Viscoelastic boundary stabilization for a transmission problem in elasticity *

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Abstract

We consider an anisotropic body which is constituted by two different types of materials supporting a memory boundary condition and we show that its energy decays uniformly as time goes to infinity with the same rate as the relaxation function $g$, that is, the energy decays exponentially when $g$ decays exponentially, and polynomially when $g$ decays polynomially.

Key words: Stability, transmission problem, viscoelasticity.

1 Introduction

In this paper, we study the asymptotic behavior of the solutions of a transmission problem in elasticity supporting a boundary condition of memory type. Let $\Omega$ be bounded open set in $\mathbb{R}^n$ with boundary $\partial \Omega = \Gamma_1 \cup \Gamma_2$ where $\Gamma_1$, $\Gamma_2$ are two smooth surfaces such that $\bar{\Gamma}_1 \cap \bar{\Gamma}_2 = \emptyset$. Let us consider an anisotropic body which has as reference configuration the region $\Omega$. The body is made of two types of materials, this means that the material in a part $\Omega_1$ of $\Omega$ is different of its complementary part $\Omega_2 := \Omega \setminus \bar{\Omega}_1$, consequently, due to this configuration, the elastic properties of the body can experience a discontinuity on the common surface $\Gamma_0$ of $\Omega_1$ and $\Omega_2$. If we denote by $u(x,t)$ and $v(x,t)$ the displacements vectors in $\Omega_1$ and $\Omega_2$ at time $t$, the system that models this problem is given by the equations

$$\begin{align*}
\rho_1 u''_{ij} - \{C_{ijkl}u_{k,l}\}_j &= 0 \quad \text{in} \quad \Omega_1 \times \mathbb{R}^+, \\
\rho_2 v''_{ij} - \{G_{ijkl}v_{k,l}\}_j &= 0 \quad \text{in} \quad \Omega_2 \times \mathbb{R}^+, 
\end{align*}$$

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satisfying the boundary conditions

\[ u = 0 \quad \text{on} \quad \Gamma_1 \times \mathbb{R}^+, \]

\[ u = v, \quad C_{ijkl}u_{k,l}v_{j} = G_{ijkl}v_{k,l}u_{j} \quad \text{on} \quad \Gamma_0 \times \mathbb{R}^+, \]

\[ v_i + \int_0^t g(t-s)G_{ijkl}v_{k,l}(s)v_{j} \, ds = 0 \quad \text{on} \quad \Gamma_2 \times \mathbb{R}^+, \]

and initial data

\[ u(x,0) = u^0(x), \quad u_t(x,0) = u^1(x) \quad \text{in} \quad \Omega_1, \]

\[ v(x,0) = v^0(x), \quad v_t(x,0) = v^1(x) \quad \text{in} \quad \Omega_2. \]

Here, the mass densities \( \rho_i \) are positive constants, the relaxation \( g \) is a non negative decreasing function and the elasticity tensors \( C_{ijkl}, G_{ijkl} \) are symmetric, that is

\[ C_{ijkl} = C_{klij} = C_{jikl}, \quad G_{ijkl} = G_{klij} = G_{jikl}. \]

Moreover, we have the following ellipticity conditions

\[ C_{ijkl}\epsilon_{kl}\epsilon_{ij} \geq C_0\epsilon_{ij}\epsilon_{ij}, \quad G_{ijkl}\epsilon_{kl}\epsilon_{ij} \geq G_0\epsilon_{ij}\epsilon_{ij}, \]

for any symmetric tensors \( \epsilon_{ij} \) and some positive constants \( C_0 \) and \( G_0 \).

The set \( \Omega \)

Controllability for transmission problems were studied by several authors, and we mention a few works. The transmission problem for the wave equation was studied by Lions [8], where he applied the Hilbert Uniqueness Method (HUM) to show exact controllability. Later, Lagnese [7], also applying HUM, extended this result, he showed the exact controllability for a class of hyperbolic systems which include the transmission problem for homogeneous anisotropic materials. The exact controllability for the plate equation was proved by Liu and Williams [10] and Aassila [1]. Some results about existence, uniqueness and regularity for elliptic stationary transmission problem can be found in \([2, 6]\).

Concerning stability, Liu and Williams, in [9], studied a transmission problem for the wave equation and showed exponential decay of the energy provided a linear feedback velocity is applied at the boundary.

For the memory condition on the boundary we can cite the following works: In [3], Ciarletta established theorems of existence, uniqueness and asymptotic stability for a
linear model of heat conduction. In this case the memory condition describes a boundary that can absorb heat and due to the hereditary term, can retain part of it. In [4] Fabrizio and Morro considered a linear electromagnetic model and proved the existence, uniqueness and asymptotic stability of the solutions. In [12] Muñoz Rivera and Andrade showed exponential stability for a non homogeneous anisotropic system when the resolvent kernel of the memory is of exponential type. They used multiplier techniques and a compactness argument. A polynomial resolvent kernel was not considered in that work.

Nonlinear one-dimensional wave equation with memory condition on the boundary was studied by Qin [13], he showed existence, uniqueness and stability of global solutions provided the initial data was small in $H^3 \times H^2$, this result was improved by Muñoz and Andrade [11], they only supposed small initial data in $H^2 \times H^1$.

The aim of this paper is to study the asymptotic behavior of solutions of system (1.1)-(1.4). We shall show that the solution decays exponentially to zero provided the relaxation function $g$ decays exponentially to zero. Moreover, if $g$ decays polynomially, then we shall show that the corresponding solution also decays to zero with the same rate of decay.

