \) be such that \( \psi = 1 \) in \( B_\delta(\Omega_1) \) let us introduce the following functional
\[
\mathcal{R}_1(t) := \int_{\Omega_1} \rho_1 u_t [r \cdot \nabla u] \, dx + \int_{\Omega_2} \rho_2 v_t [r \cdot \nabla v] \, dx.
\]

The following lemma retrieves the energy for the transversal displacement \( u \).
Lemma 2.3 If inequalities (2.1)-(2.2) hold, then there exists a constant $C_1 > 0$ such that

$$\frac{d}{dt} R_1(t) \leq - \int_{\Omega_1} (\rho_1|u_t|^2 + \alpha_1 D(u,u)) \, dx - \frac{1}{2} \int_{\Omega_2} \alpha_2 D(v,v) \, dx + C_1 \left\{ \int_{\Omega_2} (\rho_2|v_t|^2 + \alpha_2 D(v,v)) \, dx + \int_{\Omega_2} |g(v_t)|^2 \, dx \right\},$$

for any strong solutions of (1.1)-(1.4).

Proof: Let us introduce the following trilinear form

$$S(w, \sigma, \varrho) := 2 \left\{ \frac{\partial^2 w}{\partial x^2} \left[ \frac{\partial \sigma}{\partial x} \frac{\partial \varrho}{\partial x} \right] + \frac{\partial^2 w}{\partial y^2} \left[ \frac{\partial \sigma}{\partial y} \frac{\partial \varrho}{\partial y} \right] \right\} + 2\mu \left\{ \frac{\partial^2 w}{\partial x^2} \left[ \frac{\partial \sigma}{\partial y} \frac{\partial \varrho}{\partial y} \right] + \frac{\partial^2 w}{\partial y^2} \left[ \frac{\partial \sigma}{\partial x} \frac{\partial \varrho}{\partial x} \right] \right\}$$

$$+ 2(1 - \mu) \left\{ \frac{\partial^2 w}{\partial x \partial y} \left[ \frac{\partial \sigma}{\partial x} \frac{\partial \varrho}{\partial y} + \frac{\partial \sigma}{\partial y} \frac{\partial \varrho}{\partial x} \right] \right\}.$$

A simple calculation shows that

$$D(w, \sigma \varrho) = \sigma D(w, \varrho) + \varrho D(w, \sigma) + S(w, \sigma, \varrho).$$

Multiplying equation (1.1) by $[r \cdot \nabla u]$, integrating by parts and using the boundary conditions (1.3) we obtain

$$\frac{d}{dt} \int_{\Omega_1} \rho_1 [r \cdot \nabla u] \, dx = \frac{1}{2} \int_{\Gamma_1} (r \cdot \nu) \rho_1 |u_t|^2 \, d\Gamma + \int_{\Gamma_1} \alpha_1 B_1 u \frac{\partial}{\partial \nu} [r \cdot \nabla u] \, d\Gamma$$

$$- \frac{1}{2} \int_{\Gamma_1} r \cdot \nu D(u,u) \, d\Gamma - \int_{\Omega_1} (\rho_1|u_t|^2 + \alpha_1 D(u,u)) \, dx.$$

Similarly, multiplying equation (1.2) by $\psi [r \cdot \nabla v]$ we have that

$$\frac{d}{dt} \int_{\Omega_2} \rho_2 v_t \psi [r \cdot \nabla v] \, dx$$

$$= - \frac{1}{2} \int_{\Gamma_1} (r \cdot \nu) \rho_2 |v_t|^2 \, d\Gamma - \int_{\Gamma_1} \alpha_2 B_1 v \frac{\partial}{\partial \nu} [r \cdot \nabla v] \, d\Gamma$$

$$+ \frac{1}{2} \int_{\Gamma_1} r \cdot \nu D(v,v) \, d\Gamma - \int_{\Omega_1} (\rho_1|u_t|^2 + \alpha_1 D(u,u)) \, dx$$

$$- \int_{\Omega_2} (\rho_2|v_t|^2 + \alpha_2 D(v,v)) \, dx - \frac{1}{2} \int_{\Omega_2} \text{div}(\psi r)(\rho_2|v_t|^2 - \alpha_2 D(v,v)) \, dx$$

$$- \int_{\Omega_2} (\alpha_2 \partial_k v D(v, \psi_{\partial k}) + \alpha_2 S(v, \psi_{\partial k}, \partial_k v)) \, dx - \int_{\Omega_2} g(v_t) \psi [r \cdot \nabla v] \, dx.$$

