CHAOS

FIRST EDITION (2003), FIRST PRINTING

Banks, Dragan and Jones

PARTIAL SCRUTINY,

SOLUTIONS OF SELECTED EXERCISES,

COMMENTS, SUGGESTIONS AND ERRATA

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Ex. 3.1.4, p. 48
Consider $S = \{x \in \mathbb{R} \mid x \neq 0\}$ and $S \ni x \mapsto f(x) = -x$.
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Ex. 3.1.5, p. 48
(a) Nope since

$$\begin{aligned} f^3(x_0) &= f\left(f^2(x_0)\right) \\ &= f^1(x_0) \quad \text{(since } x_0 \text{ is a period-2 point)} \\ &\neq x_0 \quad \text{(since 2 is the prime period of } x_0\text{).} \end{aligned}$$

(b) Yep since

$$\begin{aligned} f^4(x_0) &= f^2\left(f^2(x_0)\right) \\ &= f^2(x_0) \\ &= x_0. \end{aligned}$$

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Ex. 3.1.10, p. 49
Since

$$\begin{aligned} x_0 &= f^n(x_0) \quad \text{(since } x_0 \text{ is a period-}n\text{ point)} \\ &= f^r\left(\left(f^p\right)^q(x_0)\right) \quad \text{(since } n = pq + r\text{)} \\ &= f^r(x_0) \quad \text{(since } x_0 \text{ is a period-}p\text{ point),} \end{aligned}$$

p is the prime period of x_0 and $0 \leq r < p$, it follows that $r = 0$. Then $n = qp$.
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Ex. 3.1.12, p. 49
For $i \in \{0, \ldots, n-1\}$, one has

$$\begin{aligned} f^n(x_i) &\underbrace{\textbf{Ex. 2.3.5}\text{, p. 42}}_{=} f^i(x_n) \\ &\underbrace{x_0 \text{ is a period-}n \text{ point; } \textbf{Theo. 2.3.3}\text{, p. 41}}_{=} f^i(x_0) \\ &\underbrace{\textbf{Theo. 2.3.3}\text{, p. 41}}_{=} x_i. \end{aligned}$$

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Ex. 3.2.6, p. 56
Since $f(a_i) = a_i$ for each $i \in \{1, \ldots, k\}$, it follows that $f^n(a_i) - a_i = 0$ for each $n \in \mathbb{N}_0$. Hence there exists some $g(x) \in \mathbb{R}[x]$ such that $f^n(x) - x = (x - a_1) \cdots (x - a_k)\, g(x)$. Thus $\text{degree}(g(x)) + k = \text{degree}\,(f^n(x) - x) = nd$.
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Ex. 3.3.1, p. 59
On the one hand, $f(0) = 0$. On the other hand:

$$\begin{aligned} f(x) = 0 &\Longrightarrow x \in \{0, 1\} \\ &\Longrightarrow f(1) = 0; \end{aligned}$$

$$f(x) = 1 \Longrightarrow x = \frac{1}{2}$$
$$\Longrightarrow f(1/2) = 1;$$

$$f(x) = \frac{1}{2} \Longrightarrow x = \frac{1}{2} \pm \frac{1}{2\sqrt{2}}$$
$$\Longrightarrow f\left(\frac{1}{2} \pm \frac{1}{2\sqrt{2}}\right) = \frac{1}{2}.$$

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Ex. 3.3.2, p. 59

(b) *Each $x_0 \in S$ is an eventually periodic point of $f : S \to S \Rightarrow S$ is a set with only finitely many elements.*

(c) False! Consider, for instance, $\mathrm{id} : \mathbb{R} \to \mathbb{R}$.

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Ex. 3.3.3, p. 59

(a) See **Def. 3.3.1**, p. 57.

(b) Use the contrapositive of $\boxed{x_{k-1} = x_{k-1+n} \Rightarrow k \neq \min\left\{l \in \mathbb{N}_0 \;:\; f^l(x_0) = f^{l+n}(x_0)\right\}}$.

(c) Use that f is 1-1.

(d) $k = 0$.

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Erratum, **Fig. 4.3.2**, p. 70

The arrows are reversed!

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Ex. 5.1.3, p. 82

See Figure 1.

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Ex. 5.2.1, p. 85

Suppose $f : S \to S$ is differentiable at $p \in S$ and $|f'(p)| > 1$. Then there is a number $a > 1$ and an open-in-S interval I such that, for all $x \in I$,

$$|f(x) - f(p)| \geq a|x - p|.$$

Proof:

Choose $1 < a < |f'(p)|$. Hence either $f'(p) < -a$ or $f'(p) > a$.

Consider $\epsilon > 0$. Thus there is an interval I containing p such that for all $x \in I - \{p\}$,

$$\frac{f(x) - f(p)}{x - p} \in (f'(p) - \epsilon, f'(p) + \epsilon).$$

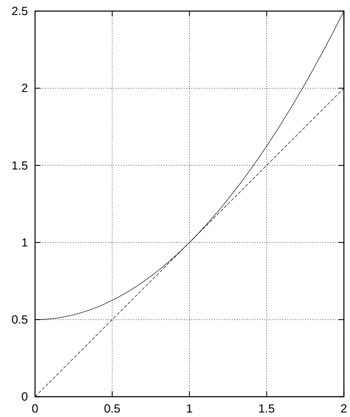

Figure 1: Graphs of $f(x) = \frac{1}{2}(x^2+1)$ and its tangent mapping $\tau_1(x) = f(1) + f'(1)(x-1) = 1 + 1(x-1) = x$ at the only one fixed point 1.

Take $\epsilon < f'(p) - a$ for $f'(p) > a$. Therefore $\frac{f(x)-f(p)}{x-p} > f'(p) - \epsilon > a$.
Take $\epsilon < -f'(p) - a$ for $f'(p) < -a$. Therefore $\frac{f(x)-f(p)}{x-p} < f'(p) + \epsilon < -a$.
Thus, for all $x \in I - \{p\}$,

$$\left| \frac{f(x) - f(p)}{x - p} \right| > a.$$

Thus, for all $x \in I$,

$$|f(x) - f(p)| \geq a \, |x - p| \, .$$

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Ex. 5.2.2, p. 85
Ex. 5.2.1 gives an $a > 1$ and an open-in-S bounded interval I containing $p = f(p)$ such that:

- $\forall x_k \in I - \{p\}$, $|x_{k+1} - p| > a|x_k - p|$, in other words, x_{k+1} is further from p than x_k is;
- $\exists b > 0$ such that $I \subsetneq [-b, b]$ and, $\forall x_k \in I$, $|x_k - p| < b$.

Suppose that $x_k \in I - \{p\}$, $k = 0, 1, \ldots$. Therefore, by the first item, $|x_k - p| > a^k|x_0 - p|$, $k = 1, 2, \ldots$. Since $|x_0 - p|$ is a constant and $\lim_{k\to\infty} a^k = \infty$, by page 64 (line -9 to line -8), there is a positive integer k_0 such that each element of $(|x_{k_0} - p|, |x_{k_0+1} - p|, \ldots)$ exceeds b! This is a contradiction by the last item!
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Ex. 6.1.8, p. 96
(a) $\frac{1}{2}(1 - y_{n+1}) = \frac{1}{2}(1 - y_n)(1 + y_n) \Longrightarrow y_{n+1} = y_n^2$.
(b) $y_n = y_0^{2^n}$, by induction.
(c) $1 - 2x_n = y_n = y_0^{2^n} = (1 - 2x_0)^{2^n} = (2x_0 - 1)^{2^n}$.
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Ex. 6.1.9, p. 96
(a)

$$\begin{aligned}\sin^2(\theta_{n+1}) &= x_{n+1}\\ &= 4x_n(1-x_n)\\ &= 4\sin^2(\theta_n)(1-\sin^2(\theta_n))\\ &= (2\sin(\theta_n)\cos(\theta_n))^2\\ &= \sin^2(2\theta_n).\end{aligned}$$

(b)

$$\begin{aligned}\sqrt{\sin^2(\theta_{n+1})} = \sqrt{\sin^2(2\theta_n)} &\Longrightarrow \sin(\theta_{n+1}) = \sin(2\theta_n)\\ &\Longrightarrow \theta_{n+1} = \arcsin\sin(2\theta_n) = 2\theta_n\\ &\Longrightarrow \theta_n = 2^n\theta_0.\end{aligned}$$