The remainder part of this paper is organized as follows. In the next section we establish the existence, uniqueness and regularity of solutions for the transmission problem. In section 3 we show the exponential decay of the first order energy and, in section 4 the polynomial decay. Finally, in section 5 we comment on generalizations of this problem.

## 2 Existence of Solutions

In this section the existence and uniqueness of strong solutions for system (1.1)-(1.4) will be established. First, we can rewrite condition (1.3) in an another way e differentiate this condition to obtain the following Volterra equation

$$G_{ijkl}v_{k,l}v_j + \frac{1}{g(0)} g' * G_{ijkl}v_{k,l}v_j = -\frac{1}{g(0)} v_i',$$

where we are denoting by $*$ the convolution product $(g * h)(t) := \int_0^t g(t - s)h(s)\, ds$. Applying the inverse Volterra operator, we obtain

$$G_{ijkl}v_{k,l}v_j = -\frac{1}{g(0)} \{v_i' + k * v_i'\},$$

where $k$ is the resolvent kernel of $-g'/g(0)$, that is, $k$ satisfies

$$k + \frac{1}{g(0)} g' * k = -\frac{1}{g(0)} g'.$$
Therefore the condition \((1.3)_3\) implies that
\[
G_{ijkl}v_{k,l}v_j = -\frac{1}{g(0)} \left\{ v'_i + k(0)v_i + k'v_i - k(t)v_i^0 \right\}.
\]
(2.1)

Note that we can retrieve condition \((1.3)_3\) provided the initial data \(v^0\) vanishes on \(\Gamma_2\). To see this, we integrate (2.1) to obtain
\[
v_i + \int_0^t g(t-s)G_{ijkl}v_{k,l}(s)v_j ds = v_i^0 = 0 \quad \text{on} \quad \Gamma_2.
\]

Hence, we can use condition (2.1) instead of \((1.3)_3\). Since we are interested in relaxation functions of exponential or polynomial type and the condition (2.1) involves the resolvent kernel \(k\), we need to know if \(k\) has the same properties. The following Lemma answers this question. Let \(h\) be a relaxation function and \(k\) the resolvent kernel, that is
\[
k(t) - k * h(t) = h(t).
\]
(2.2)

**Lemma 2.1** If \(h\) is a positive continuous function, then \(k\) is also a positive continuous function. Moreover,

1. If there exist positive constants \(c_0\) and \(\gamma\) with \(c_0 < \gamma\) such that
   \[
   h(t) \leq c_0 e^{-\gamma t},
   \]
   then, the function \(k\) satisfies
   \[
k(t) \leq \frac{c_0(\gamma - \epsilon)}{\gamma - \epsilon - c_0} e^{-\epsilon t},
   \]
   for all \(0 < \epsilon < \gamma - c_0\).

2. Given \(p > 1\), let us denote by \(c_p := \sup_{t \in \mathbb{R}^+} \int_0^t (1+t)^p (1+t-s)^{-p} (1+s)^{-p} \, ds\). If there exists a positive constant \(c_0\) with \(c_0 c_p < 1\) such that
   \[
h(t) \leq c_0 (1+t)^{-p},
   \]
   then, the function \(k\) satisfies
   \[
k(t) \leq \frac{c_0}{1 - c_0 c_p} (1+t)^{-p}.
   \]

**Proof:** Note that \(k(0) = h(0) > 0\). Now, we take \(t_0 = \inf\{t \in \mathbb{R}^+ : k(t) = 0\}\), so \(k(t) > 0\) for all \(t \in [0,t_0]\). If \(t_0 \in \mathbb{R}^+\), from equation (2.2) we get that \(-k * h(t_0) = h(t_0)\) but this is a contradiction. Therefore \(k(t) > 0\) for all \(t \in \mathbb{R}_0^+\). Now, let us fix \(\epsilon\), such that \(0 < \epsilon < \gamma - c_0\) and denote by
\[
k_\epsilon(t) := e^{\epsilon t}k(t), \quad h_\epsilon(t) := e^{\epsilon t}h(t).
\]
Multiplying equation (2.2) by $e^{\epsilon t}$ we get $k_\epsilon(t) = h_\epsilon(t) + k_\epsilon \ast h_\epsilon(t)$, hence
\[
\sup_{s \in [0,t]} k_\epsilon(s) \leq \sup_{s \in [0,t]} h_\epsilon(s) + \left( \int_0^\infty c_0 e^{(\epsilon - \gamma)s} ds \right) \sup_{s \in [0,t]} k_\epsilon(s) \leq c_0 + \frac{c_0}{(\gamma - \epsilon)} \sup_{s \in [0,t]} k_\epsilon(s).
\]
Therefore
\[
k_\epsilon(t) \leq \frac{c_0(\gamma - \epsilon)}{\gamma - \epsilon - c_0},
\]
which implies our first assertion. To show the second part let us introduce the following notations
\[
k_p(t) := (1 + t)^p k(t), \quad h_p(t) := (1 + t)^p h(t).
\]
Multiplying equation (2.2) by $(1 + t)^p$ we get $k_p(t) = h_p(t) + \int_0^t k_p(t - s)(1 + t - s)^{-p}(1 + t)^p h(s) \, ds$, hence
\[
\sup_{s \in [0,t]} k_p(s) \leq \sup_{s \in [0,t]} h_p(s) + c_0 c_p \sup_{s \in [0,t]} k_p(s) \leq c_0 + c_0 c_p \sup_{s \in [0,t]} k_p(s).
\]
Therefore
\[
k_p(t) \leq \frac{c_0}{1 - c_0 c_p},
\]
which proves our second assertion. \(\square\)