Here, repeated indexes indicate summation. Summing the two last identities and using
the boundary conditions (1.3) we arrive at

\[
\frac{d}{dt} \mathcal{R}_1(t) = \frac{1}{2} \int_{\Gamma_1} r \cdot \nu (\rho_1 - \rho_2) |u_t|^2 \, d\Gamma + \int_{\Gamma_1} \alpha_2 B_1 v \frac{\partial}{\partial \nu} [r \cdot \nabla (u - v)] \, d\Gamma \\
- \frac{1}{2} \int_{\Gamma_1} r \cdot \nu [D(u, u) - D(v, v)] \, d\Gamma - \int_{\Omega_1} (\rho_1 |u_t|^2 + \alpha_1 D(u, u)) \, dx \\
- \int_{\Omega_2} (\rho_2 |v_t|^2 + \alpha_2 D(v, v)) \, dx - \frac{1}{2} \int_{\Omega_2} \text{div}(\psi_t)(\rho_2 |v_t|^2 - \alpha_2 D(v, v)) \, dx \\
- \int_{\Omega_2} (\alpha_2 \partial_k v D(v, \psi_{\tau_k}) + \alpha_2 S(v, \psi_{\tau_k}, \partial v)) \, dx - \int_{\Omega_2} g(v_t) \psi |r \cdot \nabla v| \, dx.
\]

Applying Young and Korn’s inequalities we have that

\[
\frac{d}{dt} \mathcal{R}_1(t) \leq \frac{1}{2} \int_{\Gamma_1} r \cdot \nu (\rho_1 - \rho_2) |u_t|^2 \, d\Gamma + \int_{\Gamma_1} \alpha_2 B_1 v \frac{\partial}{\partial \nu} [r \cdot \nabla (u - v)] \, d\Gamma \\
- \frac{1}{2} \int_{\Gamma_1} r \cdot \nu [\alpha_1 D(u, u) - \alpha_2 D(v, v)] \, d\Gamma - \int_{\Omega_1} (\rho_1 |u_t|^2 + \alpha_1 D(u, u)) \, dx \\
- \frac{1}{2} \int_{\Omega_2} \alpha_2 D(v, v) \, dx + C \left\{ \int_{\Omega_2} (\rho_2 |v_t|^2 + \alpha_2 D(v, v)) \, dx + \int_{\Omega_2} |g(v_t)|^2 \, dx \right\},
\]

for some \( C > 0 \). Now, we shall estimate the second term of the right hand of this inequality. The boundary conditions (1.3) imply that for \( x \in \Gamma_1 \) we have \( \partial_j \partial_i (u - v) = \Delta (u - v) \nu_i \nu_j \) and \( B_1 v \Delta (u - v) = D(u - v, v) \). Hence, it follows that

\[
\int_{\Gamma_1} \alpha_2 B_1 v \frac{\partial}{\partial \nu} [r \cdot \nabla (u - v)] \, d\Gamma = \int_{\Gamma_1} \alpha_2 B_1 v \frac{\partial}{\partial \nu} [r_i \partial_j (u - v)] \, d\Gamma \\
= \int_{\Gamma_1} \alpha_2 B_1 v [r_i \partial_j (u - v) \nu_j] \, d\Gamma \\
= \int_{\Gamma_1} r \cdot \nu \alpha_2 B_1 v \Delta (u - v) \, d\Gamma \\
= \int_{\Gamma_1} r \cdot \nu \alpha_2 [D(u, v) - D(v, v)] \, d\Gamma.
\]

Therefore, from hypothesis (2.2) we conclude that

\[
\int_{\Gamma_1} \alpha_2 B_1 v \frac{\partial}{\partial \nu} [r \cdot \nabla (u - v)] \, d\Gamma \leq \frac{1}{2} \int_{\Gamma_1} r \cdot \nu \alpha_2 [D(u, u) - D(v, v)] \, d\Gamma.
\]