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Ex. 6.1.15-16, p. 97
$\mu > 0 \Longrightarrow C_\mu(0) = C_\mu(1) = 0$ and $C_\mu(1/2) = 1$;
$x \in (0,1) - \{1/2\} \Longrightarrow 0 < |2x-1| < 1 \Longrightarrow \lim_{\mu\to a} C_\mu(x) = 1 - \lim_{\mu\to a}|2x-1|^\mu = \; 0 \text{ for } a = 0 \text{ and } 1 \text{ for } a = \infty.$
$\therefore$ graph of $f = \{(x,0) : x \in [0,1] - \{1/2\}\} \cup \{(1/2\,,\,1)\}$ and graph of $g = \{(x,1) : x \in (0,1)\} \cup \{(0,0),(1,0)\}$.
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Ex. 6.1.17, p. 97
(a) Proof by Induction:

- $x_0 = x_0\mu^0$;
- $-\mu x_0^2 < 0 \Longrightarrow \mu x_0 - \mu x_0^2 < \mu x_0 \Longrightarrow x_1 < x_0\mu^1$;
- $x_{n+1} = \mu x_n - \mu x_n^2 \underbrace{\leq}_{-\mu x_n^2 \leq 0} \mu x_n \underbrace{\Longrightarrow}_{x_n \leq x_0\mu^n \text{ (Induction Hypothesis)}} x_{n+1} \leq x_0\mu^{n+1}.$

(b) $\lim_{n\to\infty}\mu^n \underbrace{=}_{\mu > 1} \infty$ and x_0 is a negative constant $\underbrace{\Longrightarrow}_{x_n \leq x_0\mu^n} \lim_{n\to\infty} x_n = -\infty$.
(c) $x_0 > 1 \Longrightarrow x_0^2 > x_0 \Longrightarrow \mu x_0^2 > \mu x_0 \Longrightarrow x_1 = \mu x_0 - \mu x_0^2 < 0 \overset{\text{(b)}}{\Longrightarrow} (x_0, x_1, x_2\ldots)$ diverges to $-\infty$.
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Erratum, p. 101
After **Stability**, "Recall from Definition 5.1.5 ..." should be "Recall from Definition 5.1.3 ...".
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Ex. 6.2.2, p. 104
(a) $f'_\mu(x) = 2\mu x(1-x) - \mu x^2 = -\mu x(3x-2) = 0$ iff $x \in \{0, \frac{2}{3}\}$.
(b) $f''_\mu(x) = -\mu(3x-2) - \mu x \cdot 3 = -2\mu(3x-1)$. Thus $x = \frac{1}{3}$ is the inflection point and, since $f''_\mu(0) > 0$ and $f''_\mu(2/3) < 0$, 0 is a minimum and 2/3 is a maximum. Therefore, since $f_\mu(0) = 0$ and $f_\mu(2/3) = \frac{4\mu}{27}$, $0 \leq f_\mu(x) \leq 1$ for $0 \leq \mu \leq \frac{27}{4}$.
(c) $\mu x^2(1-x) = x$ iff $-x(\mu x^2 - \mu x + 1) = 0$ iff $x \in \left\{0, \frac{\mu\pm\sqrt{\mu^2-4\mu}}{2\mu}\right\}$.
(d)

$$\begin{aligned}f'_\mu\left(\frac{1\pm\sqrt{1-4/\mu}}{2}\right) &= -3\mu\left(\frac{\mu\pm\sqrt{\mu^2-4\mu}}{2\mu}\right)^2 + 2\mu\cdot\frac{\mu\pm\sqrt{\mu^2-4\mu}}{2\mu}\\ &= -3\cdot\frac{\mu\pm2\sqrt{\mu^2-4\mu}+\mu-4}{4} + \mu\pm\sqrt{\mu^2-4\mu}\\ &= 3 - \frac{\mu}{2} \mp \frac{\sqrt{\mu^2-4\mu}}{2}.\end{aligned}$$

(e) Notice that, $\forall \mu \in \left[0, \frac{27}{4}\right]$, 0 is an attractor since $|f'_\mu(0)| = 0 < 1$. Furthermore, for some time after $\mu = 4$, $(1 - \sqrt{1 - 4/\mu})/2$ is an atractor and $(1 + \sqrt{1 - 4/\mu})/2$ is a repellor. In fact, suppose

$$4 < \mu \leq \frac{27}{4}.$$

Then, on the one hand,

$$0 < \mu^2 - 4\mu \leq \frac{11 \cdot 27}{16} \Longrightarrow 0 < \frac{\sqrt{\mu^2 - 4\mu}}{2} \leq \frac{\sqrt{18.5625}}{2} \approx 2.154.$$

On the other hand,

$$-\frac{3}{8} = 3 - \frac{27}{8} \leq 3 - \frac{\mu}{2} < 3 - 2 = 1.$$

Now consider the previous item (d).

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Ex. 6.2.4, p. 105

(a) Since

$$g'_\mu(x) = \mu \left(\frac{1-x}{1+x} \right) + \mu x \left(-\frac{2}{(1+x)^2} \right) = -\mu \left(\frac{x^2 + 2x - 1}{(1+x)^2} \right) = 0 \Longleftrightarrow$$

$$p(x) = x^2 + 2x - 1 = 0 \Longleftrightarrow x = \frac{-2 \pm \sqrt{8}}{2} \overbrace{\Longleftrightarrow}^{x \in [0,1]} x = \sqrt{2} - 1,$$

and

$$g''_\mu(x) = -\mu \left(\frac{2(x+1)((1+x)^2 - (x^2 + 2x - 1))}{(1+x)^4} \right) \overbrace{\Longrightarrow}^{p(\sqrt{2}-1) = 0} g''_\mu(\sqrt{2} - 1) < 0,$$

we have that g_μ has its maximum value at $x = \sqrt{2} - 1$, where

$$g_\mu(\sqrt{2} - 1) = \mu(\sqrt{2} - 1) \left(\frac{2 - \sqrt{2}}{\sqrt{2}} \right) = \mu(\sqrt{2} - 1)(\sqrt{2} - 1) = \mu(3 - 2\sqrt{2}).$$

(b) Since $g_\mu(0) = g_\mu(1) = 0$ and $g_\mu(x) > 0$ for $x \in (0,1)$, if $0 \leq \mu \leq 3 + 2\sqrt{2}$, then $0 \leq \mu(3 - 2\sqrt{2}) \leq (3 + 2\sqrt{2})(3 - 2\sqrt{2})$, that is, $0 \leq g_\mu(\sqrt{2} - 1) \leq 1$.

(c) $g_\mu(0) = 0$ and if $x \neq 0$ then

$$\mu x \left(\frac{1-x}{1+x} \right) = x \Longrightarrow \mu \left(\frac{1-x}{1+x} \right) = 1 \Longrightarrow x = \frac{\mu - 1}{\mu + 1} = x_\mu.$$

Observe that if $0 < \mu < 1$, then $x_\mu < 0$, which is a contradiction due to $\operatorname{dom} g_\mu = [0,1]$.

(d) From the resolution of (a), we have $g'_\mu(0) = \mu$ and

$$\begin{aligned} g'_\mu \left(\frac{\mu - 1}{\mu + 1} \right) &= -\mu \left(\frac{\left(\frac{\mu-1}{\mu+1}\right)^2 + 2\left(\frac{\mu-1}{\mu+1}\right) - 1}{\left(1 + \frac{\mu-1}{\mu+1}\right)^2} \right) \\ &= -\mu \left(\frac{(\mu-1)^2 + 2(\mu^2 - 1) - (\mu+1)^2}{4\mu^2} \right) \\ &= 1 - \frac{\mu}{2} + \frac{1}{2\mu}. \end{aligned}$$

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Ex. 6.3.5, p. 111

Use long division.

