INTRODUCTION TO HILBERT SPACES
WITH APPLICATIONS
THIRD EDITION (2010)
Debnath and Mikusinski

PARTIAL SCRUTINY,
SOLUTIONS OF SOME EXERCISES,
COMMENTS, SUGGESTIONS AND ERRATA
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On the other hand, the inequality that completes the Proof

Erratum, p. 6, l. 8
‘$\sum_{k=1}^{n} |x_k||y_k|$’ should be ‘$\sum_{j=1}^{n} |x_j||y_j|$’ or ‘$\sum_{k=1}^{n} |x_k||y_k|$’.

Comment, p. 5, Theo. 1.2.7, Proof, 2nd sentence
See Ex. 8, p. 35.

Comment, pp. 6–7, Theo. 1.2.8, Proof
The second inequality holds by Theo. 1.2.7 (Hölder’s inequality) provided that

$$((x_n + y_n)^{p-1}) \in I^p!$$

So consider partial sums (and the last inequality obtained in the Proof of Theo. 1.2.7) instead:

$$\sum_{k=1}^{m} |x_k + y_k|^p = \sum_{k=1}^{m} |x_k + y_k||x_k + y_k|^{p-1}$$

$$\leq \sum_{k=1}^{m} |x_k||x_k + y_k|^{p-1} + \sum_{k=1}^{m} |y_k||x_k + y_k|^{p-1}$$

$$\leq \left( \sum_{k=1}^{m} |x_k|^p \right)^{1/p} \left( \sum_{k=1}^{m} |x_k + y_k|^{q(p-1)} \right)^{1/q} + \left( \sum_{k=1}^{m} |y_k|^p \right)^{1/p} \left( \sum_{k=1}^{m} |x_k + y_k|^{q(p-1)} \right)^{1/q}$$

$$\leq \left( \sum_{n=1}^{\infty} |x_n|^p \right)^{1/p} \left( \sum_{n=1}^{\infty} |x_n + y_n|^{q(p-1)} \right)^{1/q} + \left( \sum_{n=1}^{\infty} |y_n|^p \right)^{1/p} \left( \sum_{n=1}^{\infty} |x_n + y_n|^{q(p-1)} \right)^{1/q}$$

$$\sum_{k=1}^{m} |x_k + y_k|^p \leq \left( \sum_{n=1}^{\infty} |x_n|^p \right)^{1/p} \left( \sum_{n=1}^{\infty} |y_n|^p \right)^{1/p} \left( \sum_{k=1}^{m} |x_k + y_k|^p \right)^{1/q} \cdot \left( \sum_{k=1}^{m} |x_k + y_k|^{q(p-1)} \right)^{1/q}. \quad (1)$$

On the other hand, the inequality that completes the Proof of Theo. 1.2.8 is trivially satisfied if

$$\sum_{n=1}^{\infty} |x_n + y_n|^p = 0. \quad (2)$$

So suppose (2) is not satisfied. Then there is an index $M$ such that

$$m \geq M \implies \sum_{k=1}^{m} |x_k + y_k|^p > 0.$$ 

Therefore, by (1),

$$\left( \sum_{k=1}^{m} |x_k + y_k|^p \right)^{1-1/q} \leq \left( \sum_{n=1}^{\infty} |x_n|^p \right)^{1/p} + \left( \sum_{n=1}^{\infty} |y_n|^p \right)^{1/p}$$

for $m \geq M$. Now let $m \to \infty$.

Erratum, p. 7, l. 14
‘$X_j$’ should be ‘$E_j$’.

Comment, p. 11, Ex. 1.3.8, penultimate sentence
Consider $t \in [0, 1]$. On the one hand,

$$g_n(t) \to 0. \quad (3)$$
Let $x$ be an arbitrary vector in $\Omega$ and consider that $n$ is an arbitrary positive integer. Suppose that $g_n \in S_5 :=$ fifth set,$^2$ $g \in \mathcal{C}(\Omega)$ and $||g_n - g|| \to 0.$ $^3$ So $(g_n - g)(x) \leq (f - g)(x)$ and $(g_n - g)(x) \to 0.$ Then $g(x) \leq f(x).$ Therefore $g \in S_5.$ $^4$

Comment, p. 16, Theo. 1.3.23
Let $X$ be the RHS of the equation. It suffices to show that $X$ is closed. In fact, suppose $X$ is closed. So, on the one hand, due to the fact that $S \subset X,$

$$\text{cl} \, S \subset X.$$ 

On the other hand, if

$$X \not\subset \text{cl} \, S,$$

there exists $x \in X$ with $x \not\in \text{cl} \, S.$ Then $x \not\in C$ for some closed set $C$ containing $S.$ This leads to a contradiction since there exist $x_1, x_2, \ldots \in S \subset C$ with $x_n \to x.$ Therefore $x \in C$ by Theo. 1.3.21, p. 16.

Comment, p. 17, sentence right before Theo. 1.3.31

‘only-if-part’
Since $(||x_n||)$ is bounded and $|\lambda_n| \to 0,$ $|\lambda_n| \, ||x_n|| \to 0$ by a very well-known result from Analysis on the Real Line.

‘if-part’
Suppose $S$ is not bounded and $n$ is a positive integer. Thus $||x_n|| \geq n$ for some $x_n \in S.$ Hence $\frac{1}{n} \, ||x_n|| \geq 1,$ which contradicts the convergence (to 0) hypothesis.

$^1$That is, $(f - g)(x) > 0$ for each $x \in \Omega.$
$^2$That is, $g_n(x) \leq f(x).$
$^3$Hence $||g_n - g|| \to 0.$
$^4$Now use Theo. 1.3.21, p. 16.
Comment, p. 18, Theo. 1.3.33, Proof, 2nd sentence
Suppose $d = 0$ and consider a positive integer $n$. Hence there exists $x_n \in X$ such that $\|z - x_n\| < \tfrac{1}{n}$, which leads to a contradiction. In fact, since $E \setminus X$ is open, there is an open ball $B(z, \epsilon) \subset E \setminus X$.

Comments, pp. 18–9, Theo. 1.3.34, Proof
‘only-if-part’
A sequence in $B(0, 1)$ satisfies the condition

$$\|a_{1,n}e_1 + \cdots + a_{N,n}e_N\| = |a_{1,n}| + \cdots + |a_{N,n}| \leq 1.$$  

Furthermore, by the Bolzano-Weierstrass Theorem, $(a_{i,n})$ has a convergent subsequence, $i = 1, \ldots, N$.

‘if-part’
Note that when the 2nd sentence ends, its verification begins!

Comment, p. 21, l. 11, that is, 2nd series
By the 2nd sentence of Ex. 1.4.6, p. 20, $a_n \in l^2$ for each $n \in \mathbb{N}$. In particular, $a_{n_0} = (a_{n_0,k}) \in l^{2,5}$

Comment, p. 22, penultimate sentence
Since $\max_{[0,1]} |P_n(x) - e^x| \to 0$, the absolute convergence criterion from Def. 1.4.8 is satisfied.

Comment, p. 23, Theo. 1.4.9, Proof, penultimate sentence
$(x_p)$ is the sum of two convergent sequences:

$$(x_p - x_{p_1}) = \left( \sum_{j=1}^{k-1} (x_{p_{j+1}} - x_{p_j}) \right) \text{ and } (x_{p_1}, x_{p_2}, \ldots).$$

Comment, p. 24, 1st paragraph
A linear isometry is automatically one-to-one. So the requirement for $\Phi$ to be one-to-one in (a) is a direct consequence of (b).

Errata, p. 24, 2nd paragraph
• antepenultimate sentence
  ‘$\|\{(x_n)\}\|_1$ should be $\|\{(x_n)\}\|_1$’;

• ultimate sentence
  ‘... $[(x_n)]$ and $[(y_n)]$...’ should be ‘... $(x_n)$ and $(y_n)$...’.