Substitution of this inequality into (2.4) yields

\[
\frac{d}{dt} \mathcal{R}_1(t) \leq \frac{1}{2} \int_{\Gamma_1} r \cdot \nu (\rho_1 - \rho_2) |u_t|^2 \, d\Gamma - \frac{1}{2} \int_{\Gamma_1} r \cdot \nu (\alpha_1 - \alpha_2) D(u, u) \, d\Gamma \\
- \int_{\Omega_1} (\rho_1 |u_t|^2 + \alpha_1 D(u, u)) \, dx - \frac{1}{2} \int_{\Omega_2} \alpha_2 D(v, v) \, dx \\
+ C \left\{ \int_{\Omega_2} (\rho_2 |v_t|^2 + \alpha_2 D(v, v)) \, dx + \int_{\Omega_2} |g(v_t)|^2 \, dx \right\}.
\]

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Finally, in view of inequalities (2.1) our conclusion follows. □

Let \( \varphi \in C^1(\mathbb{R}^2) \) be a nonnegative function such that \( \varphi = 0 \) in \( B_{\delta/2}(\Omega_1) \) and \( \varphi = 1 \) in \( \mathbb{R}^2 \setminus B_{\delta}(\Omega_1) \) and consider the following functional

\[
\mathcal{R}_2(t) := \int_{\Omega_2} \rho_2 v_t \varphi v \, dx.
\]

The following lemma retrieves a part of the energy for the transversal displacement \( v \).

**Lemma 2.4** Given \( \epsilon > 0 \) there exists a positive constant \( C_\epsilon \) such that

\[
\mathcal{R}_2(t) = -\int_{\Omega_2'} \alpha_2 \Delta^2 \varphi \, dx + \epsilon \int_{\Omega_2} \alpha_2 \Delta (v, v) \, dx + C_\epsilon \int_{\Omega_2} (|v|^2 + |v_t|^2 + |g(v_t)|^2) \, dx,
\]

for any strong solutions of (1.1)-(1.4).

**Proof:** Multiplying equation (1.2) by \( \varphi v \) and integrating by parts we have the following identity

\[
\mathcal{R}_2(t) = \int_{\Omega_2} \rho_2 \varphi |v_t|^2 \, dx - \int_{\Omega_2} \alpha_2 \varphi \Delta (v, v) \, dx - \int_{\Omega_2} \alpha_2 v \Delta (v, \varphi) \, dx
\]

\[
- \int_{\Omega_2} \alpha_2 S(v, v, \varphi) \, dx - \int_{\Omega_2} \varphi g(v_t) v \, dx.
\]

Applying Young’s inequality we get, for \( \epsilon > 0 \)

\[
\mathcal{R}_2(t) = \int_{\Omega_2} \rho_2 \varphi |v_t|^2 \, dx - \int_{\Omega_2} \alpha_2 \varphi \Delta (v, v) \, dx + \epsilon \int_{\Omega_2} \alpha_2 \Delta (v, v) \, dx
\]

\[
+ \frac{C}{\epsilon} \int_{\Omega_2} (|v|^2 + |\nabla v|^2) \, dx + \int_{\Omega_2} |g(v)|^2 \, dx.
\]

Substituting the inequality

\[
\int_{\Omega_2} |\nabla v|^2 \, dx \leq \epsilon^2 \int_{\Omega_2} \alpha_2 \Delta (v, v) \, dx + C_\epsilon \int_{\Omega_2} |v|^2 \, dx
\]

into the above equation and redefining \( \epsilon \), our conclusion follows. □

Let us denote by \( (\theta, \phi) \) the solution of the following stationary problem

\[
\alpha_1 \Delta^2 \theta = 0 \quad \text{in} \quad \Omega_1, \quad \alpha_2 \Delta^2 \phi = v \quad \text{in} \quad \Omega_2,
\]

(2.5)
with the following boundary conditions

\[ \phi = 0, \quad \frac{\partial \phi}{\partial \nu} = 0 \quad \text{on} \quad \Gamma, \]

\[ \theta = \phi, \quad \frac{\partial \theta}{\partial \nu} = \frac{\partial \phi}{\partial \nu}, \quad \alpha_1 B_1 \theta = \alpha_2 B_1 \phi, \quad \alpha_1 B_2 \theta = \alpha_2 B_2 \phi \quad \text{on} \quad \Gamma_1. \]  

(2.6)

Note that the solution \((\theta, \phi)\) of this system depends linearly of displacement \(v\). Let us consider the functional

\[ R_3(t) := \int_{\Omega_1} \rho_1 u_t \theta \, dx + \int_{\Omega_2} \rho_2 v_t \phi \, dx. \]

The following lemma allows us to eliminate the term \(\int_{\Omega_2} |v|^2 \, dx\) which appears in the previous lemma.