**Remark:** The finiteness of the constant $c_p$ can be found in [14, Lemma 7.4].

In the remainder of this paper we shall use condition (2.1) instead of (1.3)$_3$. The well-posedness of the system (1.1)-(1.4) may be established by using the standard Galerkin method. We summarize this result in the following Theorem:

**Theorem 2.2** Let us suppose that $k$, $-k'$ and $k''$ are positive continuous functions. Let $(u^0, v^0)$ be in $[H^1(\Omega_1)]^n \times [H^1(\Omega_2)]^n$ and $(u^1, v^1)$ in $[L^2(\Omega_1)]^n \times [L^2(\Omega_2)]^n$ satisfying
\[
u^0 = 0 \quad \text{on} \quad \Gamma_1, \quad u^0 = v^0 \quad \text{on} \quad \Gamma_0.
\] (2.3)

Then there exists a unique weak solution of (1.1) - (1.4) satisfying
\[
(u, v) \in \bigcap_{\mu = 0}^1 W^{\mu, \infty}(\mathbb{R}_0^+; [H^{1-\mu}(\Omega_1)]^n \times [H^{1-\mu}(\Omega_2)]^n).
\]
Moreover, if $(u^0, v^0) \in [H^2(\Omega_1)]^n \times [H^2(\Omega_2)]^n$, $(u^1, v^1) \in [H^1(\Omega_1)]^n \times [H^1(\Omega_2)]^n$ and satisfy
\[
C_{ijkl} u^0_{k,i} v^0_j = G_{ijkl} u^0_{k,i} v^0_j, \quad u^1 = v^1 \quad \text{on} \quad \Gamma_0,
\]
\[
G_{ijkl} u^0_{k,i} v^0_j = -\frac{v^1_i}{g(0)} \quad \text{on} \quad \Gamma_2,
\] (2.4)
then, the solution has the following regularity property
\[
(u, v) \in \bigcap_{\mu = 0}^2 W^{\mu, \infty}(\mathbb{R}_0^+; [H^{2-\mu}(\Omega_1)]^n \times [H^{2-\mu}(\Omega_2)]^n).
\]
In this case the solution $(u, v)$ is called strong solution.
3 Exponential decay of the energy

In this section we shall prove that the solution of system (1.1)-(1.4) decays exponentially to zero provided the resolvent kernel $k$ also decays exponentially to zero. In the remainder of this paper we denote by $C$ a positive constant which takes different values in different places. We shall assume that $\Omega_1$ and $\Omega$ are star-shaped, that is to say, there exist $x_0 \in \mathbb{R}^n$ such that

$$r(x) \cdot \nu(x) \leq 0 \text{ on } \Gamma_1, \quad r(x) \cdot \nu(x) \geq 0 \text{ on } \Gamma_0, \quad r(x) \cdot \nu(x) > 0 \text{ on } \Gamma_2, \quad (3.1)$$

where $r(x) := x - x_0$. Also, we will assume that the material type in $\Omega_2$ is more dense and stiff than that in $\Omega_1$, that is

$$\rho_1 \leq \rho_2 \quad \text{and} \quad C_{ijkl}\epsilon_{kl}\epsilon_{ij} \geq G_{ijkl}\epsilon_{kl}\epsilon_{ij}, \quad (3.2)$$

for any symmetric tensor $\epsilon_{ij}$. These assumptions, introduced by Lions in [8], will allow us to eliminate derivative terms of high order on the interface $\Gamma_0 \text{ (see Lemma 3.5).}$

The first order energy associated with system (1.1)-(1.3) is given by

$$E(t) = E(t, u, v) := \frac{1}{2} \int_{\Omega_1} (\rho_1|u_t|^2 + C_{ijkl}u_{k,l}u_{i,j}) \, dx + \frac{1}{2} \int_{\Omega_2} (\rho_1|v_t|^2 + G_{ijkl}u_{k,l}v_{i,j}) \, dx + \frac{1}{2g(0)} \int_{\Gamma_2} (-k'\Box v + k(t)|v|^2) \, d\Gamma.$$

Here, we are denoting by $(k\Box h)(t) := \int_0^t k(t-s)|h(t) - h(s)|^2 \, ds$. The exponential decay of the first order energy is established by the following Theorem

**Theorem 3.1** Let us assume that the resolvent kernel $k \in C^2(\mathbb{R})$ is a non-negative function such that

$$k(0) > 0, \quad k'(t) \leq -c_1 k(t), \quad k''(t) \geq c_2 [-k'(t)], \quad (3.3)$$

where $c_1$ and $c_2$ are positive constants. Let us take $(u^0, v^0) \in [H^1(\Omega_1)]^n \times [H^1(\Omega_2)]^n$ and $(u^1, v^1) \in [L^2(\Omega_1)]^n \times [L^2(\Omega_2)]^n$ satisfying the compatibility conditions (2.3). If inequalities (3.1)-(3.2) hold, then there exist positive constants $C = C(E(0))$ and $\gamma$ such that

