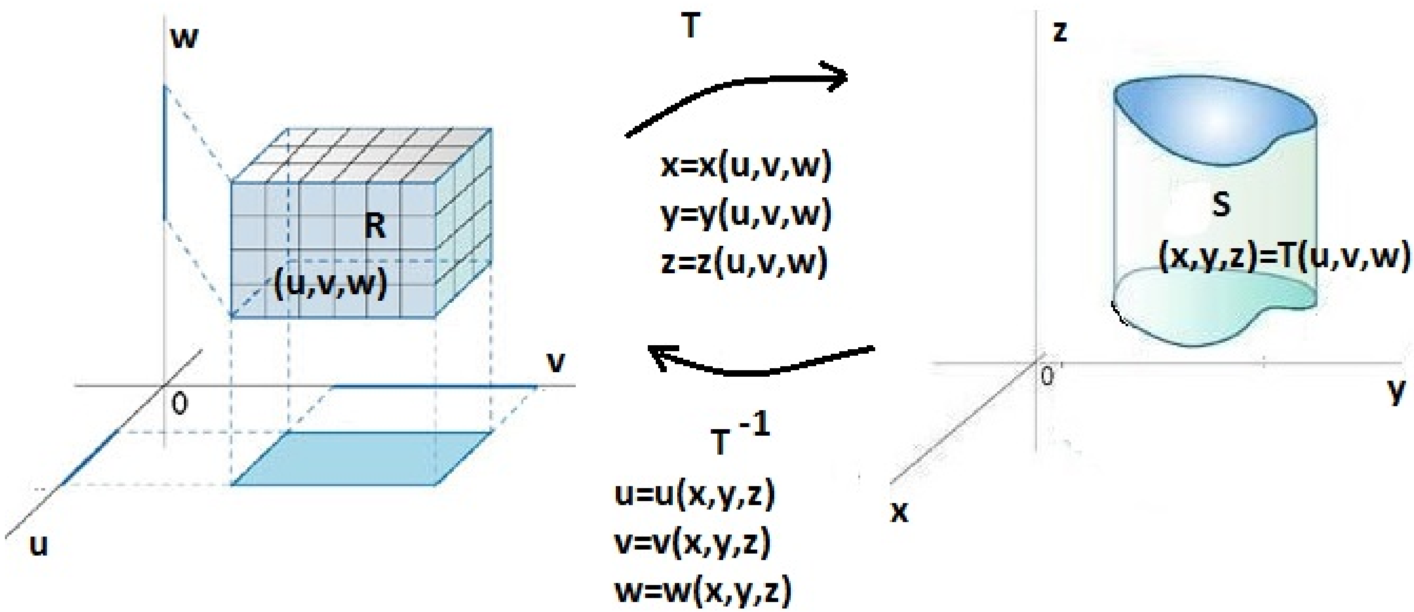
Mudança de variáveis

Para mudar das coordenadas (x, y, z) para (u, v, w), usando a função bijectiva



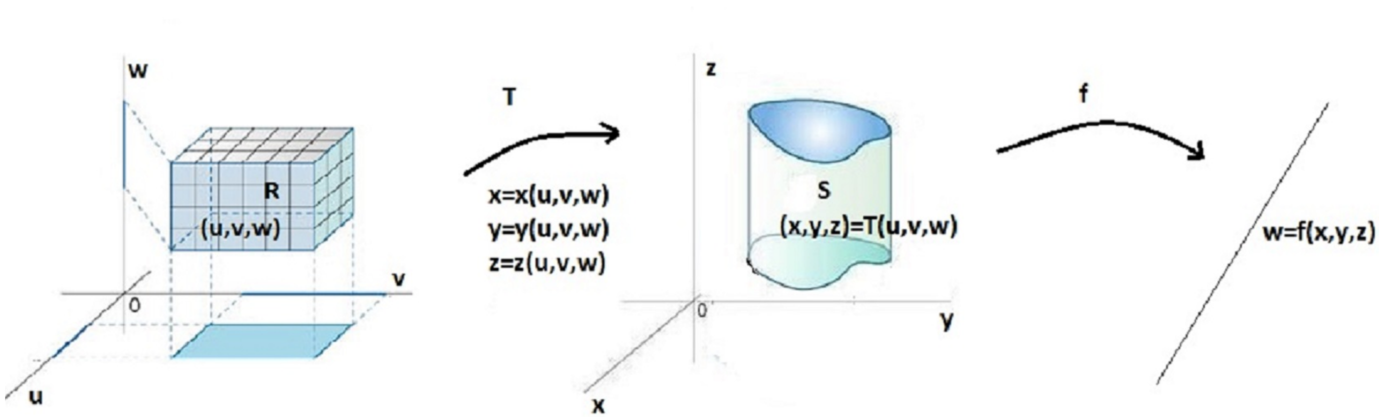
Teorema: Considere T uma aplicação definida por:

$$T(u,v,w) = (x(u,v,w), y(u,v,w), z(u,v,w))$$

onde x, y e z são funções de classe C^1 num subconjunto aberto $U \subset \mathbb{R}^3$. Seja $\mathcal{R}$ um subconjunto limitado e fechado contido em U tal que:

- T é injetora em $\mathcal{R}$.
- Suponhamos que o jacobiano de T, $J_T(u,v,w)$ seja diferente de 0 em $\mathcal{S}$, isto é,

$$J = J_T(u,v,w) = \frac{\partial(x,y,z)}{\partial(u,v,w)} = Det \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{bmatrix}_{3\times 3} \neq 0$$



Se f é integrável em $\mathcal{S} = T(R)$ então

$$\iiint_{\mathcal{S}} f(x,y,z)\, dx\, dy\, dz = \iiint_{\mathcal{R}} f(T(u,v,w)).\, |J|\, du\, dv\, dw$$

OBS.: Pelo teorema da função inversa, o jacobiano de T^{-1} é dado por

$$J_{T^{-1}}(x,y,z) = \frac{\partial(u,v,w)}{\partial(x,y,z)} = Det \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{bmatrix} = \frac{1}{J_T(u,v)}$$

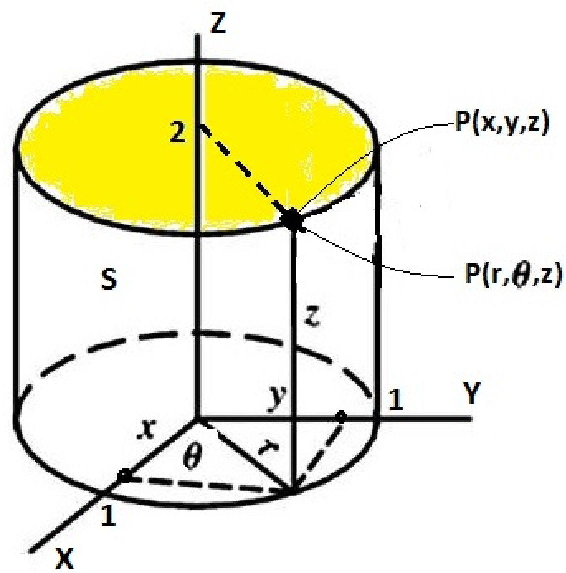
OBS.: O resultado do teorema acima ainda é válida se $\frac{\partial(x,y,z)}{\partial(u,v,w)} = 0$ ou T deixa de ser injetora em subconjuntos de $\mathcal{R}$ de volume nulo.

CASOS ESPECIAIS DE MUDANÇA DE VARIÁVEIS

(i) Mudança em coordenadas cilindricas.

