Universidade Federal do Paraná

Setor de Ciências Exatas

Departamento de Matematica Prof. Juan Carlos Vila Bravo

3^{ra} prova de cálculo II

"O Extermínio (dublado)"

Curitiba, Sexta feira 13 de Novembro de 2015

1. **Parametrização:**

   (i) Apresente uma parametrização diferenciavel para a curva $\mathcal{C}$ em $\mathbb{R}^3$, interseção das superficies dadas por:

   $$\mathcal{S}_1 : x^2 + y^2 = 1 \qquad \text{e} \qquad \mathcal{S}_2 : z = y^2 - x^2$$

   (ii) Apresente uma parametrização diferenciavel para a superficie $\mathcal{S}$ em $\mathbb{R}^3$, onde: $\mathcal{S}$ é a superficie do cilindro $x^2 + y^2 = 1$ no primeiro octante limitada pelos planos $z = 0$ e $z = x$.

2. Seja a curva $\mathcal{C}$ obtida como interseção da semiesfera $x^2 + y^2 + z^2 = 4, \;\; y \geq 0$ com o plano $x + z = 2$. Calcule o comprimento de $\mathcal{C}$.

3. Calcule a integral: $\displaystyle\int_{\mathcal{C}} (x^2 + y^2)\, dr$, onde:

   $\mathcal{C}$ é a quarta parte da circunferência, interseção das superfícies $x^2+y^2+z^2 = 4, \;\; y = x$, situada no primeiro octante.

4. Calcule a área da superfície $\mathcal{S}$ parte da esfera $x^2 + y^2 + z^2 = 1$ que está no primeiro octante entre o plano $z = 0$ e o cone $z = \sqrt{x^2 + y^2}$.

5. Calcule a integral de superfície:

   $$\int\int_{\mathcal{S}} z^2 \, d\mathcal{S},$$

   sendo $\mathcal{S}$ a parte do cone $z = \sqrt{x^2 + y^2}$ entre os planos $z = 1$ e $z = 4$

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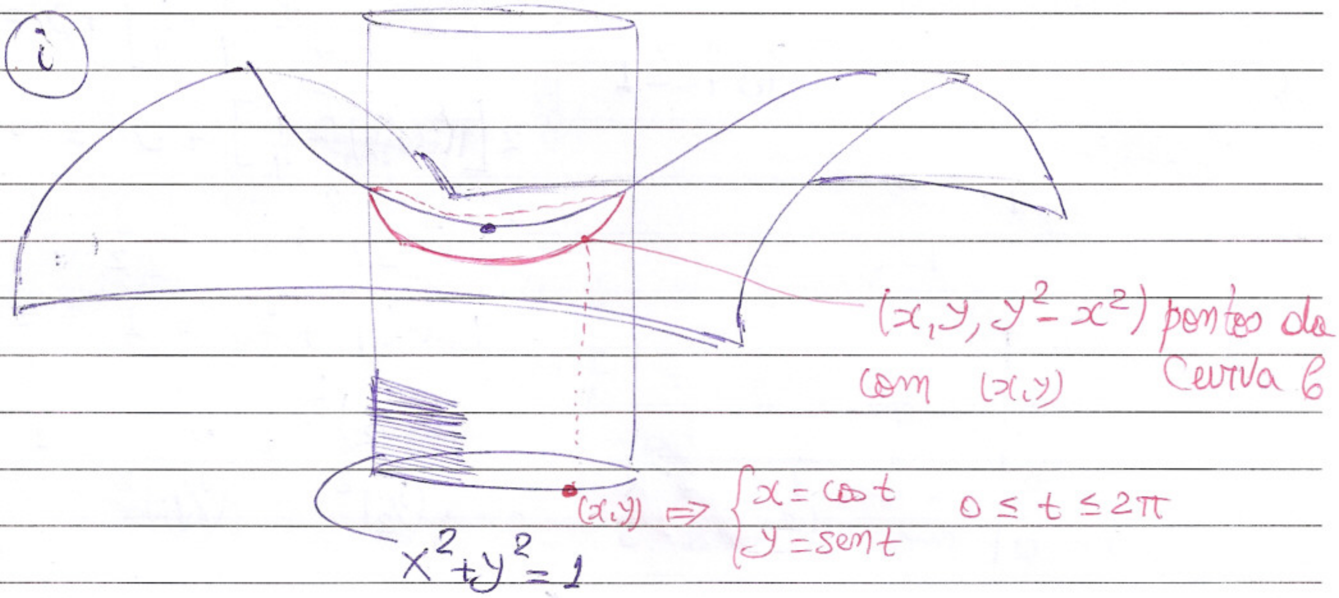
Aluno: ______________