$$E(t) \leq Ce^{-\gamma t}.$$

We shall prove this Theorem for strong solutions, that is, for solutions with initial data $(u^0, v^0) \in [H^2(\Omega_1)]^n \times [H^2(\Omega_2)]^n$ and $(u^1, v^1) \in [H^1(\Omega_1)]^n \times [H^1(\Omega_2)]^n$ satisfying the compatibility condition (2.4), then our conclusion follows by a density argument. This allows us to use the multiplier method to construct a suitable Lyapunov functional equivalent with the energy which must satisfy the differential inequality of the following Gronwall type lemma.
Lemma 3.2  Let $E$ be a real positive function of class $C^1$. If there exists positive constants $\gamma_0, \gamma_1$ and $c_0$ such that

$$E'(t) \leq -\gamma_0 E(t) + c_0 e^{-\gamma_1 t},$$

then there exist positive constants $\gamma$ and $c$ such that

$$E(t) \leq (E(0) + c)e^{-\gamma t}.$$

Proof: First, let us suppose that $\gamma_0 < \gamma_1$. Define $I(t)$ by

$$I(t) := E(t) + \frac{c_0}{\gamma_1 - \gamma_0} e^{-\gamma_1 t}.$$

Then

$$I'(t) = E'(t) - \frac{\gamma_1 c_0}{\gamma_1 - \gamma_0} e^{-\gamma_1 t} \leq -\gamma_0 I(t).$$

Integrating from 0 to $t$ we find

$$I(t) \leq I(0)e^{-\gamma_0 t} \quad \Rightarrow \quad E(t) \leq \left( E(0) + \frac{c_0}{\gamma_1 - \gamma_0} \right) e^{-\gamma_0 t}.$$

Now, we shall assume that $\gamma_0 \geq \gamma_1$. In this conditions we get

$$E'(t) \leq -\gamma_1 E(t) + c_0 e^{-\gamma_1 t} \quad \Rightarrow \quad \left[ e^\gamma E(t) \right]' \leq c_0.$$

Integrating from 0 to $t$ we obtain

$$E(t) \leq (E(0) + c_0t) e^{-\gamma t}.$$

Since $t \leq (\gamma_1 - \epsilon)e^{(\gamma_1 - \epsilon)t}$ for any $0 < \epsilon < \gamma_1$ we conclude that

$$E(t) \leq [E(0) + c_0(\gamma_1 - \epsilon)] e^{-\epsilon t}.$$

This completes the proof. \qed

Let us denote by $\diamond$ the operator given by $(g \diamond h)(t) := \int_0^t g(t-s)(h(t)-h(s))\,ds$. Some relations between the operators $\ast$, $\Box$ and $\diamond$ are given by the following Lemma.

Lemma 3.3  For any two functions $g, h \in C^1(\mathbb{R})$ and $\theta \in [0, 1]$, the following inequalities hold

$$2[g \ast h]_t' = g' \Box h - g(t)\left| h \right|^2 - \frac{d}{dt}\left\{ g \Box h - \left( \int_0^t g\,ds \right) \left| h \right|^2 \right\},$$

$$\left| (g \diamond h)(t) \right|^2 \leq \left[ \int_0^t |g(s)|^{2(1-\theta)}\,ds \right] |g|^{2\theta} \Box h.$$
Proof: Differentiating the expression
\[ g \Box h - \left( \int_0^t g \, ds \right) |h|^2, \]
the first part of our conclusion follows. The second part is a consequence of Hölder’s inequality. \qed

The first order energy of the transmission problem has the following property

Lemma 3.4 The following inequality holds
\[ \frac{d}{dt} E(t) \leq -\frac{1}{2g(0)} \int_{\Gamma_2} \left( |v_1|^2 + k'' \Box v - k'(t) |v|^2 - |k(t)v^0|^2 \right) d\Gamma. \]

Proof: Multiplying equation (1.1) by \( u_t \), equation (1.2) by \( v_t \), integrating by parts and using Lemma 3.3 we obtain
\[ \frac{d}{dt} E(t) = -\frac{1}{2g(0)} \int_{\Gamma_2} \left( 2|v_1|^2 + k'' \Box v - k'(t) |v|^2 - 2k(t)v^0v_t \right) d\Gamma. \]

Applying Young’s inequality our conclusion follows. \qed

Let us denote by \( Kw := r_m \frac{\partial}{\partial x_m} + \frac{(n-1)}{2} w \) where \( r_m \) is the \( m \)-th component of the vector \( r(x) \) and let us consider the following functionals
\[
J(t) := \int_{\Omega_1} \rho_1 u_i' K u_i \, dx + \int_{\Omega_2} \rho_2 v_i' K v_i \, dx,
\]
\[
N(t) := \int_{\Omega_1} \left( \rho_1 u_i' u_i' + C_{ijkl} u_k \cdot u_i j \right) \, dx + \int_{\Omega_2} \left( \rho_2 v_i' v_i' + G_{ijkl} v_k \cdot v_i j \right) \, dx.
\]

Lemma 3.5 If inequalities (3.1)-(3.2) hold, then there exists a constant \( C_1 > 0 \) such that
\[ \frac{d}{dt} J(t) \leq C_1 \int_{\Gamma_2} \left( |v'|^2 + |k(t)v|^2 + |k' \circ v|^2 + |k(t)v^0|^2 \right) d\Gamma - \frac{1}{4} N(t). \]

