Universidade Federal do Paraná

Setor de Ciências Exatas

Departamento de Matematica — Prof. Juan Carlos Vila Bravo

1^{ra} prova de Álgebra Linear

Curitiba, 05 de Abril de 2016

1. Determine os valores de k de modo que o seguinte sistema nas incógnitas x, y, z tenha:

$$\begin{aligned} x + y - z &= 1 \\ 2x + 3y + kz &= 3 \\ x + ky + 3z &= 2 \end{aligned}$$

   (i) Solução única.

   (ii) Nemhuma solução.

   (iii) Uma infinidade de soluções.

2. Sejam U e V os seguintes subespaços de $\mathbb{R}^4$

$$U = \Big\{(x, y, z, t) \in \mathbb{R}^4;\ y - 2z + t = 0\Big\}$$

$$W = \Big\{(x, y, z, t) \in \mathbb{R}^4;\ x = t,\ y = 2z\Big\}$$

   Ache uma base e a dimensão de U , W , $(U \cap W)$ e $(U + W)$.

3. (**)Seja V o Espaço vetorial de todas as matrizes 2×2 sobre o corpo real $\mathbb{R}$. Mostre que

$$W = \{A \in V; \det(A) = 0\}$$

   não é Subespaço vetorial de V

4. Expresse o polinômio $v = t^2 + 4t - 3$ sobre $\mathbb{R}$ como combinação linear dos polinômios $p_1 = t^2 - 2t + 5$, $p_2 = 2t^2 - 3t$, $p_3 = t + 3$.

5. Seja V o espaço das matrizes 2×2 triangulares superiores. Sejam:

$$A = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}, \qquad B = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \right\}$$

   duas bases de V.

   (i) Encontre as coordenadas de $\begin{bmatrix} 1 & 2 \\ 0 & 4 \end{bmatrix}$ em relação às bases A e à base B.

   (ii) Encontre a matriz de mudança da base A para a base B e a matriz de mudança da base B para a base A.

6. (***)Determine a dimensão do Espaço vetorial de matrizes quadradas simetricas de ordem n.

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Nota: 120.

Disciplina: Álgebra Linear

Curso: Eng. eléctrica
Professor: Eu

Aluno: Eu

Turma: ?
Data: 05 / 04 / 2016

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$$\begin{cases} x + y - z = 1 \\ 2x + 3y + kz = 3 \\ x + ky + 3z = 2 \end{cases}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & -1 & 1 \\ 2 & 3 & k & 3 \\ 1 & k & 3 & 2 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 1 & -1 & 1 \\ 0 & 1 & k+2 & 1 \\ 0 & k-1 & 4 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 1 & -1 & 1 \\ 0 & 1 & k+2 & 1 \\ 0 & 0 & 4+(k+2)(1-k) & 1+(1-k) \end{array}\right]$$

$$4 + (k+2)(1-k) = 4 + 2 - 2k + k - k^2$$
$$= 6 - k - k^2$$
$$= (3+k)(2-k)$$

$$\boxed{\left[\begin{array}{ccc|c} 1 & 1 & -1 & 1 \\ 0 & 1 & k+2 & 1 \\ 0 & 0 & (3+k)(2-k) & (2-k) \end{array}\right]}$$

(i) Solução única, se $k \neq 2$ e $k \neq -3$

(ii) Nenhuma solução, se $k = -3$ $(P_c = 2$ e $P_A = 3)$

(iii) infinitas soluções, se $k = 2$ $(P_c = 2$ e $P_A = 2)$

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$\boxed{2}$

$$U = \{(x,y,z,t) \in \mathbb{R}^4 ;\ y - 2z + t = 0\}$$

$$\begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} = \begin{pmatrix} x \\ 2z - t \\ z \\ t \end{pmatrix} = x \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + z \begin{pmatrix} 0 \\ 2 \\ 1 \\ 0 \end{pmatrix} + t \begin{pmatrix} 0 \\ -1 \\ 0 \\ 1 \end{pmatrix}$$

$$\therefore \quad U = \Big[\underbrace{\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 0 \\ 1 \end{pmatrix}}_{} \Big]$$