Turma: ______________ Data: ___ / ___ / ___

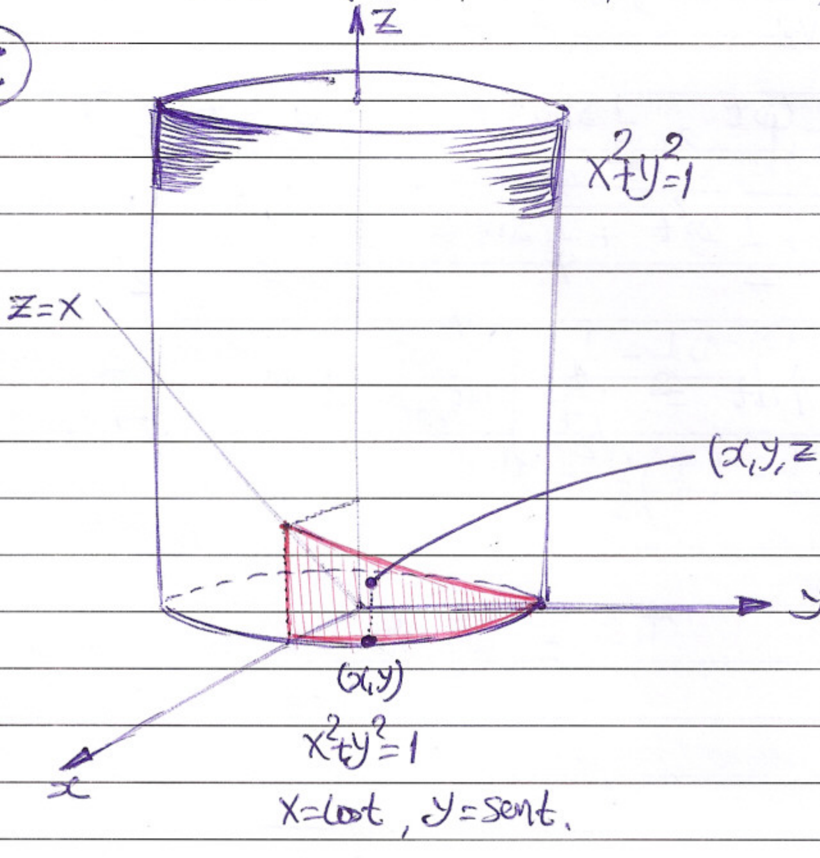
$\boxed{1}$ (i)



$$C : \gamma(t) = (\cos t, \operatorname{sen} t, \operatorname{sen}^2 t - \cos^2 t), \quad 0 \le t \le 2\pi$$

$$\gamma(t) = (\cos t, \operatorname{sen} t, -\cos(2t)), \text{ com } 0 \le t \le 2\pi$$

(ii)

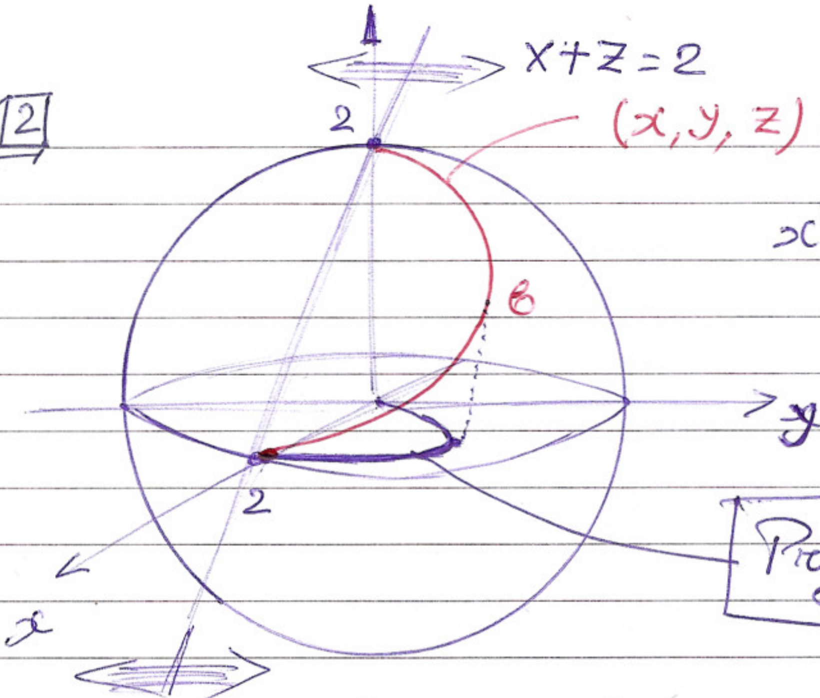


$$(x,y,z) = (\cos t, \operatorname{sen} t, z) = \varphi(t,z)$$

com $\begin{cases} 0 \le t \le \frac{\pi}{2} \\ 0 \le z \le \cos t \end{cases}$

$x = \cos t$, $y = \operatorname{sen} t$.

