A Survival Guide to

Linear Algebra Done Right

2015 Springer Edition

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Partial scrutiny,
Comments, Suggestions and Errata

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EX. 20, p. **25**
On the one hand, note that each vector $u \in U$ can be written as $u = xu_1 + yu_2$ with $u_1 = (1,1,0,0)$, $u_2 = (0,0,1,1)$ and $x, y \in \mathbf{F}$. Then, since u_1 and u_2 are linearly independent (because none of them is a scalar multiple of the other), $\{u_1, u_2\}$ is a basis of U.[1] On the other hand, suppose that $w_1 = (0,1,0,0)$ and $w_2 = (0,0,1,0)$ spans W.[2] So w_1 and w_2 are linearly independent since none of them is a scalar multiple of the other. Now consider $B = \{u_1, u_2, w_1, w_2\}$. Therefore, on the one hand, the equality $(x, x, y, y) = (0, a, b, 0)$ implies that $x = y = a = b = 0$ and thus $U \cap W = \{(0,0,0,0)\}$. On the other hand, by considering the matrix whose columns are u_1, u_2, w_1 and w_2 written as column vectors, one has

$$\begin{aligned}\det\begin{pmatrix}1&0&0&0\\1&0&1&0\\0&1&0&1\\0&1&0&0\end{pmatrix} &= -\det\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&1&0&1\\1&0&1&0\end{pmatrix}\\ &= \det\begin{pmatrix}1&0&0&0\\0&1&0&0\\1&0&1&0\\0&1&0&1\end{pmatrix}\\ &= 1\end{aligned}$$

and thus u_1, u_2, w_1 and w_2 are linearly independent, that is, B is a basis of $U + W$. Hence $\dim(U + W) = 4 = \dim \mathbf{F}^4$.
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EX. 24, p. **26**
On the one hand, if $f \in U_{\mathrm{e}} \cap U_{\mathrm{o}}$, then

$$\begin{aligned}f(x) &= f(-x)\\ &= -f(x)\end{aligned}$$

for each $x \in \mathbf{R}$ and thus $f = 0$. On the other hand, if $f \in \mathbf{R}^{\mathbf{R}}$ is arbitrary, then

$$f(x) = \frac{f(x) + f(-x)}{2} + \frac{f(x) - f(-x)}{2}$$

for each $x \in \mathbf{R}$ with the first summand in U_{e} and the second summand in U_{o}.
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[1] See CHAPTER 2.
[2] Idem.

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EX. 1, p. **37**
On the one hand,

$$\operatorname{span}(v_1 - v_2, v_2 - v_3, v_3 - v_4, v_4) \subset \operatorname{span}(v_1, v_2, v_3, v_4)$$

since

$$\begin{aligned} v_1 - v_2 &= 1 \cdot v_1 - 1 \cdot v_2 + 0 \cdot v_3 + 0 \cdot v_4 \\ v_2 - v_3 &= 0 \cdot v_1 + 1 \cdot v_2 - 1 \cdot v_3 + 0 \cdot v_4 \\ v_3 - v_4 &= 0 \cdot v_1 + 0 \cdot v_2 + 1 \cdot v_3 - 1 \cdot v_4 \\ v_4 &= 0 \cdot v_1 + 0 \cdot v_2 + 0 \cdot v_3 + 1 \cdot v_4. \end{aligned}$$

On the other hand,

$$\operatorname{span}(v_1, v_2, v_3, v_4) \subset \operatorname{span}(v_1 - v_2, v_2 - v_3, v_3 - v_4, v_4)$$

since

$$\begin{aligned} v_1 &= 1 \cdot (v_1 - v_2) + 1 \cdot (v_2 - v_3) + 1 \cdot (v_3 - v_4) + 1 \cdot v_4 \\ v_2 &= 0 \cdot (v_1 - v_2) + 1 \cdot (v_2 - v_3) + 1 \cdot (v_3 - v_4) + 1 \cdot v_4 \\ v_3 &= 0 \cdot (v_1 - v_2) + 0 \cdot (v_2 - v_3) + 1 \cdot (v_3 - v_4) + 1 \cdot v_4 \\ v_4 &= 0 \cdot (v_1 - v_2) + 0 \cdot (v_2 - v_3) + 0 \cdot (v_3 - v_4) + 1 \cdot v_4. \end{aligned}$$

Therefore

$$\begin{aligned} \operatorname{span}(v_1 - v_2, v_2 - v_3, v_3 - v_4, v_4) &= \operatorname{span}(v_1, v_2, v_3, v_4) \\ &= V. \end{aligned}$$

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EX. 6, p. **37**
Consider $a_i \in \mathbf{F}$, $i = 1, 2, 3, 4$, such that

$$a_1 (v_1 - v_2) + a_2 (v_2 - v_3) + a_3 (v_3 - v_4) + a_4 v_4 = 0. \tag{1}$$

So

$$a_1 v_1 + (a_2 - a_1) v_2 + (a_3 - a_2) v_3 + (a_4 - a_3) v_4 = 0.$$

Thus, due to the fact that v_1, v_2, v_3, v_4 is linearly independent,

$$\begin{aligned} a_1 &= 0, \\ a_2 - a_1 &= 0, \\ a_3 - a_2 &= 0, \\ a_4 - a_3 &= 0. \end{aligned}$$

Then

$$a_i = 0, \ i = 1, 2, 3, 4. \tag{2}$$

Therefore, since (1) $\Longrightarrow$ (2), $v_1 - v_2, v_2 - v_3, v_3 - v_4, v_4$ is linearly independent.
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EX. 7, p. **37**
Consider $a_i \in \mathbf{F}$, $i = 1, 2, 3, \ldots, m$, such that

$$a_1 (5v_1 - 4v_2) + a_2 v_2 + a_3 v_3 + \cdots + a_m v_m = 0. \tag{3}$$

So

$$5a_1 v_1 + (a_2 - 4a_1) v_2 + a_3 v_3 + \cdots + a_m v_m = 0.$$

Thus, due to the fact that $v_1, v_2, \ldots, v_m$ is linearly independent,

$$\begin{aligned} 5a_1 &= 0, \\ a_2 - 4a_1 &= 0, \\ a_3 &= 0, \\ &\vdots \\ a_m &= 0. \end{aligned}$$

Then

$$a_i = 0, \; i = 1, 2, 3, \ldots, m. \tag{4}$$

Therefore, since (3) $\Longrightarrow$ (4), $5v_1 - 4v_2, v_2, v_3, \ldots, v_m$ is linearly independent.

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EX. 6, p. **43**

See **EX. 1** and **EX. 6**, p. **37**.

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EX. 8, p. **43**

First, let us show that the list of $m + n$ vectors spans V. In order to do that, consider $v \in V$. Since $V = U + W$, there exist $u \in U$ and $w \in W$ for which

$$v = u + w. \tag{5}$$

On the other hand, the bases of U and W give us scalars $\alpha_1, \ldots, \alpha_m, \beta_1, \ldots, \beta_n$ in $\mathbf{F}$ such that

$$u = \alpha_1 u_1 + \cdots + \alpha_m u_m \quad \text{and} \quad w = \beta_1 w_1 + \cdots + \beta_n w_n. \tag{6}$$

Substituting (6) in (5), it folllows that v is a linear combination of the $m + n$ vectors.
Now, let us show that the list of $m + n$ vectors is linearly independent. So, consider

$$\alpha_1 u_1 + \cdots \alpha_m u_m + \beta_1 w_1 + \cdots + \beta_n w_n = 0 \tag{7}$$

with, as before, scalars in $\mathbf{F}$. Denote u and w as in (6). Then $0 = u + w$ and, due to the fact that the sum is direct,[3]

$$u = w = 0. \tag{8}$$

Therefore, substituting (8) in (6) and observing that the list of m vectors in U and the list of n vectors in W are both linearly independent, all of the $m + n$ scalars in (7) must be equal to zero.

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EX. 16, p. **49**

It is a direct consequence of the following generalization of **EX. 8**, p. **43**:

> If
> $$\mathcal{B}_i = \left\{ u_{i_1}, \ldots, u_{i_{n_i}} \right\}$$
> is a basis of U_i, $i = 1, \ldots, m$, then
> $$\mathcal{B} = \cup_{i=1}^{m} \mathcal{B}_i$$
> is a basis of $U = \oplus_{i=1}^{m} U_i$.