Comments, p. 24
• 2nd paragraph, last sentence
  Use the fact that

$$\|x_n\| - \|y_n\| \leq \|x_n - y_n\| \to 0.$$  

• 3rd paragraph, last sentence

$$\lim_{n \to \infty} \Phi(x_n) = [(x_n)] \iff \lim_{n \to \infty} \|\Phi(x_n) - [(x_n)]\|_1 = \lim_{n \to \infty} \|\{(x_n - x_1, x_n - x_2, \ldots)\}\|_1 = \lim_{n,k \to \infty} \|x_n - x_k\| = 0,$$

because $(x_n)$ is a Cauchy sequence.

\[5\]See Ex. 1.2.6, p. 4.
Comment, p. 27, 1st sentence after 2nd □

It suffices to consider that $E_1$ is finite dimensional. In fact, let $\{e_1, \ldots, e_N\}$ be a basis of $E_1$ and assume, without loss of generality, that the norm on $E_1$ is defined by

$$x = \alpha_1 e_1 + \cdots + \alpha_N e_N \mapsto \|x\| = |\alpha_1| + \cdots + |\alpha_N|.$$  

Therefore

$$\|Lx\| \leq |\alpha_1| \|Le_1\| + \cdots + |\alpha_N| \|Le_N\| \leq \alpha \|x\|$$

with $\alpha = \max \{\|Le_i\| : i = 1, \ldots, N\}$.

Comments, p. 28, Theo. 1.5.9, Proof, 2nd paragraph

- 1st sentence
  Consider $\alpha \in \mathbb{F}$ and $x_1, x_2 \in E_1$. So

  $$L(\alpha x_1 + x_2) = \lim_{n \to \infty} L_n (\alpha x_1 + x_2) = \lim_{n \to \infty} (\alpha L_n x_1 + L_n x_2) = \alpha \lim_{n \to \infty} L_n x_1 + \lim_{n \to \infty} L_n x_2 = \alpha L x_1 + L x_2.$$

- 2nd sentence
  $(L_n)$ is bounded by Lemma 1.4.4, p. 20.

- 3rd sentence
  The second equality holds by Ex. 1.5.3, p. 26.

Comments, p. 29, Theo. 1.5.10

- 1st sentence
  Note that $\text{cl} \; D(L)$ is a subspace of $E_1$. In fact, consider $\alpha \in \mathbb{F}$ and $x, y \in \text{cl} \; D(L)$, that is, there are sequences $(x_n)$ and $(y_n)$ in $D(L)$ such that $x_n \to x$ and $y_n \to y$. Therefore $\alpha x + y \in \text{cl} \; D(L)$ since $\alpha x_n + y_n \to \alpha x + y$.\footnote{Anyway, cf. p. 26, 1st paragraph.}

- 2nd sentence
  See Def. 1.3.25, p. 17.

- Proof, penultimate sentence
  Since $x_n \to x$ and $Lx_n \to Lx$, $\|x_n\| \to \|x\|$ and $\|Lx_n\| \to \|Lx\|$. In fact,

  $$\|x_n\| \leq \|x_n - x\| + \|x\| \text{ and } \|x\| \leq \|x - x_n\| + \|x_n\|$$

  imply that

  $$\|x_n - x\| \leq \|x - x_n\|.$$  

Erratum, p. 29, Theo. 1.5.11, 1st sentence

\'E\' should be \'E\'.

Comments/Erratum, p. 31

\footnote{See Theo. 1.3.13.}
• 1.3, 2nd inequality
Since $\|x_{p_i}\| \geq \epsilon$ for all $i \in \mathbb{N}$ and $\|x_{r_i}\| < \epsilon/2^{i+1}$ for all $i \neq j$,

$$\|x_{s_{ji}}\| - \sum_{i \neq j} \|x_{s_{ji}}\| > \epsilon - \sum_{i \neq j} \frac{\epsilon}{2^{i+1}} = \epsilon \left( 1 - \sum_{i \neq j} \frac{1}{2^{i+1}} \right)$$

$$= \epsilon \left( 1 - \left( \frac{1}{2^2} + \cdots + \frac{1}{2^i} \right) + \left( \frac{1}{2^{i+2}} + \frac{1}{2^{i+3}} + \cdots \right) \right)$$

$$= \epsilon \left( 1 - \left( \frac{1}{4} \left( 1 - \frac{1}{2^{i-1}} \right) + \frac{1}{2^{i+2}} \right) \right)$$

$$= \epsilon \left( 1 - \left( \frac{1}{2} - \frac{1}{2^i} + \frac{1}{2^{i+1}} \right) \right)$$

$$= \epsilon \left( \frac{1}{2} + \frac{1}{2^i} \left( 1 - \frac{1}{2} \right) \right)$$

$$= \epsilon \left( \frac{1}{2} \left( 1 + \frac{1}{2} \right) \right) > \epsilon/2$$

if $i \geq 2$, whereas

$$\|x_{s_{ji}}\| - \sum_{i \neq j} \|x_{s_{ji}}\| > \epsilon - \sum_{j=2}^{\infty} \frac{\epsilon}{2^{j+1}} = \epsilon \left( 1 - \sum_{j=2}^{\infty} \frac{1}{2^{j+1}} \right)$$

$$= \epsilon \left( 1 - \frac{1}{4} \right)$$

$$= \epsilon \left( 1 - \frac{1}{4} \right)$$

$$= \frac{3\epsilon}{4} > \frac{\epsilon}{2}$$

if $i = 1$.

• Theo. 1.5.13, Proof

- 1st and 2nd sentences
In fact, for every strictly sequence $(M_n)$ with $M_1 > 0$, there exists a sequence $(T_n)$ of elements of $T$
such that $|T_n| > M_n$ for all $n \in \mathbb{N}$. Since $T \subset B(X, Y)$, where (1.14) holds, there exists a sequence
$(x_n)$ of unit elements of $X$ such that $|T_n x_n| > M_n$ for all $n \in \mathbb{N}$.

- 5th sentence
See Theo. 1.4.9, p. 22.

- 6th sentence and 1st clause of 9th sentence
C does not depend on $i$ since $C = M_z$. Similarly, since

$$\lim_{i \to \infty} y_{ij} = 0$$

for all $j \in \mathbb{N}$.

- 8th sentence
$(y_{q,i})$ should be $(y_{q,j})$.

Comments, pp. 32–3, Ex. 1.6.3

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8See 2nd sentence of Theo. 1.5.13.
• 4th sentence
  If \( f(x) = x^3 - x - 1 \), then \( f(1) < 0 \) and \( f(2) > 0 \). So there is some \( x_0 \in (1, 2) \) such that \( f(x_0) = 0 \).

• 6th sentence
  The inequality holds since there exists some \( c \in (1, 2) \) such that
  \[
  \left| Tx - Ty \right| = \left| T'(c) \right| |x - y| = \frac{1}{3(1 + c)^{2/3}} |x - y| < \frac{1}{3(1 + 1)^{2/3}} |x - y| = \frac{1}{3} \cdot 2^{1/3} \cdot 2^{1/3} |x - y|.
  \]

• 7th/last sentence, \( Tx = x^3 - 1 \)
  On the one hand, if \( T \) is a contraction, then
  \[
  \left| \frac{x^3 - y^3}{x - y} \right| = \frac{\left| x^3 - y^3 \right|}{|x - y|} \leq \alpha < 1.
  \]
  On the other hand,
  \[
  \left| \frac{x^3 - y^3}{x - y} \right| = \left| x^2 + xy + y^2 \right| > 1.
  \]

Comment, p. 33, sentences between 2nd \( \Box \) and Ex. 1.6.5

The method is known as fixed-point iteration.

Comment, p. 34, Ex. 1.6.6, penultimate sentence

Suppose \( f \) is a contraction. So, since \( F = \mathbb{R}^+ \) is closed, \( f \) has a fixed point by Theo. 1.6.4.\(^\text{10}\)

Exercises, pp. 34–8

1. Consider \( z, z', w \in E \) with \( x + z = y = x + z' \) and \( z + w = z' \). Then \( y = x + z' = x + z + w = y + w \). So \( w = 0 \). Therefore \( z' = z + w = z \).