$\boxed{2}$



$(x, y, z) = (x, y, 2-x)$ na esfera

$x^2 + y^2 + (2-x)^2 = 4$

Projeção de $\mathcal{C}$ no plano XY

$\mathcal{C}: \quad x^2 + y^2 + (2-x)^2 = 4$

$x^2 + y^2 + 4 - 4x + x^2 = 4$

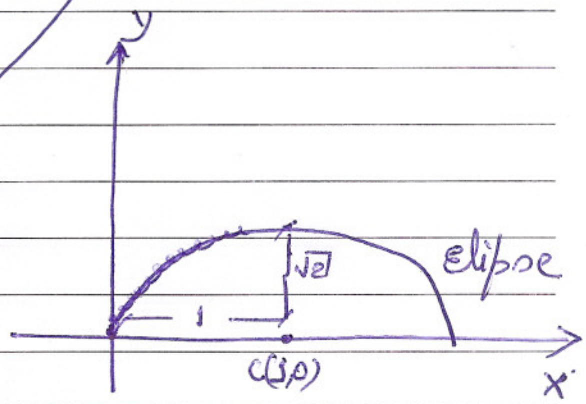
$2x^2 - 4x + y^2 = 0$

$2[x^2 - 2x + 1 - 1] + y^2 = 0$

$2[(x-1)^2 - 1] + y^2 = 0$

$2(x-1)^2 + y^2 = 2$

Elipse: $\boxed{(x-1)^2 + \dfrac{y^2}{2} = 1}$



Portanto $\begin{cases} x = 1 + \cos t \\ y = \sqrt{2}\,\text{sen}\,t \end{cases}$

com $0 \leq t \leq \pi$

Uma parametrização de $\mathcal{C}$: $\gamma(t) = (1 + \cos t, \sqrt{2}\,\text{sen}\,t, 1 - \cos t)$

com $0 \leq t \leq \pi$.

$\gamma'(t) = (-\text{sen}\,t, \sqrt{2}\cos t, \text{sen}\,t) \Rightarrow \|\gamma'(t)\| = \sqrt{2}$

$L(\mathcal{C}) = \displaystyle\int_0^{\pi} \|\gamma'(t)\|\,dt = \int_0^{\pi} \sqrt{2}\,dt = \sqrt{2}\,\pi.$

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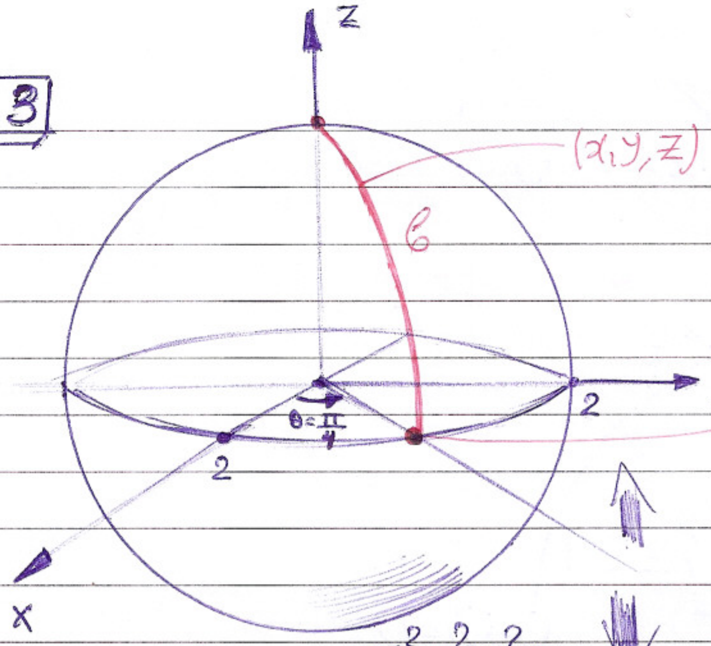


$(x, y, z) = (x, x, \sqrt{4 - 2x^2})$ com $0 \le x \le \sqrt{2}$

$\boxed{z=0} \to x^2 + x^2 = 4$

$2x^2 = 4$

$x^2 = 2$

$x = \pm\sqrt{2}$

$x^2 + y^2 + z^2 = 4$

Usando coordenadas esfericas.

