INTRODUCTION TO HILBERT SPACES WITH APPLICATIONS

THIRD EDITION (2010)

Debnath and Mikusinski

PARTIAL SCRUTINY,
SOLUTIONS OF SOME EXERCISES,
COMMENTS, SUGGESTIONS AND ERRATA

José Renato Ramos Barbosa

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Departamento de Matemática
Universidade Federal do Paraná
Curitiba - Paraná - Brasil
jrrb@ufpr.br

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Erratum, p. **6**, l. 8
'$\sum_{k=1}^{n} |x_j||y_j|$' should be '$\sum_{j=1}^{n} |x_j||y_j|$' or '$\sum_{k=1}^{n} |x_k||y_k|$'.

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Comment, p. **5**, **Theo. 1.2.7**, *Proof*, 2nd sentence
See **Ex.** 8, p. **35**.

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Comment, pp. **6–7**, **Theo. 1.2.8**, *Proof*
The second inequality holds by **Theo. 1.2.7** (Hölder's inequality) provided that

$$\left((x_n + y_n)^{p-1}\right) \in l^q!$$

So consider partial sums (and the last inequality obtained in the *Proof* of **Theo. 1.2.7**) instead:

$$\begin{aligned}
\sum_{k=1}^{m} |x_k + y_k|^p &= \sum_{k=1}^{m} |x_k + y_k||x_k + y_k|^{p-1} \\
&\leq \sum_{k=1}^{m} |x_k||x_k + y_k|^{p-1} + \sum_{k=1}^{m} |y_k||x_k + y_k|^{p-1} \\
&\leq \left(\sum_{k=1}^{m} |x_k|^p\right)^{1/p} \left(\sum_{k=1}^{m} |x_k + y_k|^{q(p-1)}\right)^{1/q} + \left(\sum_{k=1}^{m} |y_k|^p\right)^{1/p} \left(\sum_{k=1}^{m} |x_k + y_k|^{q(p-1)}\right)^{1/q} \\
&\leq \left(\sum_{n=1}^{\infty} |x_n|^p\right)^{1/p} \left(\sum_{k=1}^{m} |x_k + y_k|^{q(p-1)}\right)^{1/q} + \left(\sum_{n=1}^{\infty} |y_n|^p\right)^{1/p} \left(\sum_{k=1}^{m} |x_k + y_k|^{q(p-1)}\right)^{1/q}
\end{aligned}$$

$$\Downarrow$$

$$\sum_{k=1}^{m} |x_k + y_k|^p \leq \left\{ \left(\sum_{n=1}^{\infty} |x_n|^p\right)^{1/p} + \left(\sum_{n=1}^{\infty} |y_n|^p\right)^{1/p} \right\} \left(\sum_{k=1}^{m} |x_k + y_k|^p\right)^{1/q}. \tag{1}$$

On the other hand, the inequality that completes the *Proof* of **Theo. 1.2.8** is trivially satisfied if

$$\sum_{n=1}^{\infty} |x_n + y_n|^p = 0. \tag{2}$$

So suppose (2) is not satisfied. Then there is an index M such that

$$m \geq M \Longrightarrow \sum_{k=1}^{m} |x_k + y_k|^p > 0.$$

Therefore, by (1),

$$\left(\sum_{k=1}^{m} |x_k + y_k|^p\right)^{1-1/q} \leq \left(\sum_{n=1}^{\infty} |x_n|^p\right)^{1/p} + \left(\sum_{n=1}^{\infty} |y_n|^p\right)^{1/p}$$

for $m \geq M$. Now let $m \to \infty$.

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Erratum, p. **7**, l. 14
'X_j' should be 'E_j'.

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Comment, p. **11**, **Ex. 1.3.8**, penultimate sentence
Consider $t \in [0, 1]$. On the one hand,

$$g_n(t) \to 0. \tag{3}$$

On the other hand,

$$f_n(t) = \frac{g_n(t)}{||g_n||} \to 0$$

depending on the behavior of the sequence $(1/\,||g_n||)$ as $n \to \infty$. However, by (1.7), p. **11**, and (3),

$$\frac{1}{||g_n||} \to \infty.$$

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Comment, p. **12**, **Theo. 1.3.11**, *Proof*, penultimate sentence
The contradiction is that $||y_n||_2 \to 0$ but $||y_n||_1 \not\to 0$!