3.
  (a) \( \lambda 0 = 0 \) for each \( \lambda \) since \( \lambda 0 = \lambda (0 + 0) = \lambda 0 + \lambda 0 \). Therefore, since \( \lambda \neq 0 \),
  \[
  \lambda x = 0 \implies \lambda^{-1} (\lambda x) = \lambda^{-1} 0 \implies (\lambda^{-1} \lambda) x = 0 \implies 1x = 0 \implies x = 0.
  \]
  (b) Consider \( x \neq 0 \). Suppose \( \lambda \neq 0 \). By (a), since \( \lambda x = 0 \), it follows that \( x = 0 \), which is a contradiction.
  (c) Since \( 0x = (0 + 0)x = 0x + 0x \), it follows that \( 0x = 0 \). Then
  \[
  x + (-1)x = 1x + (-1)x = [1 + (-1)]x = 0x = 0.
  \]
  Therefore \( (-1)x = 0 - x = -x \).\(^\text{11}\)

8. Since \( h(x) := \frac{1}{p} x + \frac{1}{q} - x^{\frac{1}{q}} \) is continuous on \([0, 1]\), \( h(0) = \frac{1}{q} > 0 \), \( h'(x) = \frac{1}{p} \left( 1 - x^{-\frac{1}{q}} \right) < 0 \) for \( 0 < x < 1 \) and \( h(1) = 0 \), it follows that \( h(x) \geq 0 \) for \( 0 \leq x \leq 1 \).

\(^\text{9}\)Use the Intermediate Value Theorem.

\(^\text{10}\)See the ultimate sentence.

\(^\text{11}\)See p. 3, 2nd paragraph.
22.  
(a) Suppose \( \|x_n - x\| \to 0 \) and \( \|x_n - y\| \to 0 \). Use \( \|x - y\| \leq \|x - x_n\| + \|x_n - y\| \).
(b) Use
\[
\|\lambda_n x_n - \lambda x\| = \|\lambda_n x_n - \lambda x_n + \lambda x_n - \lambda x\| \\
\leq \|\lambda_n - \lambda\| \|x_n\| + |\lambda| \|x - x_n\| \\
\leq |\lambda_n - \lambda| (\|x_n\| + \|x\|) + |\lambda| \|x_n - x\|.
\]
(c) Use \( \|x_n + y_n - (x + y)\| \leq \|x_n - x\| + \|y_n - y\| \).

34.  
(a) \( \implies \) (b)
   The proof is trivial by Theo. 1.3.23 and Def. 1.3.25.
(b) \( \implies \) (c)
   Consider an open ball \( B(x, \epsilon) \). Since there exist \( x_1, x_2, \ldots \in S \) with \( x_n \to x \), there exists a number \( M \) such that \( x_n \in B(x, \epsilon) \) for every index \( n \geq M \).^{12}
   (c) \( \implies \) (a)
   Let \( x \in E \). Hence there exists \( x_n \in S \cap B(x, 1/n) \) for each positive integer \( n \). Therefore \( x \in \text{cl} S \).

39.  
(a) \( \implies \) (b)
   Note that \( p_n \geq n \) and \( q_n \geq n \) for each positive integer \( n \). Now consider \( \epsilon \) and \( M \) given in Def. 1.4.1, p. 19. Therefore
\[
n \geq M \implies p_n, q_n \geq M \\
\implies \|x_{p_n} - x_{q_n}\| < \epsilon.
\]
(b) \( \implies \) (c)
   Concerning (b), consider \( q_n = p_{n+1} \).
(c) \( \implies \) (a)
   Suppose (a) is false. So there is a positive \( \epsilon_0 \) such that, for each positive integer \( M \), there exist indices \( m_0 \) and \( n_0 \) where
\[
m_0, n_0 > M \quad \text{and} \quad \|x_{m_0} - x_{n_0}\| \geq \epsilon_0.
\]
Now consider \( m_0 \geq n_0 \) and an increasing sequence of positive integers \( (p_n) \) such that \( p_{n_0} = n_0 \) and \( p_{n_0+1} = m_0 \). Therefore
\[
n_0 > M \quad \text{and} \quad \|x_{p_{n+1}} - x_{p_n}\| \geq \epsilon_0,
\]
which contradicts (c).

41.  
As in Ex. 1.4.6, pp. 20–1, the same argument applies if 2nd powers and square roots are replaced with \( p \)th powers and \( p \)th roots, respectively.

48.  
(a) \( \iff \) (b)
   Via Ex. 35, p. 37, \( F \) is continuous iff for every \( x \in E_1 \) and \( \epsilon > 0 \) there exists a \( \delta > 0 \) such that \( F(B(x, \delta)) \subset B(F(x), \epsilon) \).
   (a) \( \implies \) (b)
   Let \( x \in F^{-1}(U) \) and take \( \epsilon > 0 \) and \( \delta > 0 \) with
   \[
   \begin{align*}
   F(B(x, \delta)) & \subset B(F(x), \epsilon) \\
   U & \quad \subset \quad \text{open in } E_2
   \end{align*}
   \]
   Hence \( B(x, \delta) \subset F^{-1}(U) \).

^{12}\text{See Def. 1.3.6, p. 10.}
For $x \in E_1$ and $\varepsilon > 0$, $F^{-1}(B(F(x), \varepsilon))$ is open in $E_1$. Therefore there is a $\delta > 0$ for which $B(x, \delta) \subset F^{-1}(B(F(x), \varepsilon))$. Thus $F(B(x, \delta)) \subset B(F(x), \varepsilon)$.

(b) $\iff$ (c)

Use that complements of open (resp. closed) sets are closed (resp. open) sets and inverse images commute with complements.

49. Concerning the 1st sentence, use that $N(L) = L^{-1}(\{0\})$ and Theo. 1.5.4.

51. Uniform convergence is the one with respect to (1.14). That being said, on the one hand, suppose $\|L_n - L\| \to 0$ as $n \to \infty$. Therefore $\|L_n x - L x\| \leq \|L_n - L\| \|x\| \to 0$ for every $x \in E_1$. Now, on the other hand, consider $E_1 = E_2 = l^2$ and the projection $x = (x_1, x_2, \ldots) \mapsto L_n x = (x_1, \ldots, x_n, 0, 0, \ldots)$. Then $\|L_n - L_m\| = 1$ for $n \neq m$. So, since $(L_n)$ is not a Cauchy sequence, it does not converge (uniformly). However, for $x \in l^2$, we have $L_n x \to x$ as $n \to \infty$. Thus $L_n \to I$ strongly.

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13See p. 28, sentence that precedes Theo. 1.5.9.
14See p. 28, sentence that follows $\Box$.
15Without loss of generality, assume $n < m$. Thus

$$
\|L_n - L_m\| = \sup_{|x|=1} \|(L_n - L_m)x\|
= \sup_{|x|=1} \|(0, \ldots, 0, x_{n+1}, \ldots, x_m, 0, 0, \ldots)\|
= \sup_{|x|=1} \sqrt{\sum_{i=n+1}^{m} x_i^2}
= 1.
$$

In fact, on the one hand, $\sqrt{\sum_{i=n+1}^{m} x_i^2} \leq \sqrt{\sum_{i=1}^{\infty} x_i^2} = |x| = 1$ for each unit vector $x$. On the other hand, consider $x = (0, \ldots, 0, 1, 0, 0, \ldots)$ with $1 = x_i$, $i \in \{n + 1, \ldots, m\}$. 

9
Errata

p. 71, (2.26)
‘\( f(x) \)’ should be ‘\( f' \). ‘\( b_n \)’ should be ‘\( b_n' \).

p. 73, the if part of the Proof of T. 2.12.2
It seems that the representations of \( \text{Re} f \) and \( \text{Im} f \) are inverted.

p. 76, l. 5
Remove the preposition ‘in’.

Comments, p. 42

• (2.5)
\( b_{s,m} = \min \{ b_m, s \} \). However, \( s \in S \) holds right after (2.5).