[redacted] $x = 2\,\text{sen}\,\varphi \cdot \cos\frac{\pi}{4} = \frac{2}{\sqrt{2}}\,\text{sen}\,\varphi$

$y = 2\,\text{sen}\,\varphi \cdot \text{sen}\,\frac{\pi}{4} = \frac{2}{\sqrt{2}}\,\text{sen}\,\varphi \qquad 0 \le \varphi \le \frac{\pi}{2}$

$z = 2\cos\varphi = 2\cos\varphi$

$\mathcal{C}: \gamma(t) = \left(\frac{2}{\sqrt{2}}\,\text{sen}\,t, \frac{2}{\sqrt{2}}\,\text{sen}\,t, 2\cos t\right)$ com $0 \le t \le \frac{\pi}{2}$

$\gamma'(t) = (\sqrt{2}\cos t, \sqrt{2}\cos t, -2\,\text{sen}\,t)$

$\|\gamma'(t)\| = \sqrt{2\cos^2 t + 2\cos^2 + 4\,\text{sen}^2 t} = \sqrt{4} = 2$

$$\int_{\mathcal{C}} (x^2 + y^2)\,dr = \int_0^{\pi/2} (2\,\text{sen}^2 + 2\,\text{sen}^2 t) \cdot 2\,dt = 8\int_0^{\pi/2} \text{sen}^2 t\,dt$$