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Comment, p. **13**, **Theo. 1.3.13**, *Proof*
Since the equivalence of norms is an equivalence relation, if two norms are equivalent to $||\cdot||_0$, then they are equivalent to each other.

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Comments, p. **15**, **Ex. 1.3.19**, 1st and 5th sets
Consider that $g \in S_1 :=$ first set,[1] $r := \min\{(f-g)(x) \,:\, x \in \Omega\}$ and $h \in B(g,r)$. So, for each $x \in \Omega$,

$$\begin{aligned}(h-g)(x) &\le |(h-g)(x)| \\ &< r \\ &< (f-g)(x)\end{aligned}$$

$$\Downarrow$$

$$h(x) < f(x).$$

Therefore $h \in S_1$.
Now let x be an arbitrary vector in Ω and consider that n is an arbitrary positive integer. Suppose that $g_n \in S_5 :=$ fifth set,[2] $g \in \mathcal{C}(\Omega)$ and $||g_n - g|| \to 0$.[3] So $(g_n - g)(x) \le (f-g)(x)$ and $(g_n - g)\,(x) \to 0$. Then $g(x) \le f(x)$. Therefore $g \in S_5$.[4]

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Comment, p. **16**, **Theo. 1.3.23**
Let X be the RHS of the equation. It suffices to show that X is closed. In fact, suppose X is closed. So, on the one hand, due to the fact that $S \subset X$,

$$\mathrm{cl}\, S \subset X.$$

On the other hand, if

$$X \not\subset \mathrm{cl}\, S,$$

there exists $x \in X$ with $x \notin \mathrm{cl}\, S$. Then $x \notin C$ for some closed set C containing S. This leads to a contradiction since there exist $x_1, x_2, \ldots \in S \subset C$ with $x_n \to x$. Therefore $x \in C$ by **Theo. 1.3.21**, p. **16**.

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Comments, p. **17**, sentence right before **Theo. 1.3.31**

'only-if-part'
Since $(\|x_n\|)$ is bounded and $|\lambda_n| \to 0$, $|\lambda_n|\,\|x_n\| \to 0$ by a very well-known result from Analysis on the Real Line.

'if-part'
Suppose S is not bounded and n is a positive integer. Thus $\|x_n\| \ge n$ for some $x_n \in S$. Hence $\left\|\frac{1}{n} x_n\right\| \ge 1$, which contradicts the convergence (to 0) hypothesis.

[1] That is, $(f-g)(x) > 0$ for each $x \in \Omega$.
[2] That is, $g_n(x) \le f(x)$.
[3] Hence $|(g_n - g)(x)| \to 0$.
[4] Now use **Theo. 1.3.21**, p. **16**.

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Comment, p. **18**, **Theo. 1.3.33**, *Proof*, 2nd sentence
Suppose $d = 0$ and consider a positive integer n. Hence there exists $x_n \in X$ such that $||z - x_n|| < \frac{1}{n}$, which leads to a contradiction. In fact, since $E\backslash X$ is open, there is an open ball $B(z, \varepsilon) \subset E\backslash X$.
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Comments, pp. **18–9**, **Theo. 1.3.34**, *Proof*

'only-if-part'
A sequence in $\overline{B}(0,1)$ satisfies the condition

$$\begin{aligned}||\alpha_{1,n}e_1 + \cdots + \alpha_{N,n}e_N|| &= |\alpha_{1,n}| + \cdots + |\alpha_{N,n}| \\ &\leq 1.\end{aligned}$$

Furthermore, by the Bolzano-Weierstrass Theorem, $(\alpha_{i,n})$ has a convergent subsequence, $i = 1, \ldots, N$.

'if-part'
Note that when the 2nd sentence ends, its verification begins!