Proof: Performing an integration by parts and using equation (1.1) and boundary conditions (1.3) we obtain
\[
\frac{d}{dt} \int_{\Omega_1} \rho_1 u_i' K u_i \, dx = \int_{\Omega_1} \rho_1 u_i' (K u_i)' \, dx + \int_{\Omega_1} \rho_1 u_i'' K u_i \, dx
\]
\[ = \frac{1}{2} \int_{\Gamma_0} \rho_1 (r \cdot \nu) u_i' u_i' \, d\Gamma - \frac{1}{2} \int_{\Omega_1} \rho_1 u_i'' u_i' \, dx
\]
\[ + \int_{\Gamma_1} C_{ijkl} u_k \cdot u_i j K u_i \, d\Gamma - \frac{1}{2} \int_{\Gamma_0} (r \cdot \nu) C_{ijkl} u_k \cdot u_i j \, d\Gamma
\]
\[ + \frac{1}{2} \int_{\Gamma_1} (r \cdot \nu) C_{ijkl} u_k \cdot u_i j \, d\Gamma - \frac{1}{2} \int_{\Omega_1} C_{ijkl} u_k \cdot u_i j \, dx. \quad (3.4) \]
Similarly, using equation (1.2) instead of (1.1) we get
\[
\frac{d}{dt} \int_{\Omega_2} \rho_2 v'_i K v_i \, dx = \frac{1}{2} \int_{\Gamma_2} \rho_2 (r \cdot \nu) v'_i v'_i \, d\Gamma - \frac{1}{2} \int_{\Gamma_0} \rho_2 (r \cdot \nu) v'_i v'_i \, d\Gamma - \frac{1}{2} \int_{\Gamma_2} \rho_2 v'_i v'_i \, dx + \int_{\Gamma_2} G_{ijkl} v_{k,l} v_{i,j} K v_i \, d\Gamma - \frac{1}{2} \int_{\Gamma_0} (r \cdot \nu) G_{ijkl} v_{k,l} v_{i,j} \, d\Gamma - \frac{1}{2} \int_{\Omega_2} (r \cdot \nu) G_{ijkl} v_{k,l} v_{i,j} \, dx. \tag{3.5}
\]

Summing (3.4) and (3.5) and using boundary conditions (1.3) again we arrive at
\[
\frac{d}{dt} J(t) = \frac{1}{2} \int_{\Gamma_2} r \cdot \nu (\rho_2 v'_i v'_i - G_{ijkl} v_{k,l} v_{i,j}) \, d\Gamma + \int_{\Gamma_2} G_{ijkl} v_{k,l} v_{i,j} K v_i \, d\Gamma
+ \frac{1}{2} \int_{\Gamma_0} (r \cdot \nu (\rho_1 - \rho_2) u_i' u_i') \, d\Gamma + \int_{\Gamma_0} G_{ijkl} v_{k,l} v_{i,j} \tau_m (u_{i,m} - v_{i,m}) \, d\Gamma
- \frac{1}{2} \int_{\Omega_2} (r \cdot \nu (C_{ijkl} u_{k,l} u_{i,j} - G_{ijkl} v_{k,l} v_{i,j})) \, dx - \frac{1}{2} \int_{\Omega_2} (\rho_2 v'_i v'_i + G_{ijkl} v_{k,l} v_{i,j}) \, dx. \tag{3.6}
\]

From hypothesis (2.1) we get
\[
l = - \int_{\Gamma_2} \frac{1}{g(0)} (v'_i + k(t) v_i - k' \circ v_i - k(t) v^0_i) \left( r_m v_{i,m} + \frac{(n-1)}{2} v_i \right) \, d\Gamma.
\]

Applying Young’s inequality we obtain
\[
I_1 \leq \frac{G_0 \delta_0}{2} \int_{\Gamma_2} |\nabla v|^2 \, d\Gamma + \frac{G_0}{4C} \int_{\Gamma_2} |v|^2 \, d\Gamma + C \int_{\Gamma_2} \left( |v' + k(t) v - k' \circ v - k(t) v^0|^2 \right) \, d\Gamma,
\]

where \( G_0 \) is given by (1.5), and \( \delta_0 \) and \( C_\sigma \) are positive constants such that
\[
\inf_{x \in \Gamma_2} r(x) \cdot \nu(x) \geq \delta_0, \quad \int_{\Gamma_2} |v|^2 \, d\Gamma \leq C_\tau \left\{ \int_{\Omega_1} |
abla u|^2 \, dx + \int_{\Omega_2} |
abla v|^2 \, dx \right\}.
\]

Using these inequalities and hypothesis (1.5) we arrive at
\[
I_1 \leq \frac{1}{2} \int_{\Gamma_2} r \cdot \nu G_{ijkl} v_{k,l} v_{i,j} \, d\Gamma + \frac{1}{4} \int_{\Omega_1} C_{ijkl} u_{k,l} u_{i,j} \, dx + \frac{1}{4} \int_{\Omega_2} G_{ijkl} v_{k,l} v_{i,j} \, dx
+ C \int_{\Gamma_2} \left( |v'|^2 + |k(t) v|^2 + |k' \circ v|^2 + |k(t) v^0|^2 \right) \, d\Gamma. \tag{3.7}
\]

Since \( u_i - v_i = 0 \) on \( \Gamma_0 \) we have that \( \nu_j (u_i - v_i)_m = \nu_m (u_i - v_i)_j \) on \( \Gamma_0 \), therefore
\[
I_2 = \int_{\Gamma_0} r \cdot \nu (G_{ijkl} v_{k,l} u_{i,j} - G_{ijkl} v_{k,l} v_{i,j}) \, d\Gamma
= \int_{\Gamma_0} r \cdot \nu (G_{ijkl} v_{k,l} u_{i,j} - G_{ijkl} v_{k,l} v_{i,j}) \, d\Gamma.
\]