• l. -2
\( b_k \in S \) since \( b_{b_k,n} = \min \{ b_n, b_k \} \) and \( b_{s,n} = \min \{ b_n, s \} \) imply that
\[
\sum_{a_n < b_{b_k,n}} (b_{b_k,n} - a_n) = (b_k - a_k) + \left\{ \sum_{a_n < b_{b_k,n}} (b_{b_k,n} - a_n) \right\} - (s - a_k) \\
= b_k - a_k + s - a - s + a_k \\
= b_k - a.
\]

Comment, p. 44, (2.8)
\( g \) is a step function whose support is contained in the union of

\[
[a_{1,1}, b_{1,1}) \cup \cdots \cup [a_{1,k_1}, b_{1,k_1}) \cup \cdots \cup [a_{n_0,1}, b_{n_0,1}) \cup \cdots \cup [a_{n_0,k_{n_0}}, b_{n_0,k_{n_0}}).
\]

Therefore
\[
\int g \leq \alpha \sum_{n=1}^{n_0} \sum_{k=1}^{k_n} (b_{n,k} - a_{n,k}) \\
< \alpha (b - a).
\]

Erratum, p. 44, Cor. 2.2.7
“... be nondecreasing sequences ...” should be “... be a nondecreasing sequence ...”.

Comment, p. 46, l. 2
For every \( x \in \mathbb{R} \) such that \( \sum_{n=1}^{\infty} |f_n(x)| < \infty \),
\[
\lim_{n \to \infty} g_n(x) = f_1(x) + \cdots + f_n(x) \geq \sum_{n=1}^{\infty} |f_{n_0+n}(x)| \geq f(x) \geq 0.
\]

For \( x \in \mathbb{R} \) such that \( \sum_{n=1}^{\infty} |f_n(x)| \) does not converge,
\[
\lim_{n \to \infty} g_n(x) = f_1(x) + \cdots + f_n(x) + \sum_{n=1}^{\infty} |f_{n_0+n}(x)| = +\infty.
\]
Comment, p. 48, the sentence right before Theo. 2.4.1
If \( z = 0 \) is a simple pole of \( \gamma(z) \), then \( \lim_{\varepsilon \to 0} \int_{\gamma} \gamma(z) dz = \pi i \text{Res}(\gamma, 0) \), \( \gamma(e) \) being a semicircle of small radius \( \varepsilon > 0 \), centered at the origin, situated in the upper half-plane and described in the direction of increasing argument.\(^{16}\) Hence, since \( \frac{\sin x}{x} = \frac{e^{ix} - \cos x}{ix} \) and \( \cos \frac{x}{x} \) is an odd function,

\[
\int_{-\infty}^{\infty} \frac{\sin x}{x} dx = \lim_{\varepsilon \to 0} \left( \int_{-\infty}^{\varepsilon} \frac{e^{ix}}{x} dx + \int_{\varepsilon}^{\infty} \frac{e^{ix}}{x} dx \right) = \lim_{\varepsilon \to 0} \left( \int_{\gamma} \frac{e^{iz}}{z} dz \right) = \pi \text{Res} \left( \frac{e^{iz}}{z}, 0 \right) = \pi.
\]

On the other side, \( \frac{\sin x}{x} \) is not absolutely integrable over \([0, \infty)\) since

\[
\int_{0}^{\infty} \left| \frac{\sin x}{x} \right| dx = \sum_{k=0}^{\infty} \int_{k\pi}^{(k+1)\pi} \left| \frac{\sin x}{x} \right| dx \geq \sum_{k=0}^{\infty} \frac{1}{(k+1)^{\pi}} \int_{k\pi}^{(k+1)\pi} \sin x dx = s
\]

with

\[
s = \frac{1}{\pi} \int_{0}^{\pi} \sin x dx + \frac{1}{2\pi} \int_{\pi}^{2\pi} (-\sin x) dx + \frac{1}{3\pi} \int_{2\pi}^{3\pi} \sin x dx + \frac{1}{4\pi} \int_{3\pi}^{4\pi} (-\sin x) dx + \cdots
\]

\[
= \frac{1}{\pi} \cos x \bigg|_{0}^{\pi} + \frac{1}{2\pi} \cos x \bigg|_{\pi}^{2\pi} + \frac{1}{3\pi} \cos x \bigg|_{2\pi}^{3\pi} + \frac{1}{4\pi} \cos x \bigg|_{3\pi}^{4\pi} + \cdots
\]

\[
= \frac{2}{\pi} \left( \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots \right) = \infty.
\]

(Note that \( \int_{-\infty}^{0} \left| \frac{\sin x}{x} \right| dx = \int_{0}^{\infty} \left| \frac{-\sin x}{x} \right| dx = -\int_{0}^{\infty} \sin u \frac{du}{u} = -\int_{0}^{\infty} \sin u \left| \frac{du}{u} \right. \)

Now consider Def. 2.3.1(a), p. 45, and Theo. 2.4.1.

Comment, p. 50, 1st sentence of the Proof
See Ex. 8, p. 85.

Comment/Erratum, p. 52, Theo. 2.5.3, Proof

- \( f \simeq \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} f_{n,k} \) since:

  (I) \( \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \int |f_{n,k}| \leq \sum_{n=1}^{\infty} \int |f_{n}| + \sum_{n=1}^{\infty} 2^{-n} < \infty \)

(II) \( f(x) = \sum_{n=1}^{\infty} f_{n}(x) = \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} f_{n,k}(x) \) for each \( x \in \mathbb{R} \) such that \( \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} |f_{n,k}(x)| < \infty \).

In fact, (*) implies that \( \sum_{k=1}^{\infty} |f_{n,k}(x)| < \infty \) for each \( n \in \mathbb{N} \). Hence

\[
\sum_{n=1}^{\infty} |f_{n}(x)| = \sum_{n=1}^{\infty} \left( \sum_{k=1}^{\infty} |f_{n,k}(x)| \right) \leq (*)
\]

- Change ‘\( g_{n,k} \)’ to ‘\( f_{n,k} \).

\(^{16}\)Refer to Elementary Theory of Analytic Functions of One or Several Complex Variables by Henri Cartan, p. 104.
The restriction of $f = \chi_{\{0\}}$ to each $[a, b]$ containing $\{0\}$ is Riemann integrable and its Riemann integral is 0. Now use Theo. 2.10.1, p. 64.\footnote{See also Ex. 9, p. 85.}

As a matter of fact, see the conclusion of the Proof of Theo. 2.6.3.

As usual, $f_1 + f_2 + \cdots$ converges to $f$ in norm means $f_1 + f_2 + \cdots + f_n \to f$ i.n.\footnote{Which is denoted by $f_1 + f_2 + \cdots = f$ i.n.} The same holds true if we change in norm (i.n.) to almost everywhere (a.e.)\footnote{See p. 56.}

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The sequence of functions $f_1, f_2, \ldots$ defined on $X \subset \mathbb{R}$ converges uniformly to $f$ if

$$\sup_{x \in X} |f_n(x) - f(x)| \to 0 \quad \text{as} \quad n \to \infty.$$ 

On the one hand, concerning Ex. 2.7.8, $f_n \to 0$ uniformly since

$$\sup_{x \in \mathbb{R}} |f_n(x)| = \frac{1}{\sqrt{n}} \quad \forall n \in \mathbb{N}.$$ 

On the other hand, the last line follows from

$$|f_n - f| \leq \sup_{x \in \mathbb{R}} |f_n(x) - f(x)| \chi_{[a,b]} \quad \forall n \in \mathbb{N}.$$ 

Recall that $\int |f_n| = \|f_n\|$, where $\| \cdot \|$ is the $L^1$-norm.\footnote{See p. 52.}

- $f_{p_n} \to g$ a.e. since $f_{p_1} + (f_{p_2} - f_{p_1}) + (f_{p_3} - f_{p_2}) + \cdots + (f_{p_n} - f_{p_{n-1}}) \to g$ a.e.;

- $f_{p_n} \to g$ i.n. since

$$g = f_{p_1} + (f_{p_2} - f_{p_1}) + (f_{p_3} - f_{p_2}) + \cdots \text{ i.n.}$$

by Theo. 2.7.12, p. 58.