In fact, firstly, let $u \in U$. Then $u = \sum_{i=1}^{m} u_i$ with $u_i \in U_i$, $i = 1, \ldots, m$. So, since u_i is a linear combination of the elements of $\mathcal{B}_i$, $i = 1, \ldots, m$, $\mathcal{B}$ clearly spans U. Now, it remains to show that the elements of $\mathcal{B}$ are linearly independent. Hence we first write

$$c_{1_1} u_{1_1} + \cdots + c_{1_{n_1}} u_{1_{n_1}} + \cdots + c_{m_1} u_{m_1} + \cdots + c_{m_{n_m}} u_{m_{n_m}} = 0$$

with $c_{i_1} u_{i_1} + \cdots + c_{i_{n_i}} u_{i_{n_i}} \in U_i$ and $c_{i_j} \in \mathbf{F}$, $i = 1, \ldots, m$ and $j = 1, \ldots, n_i$. Then $c_{i_1} u_{i_1} + \cdots + c_{i_{n_i}} u_{i_{n_i}} = 0$, $i = 1, \ldots, m$, by 1.44, p. **23**. Therefore, since $\mathcal{B}_i$ is a basis for U_i, $i = 1, \ldots, m$, $c_{i_j} = 0$ for every i and j.

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[3] See 1.44, p. **23**.

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EX. 3, p. **57**
Let e_k denote the k-th vector of the standard basis of $\mathbf{F}^n$. Substitute

$$T(e_k) = (A_{1,k}, \ldots, A_{m,k})$$

in

$$T\left(\sum_{k=1}^{n} x_k e_k\right) = \sum_{k=1}^{n} x_k T(e_k).$$

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EX. 4, p. **57**
Consider $a_1, \ldots, a_m \in \mathbf{F}$ such that

$$\sum_{j=1}^{m} a_j v_j = 0.$$

So

$$\begin{aligned}\sum_{j=1}^{m} a_j T(v_j) &= T\left(\sum_{j=1}^{m} a_j v_j\right)\\ &= T(0)\\ &= 0\end{aligned}$$

by the linearity of T and 3.11. Now use the hypothesis that $Tv_1, \ldots, Tv_m$ is linearly independent.
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EX. 6, p. **67**
$\dim \mathbf{R}^5 = \dim \operatorname{null} T + \dim \operatorname{range} T$ by 3.22, p. **63**, which implies that

$$5 = 2(\dim \operatorname{range} T),$$

contradicting the fact that $\dim \operatorname{range} T$ is a non-negative integer.
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EX. 9, p. **67**
Consider $a_1, \ldots, a_n \in \mathbf{F}$ such that

$$\sum_{j=1}^{n} a_j T(v_j) = 0.$$

Thus

$$T\left(\sum_{j=1}^{n} a_j v_j\right) = T(0)$$

by the linearity of T and 3.11, p. **57**. Then, since T is injective,

$$\sum_{j=1}^{n} a_j v_j = 0.$$

Now use the hypothesis that $v_1, \ldots, v_n$ is linearly independent.
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EX. 10, p. **68**
Let $w \in \operatorname{range} T$. Thus there exist $a_1, \ldots, a_n \in \mathbf{F}$ such that

$$\begin{aligned}w &= T\left(\sum_{j=1}^{n} a_j v_j\right)\\ &= \sum_{j=1}^{n} a_j T(v_j)\end{aligned}$$

by the linearity of T.

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EX. 14, p. **68**
$\dim \mathbf{R}^8 = \dim U + \dim \operatorname{range} T$ by 3.22, p. **63**, which implies that

$$\begin{aligned}\dim \operatorname{range} T &= 8 - 3 \\ &= 5.\end{aligned}$$

Therefore $\operatorname{range} T = \mathbf{R}^5$, which implies that T is surjective.

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Comment, p. **83**, l. -1
We can use 3.59 to prove 3.61 provided that both $\mathbf{F}^{m,n}$ and $\mathcal{L}(V, W)$ are finite-dimensional. The former satisfies this condition by 3.40, p. **74**. Concerning the latter, use 3.22, p. **63**, with $\mathcal{M}^{-1}$ in place of T. Therefore

$$\operatorname{range} \mathcal{M}^{-1} = \mathcal{L}(V, W)$$

is finite-dimensional.

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Erratum, p. **85**, ll. 1–9
$\mathcal{M}(v_k)$ should be $\mathcal{M}(Tv_k)$, four times.

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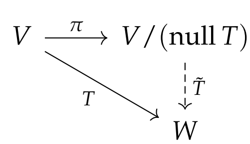
Comment, p. **97**, 3.90
Note that $\tilde{T} \circ \pi = T$.



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Comment, p. **98**, 3.91, Proof, (b)
$\operatorname{null} \tilde{T} = 0$ is an abuse of notation. It means $\operatorname{null} \tilde{T} = \{\operatorname{null} T\}$.

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EX. 15, p. **100**
The condition $\varphi \neq 0$ implies that $\operatorname{range} \varphi = \mathbf{F}$, which is a one-dimensional space. Now use 3.91(d), p. **98**.

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Erratum, p. **102**, 3.98, Proof, l. -6
F should be $\mathbf{F}$.

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Comments, p. **112**, 3.117, Proof

- 1st paragraph, 2nd sentence
  Based on (the Proof of) 2.31, pp. **40–41**, let

  $$Tv_1, \ldots, Tv_r, r \leq n,$$

  be a basis of $\operatorname{span}(Tv_1, \ldots, Tv_n)$. Thus

  $$\operatorname{span}(Tv_1, \ldots, Tv_r) = \operatorname{span}(Tv_1, \ldots, Tv_n)$$

  and, if

  $$w = a_1 Tv_1 + \cdots + a_r Tv_r$$

  with $a_1, \ldots, a_r \in \mathbf{F}$, then, by 3.62, p. **84**,

  $$\mathcal{M}(w) = \begin{pmatrix} a_1 \\ \vdots \\ a_r \end{pmatrix}.$$

  Now, clearly,

  $$\operatorname{span}(\mathcal{M}(Tv_1), \ldots, \mathcal{M}(Tv_r)) = \operatorname{span}(\mathcal{M}(Tv_1), \ldots, \mathcal{M}(Tv_n))$$

  and

  $$\operatorname{span}(Tv_1, \ldots, Tv_r) \ni w \mapsto \mathcal{M}(w) \in \mathbf{F}^{r,1}$$

  is an isomorphism.

- 2nd paragraph, 1st sentence
  See **EX. 10**, p. **68**.

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EX. 32, p. **115**

- (a) $\Longrightarrow$ (c) (and (b))
  By 3.69,[4] T is injective. So, by 3.16,[5] $\operatorname{null} T = \{0\}$. Then, by 3.22,[6] $\dim \operatorname{range} T = n$. Thus, by 3.117 and 3.40,[7] the column rank of $\mathcal{M}(T) = \dim \mathbf{F}^{n,1}$. Therefore, by 3.115 and 3.64,[8] $\mathcal{M}(Tu_1), \ldots, \mathcal{M}(Tu_n)$ is a basis of $\mathbf{F}^{n,1}$.

- (c) $\Longrightarrow$ (b)
  Use 2.42.[9]

- (b) $\Longrightarrow$ (a)
  By 3.115 and 3.117,[10] $\dim \operatorname{range} T = n$. So, by 3.22 and 3.16,[11] T is injective. Then, by 3.69,[12] T is invertible.

- (a) $\Longrightarrow$ (e) $\Longrightarrow$ (d) $\Longrightarrow$ (a)
  It follows from:
  * T is invertible $\Longleftrightarrow$ T' is invertible;[13]
  * $\mathcal{M}(T') = (\mathcal{M}(T))^t$ by 3.114;[14]
  * (a) $\Longrightarrow$ (c) $\Longrightarrow$ (b) $\Longrightarrow$ (a) with T' in the role of T and $\mathcal{M}(T')$ in the role of $\mathcal{M}(T)$.

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[4] See p. **87**.
[5] See p. **61**.
[6] See p. **63**.
[7] See p. **112** and p. **74**.
[8] See p. **111** and p. **85**.
[9] See p. **46**.
[10] See pp. **111–112**.
[11] See p. **63** and p. **61**.
[12] See p. **87**.
[13] See 3.108 and 3.110, pp. **107–108**.
[14] See p. **110**.

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Erratum, p. **123**, last paragraph, 1st sentence
"..., uses analysis its proof." should be "..., uses analysis **in** its proof.".
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Erratum, p. **125**, 4.14, Proof, 3rd paragraph, 3rd sentence
"... So we need only show that ..." should be "... So we need only **to** show that ...".
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EX. 2 and **EX. 3**, p. **129**
The answer to both questions is "no"![15]
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[15]Consider, for example, the sum of $p(x) = x^2 + x$ and $q(x) = -x^2 + 1$.