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Using the inequality

\[ 2G_{ijkl}v^k_{i,j} \leq G_{ijkl}u^k_{i,j} + G_{ijkl}v^k_{i,j} \]

and hypothesis (3.1)-(3.2) we get

\[ I_2 \leq \frac{1}{2} \int_{\Gamma_0} r \cdot \nu (G_{ijkl}u^k_{i,j} - G_{ijkl}v^k_{i,j}) \, d\Gamma \]

\[ \leq \frac{1}{2} \int_{\Gamma_0} r \cdot \nu (C_{ijkl}u^k_{i,j} - G_{ijkl}v^k_{i,j}) \, d\Gamma. \]  \hfill (3.8)

Substitution of the inequalities (3.7)-(3.8) into (3.6) gives

\[ \frac{d}{dt}J(t) \leq \frac{1}{2} \int_{\Gamma_2} r \cdot \nu (|v'|^2 + |k(t)v|^2 + |k' \circ v|^2 + |k(t)v^0|^2) \, d\Gamma \]

\[ - \frac{1}{4} \int_{\Omega_1} (\rho_1 u'_i u'_i + C_{ijkl}u^k_{i,j}) \, dx - \frac{1}{4} \int_{\Omega_2} (\rho_2 v'_i v'_i + G_{ijkl}v^k_{i,j}) \, dx. \]

Hence, our conclusion follows. \[ \square \]

**Proof of Theorem 3.1:** Using hypothesis (3.3) in Lemma 3.4 we get

\[ \frac{d}{dt}E(t) \leq -\frac{1}{2G(0)} \int_{\Gamma_2} (|v|^2 - c_2 k' \triangle v + c_1 k(t) |v|^2 - |k(t)v^0|^2) \, d\Gamma. \]

Since $k$ is bounded the second inequality of Lemma 3.3 implies that

\[ |k' \circ v|^2 \leq C [-k'] \triangle v, \quad |k(t)v|^2 \leq C k(t)|v|^2. \]

Substitution of these terms in Lemma 3.5 yields

\[ \frac{d}{dt}J(t) \leq C \int_{\Gamma_2} (|v'|^2 + k(t)|v|^2 - k' \triangle v + |k(t)v^0|^2) \, d\Gamma - \frac{1}{4} N(t). \]

Now, let us denote by $F$ the following functional

\[ F(t) := NE(t) + J(t). \]

Taking $N$ large, the above inequality implies that

\[ \frac{d}{dt}F(t) \leq -\frac{1}{8} E(t) + 2N \int_{\Gamma_2} |k(t)v^0|^2 \, d\Gamma. \]  \hfill (3.9)

It is easy to verify that for $N$ large enough, we also have

\[ \frac{N}{2} E(t) \leq F(t) \leq 2NE(t). \]  \hfill (3.10)

Combining inequalities (3.9) and (3.10) we conclude that

\[ \frac{d}{dt}F(t) \leq -\frac{1}{8N} F(t) + 2N \int_{\Gamma_2} |k(t)v^0|^2 \, d\Gamma. \]

Since $k(t) \leq k(0)e^{-ct}$, Lemma 3.2 implies that $F$ decays exponentially. In view of (3.10) the energy also decays exponentially. This completes the proof. \[ \square \]
4  Polynomial decay of the energy

Here our attention will be focused on the uniform rate of decay when the resolvent kernel $k$ decays polynomially as $(1 + t)^{-p}$. In this case we will show that the solution also decay polynomially with the same rate. We summarize the main result of this section in the following Theorem

**Theorem 4.1** Let us assume that the resolvent kernel $k \in C^2(\mathbb{R})$ is a non-negative function such that

\begin{align}
    k(0) > 0, \quad k'(t) &\leq -c_1 k(t)^{1 + \frac{1}{p}}, \quad k''(t) \geq c_2 \left[-k'(t)\right]^{1 + \frac{1}{p}}, \\
    \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad 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where \( a_0 = \min\left\{k_0, \left(\frac{q}{2}\right)^{1+\frac{1}{p}}k_1\right\} \). It follows that

\[
\mathcal{I}(t) \leq \left(\mathcal{I}(0)^{-\frac{1}{p}} + \frac{a_0}{21/p^p t}\right)^{-p}.
\]

which yields the required inequality. \( \square \)

**Lemma 4.3** Suppose that \( g \in C([0, \infty]), h \in L^1_{loc}(0, \infty) \) and \( 0 \leq \theta \leq 1 \), then we have that

\[
\int_0^t |g(\tau)h(\tau)| \, d\tau \leq \left\{\int_0^t |g(\tau)|^{1-\theta}|h(\tau)| \, d\tau\right\}^{\frac{1}{q+1}} \left\{\int_0^t |g(\tau)|^{1-\theta} \frac{q}{q} \, h(\tau) \, d\tau\right\}^{\frac{q}{q+1}}.
\]

**Proof:** For any fixed \( t \) we have

\[
\int_0^t |g(\tau)h(\tau)| \, d\tau = \int_0^t \left\{\int_0^\tau \left[ g(\tau) \left| \frac{1}{q+1} \right| h(\tau) \right] \left[ g(\tau) \left| \frac{1}{q+1} \right| h(\tau) \right] \, d\tau \right\} \, d\tau.
\]