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Ex. 6.3.7, p. 112
(b)

$$\begin{aligned}\left(Q_\mu^2\right)'(a) &\underbrace{=}_{\text{Exercise 1(b)}} \left(Q_\mu^2\right)'(b)\\ &= \left(-1-\sqrt{(\mu+1)(\mu-3)}\right)\left(-1+\sqrt{(\mu+1)(\mu-3)}\right)\\ &= -((\mu+1)(\mu-3)-1)\\ &= -\left(\mu^2-2\mu+1-5\right).\end{aligned}$$

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Erratum, l. 3, p. 128
7.2.3 should be 7.2.1.
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Erratum, (a), p. 129
"... *tox*$'$, ..." should be "... *to x*$'$, ...".
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Ex. 7.2.3, p. 130
$f^n(z) = 0 \Longrightarrow f^{n+1}(z) = f(0) = 0.$
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Ex. 7.2.4, p. 130
$f^n(x) \underbrace{=}_{\textbf{Lemma 7.2.5}} 1 \Longrightarrow f^{n+1}(x) = f(1) = 0.$
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Ex. 7.2.6, p. 131
(b) Since

$$\begin{aligned}\sin^2(2^n\theta_0) = 0 &\Longleftrightarrow \sin(2^n\theta_0) = 0\\ &\Longleftrightarrow 2^n\theta_0 = k\pi,\ k\in\mathbb{Z},\end{aligned}$$

it follows that $f^n(x_0) = 0$ for $x_0 = \sin^2(k\pi/2^n) = z_k$, $k\in\mathbb{Z}$. As a matter of fact $k\in\left\{0,1,\ldots,2^{n-1}\right\}$: the first z_k is $z_0 = \sin^2(0\pi/2^n) = 0$ and the last one is $z_{2^{n-1}} = \sin^2(2^{n-1}\pi/2^n) = 1$, which are the end points of $[0,1]$.
(c)
For $k\in\left\{1,\ldots,2^{n-1}\right\}$,

$$[z_{k-1}, z_k] = [\sin^2((k-1)\pi/2^n), \sin^2(k\pi/2^n)]$$

is the base of the kth hump of f^n. Thus, from

$$\begin{aligned}\frac{\pi}{2^n} &= \frac{k\pi}{2^n} - \frac{(k-1)\pi}{2^n}\\ &= \arcsin\sqrt{z_k} - \arcsin\sqrt{z_{k-1}},\end{aligned}$$

it follows that

$$\lim_{n\to\infty}\arcsin\sqrt{z_k} = \lim_{n\to\infty}\arcsin\sqrt{z_{k-1}}.$$

Hence, since sin and $\sqrt{-}$ are continuous, $\lim_{n\to\infty}(z_k - z_{k-1}) = 0$ follows from

$$\begin{aligned}
\sqrt{\lim_{n\to\infty} z_k} &= \lim_{n\to\infty} \sqrt{z_k} \\
&= \lim_{n\to\infty} (\sin \arcsin \sqrt{z_k}) \\
&= \sin\left(\lim_{n\to\infty} \arcsin \sqrt{z_k}\right) \\
&= \sin\left(\lim_{n\to\infty} \arcsin \sqrt{z_{k-1}}\right) \\
&= \lim_{n\to\infty} (\sin \arcsin \sqrt{z_{k-1}}) \\
&= \lim_{n\to\infty} \sqrt{z_{k-1}} \\
&= \sqrt{\lim_{n\to\infty} z_{k-1}}.
\end{aligned}$$

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Ex. 7.4.4, p. 139

(a) The set of zeroes of T^1 is

$$\{0,1\} = \left\{\frac{i}{2^{1-1}} \,:\, 0 \le i \le 2^{1-1}\right\}.$$

Now suppose the set of zeroes of T^n is

$$\left\{\frac{i}{2^{n-1}} \,:\, 0 \le i \le 2^{n-1}\right\}.$$

Then, by **Lemma 7.4.1**, x is a zero of T^{n+1} iff $T(x) = \frac{i}{2^{n-1}}$ for $i \in \{0, \dots, 2^{n-1}\}$. Therefore:

- $x \in [0, 1/2] \implies 2x = \frac{i}{2^{n-1}}$, that is, $x = \frac{i}{2^n}$ for $i \in \{0, \dots, 2^{n-1}\}$;
- $x \in (1/2, 0] \implies 2 - 2x = \frac{i}{2^{n-1}}$, that is, $x = \frac{2^n - i}{2^n} = \frac{j}{2^n}$ for $j \in \{2^{n-1}, \dots, 2^n\}$.
  In fact, if $j = 2^n - i$, then

$$\begin{aligned}
0 \le i \le 2^{n-1} &\implies -2^{n-1} \le -i \le 0 \\
&\implies 2^n - 2^{n-1} \le 2^n - i \le 2^n \\
&\implies (2-1)2^{n-1} \le j \le 2^n.
\end{aligned}$$

Therefore the set of zeroes of T^{n+1} is

$$\left\{\frac{i}{2^n} \,:\, 0 \le i \le 2^n\right\}.$$

(b) Each hump of T^n has base lenght $\frac{i+1}{2^{n-1}} - \frac{i}{2^{n-1}} = \frac{1}{2^{n-1}}$.

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Ex. 8.2.6, p. 147

(I) reads as (1), p. 145.

(II) reads as $\boxed{(\exists \delta > 0)\left(\exists y \in \mathbb{R}^{\mathbb{N}}\right)\left(\exists \lim_{k\to\infty} y_k \in \{x\}\right)(\forall k \in \mathbb{N})(\exists n \in \mathbb{N}) \quad |f^n(x) - f^n(y_k)| \ge \delta}$.

(III) reads as $\boxed{(\exists \delta > 0)(\exists n \in \mathbb{N})(\forall I \ni x)(\exists y \in I) \quad |f^n(x) - f^n(y)| \ge \delta}$.

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Ex. 8.2.7, p. 148
(a)
(I), as seen in the previous exercise!
(b)
(III) since it is the negation of $\boxed{(\forall n \in \mathbb{N})(\forall \delta > 0)(\exists I \ni x)(\forall y \in I) \quad |f^n(x) - f^n(y)| < \delta}$, which means that f^n is continuos at x for every $n \in \mathbb{N}$.
(c)
(II) $\overset{\textbf{Ex. 8.2.9}}{\overbrace{\Rightarrow}}$ $\boxed{(\exists \delta > 0)(\forall I \ni x)(\exists y \in I - \{x\})(\exists n \in \mathbb{N}) \quad |f^n(x) - f^n(y)| \geq \delta}$ $\Rightarrow$ (I).
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Ex. 8.2.8, p. 148
(a)
$\boxed{(\exists \delta > 0)(\forall y \in [0,1] - \{x\})(\exists n \in \mathbb{N}) \quad |f^n(x) - f^n(y)| \geq \delta}$.
(b)
sensitivity is a local concept whereas *expansiveness* is a global one!
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Ex. 8.2.9, p. 148
It seems the assumption must end with "... a point of $Y - \{x\}$." and the deduction must have "... $(y_k)_{k=1}^{\infty}$ in $Y - \{x\}$ ...". Otherwise, not only the assumption is always true, but also the deduction follows trivially from $y_k = x$ for $k \in \mathbb{N}$. Hence, by the assumption, for $x \in Y$ and $k \in \mathbb{N}$, if $I_k = Y \cap \left(x - \frac{1}{k}, x + \frac{1}{k}\right)$, there is an $y_k \in I_k - \{x\}$. Thus, since $0 < |y_k - x| < \frac{1}{k}$ for every $k \in \mathbb{N}$, it follows that $y_k \to x$. For the converse, assume there is $y_k \in Y - \{x\}$ for every $k \in \mathbb{N}$ such that $y_k \to x$. Thus, if J is an open interval such that $x \in I = Y \cap J$, there is an index k_0 such that $y_k \in J$ for every $k > k_0$. Hence $y_k \in I - \{x\}$ for $k > k_0$.
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Ex. 8.2.10, p. 148
First observe that the graph of f is given by the union of $[0, 1/2] \times \{0\}$ and the second hump of the graph of Q_4^2 (see **Figure 7.2.1**). Now let $\delta > 0$ and consider $I = [0, 1/2]$, which is not an open-in-$[0,1]$ interval. Thus $|f^n(1/2) - f^n(y)| = 0 < \delta$ for all $y \in I$. (For an open-in-$[0,1]$ interval I containing $\frac{1}{2}$, there is $y \in I \cap (\frac{1}{2}, 1)$ such that $|f^1(y) - f^1(1/2)| = |f(y)| > \delta$, depending on the δ taken!)
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Ex. 8.3.1, p. 154
Choose $\delta = \frac{3}{4}$. Let I be an open-in-$[0,1]$ interval containing 1. Choose $y \in I - \{1\}$. Thus choose n so large that $y^{2^n} \leq \frac{1}{4}$. This gives $|f^n(1) - f^n(y)| = |1 - y^{2^n}| \geq \frac{3}{4} = \delta$.
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Ex. 8.3.2, p. 154
Do the same as Example 8.3.1, replacing y^{2^n} with y^{3^n}.
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Ex. 8.3.3, p. 154
The points are -1 and 1:
Do the same as the last exercise for 1, choosing $y \in I$, which is an open-in-$[-1,1]$, such that $0 < y < 1$.
For -1, choose $\delta = \frac{1}{2}$, $y \in I$ such that $-1 < y < 0$, and $n \in \mathbb{N}$ such that $-\frac{1}{2} \leq y^{3^n} < 0$. Thus $\frac{1}{2} \leq y^{3^n} + 1 < 1$, that is, $\frac{1}{2} \leq |y^{3^n} - (-1)^{3^n}| < 1$.
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Ex. 8.3.4, p. 154