- $\ker f_{p_n} \to \ker g$ a.e. by Theo. 2.6.5, p. 54.
This equality is known as passage of the limit under the integral sign.

Comment/Erratum, p. 60

1.1
See Ex. 2.7.8, pp. 56–7.

Theo. 2.8.3, Proof, last equality
Change the last ‘−’ to ‘+’.

Comments/Errata, p. 61

1.6
For a fixed \( m \in \mathbb{N} \), define
\[
u_n = g_{m,n+1} = \max \{|f_m|, \ldots, |f_{m+n}|\} \quad \text{and} \quad v_n = g_{m+1,n} = \max \{|f_{m+1}|, \ldots, |f_{m+n+1}|\}.
\]
Then \( u_n \geq v_n \) for every \( n \in \mathbb{N} \). Therefore \( g_m = \lim_{n \to \infty} u_n \geq \lim_{n \to \infty} v_n = g_{m+1} \).

1.7
Change \(|f_1|\) to \(h\).

1.11
Change \(f_n\) to \(g_n\).

1.18
See Theo. 1.4.2, pp. 19 and 20.

Erratum/Comments, p. 62

3rd l. before Def. 2.9.1
Change \(f_\mathbb{R}\) to \(f_\mathbb{R}'\).

Theo. 2.9.2, Proof
Note that \(f\chi_{[a,b]} \simeq \sum_{n=1}^\infty f_n \chi_{[a,b]} = \sum_{n=1}^\infty g_n\).

ll. between \(\square\) and Def. 2.9.3
By Ex. 25, p. 87, the constant function \(f = 1\) does not belong to \(L^1(\mathbb{R})\). By Theo. 2.10.1, p. 64, \(f_a^b\) exists for every \(-\infty < a < b < \infty\).

Comment/Erratum, p. 63

1.3
Consider \([a,b] \subset [-n,n]\) and the Proof of Theo. 2.9.2, but now change \(f\) to \(f\chi_{[-n,n]}\). At the very end, we have
\[
f\chi_{[a,b]} = f\chi_{[-n,n]}\chi_{[a,b]} \simeq g_1 + g_2 + \cdots.
\]

1.8
“In applications it often...” should be “In applications it is often...”.

Comment, p. 64, ll. -7 to -1
Since \(g_n(\mathbb{R}) \subset f(\mathbb{R})\) for every \(n \in \mathbb{N}\) and \(|f| < M\), if
\[
\varphi(x) := \begin{cases} M & \text{if } x \in [a,b), \\ 0 & \text{otherwise}, \end{cases}
\]
then \(|g_n| < \varphi\) for every \(n \in \mathbb{N}\), which is what was missing in order to properly use Theo. 2.8.4.\(^{21}\)

Comments, p. 65

- l. 2
  See Def. 2.2.1, p. 41.

- l. 5
  \(g = h\) a.e. by Theo. 2.7.4, p. 55.

- l. 6
  By Theo. 2.7.4, p. 55, \(|f - g| = 0\). Hence, by the last sentence before Theo. 2.6.3, p. 53, \(f - g \in L^1(\mathbb{R})\).

- Theo. 2.10.2
  To be Lebesgue integrable means to be Lebesgue integrable on \(\mathbb{R}\). Then \(f\) is Lebesgue integrable on \((a, b)\) if \(f \chi_{(a,b)}\) is Lebesgue integrable, that is, \(f\) is integrable over \((a, b)\).\(^{22}\)

Comments

p. 68

D. 2.11.1

\(S\) is measurable if \(\chi_S \chi_{I_{a,b}}\) is integrable for each bounded interval \(I_{a,b}\) with finite endpoints \(a\) and \(b\), \(a < b\).

- l. -11
  Since \(f_1 + f_2 + \cdots \simeq \chi_S\) and \(f_1 + \cdots + f_n \leq |f_1| + \cdots + |f_n|\) for each \(n \in \mathbb{N}\), there exists some \(n_0\) such that \(A_{n_0} \neq \emptyset\).

- l. -8
  Without loss of generality, consider a disjoint union.

- l. -7
  \[
  \sum_{k=1}^{n} (b_{n,k} - a_{n,k}) = \int \chi_{A_n} \\
  \leq 2 \int \sum_{i=1}^{n} |f_i| = 2 \sum_{i=1}^{n} \int |f_i| \\
  < 2 \sum_{n=1}^{\infty} \int |f_n| \\
  < \frac{2\varepsilon}{3},
  \]

  where l. -11 is used in connection with \(\leq\).

p. 69

- l. 8
  Use C. 2.5.4, p. 52.

- ll. 12-3
  Since \(h_n \to h\) i.n., use the passage of the limit under the integral sign.\(^{23}\)

Proof of T. 2.11.4, 1st part

On the one hand, since \(S = \bigcup_{n=1}^{\infty} S_n\) is a disjoint union, for every \(x \in \mathbb{R}\),

\[
\chi_S(x) = (\chi_{S_1} + \chi_{S_2} + \cdots)(x),
\]

\(^{21}\) The same holds true for \(h_n\) in place of \(g_n\).

\(^{22}\) See Theo. 2.10.3 and Def. 2.9.1, p. 62.

\(^{23}\) See the very end of p. 59.
in other words, \((\chi_{S_1} + \cdots + \chi_{S_n}) (x) \to \chi_S(x)\). On the other hand, since \(\chi_{S_n}\) is a locally integrable function which is \(\leq \chi_{[a,b]}\) for every \(n \in \mathbb{N}\), every \(\chi_{S_n}\) is an integrable function by T. 2.9.5, p. 63. Then, since \(\chi_{S_1} + \cdots + \chi_{S_n} \leq \chi_{[a,b]}\) for every \(n \in \mathbb{N}\), \(\chi_S\) is integrable and \(\chi_{S_1} + \cdots + \chi_{S_n} \to \chi_S\) i.n. by T. 2.8.4, p. 60. Hence, by the passage of the limit under the integral sign,

\[
\lim_{n \to \infty} \int \chi_{S_n} = \lim_{n \to \infty} \sum_{k=1}^{n} \int \chi_{S_k} = \int \lim_{n \to \infty} \sum_{k=1}^{n} \chi_{S_k} = \int \chi_S < \infty.
\]

Therefore \(\chi_S \simeq \chi_{S_1} + \chi_{S_2} + \cdots\)

p. 70

**Proof of T. 2.11.4, 2nd part**

(Firstly, note that each \(\chi_{S_n}\) is locally integrable.)

"Note that, by the first part of the proof, \(S\) is measurable."

In fact, for each \([a, b]\),

\[
\chi_S = \chi_{S_1} + \chi_{S_2} + \cdots \text{ pointwise } \implies \chi_S \chi_{[a,b]} = \chi_{S_1 \chi_{[a,b]}} + \chi_{S_2 \chi_{[a,b]}} + \cdots \text{ pointwise}
\]

and, by T. 2.9.4, each \(\chi_{S_1 \chi_{[a,b]}}\) is locally integrable. Therefore, a similar argument as above will show that, for each \([a, b]\),

\[
\chi_S \chi_{[a,b]} \simeq \chi_{S_1 \chi_{[a,b]}} + \chi_{S_2 \chi_{[a,b]}} + \cdots.
\]

Then \(\chi_S\) is locally integrable.

**Case 1**

Each \(\chi_{S_n}\) is integrable by T. 2.9.5, p. 63. For the rest of Case 1, note that since \(\chi_S\) is integrable,

\[
\chi_S(x) = \chi_{S_1}(x) + \chi_{S_2}(x) + \cdots \text{ for every } x \in \mathbb{R},
\]

and \(\chi_{S_1} + \cdots + \chi_{S_n} \leq \chi_S\) for every \(n \in \mathbb{N}\), we have \(\chi_{S_1} + \cdots + \chi_{S_n} \to \chi_S\) i.n. by T. 2.8.4, p. 60. Then, as above, by the passage of the limit under the integral sign,

\[
\sum_{n=1}^{\infty} \int \chi_{S_n} < \infty.
\]

Therefore \(\chi_S \simeq \chi_{S_1} + \chi_{S_2} + \cdots\).