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Erratum, p. **134**, 5.6
F should be $\mathbf{F}$.
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EX. 2, p. **138**

$$\begin{aligned} v \in \operatorname{null} S &\Longrightarrow S(v) = 0 \\ &\Longrightarrow T(S(v)) = T(0) \\ &\Longrightarrow TS(v) = 0 \\ &\Longrightarrow ST(v) = 0 \\ &\Longrightarrow S(T(v)) = 0 \\ &\Longrightarrow T(v) \in \operatorname{null} S. \end{aligned}$$

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EX. 3, p. **139**

$$\begin{aligned} v \in \operatorname{range} S &\Longrightarrow \exists w \in V \text{ such that } v = S(w) \\ &\Longrightarrow T(v) = T(S(w)) \\ &\Longrightarrow T(v) = TS(w) \\ &\Longrightarrow T(v) = ST(w) \\ &\Longrightarrow T(v) = S(T(w)) \\ &\Longrightarrow T(v) = S(u) \text{ with } u = T(w) \\ &\Longrightarrow \exists u \in V \text{ such that } T(v) = S(u) \\ &\Longrightarrow T(v) \in \operatorname{range} S. \end{aligned}$$

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EX. 4, p. **139**

$$\begin{aligned} v \in \sum_{i=1}^{m} U_i &\Longrightarrow v = \sum_{i=1}^{m} u_i \text{ with } u_i \in U_i,\, i = 1, \dots, m \\ &\Longrightarrow T(v) = T\left(\sum_{i=1}^{m} u_i\right) \\ &\Longrightarrow T(v) = \sum_{i=1}^{m} T(u_i) \ \text{ with } T(u_i) \in U_i,\, i = 1, \dots, m, \text{ since each } U_i \text{ is invariant under } T \\ &\Longrightarrow T(v) \in \sum_{i=1}^{m} U_i. \end{aligned}$$

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EX. 33, p. **142**
First, since range T is invariant under T,[16] we can consider $T/(\operatorname{range} T)$.[17] So, for $v \in V$,

$$\begin{aligned} (T/(\operatorname{range} T))(v + \operatorname{range} T) &= T(v) + \operatorname{range} T \\ &= \operatorname{range} T \end{aligned}$$

since $T(v) \in \operatorname{range} T$.
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Comment, p. **145**, 5.21, Proof, penultimate sentence
If the m operators are injective, the image of a nonzero vector under $T - \lambda_j I$ is obtained for each index j. Therefore

$$\left(\Pi_{j=1}^{n} (T - \lambda_j I)\right) v \neq 0.$$

[16]See 5.3(d), p. **132**.
[17]See 5.14, p. **137**.

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Comment, p. **149**, Proof 1, l. 7
To be more specific, '... (see 3.69) ...' should be '... (see 5.6) ...'.[18]
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Comment, p. **150**, Proof 2, sentence that begins with 'Unraveling'
Consider $a_2, \ldots, a_j \in \mathbf{F}$, $j = 2, \ldots, n$, such that

$$\begin{aligned} Tv_j + U &= (T/U)\big(v_j + U\big) \\ &= a_2(v_2 + U) + \cdots + a_j\big(v_j + U\big) \\ &= ((a_2 v_2) + U) + \cdots + \big((a_j v_j) + U\big) \\ &= \big(a_2 v_2 + \cdots + a_j v_j\big) + U, \end{aligned}$$

where the first equality comes from 5.14, p. **137**, and the last two equalities come from 3.86, p. **96**.
Thus, for each $j \in \{2, \ldots, n\}$, since

$$Tv_j - \big(a_2 v_2 + \cdots + a_j v_j\big) \in U,\text{[19]}$$

there exists $a_1 \in \mathbf{F}$ such that

$$Tv_j = a_1 v_1 + a_2 v_2 + \cdots + a_j v_j.$$

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Comment - Erratum, p. **151**

- 5th paragraph ("To prove the other direction, ...")

$$\begin{aligned} Tv_1 = 0 &\Longrightarrow \text{null}\, T \neq \{0\} \\ &\Longrightarrow T \text{ is not injective (by 3.16, p. 61)} \\ &\Longrightarrow T \text{ is not invertible (by 3.69, p. 87)}, \end{aligned}$$

  which contradicts the assumption that T is invertible.

- 6th paragraph: ("Let $1 < j \leq n$, ...")
  "... T restricted to dim span$\big(v_1, \ldots, v_j\big)$ ..." should be "... T restricted to span$\big(v_1, \ldots, v_j\big)$ ...".

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Comment, p. **152**, 5.32, Proof, last sentence
Consider 5.6, p. **134**.
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EX. 3, p. **153**
Suppose that $T - I \neq 0$. So there is a nonzero vector $v \in V$ such that

$$\begin{aligned} w &:= (T - I)(v) \\ &\neq 0. \end{aligned}$$

Then, by 5.20, p. **144**,

$$\begin{aligned} (T + I)(w) &= (T + I)(T - I)(v) \\ &= (T^2 - I)(v) \\ &= 0 \end{aligned}$$

since $T^2 = I$. Therefore $Tw = -w$, which implies that -1 is an eigenvalue of T!
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EX. 7, p. **153**
By 5.20, p. **144**,

$$\begin{aligned} T^2 - 9I &= (T + 3I)(T - 3I) \\ &= (T - 3I)(T + 3I). \end{aligned}$$

[18] See p. **134**.
[19] See 3.85, p. **95**.

Therefore, by 5.6, p. **134**,

$$\begin{aligned} 9 \text{ is an eigenvalue of } T^2 &\iff T^2 - 9I \text{ is not injective} \\ &\iff (T-3I) \text{ or } (T+3I) \text{ is not injective} \\ &\iff 3 \text{ or } -3 \text{ is an eigenvalue of } T. \end{aligned}$$

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EX. 14, p. **154**
Define $T \in \mathcal{L}(\mathbf{F}^2)$ by $T(x,y) = (y,x)$. Obviously, $T^2 = I$ and the matrix of T with respect to the standard basis is

$$\mathcal{M}(T) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

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Comment, p. **155**, sentence immediately preceding 5.37
See 5.6, p. **134**, and 3.16, p. **61**.

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Erratum, p. **156**, 5.38, Proof, 1st sentence
'$E(\lambda, T)$' should be '$E(\lambda_j, T)$'.

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EX. 6, p. **160**
Consider 2.39, p. **45**, and $m = \dim V$ in 5.10, p. **136**. Now let λ_i' be an eigenvalue of S with $Sv_i = \lambda_i' v_i$, $i = 1, \ldots, \dim V$. Therefore, for each basis vector v_i,

$$\begin{aligned} STv_i &= S(Tv_i) \\ &= S(\lambda_i v_i) \\ &= \lambda_i S v_i \\ &= \lambda_i \lambda_i' v_i \\ &= \lambda_i' \lambda_i v_i \\ &= \lambda_i' T v_i \\ &= T(\lambda_i' v_i) \\ &= T(Sv_i) \\ &= TSv_i. \end{aligned}$$

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EX. 8, p. **160**
Suppose $T - 2I$ and $T - 6I$ are not invertible. Thus 2 and 6 are eigenvalues of T by 5.6, p. **134**. So, since $\dim E(8,T) = 4$, $\dim \mathbf{F}^5 = 5$ and

$$\dim E(2,T) + \dim E(6,T) + \dim E(8,T) \leq \dim \mathbf{F}^5$$

by 5.38, p. **156**, it follows that

$$E(2,T) = \{0\} \text{ or } E(6,T) = \{0\},$$

which is a contradiction.[20]

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EX. 9, p. **161**
0 is in both eigenspaces and

$$\begin{aligned} Tv = \lambda v &\iff T^{-1}(Tv) = T^{-1}(\lambda v) \\ &\iff v = \lambda T^{-1} v \\ &\iff T^{-1} v = \lambda^{-1} v. \end{aligned}$$

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[20] See the sentence immediately preceding 5.37, p. **155**.