**Lemma 2.5** Given \(\epsilon > 0\) there exists a positive constant \(C_\epsilon\) such that

\[ \frac{d}{dt} R_3(t) \leq -\frac{1}{2} \int_{\Omega_2} |v|^2 \, dx + \epsilon \int_{\Omega_1} \rho_1 |u_t|^2 \, dx + C_\epsilon \int_{\Omega_2} \left( \rho_2 |v_t|^2 + |g(v_t)|^2 \right) \, dx, \]

for any strong solutions of (1.1)-(1.4).

**Proof:** Multiplying equation (1.1) by \(\theta\), equation (1.2) by \(\phi\) and performing an integration by parts we get

\[ \frac{d}{dt} R_3(t) = \int_{\Omega_1} \rho_1 u_t \theta_t \, dx + \int_{\Omega_2} \rho_2 v_t \phi_t \, dx - \int_{\Omega_2} |v|^2 \, dx - \int_{\Omega_2} g(v_t) \phi \, dx. \]  

(2.7)

On the other hand, multiplying equations (2.5) by \(\theta\) and \(\phi\) respectively, integrating by parts and using Korn’s inequality we have that there exist a positive constant \(C\) such that

\[ \int_{\Omega_1} |\theta|^2 \, dx + \int_{\Omega_2} |\phi|^2 \, dx \leq C \int_{\Omega_2} |v|^2 \, dx. \]  

(2.8)

Similarly, since the system (2.5)-(2.6) is linear we obtain

\[ \int_{\Omega_1} |\theta_t|^2 \, dx + \int_{\Omega_2} |\phi_t|^2 \, dx \leq C \int_{\Omega_2} |v_t|^2 \, dx. \]  

(2.9)

Applying Young's inequality to equation (2.7) and using estimates (2.8)-(2.9) we have that, for \(\epsilon > 0\)

\[ \frac{d}{dt} R_3(t) \leq \epsilon \int_{\Omega_1} \rho_1 |u_t|^2 \, dx + C_\epsilon \int_{\Omega_2} \rho_2 |v_t|^2 \, dx - \frac{1}{2} \int_{\Omega_2} |v|^2 \, dx + C \int_{\Omega_2} |g(v_t)|^2 \, dx, \]

from where our conclusion follows. \(\square\)
Let us consider the following functional

\[ R(t) := R_1(t) + (C_1 + 1/2)R_2(t) + 2C_2R_3(t), \]

where \( C_1 \) is the constant considered in Lemma 2.3 and \( C_2 \) is a positive constant which will be determined later (see (2.10)). Finally, the following lemma retrieves the global energy for system (1.1)-(1.3)

**Lemma 2.6** There exists a positive constant \( C \) such that

\[
\frac{d}{dt} R(t) \leq -\frac{1}{2}E(t) + C \int_{\Omega_2} \left( |v_t|^2 + |g(v_t)|^2 \right) dx,
\]

for any strong solutions of (1.1)-(1.4).

**Proof:** Let us take \( \epsilon_1 \), the solution of the following equation

\[(C_1 + 1/2)\epsilon_1 = \frac{1}{4}.
\]

Using Lemma 2.3 and fixing \( \epsilon = \epsilon_1 \) in Lemma 2.4 we have that there exists a positive constant \( C_2 \) such that

\[
\frac{d}{dt} [R_1 + (C_1 + 1/2)R_2](t) \leq -\int_{\Omega_1} (\rho_1 |u_t|^2 + \alpha_1 D(u, u)) \, dx - \frac{1}{4} \int_{\Omega_2} \alpha_2 D(v, v) \, dx
\]

\[
+ C_2 \int_{\Omega_2} (|v|^2 + |v_t|^2 + |g(v_t)|^2) \, dx. \tag{2.10}
\]