**Case 2**

- \(\sum_{n=1}^{\infty} \int \chi_{S_n} < \infty \implies \int \chi_{S_n} < \infty\) for every \(n \in \mathbb{N}\) \(\implies \chi_{S_1}, \chi_{S_2}, \ldots \in L^1(\mathbb{R})\);

- \(\sum_{n=1}^{\infty} \chi_{S_n} = f\) a.e. by T. 2.7.10, p. 58;

- \(\chi_S\) is integrable by T. 2.7.4, p. 55.

"..., \(L^1(\Omega)\) is a Banach space.," 7 ll. after \(\square\)

Follow the Proof of T. 2.8.1, p. 58, with \(f_n \in L^1(\Omega), n = 1, 2, \ldots\). Note that \(f \in L^1(\mathbb{R})\) and \(f = \sum_{n=1}^{\infty} f_n\) a.e.\(^{25}\) Therefore \(f \in L^1(\Omega)\) since

\[
g(x) := \begin{cases} 
\sum_{n=1}^{\infty} f_n(x) & \text{for every } x \in \mathbb{R} \text{ such that } \sum_{n=1}^{\infty} |f_n(x)| < \infty \\
0 & \text{otherwise}
\end{cases}
\]

implies that \(f \simeq g\) a.e. and \(g \in L^1(\Omega)\).

\(^{24}\)See p. 63.

\(^{25}\)See D. 2.7.3, p. 55, and ll. -17 to -5, p. 47.
“If \( f \) is a measurable function, then \( |f| \in L^1(\mathbb{R}) \) implies \( f \in L^1(\mathbb{R}) \),” right before D. 2.11.5

Let \( g = |f| \) and consider T. 2.11.7 and T. 2.9.5, pp. 71 and 63, respectively.

“Obviously, every integrable function is measurable.”, right after D. 2.11.5

Let \( f \simeq f_1 + f_2 + \cdots \) be as in D. 2.3.1, p. 45. Then \( f_1 + \cdots + f_n \to f \) a.e. by C. 2.7.11, p. 58.

p. 71, Proof of T. 2.11.6

Consider Exercises 21 and 19.(b), pp. 86-7.

p. 73, Proof of T. 2.12.2

Concerning the only if part, consider \( C := \{ x \in \mathbb{R} \mid \sum_{n=1}^{\infty} |f_n(x)| < \infty \} \). Then, for every \( x \in C \), since

\[
\sum_{n=1}^{\infty} |\text{Re} f_n(x)| < \infty \quad \text{and} \quad \sum_{n=1}^{\infty} |\text{Im} f_n(x)| < \infty,
\]

it follows that

\[
\text{Re} f(x) + i \text{Im} f(x) = f(x)
\]

Thus \( \text{Re} f(x) = \sum_{n=1}^{\infty} \text{Re} f_n(x) \) and \( \text{Im} f(x) = \sum_{n=1}^{\infty} \text{Im} f_n(x) \) for each \( x \in \mathbb{C} \). However, we cannot infer that \( \text{Re} f \simeq \text{Re} f_1 + \text{Re} f_2 + \cdots \) and \( \text{Im} f \simeq \text{Im} f_1 + \text{Im} f_2 + \cdots \) since we have to consider the possible existence of some \( x \in \mathbb{Z} := \mathbb{R} \setminus C \) for which either \( \sum_{n=1}^{\infty} |\text{Re} f_n(x)| < \infty \) or \( \sum_{n=1}^{\infty} |\text{Im} f_n(x)| < \infty \). Nevertheless, since \( \mathbb{Z} \) is a null set and \( f(x) = f_1(x) + f_2(x) + \cdots \) for every \( x \) except \( \mathbb{Z}, \)

it follows that \( \text{Re} f(x) = \text{Re} f_1(x) + \text{Re} f_2(x) + \cdots \) and \( \text{Im} f(x) = \text{Im} f_1(x) + \text{Im} f_2(x) + \cdots \) for every \( x \) except \( \mathbb{Z} \). Now set

\[
r_j = \begin{cases} 
\text{Re} f_k & \text{if } j = 3k - 2, \\
\text{Im} f_k & \text{if } j = 3k - 1, \\
-\text{Im} f_k & \text{if } j = 3k,
\end{cases}
\]

\( k = 1, 2, \ldots \). Then, on the one hand,

\[
\sum_{j=1}^{\infty} \int |r_j| = \sum_{k=1}^{\infty} \int |\text{Re} f_k| + 2 \sum_{k=1}^{\infty} \int |\text{Im} f_k| < \infty.
\]

On the other hand, for \( x \in \mathbb{R} \), since

\[
\sum_{j=1}^{\infty} |r_j(x)| = \sum_{k=1}^{\infty} |\text{Re} f_k(x)| + 2 \sum_{k=1}^{\infty} |\text{Im} f_k(x)|,
\]

one has

\[
\sum_{j=1}^{\infty} |r_j(x)| < \infty \iff \sum_{k=1}^{\infty} |\text{Re} f_k(x)| < \infty \quad \text{and} \quad \sum_{k=1}^{\infty} |\text{Im} f_k(x)| < \infty
\]

\[\iff \sum_{k=1}^{\infty} |f_k(x)| < \infty.\]

(Concerning the \( \Rightarrow \) part of the first \( \iff \), recall that the sequence of partial sums of a series with positive terms is increasing. Therefore, either the partial sums stay bounded and the series converges or they go off to infinity and the series diverges.)

Hence \( C = \{ x \in \mathbb{R} \mid \sum_{j=1}^{\infty} |r_j(x)| < \infty \} \) and, since \( \text{Re} f(x) = \sum_{k=1}^{\infty} \text{Re} f_k(x) \) for each \( x \) except \( \mathbb{Z} \), \( \text{Re} f(x) =

\[\text{See Ill. -17 to -5, p. 47, but now consider that the functions involved are complex-valued. The same arguments holds!}\]
\[ \sum_{j=1}^{\infty} r_j(x) \text{ for each } x \text{ except } Z. \text{ Therefore } Re f \simeq \sum_{j=1}^{\infty} r_j. \text{ Now, note that a similar argument holds for } Im f. \]

Concerning the \textit{if} part, it seems that the representations of \( Re f \) and \( Im f \) are inverted. In fact, by inverting the representations, on the one hand, since

\[ |f_n + ig_n| \leq |f_n| + |g_n| \]

for each index \( n \), then

\[ \sum_{n=1}^{\infty} \int |f_n + ig_n| \leq \sum_{n=1}^{\infty} \int |f_n| + \sum_{n=1}^{\infty} \int |g_n| < \infty. \]

On the other hand, for each \( x \in \mathbb{R} \) such that \( \sum_{n=1}^{\infty} |(f_n + ig_n)(x)| < \infty \), since

\[ |f_n(x)|, |g_n(x)| \leq |(f_n + ig_n)(x)|, \]

it follows that \( \sum_{n=1}^{\infty} |f_n(x)| < \infty \) and \( \sum_{n=1}^{\infty} |g_n(x)| < \infty \). Hence

\[
\begin{align*}
    f(x) &= Re(f(x)) + iIm(f(x)) \\
         &= \sum_{n=1}^{\infty} f_n(x) + i \sum_{n=1}^{\infty} g_n(x) \\
         &= \sum_{n=1}^{\infty} (f_n + ig_n)(x).
\end{align*}
\]

pp. 74-5

\textbf{Proof of Hölder’s inequality}

1st sentence

If \( \|f\|_{p} = 0 \), then \( f^p = 0 \) a.e. by T. 2.7.4, p. 55. Hence \( f = 0 \) a.e.. Thus \( fg = 0 \) a.e.. Therefore \( \|fg\|_{1} = 0 \) by T. 2.7.4.