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EX. 13 and **EX. 14**, p. **176**
Let V be an arbitrary inner product space. Consider $u, v \in V$ are nonzero. Thus, by the Cauchy-Schwarz Inequality,[21]

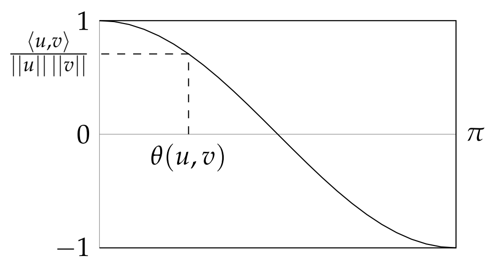
$$-||u||\,||v|| \leq \langle u, v\rangle \leq ||u||\,||v||.$$

Then, since $||u||$ and $||v||$ are positive,

$$-1 \leq \frac{\langle u, v\rangle}{||u||\,||v||} \leq 1.$$

Now let $\theta = \theta(u, v)$ be the unique number in $[0, \pi]$ (measured in radians) such that

$$\cos\theta = \frac{\langle u, v\rangle}{||u||\,||v||}.$$



Therefore $\langle u, v\rangle = ||u||\,||v||\cos\theta$ and $\theta = \arccos\frac{\langle u,v\rangle}{||u||\,||v||}$.

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EX. 16, p. **176**
Use the Parallelogram Equality.[22]

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EX. 17, p. **177**
If $x < 0$ and $y < 0$, then $||(x, y)|| < 0$. However, by 6.8 and 6.3,[23] a norm assigns to any $v \in V$ a non-negative real number $||v||$.

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EX. 30, p. **179**
See 3.69, p. **87**.

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EX. 2, p. **189**
Consider

$$||v||^2 = \sum_{j=1}^{m} |\langle v, e_j\rangle|^2$$

and

$$u = \sum_{j=1}^{m}\langle v, e_j\rangle e_j.$$

[21] Cf. 6.15, p. **172**.
[22] Cf. p. **174**.
[23] See pp. **168** and **166**.

Thus

$$\begin{aligned}\langle u, v\rangle &= \langle \sum_{j=1}^{m} \langle v, e_j\rangle e_j, v\rangle \\ &= \sum_{j=1}^{m} \langle v, e_j\rangle \langle e_j, v\rangle \\ &= \sum_{j=1}^{m} \langle v, e_j\rangle \overline{\langle v, e_j\rangle} \\ &= \sum_{j=1}^{m} |\langle v, e_j\rangle|^2 \\ &= ||v||^2 \\ &= \langle v, v\rangle.\end{aligned}$$

So $\langle u - v, v\rangle = 0$. Then, by 6.13,[24]

$$||u||^2 = ||u - v||^2 + ||v||^2.$$

Therefore, since $||u||^2 = ||v||^2$ by 6.25,[25]

$$||u - v||^2 = 0,$$

that is $u = v$, which implies that

$$v \in \text{span}(e_1, \ldots, e_m).$$

For the converse, since $e_1, \ldots, e_m$ is orthonormal, it is also linearly independent in $U = \text{span}(e_1, \ldots, e_m)$. Then $e_1, \ldots, e_m$ is an orthonormal basis of U. Therefore, if $v \in U$,

$$||v||^2 = \sum_{j=1}^{m} |\langle v, e_j\rangle|^2$$

by 6.30, p. **182**.

==

Comments, pp. **193–194**

Concerning the example (where U is a line or a plane in $\mathbf{R}^3$), since U is not necessarily a subspace of V by 6.45, it is not necessary for U to contain the zero vector. Similarly, that is why

$$U \cap U^{\perp} \supset \{0\}$$

does not necessarily hold for 6.46(d). By the way, the proof of 6.46(d) for $U \cap U^{\perp} = \emptyset$ is trivial.

==

EX. 1, p. **201**

On the one hand, from $\{v_1, \ldots, v_m\} \subset \text{span}(v_1, \ldots, v_m)$, it follows that

$$(\text{span}(v_1, \ldots, v_r))^{\perp} \subset \{v_1, \ldots, v_m\}^{\perp}$$

by 6.46(e), p. **193**. On the other hand, consider $v \in \{v_1, \ldots, v_m\}^{\perp}$ and let u be a linear combination of $v_1, \ldots, v_m$, that is, consider $c_1, \ldots, c_m \in \mathbf{F}$ such that $u = c_1 v_1 + \ldots + c_m v_m$. So

$$\begin{aligned}\langle v, u\rangle &= \overline{c_1}\langle v, v_1\rangle + \cdots + \overline{c_m}\langle v, v_m\rangle \\ &= \overline{c_1} \cdot 0 + \cdots + \overline{c_m} \cdot 0 \\ &= 0.\end{aligned}$$

Then $v \in (\text{span}(v_1, \ldots, v_m))^{\perp}$. Therefore

$$\{v_1, \ldots, v_m\}^{\perp} \subset (\text{span}(v_1, \ldots, v_m))^{\perp}.$$

==

[24]See p. **170**.

[25]See p. **180**.

EX. 4, p. **201**
For $v_1 = (1, 2, 3, -4)$ and $v_2 = (-5, 4, 3, 2)$, consider

$$u_1 = v_1 \text{ and } u_2 = v_2 - P_{\text{span}(v_1)}(v_2)$$

as in example 6.54, p. **196**. So

$$\left\{ \frac{u_1}{||u_1||}, \frac{u_2}{||u_2||} \right\}$$

is an orthonormal basis of U. On the other hand, since

$$U^\perp = \{v_1, v_2\}^\perp$$

by **EX. 1**, p. **201**, and $\dim U^\perp = 2$ by 6.50, p. **195**, if w_1, w_2 is linearly independent with

$$\langle w_i, v_j \rangle = 0,\, i, j \in \{1, 2\}\,,$$

then $\{w_1, w_2\}$ is a basis of $U^\perp$. So let $w = (a, b, c, d)$ be a vector in $\mathbf{R}^4$ for which

$$\langle w, v_j \rangle = 0,\, j = 1, 2.$$

Then, since

$$\begin{cases} a + 2b + 3c - 4d = 0 \\ -5a + 4b + 3c + 2d = 0 \end{cases}$$

and

$$\begin{cases} a = 15c - 14d \\ b = -9c + 9d \end{cases}$$

are equivalent systems,

$$\begin{aligned} w &= (15c - 14d, -9c + 9d, c, d) \\ &= c(15, -9, 1, 0) + d(-14, 9, 0, 1) \end{aligned}$$

for all real numbers c and d. Therefore, if

$$w_1 = (15, -9, 1, 0) \text{ and } w_2 = (-14, 9, 0, 1) - P_{\text{span}(w_1)}(-14, 9, 0, 1),$$

$$\left\{ \frac{w_1}{||w_1||}, \frac{w_2}{||w_2||} \right\}$$

is an orthonormal basis of $U^\perp$.
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Comment, p. **204**
Consider

$$W \ni x \overset{\langle -,w \rangle}{\longmapsto} \langle x, w \rangle.$$

So $\langle -, w \rangle \in \mathcal{L}(W, \mathbf{F})$. Then

$$\varphi = \langle -, w \rangle \circ T \in \mathcal{L}(V, \mathbf{F}). \tag{9}$$

On the other hand, by 6.42,[26] let u be the vector in V such that

$$\varphi = \langle -, u \rangle. \tag{10}$$

Therefore, denoting u by T^*w, T^* is well-defined and, via (9) and (10),

$$\langle Tv, w \rangle = \langle v, T^*w \rangle.$$

==
Comment, p. **213**, 7.21
Since $T - \lambda I$ is normal, another Proof comes from the small box in the bottom left corner of p. **212**. In fact, note that

$$\text{null}\,(T - \lambda I) = \text{null}\,\left(T^* - \overline{\lambda} I\right).$$

==
EX. 2, p. **214**
Firstly, consider the following result

> *If T is invertible, then T^* is invertible and*
> $$(T^*)^{-1} = \left(T^{-1}\right)^*.$$

In fact, by 7.6, items (d) and (e), p. **206**, if

$$\begin{aligned} TT^{-1} &= I \\ &= T^{-1}T, \end{aligned}$$

then

$$\begin{aligned} \left(T^{-1}\right)^* T^* &= I \\ &= T^* \left(T^{-1}\right)^*. \end{aligned}$$

Therefore, by 5.6, p. **134**, and by 7.6, items (a), (b) and (c), p. **206**,

$$\begin{aligned} \lambda \text{ is an eigenvalue of } T &\iff T - \lambda I \text{ is not invertible} \\ &\iff (T - \lambda I)^* = T^* - \overline{\lambda} I \text{ is not invertible} \\ &\iff \overline{\lambda} \text{ is an eigenvalue of } T^*. \end{aligned}$$

==
EX. 4, p. **214**
It is a direct consequence of 7.7, p. **207**.
==
Comments, p. **220**, Proof

[26]See p. **188**.