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\begin{aligned}
[-1,1] \ni x \mapsto f_1(x) = x^3 \in [-1,1] &\Longrightarrow [-1,1] \ni x \mapsto f_2(x) = \frac{f_1(x)}{2} = \frac{x^3}{2} \in [-1/2, 1/2] \\
&\Longrightarrow [-1,1] \ni x \mapsto f_3(x) = f_2(x) + \frac{1}{2} = \frac{x^3}{2} + \frac{1}{2} \in [0,1] \\
&\Longrightarrow [0,2] \ni x \mapsto f_4(x) = f_3(x-1) = \frac{(x-1)^3}{2} + \frac{1}{2} \in [0,1] \\
&\Longrightarrow [0,1] \ni x \mapsto f(x) = f_4(2x) = \frac{(2x-1)^3}{2} + \frac{1}{2} \in [0,1].
\end{aligned}
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Ex. 8.3.5, p. 154
Let $\delta > 0$. Thus $|y^{2^n}| < \delta$ for all $y \in I = [0,1] \cap (-\delta, \delta)$ and each $n \in \mathbb{N}$.
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Ex. 8.3.6, p. 154
(a)
Let $\delta > 0$. Thus $|f^n(y) - f^n(1)| = |y - 1| < \delta$ for all $y \in I = [0,1] \cap (1-\delta, 1+\delta)$ and each $n \in \mathbb{N}$.
(b)
Let $x \neq 1$ and $\delta > 0$. Thus $|f^n(y) - f^n(x)| = |y - x| < \delta$ for all $y \in I = [0,1] \cap (x-\delta, x+\delta)$ and each $n \in \mathbb{N}$.
==
Ex. 8.3.7, p. 154
First observe that all the iterates of f lie below the graph of id. Now use that id is not sensitive dependent anywhere. (See p. 152 and the last exercise).
==
Ex. 8.3.8, p. 154
Let $x \in [0,1]$, $\delta > 0$, $y \in I = [0,1] \cap (x-\delta, x+\delta)$ and $n \in \mathbb{N}$. Therefore, since

- $f^i(z) \in [0,1]$ and $|f'(f^i(z))| < 1$ for $i = 0, 1, \ldots, n-1$,
- a composition of differentiable functions is a differentiable function,
  and
- $(f^n)'(z) = f'(z) f'(f(z)) f'(f^2(z)) \cdots f'(f^{n-1}(z))$

for each $z \in [0,1]$, it follows from **The Mean Value Theorem** that $|f^n(x) - f^n(y)| = |(f^n)'(z_n)||x - y| < \delta$ for some $z_n \in (\min\{x,y\}, \max\{x,y\})$.
==
Ex. 8.3.9, p. 154
If f is differentiable on $[0,1]$, then there is an $x \in (0,1)$ such that $f(1) - f(0) = f'(x)(1-0)$. Thus, since $0 \leq \min\{f(0), f(1)\} \leq \max\{f(0), f(1)\} \leq 1$, there is an $x \in (0,1)$ such that $f'(x) \leq 1$.
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Ex. 9.1.1, p. 160
Use **Example 7.2.6** on p. 129.
==
Ex. 9.1.2, p. 160
(a) Take $x \in \{0,1\}$.
(b) Choose $y \in I$ and $n \in \mathbb{N}$ such that $f^n(y) = 1$.
==
Ex. 9.1.3, p. 160
Let S be the set of all zeroes of all iterates of f (**Lemma 7.3.2**), $x \in S$ and I be an open-in-$[0,1]$ interval containing x. Thus, since there is an $n_0 \in \mathbb{N}$ such that $f^n(x) = 0$ for every $n \geq n_0$, choose $y \in I$ and $n \geq n_0$ such that$f^n(y) = 1$.
==
Ex. 9.1.4, p. 160
(a) The **Spike Lemma**.
(b) It follows from the **'Wiggly $\Leftrightarrow$ dense sets of zeroes' Lemma** (see p. 133). In fact, if $I \subseteq [0,1]$ and S is the set of all zeroes of all iterates of f, since f is spreading, there is an $n \in \mathbb{N}$ and there is an $z \in I$ such that $f^n(z) = 0$, that is, $z \in S$.
==

Ex. 9.2.1, p. 164
None!
For the reasons, observe that $S \subset [0,1]$ is not dense in $[0,1]$ if S is a finite set (just take an interval $I \subset [0,1]$ such that $S \cap I = \emptyset$), and
(a) a periodic orbit is a finite set $S = \{x_0, x_1, \ldots, x_{n-1}\}$ such that $f(x_{n-1}) = x_0$,
(b) an eventually periodic orbit is a finite set $S = \{x_0, x_1, \ldots, x_k, x_{k+1}, \ldots, x_{k+n-1}\}$ such that $f(x_{k+n-1}) = x_k$,
(c) for every open interval $J \subset [0,1]$ containing a fixed point p, an orbit converging to p is the union $S \cup S'$ such that $S = \left\{x_0, x_1, \ldots, x_{n_0-1}\right\}$ and $S' = \left\{x_{n_0}, x_{n_0+1}, \ldots\right\} \subset J$. (Here, take I such that $I \cap J = \emptyset$ as well).
==
Ex. 9.2.2, p. 164
$(\exists I)(\exists J)(\forall n \in \mathbb{N}) \quad f^n(I) \cap J = \emptyset$.
Thus, to prove that f is not transitive, choose a pair of subintervals I and J of $[0,1]$ such that $f^n(I) \cap J = \emptyset$ for all $n \in \mathbb{N}$.
==
Ex. 9.2.3, p. 164
Choose a pair of subintervals I and J of $[0,1]$ such that $I \cap J = \emptyset$ and let $f = \mathrm{id}^m$ for $m \in \mathbb{N}$. Thus, since $f^n(I) \cap J = \mathrm{id}^{m+n}(I) \cap J = I \cap J = \emptyset$ for all $n \in \mathbb{N}$, $f = \mathrm{id}^m$ is not transitive for $m \in \{1, 2, \ldots\}$.
==
Ex. 9.2.4, p. 164
f is not transitive. In fact, choose $J = [0, 0.5)$ and $I = [0.5, 1]$. Thus, since $f(I) = I$, $f^n(I) \cap J = I \cap J = \emptyset$ for all $n \in \mathbb{N}$. (See Figure 2)

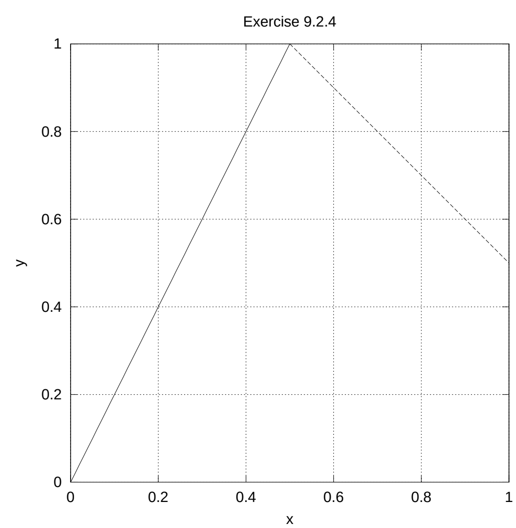

Figure 2: $f(x) = 2x$ for $x \in [0, 0.5]$ and $f(x) = 1.5 - x$ for $x \in [0.5, 1]$.