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Comment, p. **21**, l. 11, that is, 2nd series
By the 2nd sentence of **Ex. 1.4.6**, p. **20**, $a_n \in l^2$ for each $n \in \mathbb{N}$. In particular, $a_{n_0} = (\alpha_{n_0,k}) \in l^2$.[5]
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Comment, p. **22**, penultimate sentence
Since $\max_{[0,1]} |P_n(x) - e^x| \to 0$, the absolute convergence criterion from **Def. 1.4.8** is satisfied.
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Comment, p. **23**, **Theo. 1.4.9**, *Proof*, penultimate sentence
(x_{p_k}) is the sum of two convergent sequences:

$$(x_{p_k} - x_{p_1}) = \left(\sum_{j=1}^{k-1} \left(x_{p_{j+1}} - x_{p_j}\right)\right) \quad \text{and} \quad (x_{p_1}, x_{p_1}, \ldots).$$

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Comment, p. **24**, 1st paragraph
A linear isometry is automatically one-to-one. So the requirement for Φ to be one-to-one in (a) is a direct consequence of (b).
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Errata, p. **24**, 2nd paragraph

- antepenultimate sentence
  '$\|[x_n]\|_1$' should be '$\|[(x_n)]\|_1$';
- ultimate sentence
  '... $[(x_n)]$ and $[(y_n)]$ ...' should be '... (x_n) and (y_n) ...'.

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Comments, p. **24**

- 2nd paragraph, last sentence
  Use the fact that
  $$|\,\|x_n\| - \|y_n\|\,| \leq \|x_n - y_n\| \to 0.$$
- 3rd paragraph, last sentence
  $$\begin{aligned}\lim_{n\to\infty} \Phi(x_n) = [(x_n)] \iff \lim_{n\to\infty} \|\Phi(x_n) - [(x_n)]\|_1 &= \lim_{n\to\infty} \|[(x_n - x_1, x_n - x_2, \ldots)]\|_1 \\ &= \lim_{n,k\to\infty} \|x_n - x_k\| \\ &= 0,\end{aligned}$$

  because (x_n) is a Cauchy sequence.

[5]See **Ex. 1.2.6**, p. **4**.

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Comment, p. **27**, 1st sentence after 2nd □

It suffices to consider that E_1 is finite dimensional. In fact, let $\{e_1, \dots, e_N\}$ be a basis of E_1 and assume, without loss of generality,[6] that the norm on E_1 is defined by

$$x = \alpha_1 e_1 + \cdots + \alpha_N e_N \mapsto \|x\| = |\alpha_1| + \cdots + |\alpha_N| \,.$$

Therefore

$$\begin{aligned}\|Lx\| &\le |\alpha_1| \, \|Le_1\| + \cdots + |\alpha_N| \, \|Le_N\| \\ &\le \alpha\|x\|\end{aligned}$$

with $\alpha = \max\{\|Le_i\| \, : \, i = 1, \dots, N\}$.

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Comments, p. **28**, **Theo. 1.5.9**, *Proof*, 2nd paragraph

- 1st sentence
  Consider $\alpha \in \mathbb{F}$ and $x_1, x_2 \in E_1$. So

$$\begin{aligned}L(\alpha x_1 + x_2) &= \lim_{n\to\infty} L_n(\alpha x_1 + x_2) \\ &= \lim_{n\to\infty} (\alpha L_n x_1 + L_n x_2) \\ &= \alpha \lim_{n\to\infty} L_n x_1 + \lim_{n\to\infty} L_n x_2 \\ &= \alpha L x_1 + L x_2.\end{aligned}$$

- 2nd sentence
  (L_n) is bounded by **Lemma 1.4.4**, p. **20**.

- 3rd sentence
  The second equality holds by **Ex. 1.5.3**, p. **26**.

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Comments, p. **29**, **Theo. 1.5.10**

- 1st sentence
  Note that $\mathrm{cl}\,\mathcal{D}(L)$ is a subspace of E_1. In fact, consider $\alpha \in \mathbb{F}$ and $x, y \in \mathrm{cl}\,\mathcal{D}(L)$, that is, there are sequences (x_n) and (y_n) in $\mathcal{D}(L)$ such that $x_n \to x$ and $y_n \to y$. Therefore $\alpha x + y \in \mathrm{cl}\,\mathcal{D}(L)$ since $\alpha x_n + y_n \to \alpha x + y$.[7]

- 2nd sentence
  See **Def. 1.3.25**, p. **17**.

- *Proof*, penultimate sentence
  Since $x_n \to x$ and $Lx_n \to \tilde{L}x$, $\|x_n\| \to \|x\|$ and $\|Lx_n\| \to \|\tilde{L}x\|$. In fact,

$$\|x_n\| \le \|x_n - x\| + \|x\| \quad \text{and} \quad \|x\| \le \|x - x_n\| + \|x_n\|$$

  imply that

$$|\|x_n\| - \|x\|| \le \|x - x_n\| \,.$$

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Erratum, p. **29**, **Theo. 1.5.11**, 1st sentence

'E' should be 'E_1'.