são l.i, e geradores
Portanto: Base
logo. $\dim(U) = 3$

$$W = \{(x,y,z,t) \in \mathbb{R}^4 ;\ x = t,\ y = 2z\}$$

$$(x,y,z,t) = (t, 2z, z, t) = z(0,2,1,0) + t(1,0,0,1)$$

$$W = \big[\underbrace{(0,2,1,0),\ (1,0,0,1)}_{} \big]$$

São. l.i. e geradores
Portanto: é Base e $\dim(W) = 2$

$$U \cap W = \{(x,y,z,t) \in \mathbb{R}^4 ;\ y - 2z + t = 0,\ x = t,\ y = 2z\}$$

$$(x,y,z,t) = (0, 2z, z, 0) = z(0,2,1,0)$$

$$U \cap W = [(0,2,1,0)] \Rightarrow \dim(U \cap W) = 1$$

$$\begin{aligned} \dim(U + W) &= \dim(U) + \dim(W) - \dim(U \cap W) \\ &= 3 + 2 - 1 \\ &= 4 \end{aligned}$$

$$\therefore \quad \boxed{U + W = \mathbb{R}^4}$$

Dado $(x,y,z,t) \in (U+W)$, existe $\alpha, \beta, \gamma, \theta, \delta$;

$$\begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} = \alpha \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} 0 \\ 2 \\ 1 \\ 0 \end{pmatrix} + \gamma \begin{pmatrix} 0 \\ -1 \\ 0 \\ 1 \end{pmatrix} + \theta \begin{pmatrix} 0 \\ 2 \\ 1 \\ 0 \end{pmatrix} + \delta \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 2 & -1 & 2 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \\ \gamma \\ \theta \\ \delta \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix}$$

$$\left[\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & 1 & x \\ 0 & 2 & -1 & 2 & 0 & y \\ 0 & 1 & 0 & 1 & 0 & z \\ 0 & 0 & 1 & 0 & 1 & t \end{array}\right]$$

$$\left[\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & 1 & x \\ 0 & 1 & 0 & 1 & 0 & z \\ 0 & 2 & -1 & 2 & 0 & y \\ 0 & 0 & 1 & 0 & 1 & t \end{array}\right]$$

$$\left[\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & 1 & x \\ 0 & 1 & 0 & 1 & 0 & z \\ 0 & 0 & -1 & 0 & 0 & y-2z \\ 0 & 0 & 1 & 0 & 1 & t \end{array}\right]$$

$$\left[\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & 1 & x \\ 0 & 1 & 0 & 1 & 0 & z \\ 0 & 0 & -1 & 0 & 0 & y-2z \\ 0 & 0 & 0 & 0 & 1 & t+y-2z \end{array}\right]$$

$U + W = \mathbb{R}^4$

Uma base para $U+W$ poderia ser a base canonica.

$P_C = 4$, $P_A = 4$

Tem solução, sem restrição para x, y, z e t.

$\boxed{3}$

$$W = \{ A \in M_{2\times 2} \; ; \; \det(A) = 0 \}$$

(i) $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \in W$

(ii) $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \in W$, $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \in W$.

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \notin W.$$

$\therefore$ W não é sub-espaço.

$\boxed{4}$

$v = t^2 + 4t - 3$

$$\begin{cases} P_1 = t^2 - 2t + 5 \\ P_2 = 2t^2 - 3t \\ P_3 = t + 3 \end{cases}$$

Sejam α, β, γ;

$$v = \alpha P_1 + \beta P_2 + \gamma P_3$$

$$t^2 + 4t - 3 = \alpha[t^2 - 2t + 5] + \beta[2t^2 - 3t] + \gamma[t + 3]$$

$$t^2 + 4t - 3 = [\alpha + 2\beta]t^2 + [-2\alpha - 3\beta + \gamma]t + [5\alpha + 3\gamma]$$

$$\begin{cases} \alpha + 2\beta = 1 \\ -2\alpha - 3\beta + \gamma = 4 \\ 5\alpha + 3\gamma = -3 \end{cases} \quad \text{ou} \quad \begin{bmatrix} 1 & 2 & 0 \\ -2 & -3 & 1 \\ 5 & 0 & 3 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \\ -3 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 0 & 1 \\ -2 & -3 & 1 & 4 \\ 5 & 0 & 3 & -3 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 1 & 2 & 0 & 1 \\ 0 & 1 & 1 & 6 \\ 0 & -10 & 3 & -8 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 1 & 2 & 0 & 1 \\ 0 & 1 & 1 & 6 \\ 0 & 0 & 13 & 52 \end{array}\right]$$