$\cos(2t) = \cos^2 t - \text{sen}^2 t$

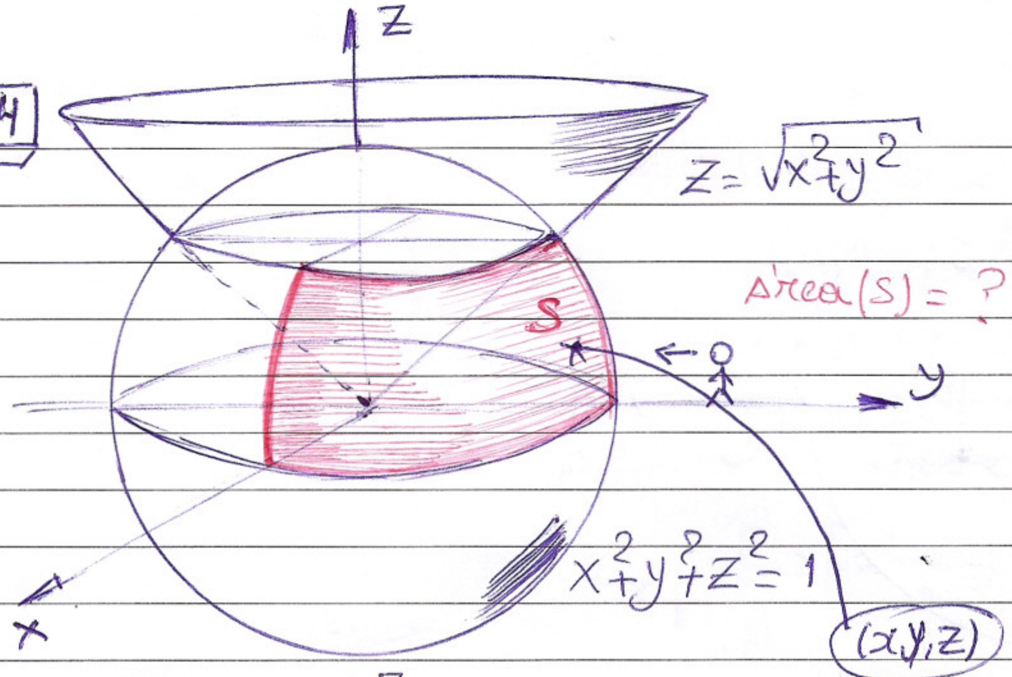
$\cos(2t) = 1 - 2\,\text{sen}^2 t$

$\text{sen}^2 t = \frac{1}{2} - \frac{1}{2}\cos(2t)$

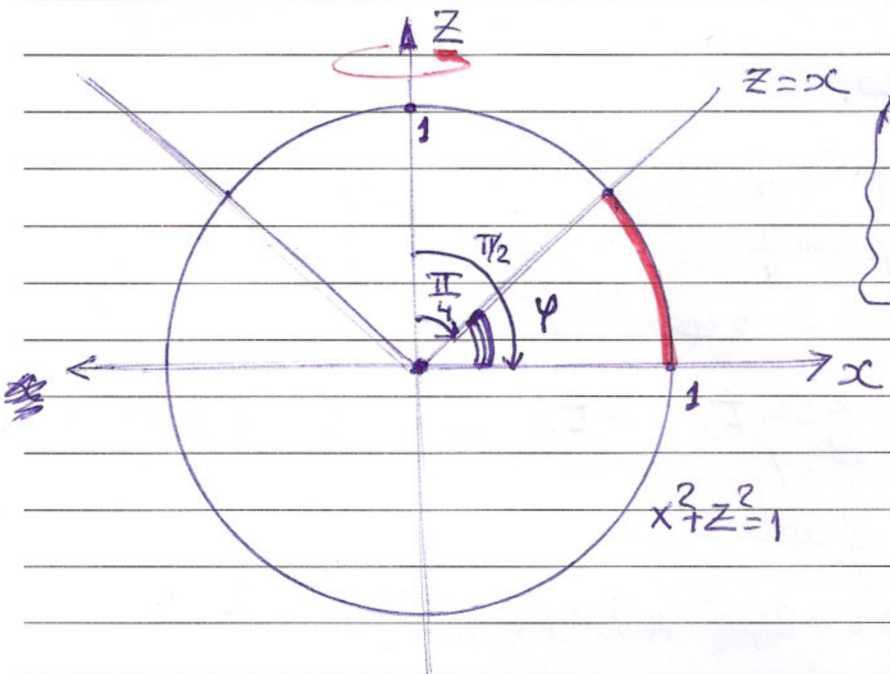
$$\int_0^{\pi/2} \text{sen}^2 t\,dt = \frac{1}{2}\left(\frac{\pi}{2}\right) - \frac{1}{2}\int_0^{\pi/2} \cos(2t)\,dt = \frac{\pi}{4} - \cancel{\left[\frac{\text{sen}(2t)}{4}\right]_0^{\pi/2}} \xrightarrow{} 0$$

$$\int_{\mathcal{C}} (x^2 + y^2)\,dr = 8 \cdot \frac{\pi}{4} = 2\pi$$

$\boxed{4}$



$$\begin{cases} x = \operatorname{sen}\varphi \cos\theta \\ y = \operatorname{sen}\varphi \operatorname{sen}\theta \\ z = \cos\varphi \end{cases}$$



com

$$\begin{cases} 0 \le \theta \le \frac{\pi}{2} \\ \frac{\pi}{4} \le \varphi \le \frac{\pi}{2} \end{cases}$$

$$S: \ \sigma(\theta,\varphi) = (\operatorname{sen}\varphi\cos\theta, \operatorname{sen}\varphi\operatorname{sen}\theta, \cos\varphi)$$

$$\sigma_\theta = (-\operatorname{sen}\varphi\operatorname{sen}\theta, \ \operatorname{sen}\varphi\cos\theta, \ 0)$$

$$\sigma_\varphi = (\cos\varphi\cos\theta, \ \cos\varphi\operatorname{sen}\theta, \ -\operatorname{sen}\varphi$$

$$\sigma_\theta \times \sigma_\varphi = \begin{vmatrix} i & j & k \\ -\operatorname{sen}\varphi\operatorname{sen}\theta & \operatorname{sen}\varphi\cos\theta & 0 \\ \cos\varphi\cos\theta & \cos\varphi\operatorname{sen}\theta & -\operatorname{sen}\varphi \end{vmatrix}$$

$$\sigma_\theta \times \sigma_\varphi = (-\operatorname{sen}^2\varphi\cos\theta, \ -\operatorname{sen}^2\varphi\operatorname{sen}\theta, \ -\operatorname{sen}\varphi\cos\varphi)$$

$$\|\sigma_\theta \times \sigma_\varphi\| = \sqrt{\operatorname{sen}^4\varphi + \operatorname{sen}^2\varphi\cos^2\varphi} = \sqrt{\operatorname{sen}^2\varphi} = |\operatorname{sen}\varphi|$$

$$\text{Área}(S) = \int_0^{\pi/2}\int_{\pi/4}^{\pi/2} |\operatorname{sen}\varphi| \, d\varphi \, d\theta = \frac{\pi}{2}\int_{\pi/4}^{\pi/2} \operatorname{sen}\varphi \, d\varphi = \frac{\pi}{2}\left(-\cos\varphi\right)\Big|_{\pi/4}^{\pi/2} = \frac{\pi}{2}\cdot\frac{1}{\sqrt{2}} = \frac{\pi}{2\sqrt{2}}$$