Note that \( w_1 \in L^p_{loc}(0, \infty), w_2 \in L^{p'}_{loc}(0, \infty) \), where \( p = q + 1 \) and \( p' = \frac{q+1}{q} \). Using Hölder’s inequality, we get

\[
\int_0^t |g(\tau)h(\tau)| \, d\tau \leq \left\{\int_0^t |g(\tau)|^{1-\theta}|h(\tau)| \, d\tau\right\}^{\frac{1}{q+1}} \left\{\int_0^t |g(\tau)|^{1-\theta} \frac{q}{q} \, h(\tau) \, d\tau\right\}^{\frac{q}{q+1}}.
\]

This completes the proof. \( \square \)

**Lemma 4.4** Let us suppose that \( v \in L^\infty(0, T; L^2(\Gamma_2)) \) and \( g \) is a continuous function. Then

\[
\int_{\Gamma_2} g \Box v \, d\Gamma \leq \sqrt{2} \left\{\int_0^t \|v(\tau)\|^2_{L^2(\Gamma_2)} \, d\tau + t\|v(\tau)\|^2_{L^2(\Gamma_2)} \right\}^{\frac{1}{p+1}} \left\{\int_{\Gamma_2} g^{1+\frac{1}{p}} \Box v \, d\Gamma \right\}^{\frac{p}{p+1}}.
\]

Moreover, If there exists \( 0 < \theta < 1 \) such that \( \int_0^\infty g^{1-\theta}(s) \, ds < \infty \), then we have

\[
\int_{\Gamma_2} g \Box v \, d\Gamma \leq \frac{4}{\theta} \left\{\int_0^\infty g^{1-\theta} \, d\tau \right\} \left\{\int_{\Gamma_2} g^{1+\frac{1}{p}} \Box v \, d\Gamma \right\}^{\frac{p}{p+1}}.
\]
Proof: From the hypothesis on \(v\) and Lemma 4.3 we get

\[
\int_{\Gamma} g \Box v \, d\Gamma = \int_{\Gamma} \int_{0}^{t} g(t - \tau) \frac{1}{\sqrt{\tau}} h(\tau) \, d\tau \, d\Gamma
\]

\[
\leq \left\{ \int_{\Gamma} \int_{0}^{t} g^{1-\theta}(t - \tau) h(\tau) \, d\tau \, d\Gamma \right\} \frac{1}{\sqrt{\tau}} \left\{ \int_{\Gamma} \int_{0}^{t} g^{1+\frac{1}{p}}(t - \tau) h(\tau) \, d\tau \, d\Gamma \right\} \frac{\theta}{\theta p + 1}.
\]

Now, for \(0 < \theta < 1\) we have

\[
\int_{\Gamma} g^{1-\theta} \Box v \, d\Gamma = \int_{0}^{t} g^{1-\theta}(t - \tau) \int_{\Gamma} (v(t) - v(\tau))(v(t) - v(\tau)) \, d\tau \, d\Gamma
\]

\[
\leq 4 \left( \int_{0}^{t} g^{1-\theta}(\tau) \, d\tau \right) \|v\|_{L^{\infty}(0, T; L^{2}(\Gamma))}^{2}.
\]

From where the second inequality of this Lemma follows. When \(\theta = 1\) we get

\[
\int_{\Gamma} 1 \Box v \, d\Gamma = \int_{0}^{t} \int_{\Gamma} (v(t) - v(\tau))(v(t) - v(\tau)) \, d\tau \, d\Gamma
\]

\[
\leq 2t \int_{\Gamma} |v(t)|^{2} \, d\Gamma + 2 \int_{0}^{t} \int_{\Gamma} |v(\tau)|^{2} \, d\Gamma \, d\tau.
\]

Substitution of this inequality into (4.2) yields the first inequality. The proof is now complete. \(\square\)

Proof of Theorem 4.1: We use similar estimates as in the previous section which are independent of the behavior of the resolvent kernel \(k\). Using hypothesis (4.1) in Lemma 3.4 yields

\[
\frac{d}{dt} E(t) \leq -\frac{1}{2g(0)} \int_{\Gamma} |v_{t}|^{2} + c_{2}[-k']^{1+\frac{1}{p}} \Box v + c_{1}k(t)^{1+\frac{1}{p}} |v|^{2} - |k(t)v_{0}|^{2} \, d\Gamma. \tag{4.3}
\]

Applying the second inequality of Lemma 3.3 to \(k'\) and \(v\) with \(\theta = \frac{1}{2} \left(1 + \frac{1}{p}\right)\) we get

\[
|k' \diamond v|^{2} \leq \left[ \int_{0}^{t} [-k'(s)]^{1-\frac{1}{p}} \, ds \right] [-k']^{1+\frac{1}{p}} \Box v.
\]

Hypothesis (4.1) implies that \(-k'(t) \leq C(1 + t)^{-p}\). Since \(p > 2\) we conclude that