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Comments/Erratum, p. **31**

[6] See **Theo. 1.3.13**.

[7] Anyway, cf. p. **26**, 1st paragraph.

- l. 3, 2nd inequality
  Since $\left\|x_{p_i p_i}\right\| \geq \varepsilon$ for all $i \in \mathbb{N}$ and $\left\|x_{r_i r_j}\right\| < \varepsilon/2^{j+1}$ for all $i \neq j$,

$$
\begin{aligned}
\|x_{s_i s_i}\| - \sum_{i \neq j} \left\|x_{s_i s_j}\right\| > \varepsilon - \sum_{i \neq j} \frac{\varepsilon}{2^{j+1}} &= \varepsilon \left(1 - \sum_{i \neq j} \frac{1}{2^{j+1}}\right) \\
&= \varepsilon \left\{1 - \left[\left(\frac{1}{2^2} + \cdots + \frac{1}{2^i}\right) + \left(\frac{1}{2^{i+2}} + \frac{1}{2^{i+3}} + \cdots\right)\right]\right\} \\
&= \varepsilon \left[1 - \left(\frac{\frac{1}{4}\left(1 - \frac{1}{2^{i-1}}\right)}{1 - \frac{1}{2}} + \frac{\frac{1}{2^{i+2}}}{1 - \frac{1}{2}}\right)\right] \\
&= \varepsilon \left[1 - \left(\frac{1}{2} - \frac{1}{2^i} + \frac{1}{2^{i+1}}\right)\right] \\
&= \varepsilon \left[\frac{1}{2} + \frac{1}{2^i}\left(1 - \frac{1}{2}\right)\right] \\
&= \varepsilon \left[\frac{1}{2}\left(1 + \frac{1}{2^i}\right)\right] > \frac{\varepsilon}{2}
\end{aligned}
$$

  if $i \geq 2$, whereas

$$
\begin{aligned}
\|x_{s_i s_i}\| - \sum_{i \neq j} \left\|x_{s_i s_j}\right\| > \varepsilon - \sum_{j=2}^{\infty} \frac{\varepsilon}{2^{j+1}} &= \varepsilon \left(1 - \sum_{j=2}^{\infty} \frac{1}{2^{j+1}}\right) \\
&= \varepsilon \left(1 - \frac{\frac{1}{8}}{1 - \frac{1}{2}}\right) \\
&= \varepsilon \left(1 - \frac{1}{4}\right) \\
&= \frac{3\varepsilon}{4} > \frac{\varepsilon}{2}
\end{aligned}
$$

  if $i = 1$.

- **Theo. 1.5.13**, *Proof*

  - 1st and 2nd sentences
    In fact, for every strictly sequence (M_n) with $M_1 > 0$, there exists a sequence (T_n) of elements of $\mathcal{T}$ such that $||T_n|| > M_n$ for all $n \in \mathbb{N}$. Since $\mathcal{T} \subset \mathcal{B}(X, Y)$, where (1.14) holds, there exists a sequence (x_n) of unit elements of X such that $||T_n x_n|| > M_n$ for all $n \in \mathbb{N}$.
  - 5th sentence
    See **Theo. 1.4.9**, p. **22**.
  - 6th sentence and 1st clause of 9th sentence
    C does not depend on i since $C = M_z$.[8] Similarly, since

$$
||y_{ij}|| = \frac{1}{i} \left|\left| T_{p_i} \frac{x_{p_j}}{2^j} \right|\right| \underbrace{\leq}_{} \!\!\!\!\!\!\!\!\!\!\!\!\overbrace{}^{\frac{x_{p_j}}{2^j} := x_j} \frac{M_{x_j}}{i}, \quad i, j \in \mathbb{N},
$$

    $\lim_{i \to \infty} y_{ij} = 0$ for all $j \in \mathbb{N}$.
  - 8th sentence
    $\boxed{(y_{q_i q_i})}$ should be $\boxed{(y_{q_i q_j})}$.

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Comments, pp. **32–3**, **Ex. 1.6.3**

[8]See 2nd sentence of **Theo. 1.5.13**.