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Ex. 9.2.5, p. 164
Choose a pair of subintervals I and J of $[0,1]$ such that I is a small enough interval containing the fixed point and $I \cap J = \emptyset$. Thus, since $I \supseteq f(I) \supseteq f^2(I) \supseteq \cdots$, it follows that $f^n(I) \cap J = \emptyset$ for all $n \in \mathbb{N}$.
==
Ex. 9.2.6, p. 164
(a) Let $S_0 = D - \{x_0\}$ with $x_0 \in D$ and let I be a subinterval of $[0,1]$. If $x_0 \notin I$, then $I \cap S_0 \neq \emptyset$ since $I \cap D \neq \emptyset$. Otherwise, if $x_0 \in I$, take another subinterval J of $[0,1]$ such that $J \subset I - \{x_0\}$. Thus $J \cap S_0 \neq \emptyset$ since $J \cap D \neq \emptyset$. Therefore, $I \cap S_0 \neq \emptyset$.
(b) Suppose $S_k = D - \{x_0, \ldots, x_k\}$ is dense in $[0,1]$ if $\{x_0, \ldots, x_k\} \subset D$. Let $S_{k+1} = D - \{x_0, \ldots, x_{k+1}\}$ with $\{x_0, \ldots, x_{k+1}\} \subset D$ and let I be a subinterval of $[0,1]$. If $x_{k+1} \notin I$, then $I \cap S_{k+1} \neq \emptyset$ since $I \cap S_k \neq \emptyset$. Otherwise, if $x_{k+1} \in I$, take another subinterval J of $[0,1]$ such that $J \subset I - \{x_{k+1}\}$. Thus $J \cap S_{k+1} \neq \emptyset$ since $J \cap S_k \neq \emptyset$. Therefore, $I \cap S_{k+1} \neq \emptyset$.
==
Ex. 9.2.7, p. 165
Consider $y_0 \in D = \{x_n \;/\; n \in \mathbb{N}_0\}$ such that $(x_0, x_1, \ldots)$ is a dense orbit and, for $y_0 = x_{n_0}$, let $S = D -$

$\{x_0, \ldots, x_{n_0-1}\}$, which is a dense set from Exercise 6. Thus, if $y_n = x_{n_0+n}$ for all $n \in \mathbb{N}_0$, it follows that $(y_0, y_1, \ldots)$ is a dense orbit.

==

Ex. 9.2.8, p. 165
Let I and J be a pair of subintervals of $[0,1]$ and $(x_0, x_1, \ldots)$ a dense orbit in $[0,1]$. Therefore I contains an element x_{n_0} of the dense orbit. Thus, since $f^k(x_{n_0}) \in f^k(I)$ for all $k \in \mathbb{N}$ and, from Exercise 7, $f^n(x_{n_0}) \in J$ for some $n \in \mathbb{N}$, it follows that $f^n(I) \cap J \neq \emptyset$.

==

Ex. 9.3.1, p. 167
(a) Consider T_4.
(b) See Figure 3.

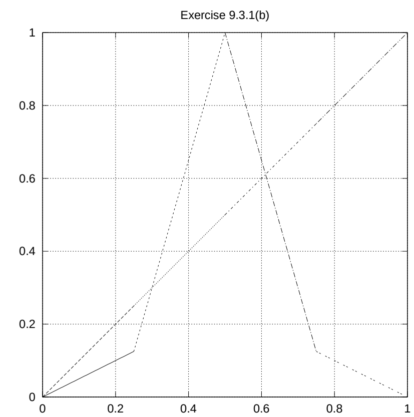

Figure 3: $f(x) = \frac{x}{2}$ for $x \in \left[0, \frac{1}{4}\right]$, $f(x) = \frac{7x}{2} - \frac{3}{4}$ for $x \in \left[\frac{1}{4}, \frac{1}{2}\right]$, $f(x) = -\frac{7x}{2} + \frac{11}{4}$ for $x \in \left[\frac{1}{2}, \frac{3}{4}\right]$, and $f(x) = \frac{-x+1}{2}$ for $x \in \left[\frac{3}{4}, 1\right]$.

(c) See Figure 4.

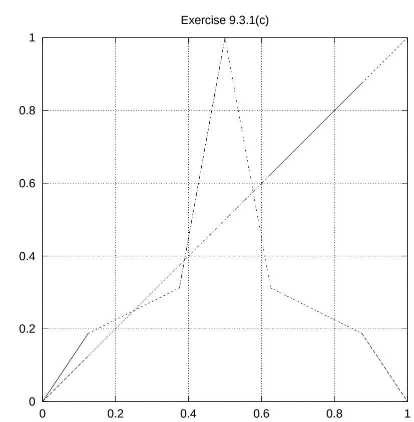

Figure 4: $f(x) = \frac{3x}{2}$ for $x \in \left[0, \frac{1}{8}\right]$, $f(x) = \frac{x}{2} + \frac{1}{8}$ for $x \in \left[\frac{1}{8}, \frac{3}{8}\right]$, $f(x) = \frac{11x}{2} - \frac{7}{4}$ for $x \in \left[\frac{3}{8}, \frac{1}{2}\right]$, $f(x) = -\frac{11x}{2} + \frac{15}{4}$ for $x \in \left[\frac{1}{2}, \frac{5}{8}\right]$, $f(x) = -\frac{x}{2} + \frac{5}{8}$ for $x \in \left[\frac{5}{8}, \frac{7}{8}\right]$, and $f(x) = \frac{-3x+3}{2}$ for $x \in \left[\frac{7}{8}, 1\right]$.

The least possible number of fixed points for a one-hump maps is two.

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Ex. 9.3.2, p. 167
Since both Q_4 and T_4 have wiggly iterates (see **Example 7.2.6**), they are mappings whose periodic points are dense in $[0,1]$. Thus

$$[0,1] \times [0,1] \ni (x_1, x_2) \overset{(Q_4,T_4)}{\mapsto} (Q_4(x_1), T_4(x_2)) \in [0,1] \times [0,1]$$

is a mapping whose periodic points are dense in $[0,1] \times [0,1]$. In fact, let S_{Q_4} and S_{T_4} be the sets of periodic points of Q_4 and T_4, $(x_1, x_2) \in [0,1] \times [0,1]$ and $B((x_1, x_2), r) \subset [0,1] \times [0,1]$ be an open ball of radius r centered at (x_1, x_2). Consider that I_1 and I_2 are intervals such that $(x_1, x_2) \in (I_1 \times I_2) \subset B((x_1, x_2), r)$. Therefore, since $(I_1 \times I_2) \cap \left(S_{Q_4} \times S_{T_4}\right) \neq \emptyset$, it follows that $B((x_1, x_2), r) \cap \left(S_{Q_4} \times S_{T_4}\right) \neq \emptyset$. Finally, $B((x_1, x_2), r) \cap \left(S_{Q_4} \times S_{T_4}\right) \ni (p_1, p_2)$ is a period-$n_1 n_2$ point of (Q_4, T_4) if p_1 is a period-n_1 point of Q_4 and p_2 is a period-n_2 point of T_4.

==

Ex. 9.3.3, p. 167

Since $f^n - \text{id} : [z_i, y_i] \to \mathbb{R}$ is a continuos function, $(f^n - \text{id})(z_i) = -z_i \leq 0$ (since $0 \leq z_i < 1$) and $(f^n - \text{id})(y_i) = 1 - y_i > 0$ (since $0 < y_i < 1$), it follows that there is $x_i \in [z_i, y_i]$ such that $(f^n - \text{id})(x_i) = 0$, that is, $f^n(x_i) = \text{id}(x_i) = x_i$. Since $f^n - \text{id} : [y_i, z_{i+1}] \to \mathbb{R}$ is a continuos function, $(f^n - \text{id})(y_i) = 1 - y_i > 0$ (since $0 < y_i < 1$) and $(f^n - \text{id})(z_{i+1}) = -z_{i+1} < 0$ (since $0 < z_{i+1} \leq 1$), it follows that there is $x_{i+1} \in [y_i, z_{i+1}]$ such that $(f^n - \text{id})(x_{i+1}) = 0$, that is, $f^n(x_{i+1}) = \text{id}(x_{i+1}) = x_{i+1}$.