- $\boxed{m + M \geq 1}$
  In fact, since $V \neq \{0\}$,

$$\begin{aligned} n &= \dim V \\ &\geq 1. \end{aligned}$$

  Therefore $a_0 + a_1 x + \cdots + a_n x^n$ is a nonconstant polynomial.

- $\boxed{m > 0}$
  In fact, if there is no $(T - \lambda_i I)$ factor, then $M > 0$ and the invertible operator

$$c \prod_{j=1}^{M} \left(T^2 + b_j T + c_j I \right)$$

  is the zero vector at $v \neq 0$, which is a violation of 3.69, p. **87**.

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Comment, p. **222**, Proof, (c) $\Longrightarrow$ (a)
Suppose $\dim V = n$ and let $\mathcal{B}$ be the orthonormal basis mentioned in the 1st paragraph. Then, since $\mathcal{M}(T, \mathcal{B}) \in \mathbf{R}^{n,n}$ is diagonal,

$$\mathcal{M}(T^*, \mathcal{B}) = \mathcal{M}(T, \mathcal{B})$$

by 7.10, p. **208**. Furthermore, from the fact that

$$\begin{array}{rcccc} \mathcal{M}(-, \mathcal{B}) & : & \mathcal{L}(V) & \longrightarrow & \mathbf{R}^{n,n} \\ & & S & \longmapsto & \mathcal{M}(S, \mathcal{B}) \end{array}$$

is an isomorphism,[27] it follows that $T^* = T$.

==

EX. 6, p. **223**
Let T be a normal operator on a complex inner product space and, firstly, suppose that

$$T^* = T.$$

Furthermore, consider that λ is an eigenvalue of T. Then, by 7.21, p. **213**, $\overline{\lambda}$ is an eigenvalue of T and

$$\begin{aligned} E(\lambda, T) &= E\left(\overline{\lambda}, T^*\right) \\ &= E\left(\overline{\lambda}, T\right). \end{aligned}$$

So, by the Complex Spectral Theorem, p. **218**, T is diagonalizable. Then, by 5.41(d), p. **157**, $\lambda = \overline{\lambda}$, that is,

$$\lambda \in \mathbf{R}.$$

> THE PREVIOUS REASONING USES THE NORMALITY HYPOTHESIS. A MORE STRAIGHTFORWARD PROOF FOLLOWS FROM 7.13, P. **210**.

Now, for the converse, suppose that

$$\lambda \text{ is an eigenvalue of } T \Longrightarrow \lambda \in \mathbf{R}.$$

Hence, by the Complex Spectral Theorem, there exists an orthonormal basis $\mathcal{B}$ such that

$$D = \mathcal{M}(T, \mathcal{B})$$

is diagonal (with its (real) eigenvalues lying on the main diagonal). Then, by 7.10, p. **208**,

$$\begin{aligned} \mathcal{M}(T^*, \mathcal{B}) &= \overline{D}^t \\ &= D, \end{aligned}$$

[27] Cf. p. **83**.

which implies that

$$T^* = T.$$

==

EX. 13, p. **224**

Consider the following modifications in the Proof of page **222**:

- In the 1st paragraph, right before the last sentence, insert the sentence: $\boxed{\text{Then } T \text{ is normal.}}$;[28]
- In the 2nd paragraph, replace the word $\boxed{\text{real}}$ with the word $\boxed{\text{complex}}$;
- In the 3rd paragraph, replace:
  - the word $\boxed{\text{self-adjoint}}$ with the word $\boxed{\text{normal}}$, twice;
  - the result $\boxed{7.27}$ with the result $\boxed{5.21}$;
  - the result $\boxed{7.28(c)}$ with the result $\boxed{9.30(d)}$.

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Comment, p. **226**, Proof, (b) $\Longrightarrow$ (c)

Let $\mathcal{B} = \{e_1, \ldots, e_n\}$. Then, since $\mathcal{M}(R, \mathcal{B}) \in \mathbf{R}^{n,n}$ is diagonal,[29]

$$\mathcal{M}(R^*, \mathcal{B}) = \mathcal{M}(R, \mathcal{B})$$

by 7.10, p. **208**. Furthermore, from the fact that

$$\begin{array}{rcccc} \mathcal{M}(-, \mathcal{B}) & : & \mathcal{L}(V) & \longrightarrow & \mathbf{F}^{n,n} \\ & & S & \longmapsto & \mathcal{M}(S, \mathcal{B}) \end{array}$$

is an isomorphism,[30] it follows that $R^* = R$, that is, R is self-adjoint.
Now, the positivity of R follows from the fact that

$$\langle Re_i, e_j \rangle = \sqrt{\lambda_i} \langle e_i, e_j \rangle$$

is nonnegative for each indices i, j.

==

EX. 4, p. **231**

Firstly, T^*T and TT^* are self-adjoint operators since, by 7.6, (c) and (e).[31]

$$\begin{aligned} (T^*T)^* &= T^* (T^*)^* \\ &= T^*T \end{aligned}$$

and

$$\begin{aligned} (TT^*)^* &= (T^*)^* T^* \\ &= TT^*. \end{aligned}$$

Now, for $v \in V$ and $w \in W$, the positivity follows from

$$\begin{aligned} \langle (T^*T) v, v \rangle &= \langle T^* (Tv), v \rangle \\ &= \langle Tv, Tv \rangle \\ &\geq 0 \end{aligned}$$

and

$$\begin{aligned} \langle (TT^*) w, w \rangle &= \langle T (T^*w), w \rangle \\ &= \langle T^*w, T^*w \rangle \\ &\geq 0. \end{aligned}$$

[28] Every self-adjoint operator is normal!

[29] For each $j \in \{1, \ldots, n\}$, the diagonal entry $\sqrt{\lambda_j}$ is the square root of a nonnegative number. Now see the last paragraph of p. **225**.

[30] Cf. p. **83**.

[31] See p. **206**.

==
EX. 10, p. **232**
By 7.42, p. **229**, (a) $\Longleftrightarrow$ (g). Then one can replace S by S^* in (a), (b), (c) and (d) of 7.42.
==
Comment, p. **233**
Concerning the sentence preceding 7.45, see **EX. 4**, p. **231**.
==
Comment, p. **236**, 7.50, Solution

- 1st sentence

$$\begin{aligned}\left\langle (z_1, z_2, z_3, z_4), T^*(w_1, w_2, w_3, w_4) \right\rangle &= \left\langle T(z_1, z_2, z_3, z_4), (w_1, w_2, w_3, w_4) \right\rangle \\ &= \left\langle (0, 3z_1, 2z_2, -3z_4), (w_1, w_2, w_3, w_4) \right\rangle \\ &= 3z_1\overline{w_2} + 2z_2\overline{w_3} - 3z_4\overline{w_4} \\ &= \left\langle (z_1, z_2, z_3, z_4), (3w_2, 2w_3, 0, -3w_4) \right\rangle\end{aligned}$$

$$\Downarrow$$

$$T^*(w_1, w_2, w_3, w_4) = (3w_2, 2w_3, 0, -3w_4)$$

$$\Downarrow$$

$$\begin{aligned}T^*T(z_1, z_2, z_3, z_4) &= T^*(0, 3z_1, 2z_2, -3z_4) \\ &= (9z_1, 4z_2, 0, 9z_4).\end{aligned}$$

- 2nd sentence
On the one hand, $\sqrt{T^*T}$ denotes the unique positive square root of the (self-adjoint) positive operator T^*T.[32] On the other hand, the operator $\sqrt{T^*T}$ presented in the solution is (self-adjoint)[33] positive since

$$\begin{aligned}\left\langle \sqrt{T^*T}(z_1, z_2, z_3, z_4), (z_1, z_2, z_3, z_4) \right\rangle &= \left\langle (3z_1, 2z_2, 0, 3z_4), (z_1, z_2, z_3, z_4) \right\rangle \\ &= 3z_1\overline{z_1} + 2z_2\overline{z_2} + 3z_4\overline{z_4} \\ &= 3|z_1|^2 + 2|z_2|^2 + 3|z_4|^2 \\ &\geq 0.\end{aligned}$$

Furthermore

$$\begin{aligned}\left(\sqrt{T^*T}\sqrt{T^*T}\right)(z_1, z_2, z_3, z_4) &= \sqrt{T^*T}\left(\sqrt{T^*T}(z_1, z_2, z_3, z_4)\right) \\ &= \sqrt{T^*T}(3z_1, 2z_2, 0, 3z_4) \\ &= (3(3z_1), 2(2z_2), 0, 3(3z_4)) \\ &= T^*T(z_1, z_2, z_3, z_4).\end{aligned}$$

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Comment, p. **238**, 7.52, Proof
If $\mathcal{B} = \{e_1, \ldots, e_n\}$, then

$$\left(\mathcal{M}\left(\sqrt{T^*T}, \mathcal{B}\right)\right)^2 = \mathcal{M}(T^*T, \mathcal{B})$$

since $\left(\sqrt{T^*T}\right)^2 = T^*T$. Therefore, for each index j,

$$\begin{aligned}\sqrt{T^*T}e_j = \alpha_j e_j &\Longrightarrow \alpha_j^2 = \lambda_j \\ &\Longrightarrow \alpha_j = \sqrt{\lambda_j}.\end{aligned}$$

[32]See p. **233**.
[33]If $\mathcal{B}$ is the standard basis, $\mathcal{M}\left(\sqrt{T^*T}, \mathcal{B}\right)$ is diagonal with real entries!