Let \( \epsilon_2 \) be a positive constant such that

\[ 2C_2\epsilon_2 = \frac{1}{2}. \]

Combining inequality (2.10) and Lemma 2.5 with \( \epsilon = \epsilon_2 \) we arrive at

\[
\frac{d}{dt} R(t) \leq -\frac{1}{2} \int_{\Omega_1} (\rho_1 |u_t|^2 + \alpha_1 D(u, u)) \, dx - \frac{1}{4} \int_{\Omega_2} \alpha_2 D(v, v) \, dx
\]

\[
+ C \int_{\Omega_2} (|v_t|^2 + |g(v_t)|^2) \, dx.
\]

Adding the term \(-\frac{1}{4} \int_{\Omega_2} \rho_2 |v_t|^2 \, dx \) to this inequality, our conclusion follows. \( \square \)
**Proof of Theorem 2.1:** From hypothesis of the function $g$ we easily we obtain
\[
\int_{\Omega_2} (|v_t|^2 + |g(v_t)|^2) \, dx \leq C \int_{\Omega_2} g(v_t)v_t \, dx,
\]
for some $C > 0$. Substitution of this inequality into Lemma 2.6 we arrive at
\[
\frac{d}{dt} R(t) \leq -\frac{1}{2} E(t) + C \int_{\Omega_2} g(v_t)v_t \, dx. \tag{2.11}
\]
Let $N$ a positive constant, let us introduce the Lyapunov functional
\[
\mathcal{F}(t) := NE(t) + R(t).
\]
It is simple to verify that, for $N$ large, we have
\[
\frac{N}{2} E(t) \leq \mathcal{F}(t) \leq 2NE(t), \quad \forall t \geq 0. \tag{2.12}
\]
In these conditions, Lemma 2.2 and inequality (2.11) imply that, for $N$ large
\[
\frac{d}{dt} \mathcal{F}(t) \leq -\frac{1}{2} E(t),
\]
from where follows, in view of (2.12), that
\[
\frac{d}{dt} \mathcal{F}(t) \leq -\frac{1}{4N} \mathcal{F}(t). \tag{2.13}
\]
This inequality implies that
\[
\mathcal{F}(t) \leq \mathcal{F}(0)e^{-\frac{t}{4N}},
\]
and from equivalence relation (2.12) our conclusion follows. \(\square\)

### 3 Polynomial decay

Here our attention will be focused on the uniform rate of decay when the function $g(v)$ is nonlinear in a neighborhood of zero as $v^p$ with $p > 1$. In this case we prove that the solution decays as $(1 + t)^{-2/(p-1)}$. This result is given by the following theorem

**Theorem 3.1** Let us assume that (2.1)-(2.2) hold and there exist positive constants $c_1, \ldots, c_4$ such that
\[
\begin{align*}
    c_1|w| &\leq |g(w)| \leq c_2|w| \quad \text{for} \quad |w| > 1, \\
    c_3|w|^p &\leq |g(w)| \leq c_4|w|^{1/p} \quad \text{for} \quad |w| \leq 1,
\end{align*}
\]
for some $p > 1$. Then, for any $(u_0, v^0) \in H^2(\Omega_1) \times H^2(\Omega_2)$ and $(u_1, v^1) \in L^2(\Omega_1) \times L^2(\Omega_2)$ satisfying the compatibility conditions (1.6) there exists a positive constant $M = M(E(0))$ such that
\[
E(t) \leq M(1 + t)^{-2/(p-1)}.
\]
Proof: Let us consider the following decomposition of $\Omega_2$

\[ \Omega_2^+ := \{ x \in \Omega_2 : |v_t(x)| > 1 \} \quad \text{and} \quad \Omega_2^- := \{ x \in \Omega_2 : |v_t(x)| \leq 1 \}. \]