- 4th sentence
  If $f(x) = x^3 - x - 1$, then $f(1) < 0$ and $f(2) > 0$. So there is some $x_0 \in (1, 2)$ such that $f(x_0) = 0$.[9]

- 6th sentence
  The inequality holds since there exists some $c \in (1, 2)$ such that

$$\begin{aligned}|Tx - Ty| &= \left|T'(c)\right| |x - y| \\ &= \frac{1}{3(1+c)^{2/3}} |x - y| \\ &< \frac{1}{3(1+1)^{2/3}} |x - y| = \frac{1 \cdot 2^{1/3}}{3 \cdot 2^{2/3} \cdot 2^{1/3}} |x - y| .\end{aligned}$$

- 7th/last sentence, $Tx = x^3 - 1$
  On the one hand, if T is a contraction, then

$$\frac{\left|x^3 - y^3\right|}{|x - y|} \leq \alpha < 1.$$

  On the other hand,

$$\frac{\left|x^3 - y^3\right|}{|x - y|} = \left|x^2 + xy + y^2\right| > 1.$$

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Comment, p. **33**, sentences betweeen 2nd □ and **Ex. 1.6.5**
The method is known as *fixed-point iteration*.

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Comment, p. **34**, **Ex. 1.6.6**, penultimate sentence
Suppose f is a contraction. So, since $F = \mathbb{R}^+$ is closed, f has a fixed point by **Theo. 1.6.4**.[10]

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Exercises, pp. **34–8**

1. Consider $z, z', w \in E$ with $x + z = y = x + z'$ and $z + w = z'$. Then $y = x + z' = x + z + w = y + w$. So $w = 0$. Therefore $z' = z + w = z$.

3.

   (a) $\lambda 0 = 0$ for each λ since $\lambda 0 = \lambda(0 + 0) = \lambda 0 + \lambda 0$. Therefore, since $\lambda \neq 0$,

$$\begin{aligned}\lambda x = 0 &\Longrightarrow \lambda^{-1}(\lambda x) = \lambda^{-1} 0 \\ &\Longrightarrow (\lambda^{-1}\lambda) x = 0 \\ &\Longrightarrow 1x = 0 \\ &\Longrightarrow x = 0.\end{aligned}$$

   (b) Consider $x \neq 0$. Suppose $\lambda \neq 0$. By (a), since $\lambda x = 0$, it follows that $x = 0$, which is a contradiction.

   (c) Since $0x = (0 + 0)x = 0x + 0x$, it follows that $0x = 0$. Then

$$\begin{aligned}x + (-1)x &= 1x + (-1)x \\ &= [1 + (-1)]x \\ &= 0x \\ &= 0.\end{aligned}$$

   Therefore $(-1)x = 0 - x = -x$.[11]

8. Since $h(x) := \frac{1}{p}x + \frac{1}{q} - x^{\frac{1}{p}}$ is continuous on $[0, 1]$, $h(0) = \frac{1}{q} > 0$, $h'(x) = \frac{1}{p}\left(1 - x^{-\frac{1}{q}}\right) < 0$ for $0 < x < 1$ and $h(1) = 0$, it follows that $h(x) \geq 0$ for $0 \leq x \leq 1$.

[9] Use the Intermediate Value Theorem.
[10] See the ultimate sentence.
[11] See p. **3**, 2nd paragraph.

22.

(a) Suppose $\|x_n - x\| \to 0$ and $\|x_n - y\| \to 0$. Use $\|x - y\| \leq \|x - x_n\| + \|x_n - y\|$.

(b) Use

$$\begin{aligned}\|\lambda_n x_n - \lambda x\| &= \|\lambda_n x_n - \lambda x_n + \lambda x_n - \lambda x\| \\ &= \|(\lambda_n - \lambda)\, x_n + \lambda\,(x_n - x)\| \\ &\leq |\lambda_n - \lambda|\, \|x_n\| + |\lambda|\, \|x_n - x\| \\ &\leq |\lambda_n - \lambda|\, (\|x_n - x\| + \|x\|) + |\lambda|\, \|x_n - x\|\,.\end{aligned}$$

(c) Use $\|x_n + y_n - (x + y)\| \leq \|x_n - x\| + \|y_n - y\|$.

34.