\[
\int_{0}^{\infty} [-k'(s)]^{1-\frac{1}{p}} \, ds \leq C \int_{0}^{\infty} \frac{1}{(1 + t)^{p-1}} \, ds < \infty.
\]
Therefore $|k' \circ v|^2 \leq C[-k']^{1+\frac{1}{p}} \Box v$. Additionally, since $k$ is bounded we have that $|k(t)v|^2 \leq Ck(t)^{1+1/p}|v|^2$. Substituting these inequalities in Lemma 3.5 we get
\[
\frac{d}{dt} J(t) \leq C \int_{\Gamma_2} |v'|^2 + k(t)^{1+\frac{1}{p}} |v|^2 + |[k']^{1+\frac{1}{p}} \Box v| + |k(t)v_0|^2 d\Gamma - \frac{1}{4} \mathcal{N}(t). \tag{4.4}
\]
Let us denote by $\mathcal{F}$ the following functional
\[
\mathcal{F}(t) := NE(t) + J(t).
\]
Taking $N$ large, from inequalities (4.3)-(4.4) we have that
\[
\frac{d}{dt} \mathcal{F}(t) \leq -\frac{1}{4} \left\{ \int_{\Gamma_2} [-k']^{1+\frac{1}{p}} \Box v + k(t)^{1+\frac{1}{p}} |v|^2 d\Gamma + \mathcal{N}(t) \right\} + 2N \int_{\Gamma_2} |k(t)v_0|^2 d\Gamma. \tag{4.5}
\]
Fix $\theta = 1/2$. Hypothesis (4.1) and $p > 2$ imply that $\int_0^\infty [-k'(t)]^{1-\theta} \, dt < \infty$. From second part of Lemma 4.4 we get
\[
\int_{\Gamma_2} [-k']^{1+\frac{1}{p}} \Box v \, d\Gamma \geq \frac{1}{C} \left[ \int_{\Gamma_2} [-k'] \Box v \, d\Gamma \right]^{\frac{\theta p+1}{\theta p}}.
\]
The resolvent kernel $k$ and the energy $E$ are bounded, therefore there exists $C > 0$ such that
\[
\int_{\Gamma_2} k(t)^{1+\frac{1}{p}} |v|^2 \, d\Gamma \geq \frac{1}{C} \left[ \int_{\Gamma_2} k(t)|v|^2 \, d\Gamma \right]^{\frac{\theta p+1}{\theta p}}, \quad \mathcal{N}(t) \geq \frac{1}{C} \mathcal{N}(t)^{\frac{\theta p+1}{\theta p}}.
\]
Substitution of the these inequalities into (4.5) gives
\[
\frac{d}{dt} \mathcal{F}(t) \leq -\frac{1}{C} E(t)^{\frac{\theta p+1}{\theta p}} + 2N \int_{\Gamma_2} |k(t)v_0|^2 d\Gamma.
\]
From inequality (3.10) there exists a positive constant $C = C(N)$ such that
\[
\frac{d}{dt} \mathcal{F}(t) \leq -\frac{1}{C} \mathcal{F}(t)^{\frac{\theta p+1}{\theta p}} + 2N \int_{\Gamma_2} |k(t)v_0|^2 d\Gamma.
\]
Hypothesis (4.1) imply that $|k(t)|^2 \leq C(1+t)^{-2p} \leq C(1+t)^{-(\theta p+1)}$. Applying Lemma 4.2 we conclude that
\[
\mathcal{F}(t) \leq \frac{C}{(1+t)^{\theta p}} \quad \text{consequently} \quad E(t) \leq \frac{C}{(1+t)^{\theta p}}.
\]
Since $p > 2$ and $\theta = 1/2$, this last inequality implies that
\[
\int_0^\infty \|v(\tau)\|_{L^2(\Gamma_2)}^2 \, d\tau + t\|v(\tau)\|_{L^2(\Gamma_2)}^2 \leq C \left\{ \int_0^\infty E(\tau, u, v) \, d\tau + tE(t, u, v) \right\} < \infty.
\]
Repeating the same reasoning as above and using the first part of Lemma 4.4 we arrive to
\[
E(t) \leq \frac{C}{(1+t)^{\theta p}}.
\]
This completes the proof. \(\square\)
5 Some generalizations

The method used in this article to show uniform stability can be also applied when:

1. The material is nonhomogeneous and anisotropic, that is, $C_{ijkl} = C_{ijkl}(x)$, $G_{ijkl} = G_{ijkl}(x)$, provided

$$r_m(x)C_{ijkl,m}\epsilon_{kl}\epsilon_{ij} \leq \gamma C_{ijkl}\epsilon_{ij}\epsilon_{ij}$$

$$r_m(x)G_{ijkl,m}\epsilon_{kl}\epsilon_{ij} \leq \gamma G_{ijkl}\epsilon_{ij}\epsilon_{ij}$$

for any symmetric tensors $\epsilon_{ij}$ and for some $\gamma \in [-\infty, 2]$ (see [5]).

2. The body is made of $N$ different types of materials, that is, for $\Omega = \bigcup_{m=1}^{N} \Omega_m$, with $\Omega_m = U_m \setminus \bar{U}_{m-1}$ for $m = 1, \ldots, N$, $U_N = \Omega$, where $\bar{U}_m \subset U_{m+1}$ and the subsets $U_m$ are star-shaped with respect to $x_0$. The density $\rho_m$ and elasticity tensor $C_{ijkl}^m$ restricted to $\Omega_m$ must satisfy

$$\rho_m \leq \rho_{m+1} \quad \text{and} \quad C_{ijkl}^m\epsilon_{kl}\epsilon_{ij} \geq C_{ijkl}^{m+1}\epsilon_{kl}\epsilon_{ij},$$

for $m = 1, \ldots, N - 1$ and any symmetric tensor $\epsilon_{ij}$.

3. The condition (1.3)3 is substituted by

$$v_i + \int_0^t g_i(t-s)G_{ijkl}v_{k,l}(s)v_j \, ds = 0 \quad \text{on} \quad \Gamma_2 \times \mathbb{R}^+,$$

provided the resolvent kernels $k_i$ of relaxation functions $g_i$ satisfy (3.3) or (4.1).

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References