==
EX. 1, p. **238**
In the exercise,

$$Tv = \langle v, u \rangle x. \tag{11}$$

So, on the one hand,

$$\begin{aligned} \langle v, T^* w \rangle &= \langle Tv, w \rangle \\ &= \langle \langle v, u \rangle x, w \rangle \\ &= \langle v, u \rangle \langle x, w \rangle \\ &= \langle v, \langle w, x \rangle u \rangle \end{aligned}$$

$$\Downarrow$$

$$T^* v = \langle v, x \rangle u \tag{12}$$

$$\Downarrow$$

$$\begin{aligned} (T^* T)\, v &= T^* (Tv) && (13) \\ &= T^* (\langle v, u \rangle x) && (14) \\ &= \langle v, u \rangle T^* x && (15) \\ &= \langle v, u \rangle \langle x, x \rangle u && (16) \\ &= ||x||^2 \langle v, u \rangle u. && (17) \end{aligned}$$

On the other hand, $\sqrt{T^*T}$ denotes the unique positive square root of the (self-adjoint) positive operator T^*T.[34] Furthermore, the operator $\sqrt{T^*T}$ presented in the exercise is self-adjoint,[35] positive (since

$$\begin{aligned} \left\langle \sqrt{T^*T} v, v \right\rangle &= \left\langle \frac{||x||}{||u||} \langle v, u \rangle u, v \right\rangle \\ &= \frac{||x||}{||u||} \langle v, u \rangle \langle u, v \rangle \\ &= \frac{||x||}{||u||} |\langle v, u \rangle|^2 \\ &\geq 0 \end{aligned}$$

for each $v \in V$) and a root of T^*T (since, for each $v \in V$,

$$\begin{aligned} \left(\sqrt{T^*T} \right)^2 v &= \sqrt{T^*T} \left(\sqrt{T^*T} v \right) \\ &= \sqrt{T^*T} \left(\frac{||x||}{||u||} \langle v, u \rangle u \right) \\ &= \frac{||x||}{||u||} \langle v, u \rangle \sqrt{T^*T} u \\ &= \frac{||x||}{||u||} \langle v, u \rangle \frac{||x||}{||u||} \langle u, u \rangle u \\ &= \frac{||x||^2}{||u||^2} \langle v, u \rangle ||u||^2 u \\ &= (T^*T)\, v \end{aligned}$$

by (17)).
==

[34] See p. **233**.
[35] Use (11) and (12)

EX. 3, p. **239**
By 7.45, p. **233**, there exists an isometry $\Sigma \in \mathcal{L}(V)$ such that

$$T^* = \Sigma\sqrt{(T^*)^* T^*} \tag{18}$$
$$= \Sigma\sqrt{TT^*}, \tag{19}$$

where (19) holds by 7.6(c).[36] So

$$\begin{aligned} T &= (T^*)^* \\ &= \left(\Sigma\sqrt{TT^*}\right)^* \\ &= \left(\sqrt{TT^*}\right)^* \Sigma^* \\ &= \sqrt{TT^*}\Sigma^*, \end{aligned}$$

where the first equality holds by 7.6(c), the second equality holds by (19) (as stated above), the third equality holds by 7.6(e),[37] and the last equality holds because each positive operator is self-adjoint by definition. Now use S to denote Σ^*.
==
EX. 17, p. **240**
(a)

$$\begin{aligned} \langle v, T^*w\rangle &= \langle Tv, w\rangle \\ &= \left\langle \sum_{j=1}^{n} s_j\langle v, e_j\rangle f_j, w\right\rangle \\ &= \sum_{j=1}^{n} s_j\langle v, e_j\rangle\langle f_j, w\rangle \\ &= \sum_{j=1}^{n} s_j\overline{\langle w, f_j\rangle}\langle v, e_j\rangle \\ &= \left\langle v, \sum_{j=1}^{n} \overline{s_j}\langle w, f_j\rangle e_j\right\rangle \end{aligned}$$

$$\Downarrow$$

$$T^*w = \sum_{j=1}^{n} s_j\langle w, f_j\rangle e_j$$

since, for each $j \in \{1, \ldots, n\}$, s_j is an eigenvalue of $\sqrt{T^*T}$, which is self-adjoint.[38]
(b)

$$\begin{aligned} (T^*T)\, v &= T^*(Tv) \\ &= T^*\left(\sum_{i=1}^{n} s_i\langle v, e_i\rangle f_i\right) \\ &= \sum_{i=1}^{n} s_i\langle v, e_i\rangle T^* f_i \\ &= \sum_{i=1}^{n} s_i\langle v, e_i\rangle s_i e_i \\ &= \sum_{i=1}^{n} s_i^2\langle v, e_i\rangle e_i. \end{aligned}$$

[36] See p. **206**.
[37] See p. **206**.
[38] Cf. 7.13, p. **210**.

(c) Consider $\mathcal{B} = \{e_1, \ldots, e_n\}$. Then

$$\mathcal{M}(T^*T, \mathcal{B}) = \begin{pmatrix} s_1^2 & & \text{zeros} \\ & \ddots & \\ \text{zeros} & & s_n^2 \end{pmatrix}$$

by (b). Now use 7.52, p. **238**.
(d) Firstly, note that, for each $j \in \{1, \ldots, n\}$, $s_j \neq 0$ if T^{-1} exists. In fact, by 3.43, p. **75**, if $\mathcal{B} = \{e_1, \ldots, e_n\}$ and $\mathcal{B}' = \{f_1, \ldots, f_n\}$, then

$$\begin{aligned} \mathcal{M}\left(T^{-1}, \mathcal{B}', \mathcal{B}\right) \mathcal{M}(T, \mathcal{B}, \mathcal{B}') &= \mathcal{M}\left(T^{-1}T, \mathcal{B}\right) \\ &= \mathcal{M}\left(I, \mathcal{B}\right) \end{aligned}$$

does not hold if some diagonal entry of the matrix $\mathcal{M}(T, \mathcal{B}, \mathcal{B}')$ shown on page **237**, last line, is zero.
Now, consider the operator 'T^{-1}' presented in the exercise. Then

$$\begin{aligned} \left(T^{-1}T\right) v &= T^{-1}(Tv) \\ &= T^{-1}\left(\sum_{j=1}^{n} s_j \langle v, e_j \rangle f_j\right) \\ &= \sum_{j=1}^{n} s_j \langle v, e_j \rangle T^{-1}(f_j) \\ &= \sum_{j=1}^{n} s_j \langle v, e_j \rangle \frac{\langle f_j, f_j \rangle e_j}{s_j} \\ &= \sum_{j=1}^{n} \langle v, e_j \rangle e_j \\ &= v \end{aligned}$$

and

$$\begin{aligned} \left(TT^{-1}\right) v &= T\left(T^{-1}v\right) \\ &= T\left(\sum_{j=1}^{n} \frac{\langle v, f_j \rangle e_j}{s_j}\right) \\ &= \sum_{j=1}^{n} \frac{\langle v, f_j \rangle T(e_j)}{s_j} \\ &= \sum_{j=1}^{n} \frac{\langle v, f_j \rangle s_j f_j}{s_j} \\ &= \sum_{j=1}^{n} \langle v, f_j \rangle f_j \\ &= v. \end{aligned}$$

Therefore $T^{-1}T = I$ and $TT^{-1} = I$.