(a) $\Longrightarrow$ (b)
The proof is trivial by **Theo. 1.3.23** and **Def. 1.3.25**.

(b) $\Longrightarrow$ (c)
Consider an open ball $B(x, \varepsilon)$. Since there exist $x_1, x_2, \ldots \in S$ with $x_n \to x$, there exists a number M such that $x_n \in B(x, \varepsilon)$ for every index $n \geq M$.[12]

(c) $\Longrightarrow$ (a)
Let $x \in E$. Hence there exists $x_n \in S \cap B(x, 1/n)$ for each positive integer n. Therefore $x \in \operatorname{cl} S$.

39.

(a) $\Longrightarrow$ (b)
Note that $p_n \geq n$ and $q_n \geq n$ for each positive integer n. Now consider ε and M given in **Def. 1.4.1**, p. **19**. Therefore

$$\begin{aligned} n \geq M &\Longrightarrow p_n, q_n \geq M \\ &\Longrightarrow \left\|x_{p_n} - x_{q_n}\right\| < \varepsilon. \end{aligned}$$

(b) $\Longrightarrow$ (c)
Concerning (b), consider $q_n = p_{n+1}$.

(c) $\Longrightarrow$ (a)
Suppose (a) is false. So there is a positive ε_0 such that, for each positive integer M, there exist indices m_0 and n_0 where

$$m_0, n_0 > M \ \text{ and } \ \|x_{m_0} - x_{n_0}\| \geq \varepsilon_0.$$

Now consider $m_0 \geq n_0$ and an increasing sequence of positive integers (p_n) such that $p_{n_0} = n_0$ and $p_{n_0+1} = m_0$. Therefore

$$n_0 > M \ \text{ and } \ \left\|x_{p_{n_0+1}} - x_{p_{n_0}}\right\| \geq \varepsilon_0,$$

which contradicts (c).

41. As in **Ex. 1.4.6**, pp. **20–1**, the same argument applies if 2nd powers and square roots are replaced with pth powers and pth roots, respectively.

48.

(a) $\Longleftrightarrow$ (b)
Via **Ex.** 35, p. **37**, F is continuous iff for every $x \in E_1$ and $\varepsilon > 0$ there exists a $\delta > 0$ such that $F(B(x, \delta)) \subset B(F(x), \varepsilon)$.

(a) $\Longrightarrow$ (b)
Let $x \in F^{-1}(U)$ and take $\varepsilon > 0$ and $\delta > 0$ with

$$F(B(x,\delta)) \overbrace{\subset}^{F\text{ is continuous}} B(F(x),\varepsilon) \overbrace{\subset}^{U\text{ is open in }E_2} U.$$

Hence $B(x, \delta) \subset F^{-1}(U)$.

[12] See **Def. 1.3.6**, p. **10**.

(a) $\Longleftarrow$ (b)
For $x \in E_1$ and $\varepsilon > 0$, $F^{-1}(B(F(x), \varepsilon))$ is open in E_1. Therefore there is a $\delta > 0$ for which $B(x, \delta) \subset F^{-1}(B(F(x), \varepsilon))$. Thus $F(B(x, \delta)) \subset B(F(x), \varepsilon)$.

(b) $\Longleftrightarrow$ (c)
Use that *complements of open (resp. closed) sets are closed (resp. open) sets* and *inverse images commute with complements*.

49. Concerning the 1st sentence, use that $\mathcal{N}(L) = L^{-1}(\{0\})$ and **Theo. 1.5.4**.

51. Uniform convergence is the one with respect to (1.14).[13] That being said, on the one hand, suppose $\|L_n - L\| \to 0$ as $n \to \infty$. Therefore $\|L_n x - Lx\| \leq \|L_n - L\| \|x\| \to 0$ for every $x \in E_1$.[14] Now, on the other hand, consider $E_1 = E_2 = l^2$ and the projection $x = (x_1, x_2, \ldots) \mapsto L_n x = (x_1, \ldots, x_n, 0, 0, \ldots)$. Then $\|L_n - L_m\| = 1$ for $n \neq m$.[15] So, since (L_n) is not a Cauchy sequence, it does not converge (uniformly). However, for $x \in l^2$, we have $L_n x \to x$ as $n \to \infty$. Thus $L_n \to I$ strongly.