==

Ex. 9.3.4, p. 167

(a) From Theorem 5.2.1, there is an open-in-$[0,1]$ interval I containing p such that

$$|f^n(x_0) - p| = |x_n - p| < \cdots < |x_2 - p| < |x_1 - p| < |x_0 - p|$$

for all $x_0 \in I - \{p\}$ and all $n \in \mathbb{N}$.

(b) Now suppose $J \subset I - \{p\}$ is an interval and $p_0 \in J$ is a fixed point of f. Thus $|f^n(p_0) - p| = |p_0 - p|$ for all $n \in \mathbb{N}$. This contradicts (a).

(c) See Exercice 9.3.1.(b)-(c).

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Ex. 9.5.1, p. 175

If $\{u, v\} \subset f(I)$ and $w \in [\min\{u, v\}, \max\{u, v\}]$, then there is a $\{x, y\} \subset I$ such that $f(x) = u$ and $f(y) = v$, and the IVT implies that there is a $z \in [\min\{x, y\}, \max\{x, y\}]$ such that $f(z) = w$. Thus $w \in f(I)$.

==

Ex. 9.5.2, p. 175

(a) $\{0, c, 1\} \cap J = \emptyset$ since $f(0) = f^2(c) = f(1) = 0$. Thus either $J \subset (0, c)$ or $J \subset (c, 1)$. Therefore, since f is strictly increasing on $(0, c)$ and strictly decreasing on $(c, 1)$, either f is strictly increasing on J or strictly decreasing on J.

(b) $f(J)$ is an interval contained in $f((0, c)) = f((c, 1)) = (0, 1)$. Thus, if $J \subset (0, c)$, that is, f is strictly increasing on J, since $f(J) \subset f((c, 1))$, then the IVT implies that there exists an interval $K \subset (c, 1)$ such that $f(K) = f(J)$ and f is strictly decreasing on K. Otherwise, if $J \subset (c, 1)$, that is, f is strictly decreasing on J, since $f(J) \subset f((0, c))$, then the IVT implies that there exists an interval $K \subset (0, c)$ such that $f(K) = f(J)$ and f is strictly increasing on K.

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Ex. 10.1.2, p. 184

$f(x) = x^3$. (See Figure 6).

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Ex. 10.1.3, p. 185

(a) Use **Ex. 9.2.5** and Lemma 9.2.2.

(b) 0 is an attracting fixed point since $f'(0) = 0$.

==

Ex. 10.1.4, p. 185

g has an attracting fixed point by **Figure 10.1.5**.

==
Ex. 10.1.6, p. 185
Consider $f(x) = \sin x$ for $x \in I = \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ and $a = 0$. Thus (see Figure 5) $f'(x) = \cos x > 0$ for $x \in I$, $f''(a) = -\sin 0 = 0$ and $f'(a)f'''(a) = (\cos a)(-\cos a) = 1 \cdot (-1) < 0$.

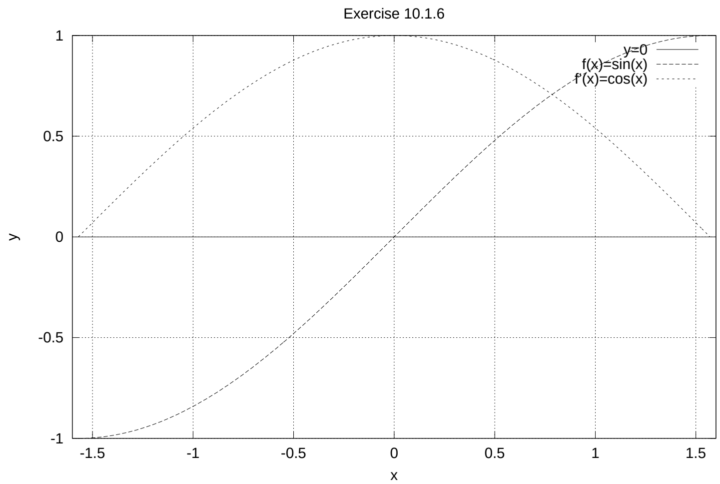

Figure 5: $f(x) = \sin x$ and $f'(x) = \cos x$ for $x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

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Ex. 10.1.7, p. 185
It's similar to the previous exercise.
==
Ex. 10.1.8, p. 185
2^{n-1} zeroes since f^n has 2^{n-1} humps and each one of the humps has its peak correspondingto the maximum of f^n.
==
Ex. 10.2.2, p. 189

$$T_4(x) \overset{\textbf{Ex. 6.1.4}}{=} \begin{cases} 2x, & 0 \le x \le \frac{1}{2} \\ 2(1-x), & \frac{1}{2} \le x \le 1 \end{cases}$$

$$\Downarrow$$

$$T_4'(x) = \begin{cases} 2, & 0 \le x < \frac{1}{2} \\ -2, & \frac{1}{2} < x \le 1 \end{cases}$$

$$\Downarrow$$

$$T_4''(x) = T_4'''(x) = 0,\ \forall x \in [0,1] - \left\{\frac{1}{2}\right\}$$

$$\Downarrow$$

$$S(T_4)(x) = 0,\ \forall x \in [0,1] - \left\{\frac{1}{2}\right\}.$$

==
Ex. 10.2.3, p. 189
Since $f'(x) = 3x^2 = 0$ iff $x = 0$, $f''(x) = 6x$ and $f'''(x) = 6$, it follows that

$$\begin{aligned} S(f)(x) &= 2 \cdot 3x^2 \cdot 6 - 3(6x)^2 \\ &= -72x^2 < 0 \end{aligned}$$

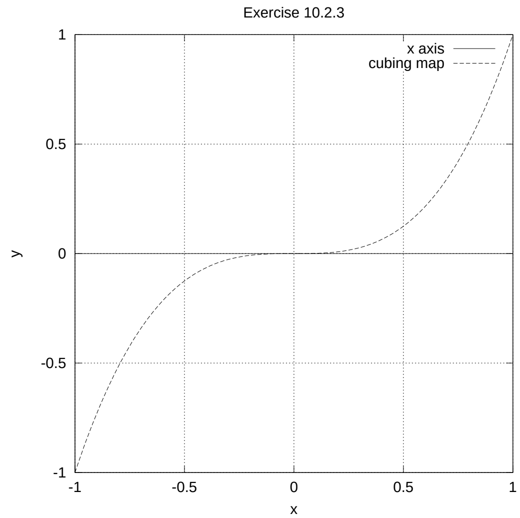

Figure 6: Cubing map.

for all $x \neq 0$. (See Figure 6).

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Ex. 10.2.4, p. 189
If

$$A = f''' \circ g \cdot f' \circ g \cdot (g')^4 + 3f'' \circ g \cdot f' \circ g \cdot g'' \cdot (g')^2 + (f' \circ g)^2 \cdot g''' \cdot g'$$

and

$$B = 2(f'' \circ g)^2 \cdot (g')^4 + 4f'' \circ g \cdot f' \circ g \cdot g'' \cdot (g')^2 + 2(f' \circ g)^2 \cdot (g'')^2,$$

then $S(f \circ g)$ would be equal to

$$A + B = S(f) \circ g \cdot (g')^4 + (f' \circ g)^2 \cdot S(g) + 7f'' \circ g \cdot f' \circ g \cdot g'' \cdot (g')^2.$$

Therefore $S(f \circ g)$ would depend on $\boxed{7f'' \circ g \cdot f' \circ g \cdot g'' \cdot (g')^2}$. However, for $\boxed{S(f) \overset{\textbf{Def. 10.2.2}}{=} f'f''' + 2(f'')^2}$, it is even hard to find an $f : [0,1] \to [0,1]$ such that $S(f) < 0$ holds at all points of $[0,1]$ for which $f' \neq 0$. For example, if $f(x) = \sin \pi x$, then $S(f)(x) = \pi(2 - 3\cos^2 \pi x) < 0$ for x sufficiently close to 0 or for x sufficiently close to 1, whereas $S(f)(x) > 0$ for x sufficiently close to $\frac{1}{2}$. As another example, if $a \neq 0$ and $f(x) = ax^2 + bx + c$, then $S(f)(x) = 4a^2 > 0$ for all x.

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Ex. 10.2.5, p. 189
Concerning the first equality of the next exercise, take $c = -1$.

==

Ex. 10.2.6, p. 189
Use that $S(cf) = c^2 S(f)$ and $S(f + c) = S(f)$.