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Comment, p. **247**, Proof, 2nd paragraph
By 5.20, p. **144**, concerning the operator

$$(T-\lambda_1 I)^k (T-\lambda_2 I)^n \cdots (T-\lambda_m I)^n ,$$

the factors $(T-\lambda_1 I)^k$, $(T-\lambda_2 I)^n$, etc, commute with each other.
==
Comment, p. **248**, Proof
There is some positive integer p such that $N^p = 0$. Therefore

$$\begin{aligned}\text{null}\left(N^{\dim V}\right) &= \text{null}\left(N^p\right)\\ &= V\end{aligned}$$

by 8.2 and 8.4, pp. **242–3**.
==
EX. 4, p. **250**
Firstly, note that

$$E(\alpha, T) \cap G(\beta, T) = \{0\}\,. \tag{20}$$

In fact, consider $v \in E(\alpha, T) \cap G(\beta, T)$. So

$$\begin{aligned}0 &= (T-\beta I)^{\dim V} v\\ &= (T-\beta I)^{\dim V - 1}(T-\beta I)v\\ &= (\alpha-\beta)(T-\beta I)^{\dim V-1} v.\end{aligned}$$

Repeated use of these steps eventually give

$$0 = (\alpha-\beta)^{\dim V} v,$$

and since $\alpha \neq \beta$, we must have $v = 0$. Then (20) holds.
Now consider

$$G(\alpha, T) \cap G(\beta, T) = W.$$

By 8.21, (b), $T|_W \in \mathcal{L}(W)$. As such, if $W \neq \{0\}$, **provided that V is a complex vector space**, $T|_W$ has at least one eigenvalue with an associated eigenvector $v \in W$, which contradicts the fact that $G(\beta, T)$ contains all the eigenvectors of T corresponding to β but no other eigenvectors (by (20)). Therefore $W = \{0\}$.
==
EX. 5, p. **250**
Firstly, note that:

- $v \neq 0$ since
$$T^{m-1} v \neq 0; \tag{21}$$
- $T^m v = 0$ implies that
$$T^{m+j} v = 0,\ j = 0, 1, 2, \ldots. \tag{22}$$

Now consider

$$\sum_{j=0}^{m-1} a_{j+1} T^j v = 0 \tag{23}$$

with $a_{j+1} \in \mathbf{F}$, $j = 0, 1, \ldots, m-1$. Then

$$\begin{aligned}T^{m-1}\left(\sum_{j=0}^{m-1} a_{j+1} T^j v\right) = T^{m-1} 0 &\Longrightarrow \sum_{j=0}^{m-1} a_{j+1} T^{m-1+j} v = 0\\ &\Longrightarrow a_1 T^{m-1} v = 0\\ &\Longrightarrow a_1 = 0\end{aligned}$$

by (22) and (21). Then (23) becomes

$$\sum_{j=1}^{m-1} a_{j+1} T^j v = 0.$$

Therefore

$$\begin{aligned} T^{m-2}\left(\sum_{j=1}^{m-1} a_{j+1} T^j v\right) = T^{m-2}0 &\Longrightarrow \sum_{j=1}^{m-1} a_{j+1} T^{m-2+j} v = 0 \\ &\Longrightarrow a_2 T^{m-1} v = 0 \\ &\Longrightarrow a_2 = 0 \end{aligned}$$

by (22) and (21). Continuing with the same reasoning, we also have

$$a_3 = \cdots = a_m = 0.$$

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EX. 7, p. **250**

Consider $\lambda \in \mathbf{F}$, $0 \neq v \in V$ and $N^p = 0$ for a positive integer p. Therefore

$$\begin{aligned} Nv = \lambda v &\Longrightarrow N^p v = \lambda^p v \\ &\Longrightarrow \lambda^p v = 0 \\ &\Longrightarrow \lambda = 0. \end{aligned}$$

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Comment, p. **252**, Proof

By 5.20, p. **144**,

$$p(T)T = Tp(T).$$

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Comment, p. **253**, Proof, 4th sentence

Firstly, note that $G(\lambda_j, T)$ is invariant under $T - \lambda_j I$ by (b). Therefore

$$(T - \lambda_j I)|_{G(\lambda_j,T)} \in \mathcal{L}(G(\lambda_j, T)).$$

Now, if $v \in G(\lambda_j, T)$, that is, if

$$(T - \lambda_j I)^{\dim V} v = 0,$$

then

$$\begin{aligned} \left((T - \lambda_j I)|_{G(\lambda_j,T)}\right)^{\dim V} v &= (T - \lambda_j I)^{\dim V} v \\ &= 0, \end{aligned}$$

which implies that $(T - \lambda_j I)|_{G(\lambda_j,T)}$ is nilpotent.

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EX. 1, p. **267**

By 8.36, the characteristic polynomial of T belongs to

$$\left\{(t-3)^2(t-5)(t-8), (t-3)(t-5)^2(t-8), (t-3)(t-5)(t-8)^2\right\}.$$

Now use 8.37 and the fact that, by commutativity,

$$\begin{aligned} (T-3I)^2(T-5I)^2(T-8I)^2 &= (T-5I)(T-8I)\left[(T-3I)^2(T-5I)(T-8I)\right] \\ &= (T-3I)(T-8I)\left[(T-3I)(T-5I)^2(T-8I)\right] \\ &= (T-3I)(T-5I)\left[(T-3I)(T-5I)(T-8I)^2\right]. \end{aligned}$$

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EX. 2, p. **267**
By 8.36, the characteristic polynomial of T belongs to

$$\left\{(t-5)^{n-i}(t-6)^{i} \;:\; i=1,\ldots,n-1\right\}.$$

Now use 8.37 and the fact that, by commutativity,

$$(T-5I)^{n-1}(T-6I)^{n-1}=(T-5I)^{i-1}(T-6I)^{n-(i+1)}\left[(T-5I)^{n-i}(T-6I)^{i}\right],\; i=1,\ldots,n-1.$$

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Erratum, p. **277**, 9.4, Proof, l. -4
'Im_n' should be '$\text{Im}\,\lambda_n$'.
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Comment, p. **279**, 9.10, Proof, 2nd sentence
Consider $p(t) = \sum_{j=0}^{m} a_j t^j$ with $a_j \in \mathbf{R}$, $j = 0, 1, \ldots, m$, and $u, v \in V$. Therefore

$$\begin{aligned}
\big(p(T)\big)_{\mathbf{C}}(u + iv) &= p(T)u + ip(T)v \\
&= \sum_{j=0}^{m} a_j T^j u + i \sum_{j=0}^{m} a_j T^j v \\
&= \sum_{j=0}^{m} a_j \left(T^j u + iT^j v\right) \\
&= \sum_{j=0}^{m} a_j \left(T_{\mathbf{C}}\right)^j (u + iv) \\
&= p(T_{\mathbf{C}})\,(u + iv).
\end{aligned}$$

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Erratum, p. **284**, 9.23, Proof, last sentence
'8.36(a)' should be '8.36(b)'.
==
EX. 8, p. **285**
On the one hand, 5 and 7 are eigenvalues of $T_{\mathbf{C}}$ by 9.11. On the other hand, the sum of the multiplicities of all the eigenvalues of $T_{\mathbf{C}}$ equals 3 by 8.26. So there are only two possibilities:

1. 5 and 7 are the eigenvalues of $T_{\mathbf{C}}$, one of multiplicity 1 and one of multiplicity 2;
2. 5, 7 and another scalar λ are the eigenvalues of $T_{\mathbf{C}}$, each with multiplicity 1.