$$x^2 + y^2 \leqslant 1$$

$$0 \leqslant z \leqslant 2$$



$$x = r\cos\theta$$

$$y = r\,\mathrm{sen}\,\theta$$

$$z = z$$

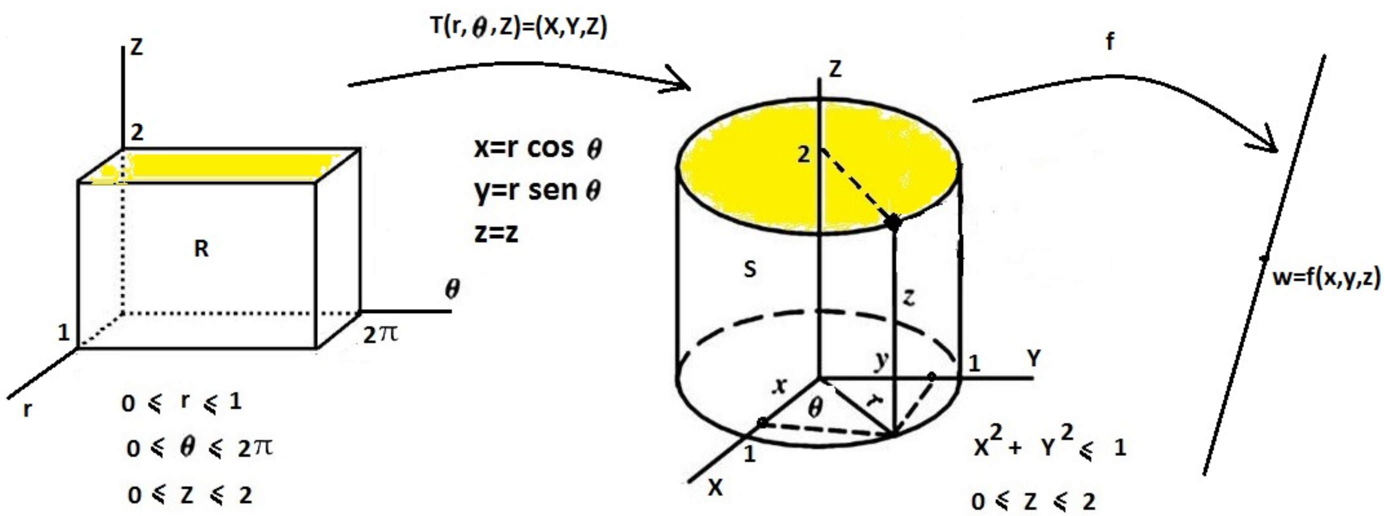
onde:

$$0 \leqslant r \leqslant 1$$

$$0 \leqslant \theta \leqslant 2\pi$$

$$0 \leqslant z \leqslant 2$$

Tem-se assim a seguinte transformação entre coordenadas cilindricas (r, θ, z) e as coordenadas cartesianas (x, y, z).



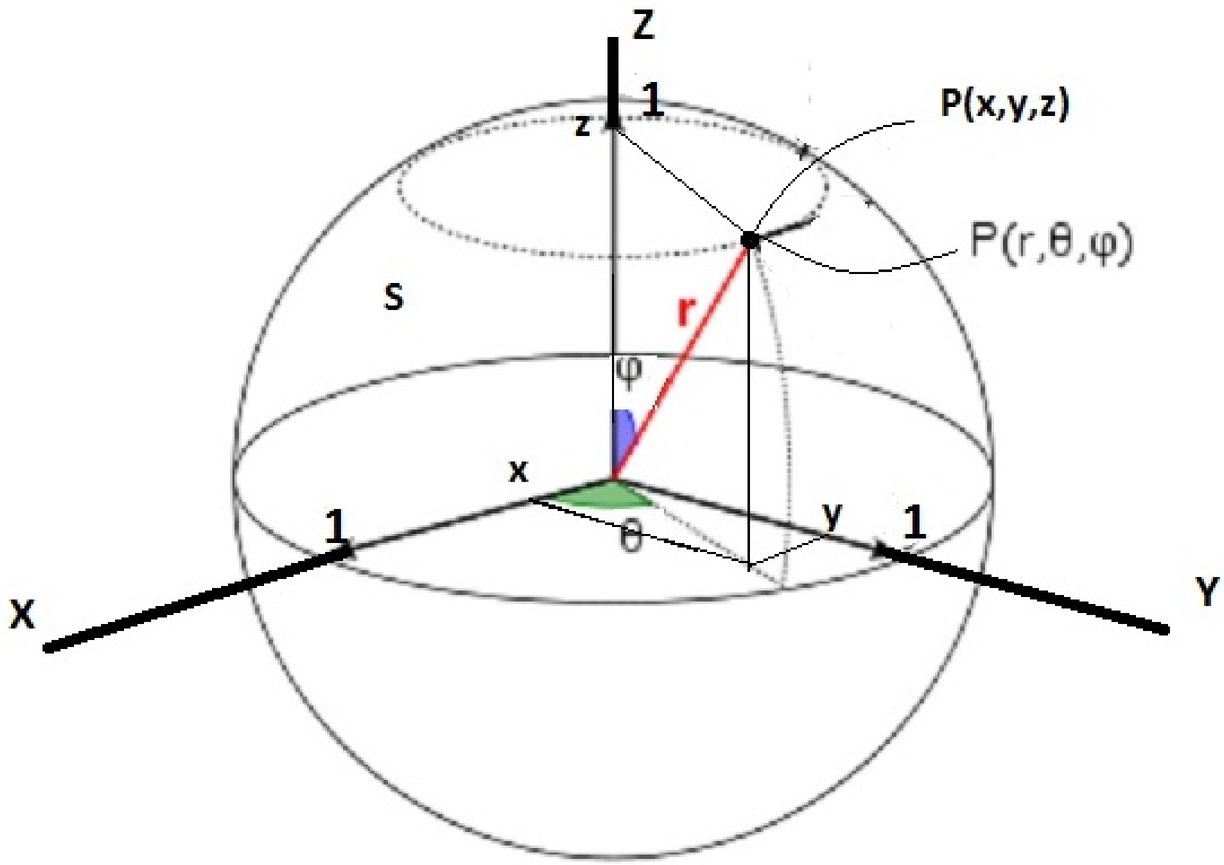
onde o jacobiano da transformação é:

$$J = \frac{\partial(x,y,z)}{\partial(r,\theta,z)} = Det \begin{bmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial z} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial z} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial z} \end{bmatrix} = r$$

Se f continua em $\mathcal{S}$, então:

$$\underset{\mathcal{S}}{\int\int\int} f(x,y,z)dxdydz = \underset{\mathcal{R}}{\int\int\int} f(r\cos\theta, r\text{sen }\theta, z)r\; drd\theta dz$$

(ii) Mudança em coordenadas esfericas.



$$X = r \operatorname{sen}\varphi \cos\theta$$

$$Y = r \operatorname{sen}\varphi \operatorname{sen}\theta$$

$$Z = r \cos\varphi$$

Onde:

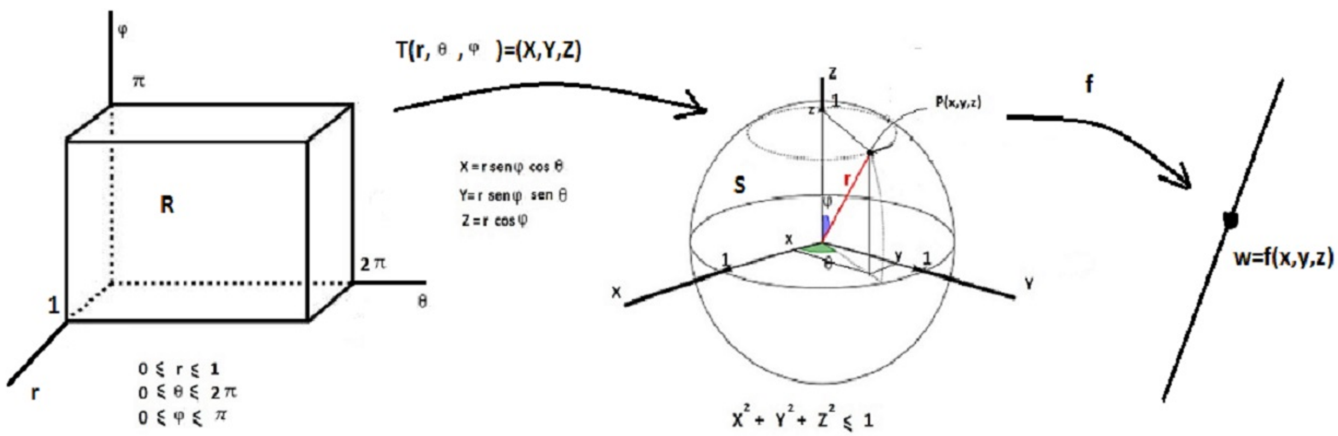
$$0 \leqslant r \leqslant 1$$

$$0 \leqslant \theta \leqslant 2\pi$$

$$0 \leqslant \varphi \leqslant \pi$$

$$X^2 + Y^2 + Z^2 \leqslant 1$$

Tem-se assim a seguinte transformação entre coordenadas cilindricas (r, θ, z) e as coordenadas cartesianas (x, y, z).