Last sentence

On the one hand, one can integrate \( |fg| \) iff \( |fg| \in L^1(\mathbb{R}) \). On the other hand, one only has that \( f \) and \( g \) are locally integrable by D. 2.13.1, p. 74. Now, in order to integrate the lhs of the inequality of the penultimate sentence, if \( f \) or \( g \) is bounded on every bounded interval, then \( fg \) is locally integrable by T. 2.9.4, p. 63.\textsuperscript{27} Thus \( fg \in L^1(\mathbb{R}) \) by that inequality and T. 2.9.5, p. 63. Therefore \( |fg| \in L^1(\mathbb{R}) \) by T. 2.4.1, p. 48.

\textbf{Proof of Minkowski’s inequality}

3rd sentence

Similarly, if \( (f + g)^p \) is locally integrable, then \( |f + g|_{p} \in L^1(\mathbb{R}) \) by T. 2.9.5, p. 63, and T. 2.4.1, p. 48. However, we only have that \( f + g \) is locally integrable.

\textbf{Suggestion}

Concerning the items ‘Last sentence’ and ‘3rd sentence’ above, everything will work out just fine if one changes ‘locally integrable’ to ‘measurable’ in D. 2.13.1, p. 74.\textsuperscript{28}

p. 79, D. 2.14.3

See the previous Suggestion.

\textbf{Exercises, pp. 84-91}

4. \( \text{supp}|f| = \text{supp} f \) is a finite union of semiopen intervals, which is contained in \( \bigcup_{k=1}^{n} [a_k, b_k) \). On the other hand, consider the step function \( g = M_{g_1} + \cdots + M_{g_n} \) where \( g_k \) is the characteristic function of \( [a_k, b_k) \), \( k = 1, \ldots, n \). So \( |f| \leq g \). Now use Theo. 2.2.2.(c).

\textsuperscript{27}Note that, if \( f \) and \( g \) are measurable, one also has that \( fg \) is locally integrable by T. 2.11.6 and T. 2.11.7, p. 71.

\textsuperscript{28}As a matter of fact, see how \( L^p(\Omega) \) is defined on p. 77.
7.

(a) If \( n \in \mathbb{N} \), then \( \tau_z |f_n| (x) = |f_n|(x-z) = |f_n(x-z)| = |\tau_z f_n| (x) = |\tau_z f_n| (x) \) for every \( x \in \mathbb{R} \). Hence

\[
\sum_{n=1}^{\infty} \int |\tau_z f_n| = \sum_{n=1}^{\infty} \int |\tau_z f_n| = \sum_{n=1}^{\infty} \int |f_n| < \infty.
\]

(b) \( \tau_z f(x) = f(x-z) = \sum_{n=1}^{\infty} f_n(x-z) = \sum_{n=1}^{\infty} \tau_z f_n(x) \) for every \( x \in \mathbb{R} \) such that \( \sum_{n=1}^{\infty} |f_n(x-z)| < \infty \). Therefore \( \tau_z f \simeq \tau_z f_1 + \tau_z f_2 + \cdots \) and

\[
\int \tau_z f = \sum_{n=1}^{\infty} \int \tau_z f_n \simeq \sum_{n=1}^{\infty} \int f_n = \int f.
\]

8. Without loss of generality, suppose \( f \) is the characteristic function of \([a, b)\) and \( z > 0 \) is sufficiently small with \([a, b) \cap [a+z, b+z) \neq \emptyset \). Therefore, since

\[
(\tau_z f - f)(x) = \begin{cases} 
-1 & \text{if } x \in [a, a+z), \\
0 & \text{if } x \in [a+z, b), \\
1 & \text{if } x \in [b, b+z),
\end{cases}
\]

if \( z \to 0 \), then

\[
\int |\tau_z f - f| = 2z \to 0.
\]

9.

Let \( (f_n) \) be as in D. 2.3.1, p. 45. Then \( f_n = \sum_{m=1}^{m(n)} \lambda_{m,n} \chi_{[a_{m,n}, b_{m,n}]} \) and \( |f_n| = \sum_{m=1}^{m(n)} |\lambda_{m,n}| \chi_{[a_{m,n}, b_{m,n}]} \) for every \( n \in \mathbb{N} \). Therefore

(a) \( \sum_{n=1}^{\infty} \sum_{m=1}^{m(n)} \int |\lambda_{m,n} \chi_{[a_{m,n}, b_{m,n}]}| = \sum_{n=1}^{\infty} \int |f_n| < \infty; \)

(b) \( f(x) = \sum_{n=1}^{\infty} f_n(x) = \sum_{n=1}^{\infty} \sum_{m=1}^{m(n)} \lambda_{m,n} \chi_{[a_{m,n}, b_{m,n}]}(x) \) whenever \( \sum_{n=1}^{\infty} \sum_{m=1}^{m(n)} |\lambda_{m,n} \chi_{[a_{m,n}, b_{m,n}]}(x)| = \sum_{n=1}^{\infty} |f_n(x)| < \infty. \)

Now arrange the family of intervals \([a_{m,n}, b_{m,n}]\) and the family of numbers \( \lambda_{m,n} \) into sequences

\([a_1, b_1], [a_2, b_2], \ldots \) and \( \lambda_1, \lambda_2, \ldots \),

respectively, so that none of them are missed. Thus \( f \simeq \lambda_1 \chi_{[a_1, b_1]} + \lambda_2 \chi_{[a_2, b_2]} + \cdots \).

10.

(a) Let \( X \) be a countable subset of \( \mathbb{R} \). If \( X \) is finite, use the item of the Comments section concerning p. 53, ll. 7-9, with \( X \) in place of \( \{0\} \). If \( X = \{x_n \ | n \in \mathbb{N} \} \) is infinite and \( f_n = \chi_{\{x_n\}} \) for each \( n \in \mathbb{N} \), since \( f_n \in L^1(\mathbb{R}) \) and \( \int |f_n| = 0 \) for each \( n \in \mathbb{N} \), there exists an \( f \in L^1(\mathbb{R}) \) such that \( f \simeq f_1 + f_2 + \cdots \).

Hence, if \( Z \) denotes the set of all points \( x \) where the series \( \sum_{n=1}^{\infty} |f_n(x)| \) diverges, then the integral of \( \chi_Z \) equals 0, that is \( Z \) is a null set. Thus, since \( X \subset Z \), \( X \) is a null set.\[ ' \]

(b) Consider \( \epsilon > 0 \) is sufficiently small, \( S_\epsilon \subset \mathbb{R} \) is a null set for each \( n \in \mathbb{N} \) and \( S = \bigcup_{n=1}^{\infty} S_n \). By T. 2.11.3, p. 68, there exist intervals \( I_{n,k} = [a_{n,k}, b_{n,k}] \) such that

\[
S_n \subset \bigcup_{k=1}^{\infty} I_{n,k} \quad \text{and} \quad \sum_{k=1}^{\infty} l(I_{n,k}) < \frac{\epsilon}{2^n} \quad \text{for each} \ n \in \mathbb{N}.
\]

29See pp. 40-1.

30Note that the converse is obvious.

31Use the item of the Comments section concerning p. 53, ll. 7-9, with \( \{x_n\} \) in place of \( \{0\} \).

32See C. 2.5.4, p. 52.

33See p. 47, ll. -17, -16 and -15.

34See T. 2.7.2, p. 55.

35Notice that \( \{n \in \mathbb{N} \ | S_n \neq \emptyset\} \) can be finite or infinite.

36If \( l = [a,b] \), then \( l(l) = b-a \).
Now arrange the family of intervals $I_{n,k}$ into a sequence $I_1, I_2, \ldots$ so that none of the $I_{n,k}$ are missed. Therefore

$$S \subset \bigcup_{n=1}^{\infty} \left( \bigcup_{k=1}^{\infty} I_{n,k} \right) = \bigcup_{i=1}^{\infty} I_i$$

and

$$\sum_{i=1}^{\infty} l(I_i) = \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} l(I_{n,k}) < \varepsilon.$$

33.

(a) The constant function $\chi_{\mathbb{R}} = 1$ is locally integrable,\(^{38}\) that is $\mathbb{R} \in \mathcal{M}$. Then $\emptyset = \mathbb{R} \setminus \mathbb{R} \in \mathcal{M}$ by (d).