By 9.16, $\lambda \in \mathbf{R}$.
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EX. 11, p. **286**
Consider $q(t) = t^2 + bt + c$ and let p denote the minimal polynomial of T. So, by 8.46, either $q = p$ or there is $s \in \mathcal{P}(\mathbf{R})$ such that $q = ps$.[39] Therefore, by 8.49,

$$\begin{aligned}
T \text{ has an eigenvalue} &\iff p \text{ has a zero} \\
&\iff b^2 - 4c \geq 0.
\end{aligned}$$

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Comment, p. **288**, penultimate paragraph before ■

$$\mathcal{M}(T, (e_1, e_2)) = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$$

$$\Downarrow$$

$$\begin{aligned}
Te_1 &= ae_1 + be_2 \\
&= ae_1 + (-b)(-e_2), \\
Te_2 &= -be_1 + ae_2 \\
&= (-1)\left(be_1 + a\left(-e_2\right)\right)
\end{aligned}$$

$$\Downarrow$$

[39] In this case, p and s are monic polynomials of degree one.

$$\mathcal{M}(T,(e_1,-e_2)) = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}.$$

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EX. 4, p. **294**

For $u, v, x, y \in V$,

$$\begin{aligned}
\big\langle T_{\mathbf{C}}(u+iv), x+iy\big\rangle &= \langle Tu + iTv, x+iy\rangle \\
&= \langle Tu, x\rangle + \langle Tv, y\rangle + (\langle Tv, x\rangle - \langle Tu, y\rangle)i \\
&= \langle u, Tx\rangle + \langle v, Ty\rangle + (\langle v, Tx\rangle - \langle u, Ty\rangle)i \\
&= \langle u+iv, Tx+iTy\rangle \\
&= \big\langle u+iv, T_{\mathbf{C}}(x+iy)\big\rangle.
\end{aligned}$$

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Comment, 10.10, p. **299**
Because the use of Determinants to obtain eigenvalues is established in the next section, **10.B**, in order to verify here that $\lambda \in \{1, 2 \pm 3i\}$ is an eigenvalue of T, we may determine a basis (v_λ) of null$(T - \lambda I)$ and verify that

$$\begin{pmatrix} 3-\lambda & -1 & -2 \\ 3 & 2-\lambda & -3 \\ 1 & 2 & -\lambda \end{pmatrix} v_\lambda = \lambda v_\lambda.$$

For example, consider $\lambda = 1$. So the reduced row echelon form of

$$\begin{pmatrix} 2 & -1 & -2 \\ 3 & 1 & -3 \\ 1 & 2 & -1 \end{pmatrix}$$

is

$$\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

and

$$v_\lambda = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}.$$

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EX. 4, p. **304**

$$\begin{aligned} \mathcal{M}(T, (v_1, \ldots, v_n)) &= \mathcal{M}(IT, (v_1, \ldots, v_n), (v_1, \ldots, v_n)) \\ &= \mathcal{M}(I, (u_1, \ldots, u_n), (v_1, \ldots, v_n)) \mathcal{M}(T, (v_1, \ldots, v_n), (u_1, \ldots, u_n)) \text{ (by 10.4)} \\ &= \mathcal{M}(I, (u_1, \ldots, u_n), (v_1, \ldots, v_n)) I. \end{aligned}$$

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EX. 13, p. **305**
By 10.9, 10.13 and 10.16, we may consider the eigenvalue λ of T such that

$$-48 + 24 + \lambda = 51 + (-40) + 1.$$

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EXERCISES 14–5, p. **305**

$$\begin{aligned} \operatorname{trace}(cT) &= \operatorname{trace} \mathcal{M}(cT) \\ &= \operatorname{trace}(c\mathcal{M}(T)) \\ &= c \operatorname{trace} \mathcal{M}(T) \\ &= c \operatorname{trace} T, \end{aligned}$$

where the first and last equalities come from 10.16, the second equality comes from 3.38, and the third equality is trivial.

$$\begin{aligned} \operatorname{trace}(ST) &= \operatorname{trace} \mathcal{M}(ST) \\ &= \operatorname{trace}(\mathcal{M}(S)\mathcal{M}(T)) \\ &= \operatorname{trace}(\mathcal{M}(T)\mathcal{M}(S)) \\ &= \operatorname{trace} \mathcal{M}(TS) \\ &= \operatorname{trace}(TS), \end{aligned}$$

where the first and last equalities come from 10.16, the second and penultimate equalities come from 3.43, and the third equality comes from 10.14.

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Comment, 10.26, Solution, 3rd paragraph, 1st sentence, p. **310**
Here $P(j)$ represents the claim
$$T^n v_j = a_1 \cdots a_n v_j \text{ for each } j.$$
So $P(1)$ is valid since $T^{n-1}v_1 = a_1 \cdots a_{n-1}v_n$ implies that
$$\begin{aligned} T^n v_1 &= a_1 \cdots a_{n-1} T v_n \\ &= a_1 \cdots a_{n-1} a_n v_1. \end{aligned}$$
Now suppose $P(j-1)$ holds for a fixed $j \in \{2, \dots, n\}$. Therefore
$$\begin{aligned} Tv_{j-1} = a_{j-1}v_j &\implies T^n v_{j-1} = a_{j-1}T^{n-1}v_j \\ &\implies a_1 \cdots a_n v_{j-1} = a_{j-1}T^{n-1}v_j \\ &\implies a_1 \cdots a_{j-2}a_j \cdots a_n v_{j-1} = T^{n-1}v_j \\ &\implies a_1 \cdots a_n v_j = T^n v_j \\ &\implies P(j) \text{ holds.} \end{aligned}$$
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Erratum, 10.54, Proof, 1st sentence, p. **326**
'V' should be '$\mathbf{R}^n$'.
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EX. 1, p. **330**
T is assumed to have no real eigenvalues. So $T_{\mathbf{C}}$ has no real eigenvalues by 9.11. However, by 5.21 and 9.16, $T_{\mathbf{C}}$ has a pair of complex conjugate eigenvalues. Therefore, by 10.20, $\det T = \det T_{\mathbf{C}}$ is a product of positive real numbers, with each number expressed in the form $\lambda\bar{\lambda}$ where λ is a complex eigenvalue of $T_{\mathbf{C}}$.
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EX. 8, p. **331**
Suppose firstly that $\lambda_1, \dots, \lambda_n$ are the eigenvalues of T (or of $T_{\mathbf{C}}$ if V is a real vector space) with each eigenvalue repeated according to its multiplicity. So
$$\begin{aligned} \overline{\det T} &= \overline{\lambda_1 \cdots \lambda_n} \\ &= \overline{\lambda_1} \cdots \overline{\lambda_n} \\ &= \det T^* \end{aligned}$$
by definition and by exercise **2** in **SECTION 7.A**.[40] Therefore
$$\begin{aligned} |\det T|^2 &= \det T\, \overline{\det T} \\ &= \det T \det T^* \\ &= \det (TT^*) \\ &= \det\left(\sqrt{TT^*}\sqrt{TT^*}\right) \\ &= \det\left(\sqrt{TT^*}\right) \det\left(\sqrt{TT^*}\right) \\ &= \left(\det\left(\sqrt{TT^*}\right)\right)^2 \end{aligned}$$
where the first equality holds because $|z|^2 = z\bar{z}$ for each complex number z, the third and fifth equalities hold by 10.44, p. **320**, and the fourth equality holds by the remarks made before 10.46, p. **322**.
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EX. 9, p. **331**
Firstly, in 10.56, p. **327**, replace T with $T_i \in \mathcal{L}(\mathbf{R}^n)$, $i = 1, 2$. Now, by the Triangle Inequality,[41]
$$||T_1 y - T_2 y|| \leq ||\sigma(x+y) - \sigma(x) - T_2 y|| + ||T_1 y - \sigma(x+y) + \sigma(x)||$$

[40] See pages **307** and **214**.
[41] 6.18, p. **173**.

for $x \in \Omega$ and $y \in \mathbf{R}^n$ such that $x + y \in \Omega$. So

$$\lim_{y \to 0} \frac{||(T_1 - T_2)(y)||}{||y||} = 0.$$

Then, in particular, for each nonzero vector y and all sufficiently small nonzero real number t,

$$\begin{aligned} 0 &= \lim_{t \to 0} \frac{||(T_1 - T_2)(ty)||}{||ty||} \\ &= \lim_{t \to 0} \left(\frac{|t|}{|t|} \right) \left(\frac{||(T_1 - T_2)(y)||}{||y||} \right) \\ &= \frac{||(T_1 - T_2)(y)||}{||y||}. \end{aligned}$$

Hence $(T_1 - T_2)(y) = 0$ for each $y \neq 0$. Therefore, along with the fact that every linear transformation maps the zero vector to the zero vector, $T_1 - T_2 = 0$.

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EX. 10, p. **331**

By 10.56, p. **327**, $T \in \mathcal{L}(\mathbf{R}^n)$ is differentiable at $x \in \mathbf{R}^n$ if

$$\lim_{y \to 0} \frac{||T(x+y) - Tx - Ly||}{||y||} = 0 \tag{24}$$

for an operator $L \in \mathcal{L}(\mathbf{R}^n)$. Furthermore, by **EX. 9**, p. **331**, this $L := T'(x)$ is unique. So, since

$$x, y \in \mathbf{R}^n \Longrightarrow ||T(x+y) - Tx - Ty|| = 0,$$

(24) holds and $T'(x) = T$ for each $x \in \mathbf{R}^n$.

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