Now, we shall use some estimates of the previous section which does not depend of the behavior of the function $g$ in a neighborhood of zero. From first hypothesis of (3.1) we get

\[
\int_{\Omega_2^+} (|v_t|^2 + |g(v_t)|^2) \, dx \leq C \int_{\Omega_2} g(v_t) v_t \, dx,
\]

for some $C > 0$. On the other hand, the second part of the hypothesis (3.1) implies that

\[
|v_t|^2 \leq C|g(v_t)v_t|^{2/(p+1)}, \quad |g(v_t)|^2 \leq C|g(v_t)v_t|^{2/(p+1)},
\]

for any $x \in \Omega_2^-$. Moreover, using Holder’s inequality we have that

\[
\int_{\Omega_2^-} |g(v_t)v_t|^{2/(p+1)} \, dx \leq C \left( \int_{\Omega_2} g(v_t)v_t \, dx \right)^{2/(p+1)}.
\]

Therefore, these two last inequalities imply that

\[
\int_{\Omega_2^-} (|v_t|^2 + |g(v_t)|^2) \, dx \leq C \left( \int_{\Omega_2} g(v_t)v_t \, dx \right)^{2/(p+1)}.
\]

Finally, summing inequalities (3.2) and (3.3) we conclude that

\[
\int_{\Omega_2} (|v_t|^2 + |g(v_t)|^2) \, dx \leq \int_{\Omega_2^+} (|v_t|^2 + |g(v_t)|^2) \, dx + \int_{\Omega_2^-} (|v_t|^2 + |g(v_t)|^2) \, dx
\]

\[
\leq C \left\{ \int_{\Omega_2} g(v_t)v_t \, dx + \left( \int_{\Omega_2} g(v_t)v_t \, dx \right)^{\frac{2}{p+1}} \right\}.
\]

Substituting this integral term in Lemma 2.6 we get

\[
\frac{d}{dt} R(t) \leq \frac{1}{2} E(t) + C \left\{ \int_{\Omega_2} g(v_t)v_t \, dx + \left( \int_{\Omega_2} g(v_t)v_t \, dx \right)^{\frac{2}{p+1}} \right\}.
\]

Since $|R(t)| \leq C_3 E(t)$ for some $C_3 > 0$, Lemma 2.2 and the above inequality imply that

\[
\frac{d}{dt} [E^{\frac{p+1}{p}} R](t) = \frac{p-1}{2} \frac{d}{dt} \frac{1}{E^{\frac{p+1}{p}}(t)} E(t) + \frac{d}{dt} R(t)
\]

\[
\leq \frac{p-1}{2} C_3 E^{\frac{p-1}{p}}(t) \int_{\Omega_2} g(v_t)v_t \, dx + E^{\frac{p-1}{p}}(t) \frac{d}{dt} R(t)
\]

\[
\leq C E^{\frac{p+1}{p}}(0) \int_{\Omega_2} g(v_t)v_t \, dx - \frac{1}{2} E^{\frac{p+1}{p}}(t) + C E^{\frac{p+1}{p}}(t) \left( \int_{\Omega_2} g(v_t)v_t \, dx \right)^{\frac{2}{p+1}}.
\]
Applying the inequality \( ab \leq \epsilon a^{p+1} + C b^{p+1} \) to the last term of the above inequality we arrive at

\[
\frac{d}{dt} [E^{\frac{p+1}{2}} R](t) \leq -\frac{1}{4} E^{\frac{p+1}{2}}(t) + C \int_{\Omega_2} g(v_t) v_t \, dx.
\] (3.4)

Let \( N \) a positive constant, let us introduce the Lyapunov functional

\[
\mathcal{F}(t) := NE(t) + E^{\frac{p+1}{2}}(t) R(t).
\]

It easy to verify that, for \( N \) large, we have

\[
\frac{N}{2} E(t) \leq \mathcal{F}(t) \leq 2NE(t), \quad \forall t \geq 0.
\] (3.5)

In these conditions, Lemma 2.2 and inequality (3.4) imply that, for \( N \) large

\[
\frac{d}{dt} \mathcal{F}(t) \leq -\frac{1}{4} E^{\frac{p+1}{2}}(t),
\]

from where follows, in view of (3.5), that

\[
\frac{d}{dt} \mathcal{F}(t) \leq -k_1 \mathcal{F}^{\frac{p+1}{2}}(t),
\]

for some \( k_1 > 0 \). Hence, we get

\[
\mathcal{F}(t) \leq M(1 + t)^{-2/(p-1)}.
\]

From equivalence relation (3.5) our conclusion follows. This complete the proof. \( \square \)

References