==

Ex. 10.2.7-8, p. 189
It's similar to the *Proof* of **Theo. 10.2.4**.

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Ex. 10.2.11, p. 190
Let $x \in I$ such that $f'(x) \neq 0$. Thus $(f'(x))^2 > 0$, $\boxed{\overline{S}(f)(x) < 0 \Longrightarrow S(f)(x) = 2(f'(x))^2 \overline{S}(f)(x) < 0}$ and $\boxed{S(f)(x) < 0 \Longrightarrow \overline{S}(f)(x) = \frac{S(f)(x)}{2(f'(x))^2} < 0}$.

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Example, p. 192
Q_μ has negative Schwarzian derivative (except at $x = \frac{1}{2}$) and $Q'_\mu(0) = \mu$. Does Q_μ have chaotic behaviour if $\mu > 1$?

In order to use the **Test for chaos**, p. 191, Q_μ must be a symmetric one-hump mapping (**Def. 7.2.1**, p. 126). Thus $Q_\mu(1/2)$ must be equal to 1, that is, $\mu = 4$.

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Ex. 10.3.1, p. 192
If $f(x) = 1 - (2x-1)^4$, then $f'(x) = -8(2x-1)^3$, $f''(x) = -48(2x-1)^2$ and $f'''(x) = -192(2x-1)$. Hence $f'(0) = 8 > 1$ and $S(f)(x) = (3072 - 6912)(2x-1)^4 < 0$ for $x \neq \frac{1}{2}$.
If $f(x) = \sin(\pi x)$, then $f'(x) = \pi\cos(\pi x)$, $f''(x) = -\pi^2\sin(\pi x)$ and $f'''(x) = -\pi^3\cos(\pi x)$. Hence $f'(0) = \pi > 1$ and $S(f)(x) = -\pi^4(2\cos^2(\pi x) + 3\sin^2(\pi x)) < 0$.

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Erratum, l. -2, p. 213
"... (see Exercise 11.3.4) ..." should be "... (see Exercise 11.3.5) ...".

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Ex. 11.3.4, p. 215
No for both items since

$$\begin{aligned} f(x) &= T_4(x) \\ &= \begin{cases} 2x & \text{if } x \in [0, 1/2] \\ 2 - 2x & \text{if } x \in (1/2, 1] \end{cases} \end{aligned}$$

has two fixed points, whereas

$$g(x) = \begin{cases} 2x + 2 & \text{if } x \in [-1, -1/2] \\ -2x & \text{if } x \in (-1/2, 0] \end{cases}$$

has only one fixed point.

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Ex. 12.1.5, p. 223
For $x \geq 0$, see **Example 12.1.6**. For $x < 0$,

$$\begin{aligned} h(f(x)) &= h\left(\frac{x}{4}\right) \\ &= -\sqrt{-\frac{x}{4}} \\ &= -\frac{\sqrt{-x}}{2} \\ &= \frac{h(x)}{2} \\ &= g(h(x)). \end{aligned}$$

==

Ex. 12.1.6, p. 223
(a) Clearly h is continuous at $x \neq 0$.

What about $\lim_{x\to 0} h(x)$?
Since $\phi(0) \geq 1$ and ϕ is either strictly increasing or strictly decreasing[1], it follows that $\phi(0) = 1$ (and ϕ is strictly increasing, which implies that h is strictly increasing[2], which implies that h is invertible if h is continuous). Thus

$$\lim_{x\to 0^+} h(x) = \lim_{x\to 0^+} \phi(x) = \phi(0) = 1 \text{ and } \lim_{x\to 0^-} h(x) = \lim_{x\to 0^-} \frac{1}{\phi(-x)} \overbrace{=}^{u=-x} \lim_{u\to 0^+} \frac{1}{\phi(u)} = \frac{1}{\phi(0)} = 1.$$

Now, since h is strictly increasing, h^{-1} is continuous[3].
(b) Let $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R}^+ \to \mathbb{R}^+$ with $f(x) = -x$ and $g(x) = \frac{1}{x}$. Then:

- Since $h(0) = \phi(0) = 1$, $h(f(0)) = h(0) = \frac{1}{h(0)} = g(h(0))$;
- If $x > 0$, then $h(f(x)) = h(-x) = \frac{1}{\phi(x)} = g(\phi(x)) = g(h(x))$;
- If $x < 0$, then $h(f(x)) = h(-x) = \phi(-x) = \frac{1}{1/\phi(-x)} = \frac{1}{h(x)} = g(h(x))$.

(c) There are infinitely many such homeomorphisms h since there are infinitely many such strictly increasing maps $\phi : [0, \infty) \to [1, \infty)$.
==
Ex. 12.1.8, p. 224

If $\boxed{f \textit{ is conjugate to } g \textit{ via } h}$ is denoted by $\boxed{f \overset{h}{\sim} g}$, then:
(i) $f \overset{id_I}{\sim} f$;
(ii) $f \overset{h}{\sim} g \Longrightarrow g \overset{h^{-1}}{\sim} f$;
(iii) $f_1 \overset{h_1}{\sim} f_2$ and $f_2 \overset{h_2}{\sim} f_3 \Longrightarrow f_1 \overset{h_2\circ h_1}{\sim} f_3$.
==
Ex. 12.1.9, p. 224
From **Example 12.1.3**,

$$h \circ f = g \circ h \Longrightarrow h \circ f^2 = g^2 \circ h.$$

Therefore

$$\begin{aligned} h \circ f^3 &= h \circ f^2 \circ f \\ &= g^2 \circ h \circ f \\ &= g^2 \circ g \circ h \\ &= g^3 \circ h. \end{aligned}$$

==
Ex. 12.1.10, p. 224
Suppose $h \circ f = g \circ h$ and $h \circ f^n = g^n \circ h$ with $n \in \mathbb{N}$. Thus

$$\begin{aligned} h \circ f^{n+1} &= h \circ f^n \circ f \\ &= g^n \circ h \circ f \\ &= g^n \circ g \circ h \\ &= g^{n+1} \circ h. \end{aligned}$$

==
Ex. 12.2.4, p. 237

$$x \in [z_i, z_{i+1}] \overbrace{\Longrightarrow}^{h \text{ is increasing}} h(x) \in [h(z_i), h(z_{i+1})] \overbrace{=}^{\textbf{Lemma 12.2.5}} [w_i, w_{i+1}].$$

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Lemma 12.3.1, p. 238

[1] From Calculus, if I is an interval and $f : I \to f(I)$ is a continuous function, then f is invertible iff f is either strictly increasing or strictly decreasing;
[2] If $0 \leq x_1 < x_2$, then $h(x_1) = \phi(x_1) < \phi(x_2) = h(x_2)$. If $x_1 < x_2 < 0$, that is $0 < -x_2 < -x_1$, then $h(x_1) = \frac{1}{\phi(-x_1)} < \frac{1}{\phi(-x_2)} = h(x_2)$;
[3] From Calculus, if I is an interval and $f : I \to \mathbb{R}$ is strictly monotone, then $f^{-1} : f(I) \to \mathbb{R}$ is continuous.

"Each time the exponent n is increased by 1,
f^n acquires one new zero between each of the old ones,
as f is a one-hump mapping."

In fact, besides the zeroes of f^n are also zeroes of f^{n+1} (see **Ex. 7.2.3**, p. 130), f^n is a 2^{n-1}-hump mapping, thus it has $2^{n-1}+1$ zeroes, whereas f^{n+1} is a 2^n-hump mapping, thus it has 2^n+1 zeroes.

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Erratum, **Def. 12.3.3**, p. 240
"... where the h^n are ..." should be "... where the h_n are ...".

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Erratum, order of compositions, p. 240 and p. 242
$f \circ h = h \circ g$ should be $h \circ f = g \circ h$.

==

Ex. 12.3.1, p. 245
(a) Let $z = z_i$ be the i$^{\text{th}}$ zero of f^n. Thus $h_n(z) = h_n(z_i) = w_i$ is the i$^{\text{th}}$ zero of g^n, by **Matching Zeroes**. Hence, by **Lemma 12.3.1**, if $m > n$ then $z = z_j$ and $h_n(z) = w_j$ are the j$^{\text{th}}$ zeroes of f^m and g^m for $j > i$, that is, $h_m(z) = h_m(z_j) = w_j = h_n(z)$ by **Matching Zeroes**.
(b) $h(z) = \lim_{n\to\infty} h_n(z)$.
(c) $n = 1, 2, \ldots \Longrightarrow h(0) = h_n(0) = 0,\ h(1) = h_n(1) = 1$.
(d) Since h is continuous, if $u \in [h(0), h(1)]$ then there is a $z \in [0, 1]$ such that $u = h(z)$, by the IVT.