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$$\begin{cases} \alpha + 2\beta = 1 \\ \beta + \gamma = 6 \\ 13\gamma = 52 \end{cases} \Rightarrow \begin{cases} \gamma = 4 \\ \beta = 2 \\ \alpha = -3 \end{cases}$$

$$t^2 + 4t - 3 = -3\left[t^2 - 2t + 5\right] + 2\left[2t^2 - 3t\right] + 4\left[t + 3\right]$$

$\boxed{5}$ i

$$\begin{bmatrix} 1 & 2 \\ 0 & 4 \end{bmatrix} = a\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + c\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 \\ 0 & 4 \end{bmatrix} = \begin{bmatrix} a & b \\ 0 & c \end{bmatrix}$$

$$\therefore \begin{bmatrix} 1 & 2 \\ 0 & 4 \end{bmatrix} = (1, 2, 4)_A$$

$$\begin{bmatrix} 1 & 2 \\ 0 & 4 \end{bmatrix} = a\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} + c\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 \\ 0 & 4 \end{bmatrix} = \begin{bmatrix} a+b+c & b+c \\ 0 & c \end{bmatrix}$$

$$a+b+c = 1$$
$$b+c = 2$$
$$c = 4 \Rightarrow b = -2 \text{ e } a = -1$$

$$\begin{bmatrix} 1 & 2 \\ 0 & 4 \end{bmatrix} = (-1, -2, 4)_B$$

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(ii) Matriz mudança da base A para base B

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = a\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} + c\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} a+b+c & b+c \\ 0 & c \end{bmatrix}$$

$$\begin{cases} a+b+c = 1 \\ b+c = 0 \\ c = 0 \end{cases} \Rightarrow \boxed{\begin{matrix} a = 1 \\ b = 0 \\ c = 0 \end{matrix}}$$

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = d\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + e\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} + f\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} d+e+f & e+f \\ 0 & f \end{bmatrix}$$

$$\begin{cases} d+e+f = 0 \\ e+f = 1 \\ f = 0 \end{cases} \Rightarrow \boxed{\begin{matrix} d = -1 \\ e = 1 \\ f = 0 \end{matrix}}$$

$$\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = g\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + h\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} + i\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

$$\begin{cases} g+h+i = 0 \\ h+i = 0 \\ i = 1 \end{cases} \Rightarrow \boxed{\begin{matrix} g = 0 \\ h = -1 \\ i = 1 \end{matrix}}$$

$$[I]^A_B = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$[I]^B_A = \left([I]^A_B\right)^{-1}$$

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\therefore \quad [I]^B_A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

$\boxed{6}$

$$A = \begin{bmatrix} a \end{bmatrix}_{1\times 1} \longrightarrow \boxed{\dim = 1}$$

$$A = \begin{bmatrix} a & b \\ b & a \end{bmatrix}_{2\times 2} \longrightarrow \boxed{\dim = 3}$$

$$A = \begin{bmatrix} a & b & c \\ b & d & e \\ c & e & f \end{bmatrix}_{3\times 3} \longrightarrow \dim = \,?$$

$$A = a\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + b\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + c\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} + d\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$+ e\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} + f\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \boxed{\dim = 6}$$

$$A = \begin{bmatrix} a & b & c & d \\ b & e & f & g \\ c & f & h & i \\ d & g & i & j \end{bmatrix}_{4\times 4} \Rightarrow \dim = 10$$

$$a \begin{bmatrix} 1 & 0 \\ 0 & \cdot \\ 0 & \\ 0 & \end{bmatrix} \dots$$

n	dim
1	1
2	3
3	6
4	10
$n-1$	a_{n-1}
n	a_n

$$\boxed{a_0 = 0}$$

$$a_n = a_{n-1} + n$$

$$a_n - a_{n-1} = n$$

$$n\text{-equações}\left\{\begin{aligned} a_1 - a_0 &= 1 \\ a_2 - a_1 &= 2 \\ a_3 - a_2 &= 3 \\ &\vdots \\ a_n - a_{n-1} &= n \end{aligned}\right.$$

$$a_n - a_0 = 1 + 2 + 3 + \dots + n$$

$$a_n = 1 + 2 + 3 + \dots + n$$

$$a_n = (n) + (n-1) + (n-2) + \dots + (1)$$

$$2a_n = n(n+1)$$

$$\boxed{a_n = \frac{n(n+1)}{2}}$$