onde o jacobiano da transformação é:

$$J = \frac{\partial(x,y,z)}{\partial(r,\theta,z)} = Det \begin{bmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \varphi} \\ \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \varphi} \\ \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \varphi} \end{bmatrix} = r^2 \text{sen } \varphi$$

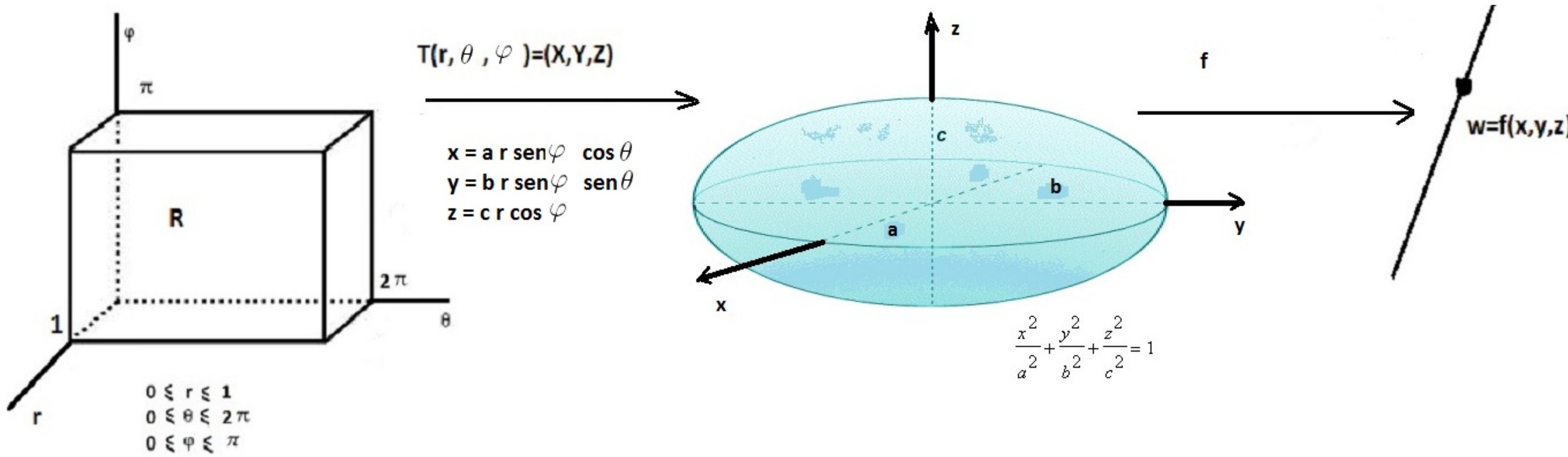
Se f continua em $\mathcal{S}$, então:

$$\underset{\mathcal{S}}{\int\int\int} f(x,y,z)dxdydz = \underset{\mathcal{R}}{\int\int\int} f(T(r,\theta\varphi))r^2 \text{sen } \varphi \, drd\theta d\varphi$$

onde:

$$T(r,\theta,\varphi) = (r\text{sen } \varphi \ \cos\theta, r\text{sen } \varphi \text{ sen } \theta, r\cos\varphi).$$

(iii) Mudança em coordenadas esfericas.



onde o jacobiano da transformação é:

$$J = \frac{\partial(x,y,z)}{\partial(r,\theta,z)} = Det \begin{bmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \varphi} \\ \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \varphi} \\ \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \varphi} \end{bmatrix} = abcr^2 \text{sen } \varphi$$

Se f continua em $\mathcal{S}$, então:

$$\underset{\mathcal{S}}{\int\int\int} f(x,y,z)dxdydz = \underset{\mathcal{R}}{\int\int\int} f(T(r,\theta,\varphi)abcr^2 \text{sen } \varphi \, drd\theta d\varphi$$

onde:

$$T(r,\theta,\varphi) = (r\cos\theta \text{sen } \varphi, r\text{sen } \theta \text{sen } \varphi, r\cos\varphi).$$