(e) Consider $I$ is an interval and let $I_{a,b}$ be a bounded interval with finite endpoints $a$ and $b$, $a < b$. Thus $\chi_I$ is locally integrable since $\chi_I \chi_{I_{a,b}}$ is Lebesgue integrable by T. 2.10.1, p. 64.

(f) Any open subset of $\mathbb{R}$ is a countable union of disjoint open intervals. Now use (e) and (b).

(g) Consider $A = \mathbb{R}$ and let $B$ be an arbitrary open subset of $A$. Now use (a), (f) and (d).

34.

(b) Consider $S = B$, $S_1 = A$, $S_2 = B \setminus A$ and $S_n = \emptyset$ for $n = 3, 4, \ldots$. Now use T. 2.11.4, p. 69.

35. Equivalence of all the conditions but (a):

To show (d)$\iff$(e), see E.33.(b,c) and note that

$$\{x \in \mathbb{R} : f(x) > \alpha\} = \bigcup_{n=1}^{\infty} \left\{x \in \mathbb{R} : f(x) \geq \alpha + \frac{1}{n}\right\}, \quad \{x \in \mathbb{R} : f(x) \geq \alpha\} = \bigcap_{n=1}^{\infty} \left\{x \in \mathbb{R} : f(x) > \alpha + \frac{1}{n}\right\}.$$

Since E.33.(a,d) hold, we have (b)$\iff$(e) and (c)$\iff$(d).

37.

See, in the Comments section, concerning p. 70, what was said about "..., $L^1(\Omega)$ is a Banach space."

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\(^{37}\)For example, set $I_1 = I_{1,1}$, $I_2 = I_{1,2}$, $I_3 = I_{2,2}$, $I_4 = I_{2,1}$, $I_5 = I_{1,3}$, $I_6 = I_{2,3}$, $I_7 = I_{3,3}$, $I_8 = I_{3,2}$, $I_9 = I_{3,1}$, etc.

\(^{38}\)See the lines between □ and D. 2.9.3, p. 62.
Errata

p. 98, l. -10
Change ‘(b)’ to ‘(a)’.

p. 100, l. 1
Remove the comma.

p. 106, l. 9
‘N’ should be ‘{1, . . . , n}’.

Comments

p. 95, E. 3.2.6 and E. 3.2.7
The problem resides in proving that the conjugate symmetry condition (D. 3.2.1(a), p. 94) holds. So, there is no problem at all since complex conjugation is a continuous function. In fact, one can see it as a linear function on $\mathbb{C}$ identified with $\mathbb{R}^2$. Its matrix is diagonal with 1 and $-1$ as its diagonal entries. As a linear operator on $\mathbb{R}^2$, it is continuous.

p. 100, E. 3.3.5,

\[ ||f_n - f_m|| \leq \left(\frac{1}{n} + \frac{1}{m}\right)^{1/2} \]

Suppose $n > m$. Concerning F. 3.1 on p. 99, in order to consider $f_n - f_m$, visualize the graphs of $f_n$ and $f_m$ simultaneously and denote the points where the oblique line segments intersect the $x$-axis by $x_1 = \frac{1}{2} + \frac{1}{2n}$ and $x_2 = \frac{1}{2} + \frac{1}{2m}$ (with $x_1 < x_2$). Hence

\[
f_n(x) - f_m(x) = \begin{cases} 
0 & \text{if } 0 \leq x \leq \frac{1}{4}, \\
2(m - n) \left(x - \frac{1}{2}\right) & \text{if } \frac{1}{2} \leq x \leq x_1, \\
2m \left(x - \frac{1}{2}\right) - 1 & \text{if } x_1 < x \leq x_2, \\
0 & \text{if } x_2 \leq x \leq 1.
\end{cases}
\]

Then $||f_n - f_m|| = \sqrt{I_{m,n}}$ with

\[
I_{m,n} = \int_0^1 (f_n(x) - f_m(x))^2 \, dx
\]

\[
= 4(m - n)^2 \int_{1/2}^{x_1} \left(x - \frac{1}{2}\right)^2 \, dx + 4m^2 \int_{x_1}^{x_2} \left(x - \frac{1}{2}\right)^2 \, dx - 4m \int_{x_1}^{x_2} \left(x - \frac{1}{2}\right) \, dx + \int_{x_2}^{1} \, dx
\]

\[
= 4(m - n)^2 \int_{1/2}^{1/2n} t^2 \, dt + 4m^2 \int_{1/2n}^{1/2n} t^2 \, dt - 4m \int_{1/2n}^{1/2m} t \, dt + \frac{1}{2m} - \frac{1}{2n}
\]

\[
= \frac{4(m - n)^2}{3} \left(\frac{1}{2n}\right)^3 + \frac{4m^2}{3} \left[\left(\frac{1}{2m}\right)^3 - \left(\frac{1}{2n}\right)^3\right] - 2m \left[\left(\frac{1}{2m}\right)^2 - \left(\frac{1}{2n}\right)^2\right] + \frac{1}{2m} - \frac{1}{2n}
\]

\[
= \frac{1}{2} \left[\frac{(m - n)^2}{3} \left(\frac{1}{n}\right)^3 + \frac{m^2}{3} \left[\left(\frac{1}{m}\right)^3 - \left(\frac{1}{n}\right)^3\right] - m \left[\left(\frac{1}{m}\right)^2 - \left(\frac{1}{n}\right)^2\right] + \frac{1}{m} - \frac{1}{n}\right]
\]

\[
= \frac{1}{2} \left[\frac{-2m}{3n^2} + \frac{1}{3n} + \frac{1}{5m} + \frac{m}{n^2} - \frac{1}{n}\right]
\]

\[
= \frac{1}{2} \left[\frac{m}{3n^2} - \frac{2}{3n} + \frac{1}{5m}\right]
\]

\[
= \frac{m^2 - 2mn + n^2}{6mn^2}
\]
Therefore, if \( \frac{(m-n)^2}{6mn^2} > \frac{m+n}{mn} \), then \( m > n + \sqrt{6n(m+n)} > n \), which is a contradiction.

p. 103
l. 9
In fact, since \( \langle \cdot, x \rangle \) and complex conjugation are continuous, \( 39 \) \( \langle x, \cdot \rangle = \langle \cdot, x \rangle \) is continuous.

l. -1
\[
\text{Re} \langle x_n, x \rangle \leq |\langle x_n, x \rangle| \\
\leq ||x_n|| ||x|| \to ||x||^2.
\]

p. 111, (3.25)
l. -3
Concerning the last summand, note that
\[
\left( \sum_{j=1}^{n} a_j x_j \right) \left( \sum_{k=1}^{n} a_k x_k \right) = \sum_{k=1}^{n} \left( \sum_{j=1}^{n} a_j x_j a_k x_k \right) \\
= \sum_{k=1}^{n} \sum_{j=1}^{n} \langle a_j x_j, a_k x_k \rangle \\
= \sum_{k=1}^{n} \langle a_k x_k, a_k x_k \rangle.
\]

l. -1
Concerning the last summand, note that
\[
\sum_{k=1}^{n} |\langle x, x_k \rangle - a_k|^2 = \sum_{k=1}^{n} \left( \langle x, x_k \rangle - a_k \right) \left( \langle x, x_k \rangle - a_k \right) \\
= \sum_{k=1}^{n} \left( \langle x, x_k \rangle \langle x, x_k \rangle - \overline{a_k} \langle x, x_k \rangle - a_k \langle x, x_k \rangle + a_k \overline{a_k} \right).
\]

p. 112, \( \lim \limits_{n \to \infty} \langle x, x_n \rangle = 0 \)
The sequence
\[
\left( \sum_{k=1}^{n} |\langle x, x_k \rangle|^2 \right)_{n=1,2,\ldots}
\]
is increasing and bounded above. Then the series of expression (3.26) is convergent. Therefore
\[
\lim \limits_{n \to \infty} |\langle x, x_n \rangle|^2 = 0.
\]

p. 113, E. 3.4.11
See E. 3.4.17.

\[39\text{For the continuity of } \tau : C \to C, \text{ see the first comment in this } \text{Comments} \text{ section.}\]