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Ex. 12.3.2, p. 245
If $\epsilon = \frac{1}{2}(h(a) - h_n(z))$ and $m \geq n$, use that h_m is strictly increasing and **Ex. 12.3.1** in order to obtain $h_m(a) < h_m(z) = h_n(z) < h(a) - \epsilon$.

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Ex. 12.4.3, p. 253
(a) **Theo. 12.4.2**.
(b) Since x_0 is a periodic point for f with prime period n and there is a one-to-one correspondence between $X = \{f^i(x_0) \mid i = 0, \ldots, n-1\}$ and $Y = \{h(f^i(x_0)) \mid i = 0, \ldots, n-1\}$, X and Y have the same number (n) of elements. Now suppose that $i < n$ is the prime period of $h(x_0)$ under g. Thus $h(f^i(x_0)) = g^i(h(x_0)) = h(x_0)$ and hence Y has less than n elements!

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Ex. 12.4.5, p. 253
It suffices to prove the result given in the box that precedes **Theo. 12.4.3**. Let D be dense in $[0,1]$ and $I \subseteq [0,1]$ be an interval. Since $h^{-1}(I) \subseteq [0,1]$ is an interval, there is a point $x \in D \cap h^{-1}(I)$. So $h(x) \in h(D) \cap I$. Therefore $h(D)$ is dense in $[0,1]$.

==

Ex. 12.4.7, p. 253
From **Ex. 9.5.1**, it follows that I' and J' are intervals as images of I and J under h^{-1}. Therefore, since the inverse image of an open set under h is open in its domain, which is a basic fact from General Topology, and $h^{-1}(I)$ and $h^{-1}(J)$ are both images under h^{-1} and inverse images under h, which comes from the fact that $h \circ h^{-1}$ and $h^{-1} \circ h$ are both identities, it follows that I' and J' are open intervals.
Let $x \in J' \cap f^n(I')$. Thus $y = h(x) \in h(J') \cap h(f^n(I'))$. Therefore, since there is a $z \in I'$ such that $y = h(f^n(z)) = g^n(h(z))$, it follows that $y \in h(J') \cap g^n(h(I'))$. Hence $y \in J \cap g^n(I)$, by $(h \circ h^{-1})|J = \mathrm{id}_J$, $(h \circ h^{-1})|I = \mathrm{id}_I$.

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$\boxed{\alpha = 1 - 1/r}$, p. 262, l. 2
In fact, $\frac{r}{1/2}$ is the slope of 'the first half of the graph of T', that is, the slope of the line segment joining the points $(0,0)$ and $(1/2, r)$, which equals the slope of 'the first half of the triangle of base α and height $r-1$'. Thus $\boxed{\frac{r}{1/2} = \frac{r-1}{\alpha/2}}$.
==
Ex. 13.1.8, p. 264
See **Ex. 13.3.3**, p. 276.
==
Ex. 13.1.9, p. 264
General Version of Theorem 13.1.5
Let $T = T_\mu$ with $\mu > 4$. The graph of T^n consists of 2^{n-1} isosceles triangles, each of height $r = \frac{\mu}{4} > 1$ and base lenght $\left(\frac{1-\alpha}{2}\right)^{n-1}$ ($\alpha + (1-\alpha) = 1$) with $\alpha = 1 - \frac{1}{r}$ ($0 < \alpha < 1$).
The domain of T^n can be obtained recursively from the results:
(a) dom $T^1 = [0,1]$;
(b) dom T^{k+1} is the result of removing the open middle fractions α of all the maximal closed intervals in dom T^k.
Proof: Concerning the lines of the original *Proof* of **Theorem 13.1.5**, put $\boxed{\textit{a fraction } \frac{1-\alpha}{2} \textit{ of the length}}$ in place of $\boxed{\textit{one third of the length}}$ (line 8), put $\boxed{\text{a fraction } \frac{1-\alpha}{2}}$ in place of $\boxed{\text{one third}}$ (line 11) and put $\boxed{\frac{1-\alpha}{2}}$ in place of $\boxed{1/3}$ (line 11).
==
Erratum, **Example 13.2.2**, p. 265
"Let x_1 and x_2 be ..." should be "Let x_0 and x_1 be ...".
==
Ex. 13.2.3, p. 270

$$\begin{aligned} C_{n+1} &= \text{dom } f^{n+1} \\ &= \text{dom } f^n \circ f \\ &= \{x \in \text{dom } f \,:\, f(x) \in \text{dom } f^n\} \\ &= \{x \in [0,1] \,:\, f(x) \in C_n\}\,. \end{aligned}$$

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Ex. 13.2.4, p. 270

- $$\begin{aligned} \cap_{n=1}^{\infty} C_n &= \cap_{i=0}^{\infty} C_{i+1} \\ &= C_1 \cap (\cap_{i=1}^{\infty} C_{i+1}) \\ &= [0,1] \cap (\cap_{i=1}^{\infty} C_{i+1}) \\ &= \cap_{i=1}^{\infty} C_{i+1} \\ &= \cap_{n=1}^{\infty} C_{n+1}; \end{aligned}$$
- $\cap_{n=1}^{\infty} C_{n+1} \subset C_{n+1}$ for $n = 1, 2, \ldots \Longrightarrow f\left(\cap_{n=1}^{\infty} C_{n+1}\right) \subset f(C_{n+1})$;
- **Ex. 13.2.3**.

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Ex. 13.2.5, p. 271
$f^1(C) \subset C$ and $f^n(C) \subset C$ is the induction hypothesis. Thus $f^{n+1}(C) = f(f^n(C)) \subset f(C) \subset C$.
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Ex. 13.2.11, p. 271
x_0 is a periodic point of $f \overbrace{\Longrightarrow}^{\textbf{Ex. 3.1.12}} x_1 = f(x_0)$ is a periodic point of f.
==
Ex. 13.2.12, p. 271
Let $x \in [0,1]$ be a zero of f^n, that is, $f^n(x) = 0$. Therefore $f^{n-1}(f(x)) = 0$, that is, $f(x)$ is a zero of f^{n-1}.

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Erratum, the line before **Fig. 13.3.2**, p. 274

"... in Exercise 13.3.5." should be "... in Exercise 13.3.6.".

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Ex. 13.3.1, p. 276

(a)$\longleftrightarrow$(e$'$),
(b)$\longleftrightarrow$(c$'$),
(c)$\longleftrightarrow$(a$'$),
(d)$\longleftrightarrow$(b$'$) and
(e)$\longleftrightarrow$(d$'$).

==

Ex. 13.3.3, p. 276

dom $T^1 = C_1$ and, for $n = 1, 2, \ldots$, dom $T^n = C_n$ is the union of the bases of 2^{n-1} isosceles triangles, which are the bases of humps for T^n, each of base lenght $(1/3)^{n-1}$. C_{n+1} is obtained from C_n by removing the open middle thirds of all the maximal closed intervals in C_n, which are the bases of humps for T^n. This leaves two closed intervals in C_{n+1}, each of lenght $(1/3)^{n-2}$, in place of each maximal closed interval in C_n. Thus $(1/3)^{n-1}$ is also the lenght of the longest interval in C_n and it $\to 0$ as $n \to \infty$.

==

Ex. 13.3.4, p. 276

See the resolution of **Ex. 13.1.9**, the resolution of the last exercise and notice that $\left(\frac{1-\alpha}{2}\right)^{n-1} \to 0$ as $n \to \infty$.

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Erratum, **Ex. 13.4.8**, p. 281

$\boxed{I \cap C_n \neq \emptyset}$ should be $\boxed{I \cap C \neq \emptyset}$.

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Ex. 13.4.8, p. 281

$S \subset C$ by **Corollary 13.2.10**. Since $I \cap C \neq \emptyset$, there is a hump of some f^n with base contained in I by the **Spike Lemma**. Since the endpoints of such a base are zeroes of f^n, we have that $S \cap (I \cap C) \neq \emptyset$.

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