[13] See p. **28**, sentence that precedes **Theo. 1.5.9**.
[14] See p. **28**, sentence that follows $\square$.
[15] Without loss of generality, assume $n < m$. Thus

$$\begin{aligned}\|L_n - L_m\| &= \sup_{\|x\|=1} \|(L_n - L_m)\, x\| \\ &= \sup_{\|x\|=1} \|(0, \ldots, 0, x_{n+1}, \ldots, x_m, 0, 0, \ldots)\| \\ &= \sup_{\|x\|=1} \sqrt{\sum_{i=n+1}^{m} x_i^2} \\ &= 1.\end{aligned}$$

In fact, on the one hand, $\sqrt{\sum_{i=n+1}^{m} x_i^2} \leq \sqrt{\sum_{i=1}^{\infty} x_i^2} = \|x\| = 1$ for each unit vector x. On the other hand, consider $x = (0, \ldots, 0, 1, 0, 0, \ldots)$ with $1 = x_i$, $i \in \{n+1, \ldots, m\}$.

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Comments, p. **42**, **Lemma 2.2.4**, *Proof*

- 1st paragraph, penultimate sentence
  Since $b_{n_0} \in (a_{n_0}, b]$ and $b_{n_0,n} = b_{n_0}$ for each positive integer n,
  $$\{n : a_n < b_{n_0,n}\} = \{n_0\}.$$
- 3rd paragraph, penultimate sentence
  $b_{b_k,n} = \min\{b_n, b_k\}$ and $b_{s,n} = \min\{b_n, s\}$ imply that
  $$\begin{aligned}\sum_{a_n < b_{b_k,n}} \left(b_{b_k,n} - a_n\right) &= (b_k - a_k) + \left\{ \left[\sum_{a_n < b_{s,n}} (b_{s,n} - a_n) \right] - (s - a_k) \right\} \\ &= b_k - a_k + s - a - s + a_k \\ &= b_k - a.\end{aligned}$$

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Comments, p. **43**, **Theo. 2.2.6**, *Proof*

- 1st sentence
  Use **Theo. 2.2.2.**(c), twice!
- 7th sentence
  $[a, b) \subset \cup_{n=1}^{\infty} A_n$. In fact, suppose otherwise. So consider $a \leq \ell < b$ such that $f_n(\ell) \geq \alpha$ for each index n. Therefore $f_n(\ell) \not\to 0$, which is a contradiction!

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Comment, p. **44**, (2.8)
g is a step function with support contained in the union of

$$[a_{1,1}, b_{1,1}), \ldots, [a_{1,k_1}, b_{1,k_1}), \ldots, [a_{n_0,1}, b_{n_0,1}), \ldots, \left[a_{n_0,k_{n_0}}, b_{n_0,k_{n_0}}\right).$$

Therefore

$$\begin{aligned}\int g &\leq \alpha \sum_{n=1}^{n_0} \sum_{k=1}^{k_n} (b_{n,k} - a_{n,k}) \\ &< \alpha(b - a).\end{aligned}$$

==

Erratum, p. **44**, **Cor. 2.2.7**
"... be nondecreasing sequences ..." should be *"... be a nondecreasing sequence ..."*.

==

Comment, p. **46**, l. 2
For every $x \in \mathbb{R}$ such that $\sum_{n=1}^{\infty} |f_n(x)| < \infty$,

$$\begin{aligned}\lim_{n\to\infty} g_n(x) &= f_1(x) + \cdots + f_{n_0}(x) + \sum_{n=1}^{\infty} |f_{n_0+n}(x)| \\ &\geq f_1(x) + \cdots + f_{n_0}(x) + \sum_{n=1}^{\infty} f_{n_0+n}(x) \overbrace{=}^{\text{(b), p. }\mathbf{45}} f(x) \\ &\geq 0.\end{aligned}$$

For $x \in \mathbb{R}$ such that $\sum_{n=1}^{\infty} |f_n(x)|$ does not converge,

$$\lim_{n\to\infty} g_n(x) = f_1(x) + \cdots + f_{n_0}(x) + \sum_{n=1}^{\infty} |f_{n_0+n}(x)| = +\infty.$$

==

Comments, p. **47**, paragraph right after □

- Penultimate sentence
  Since $f+g$ and (f_n) satisfy **Def. 2.3.1**, both f and $f+g$ have the same representation and, by (2.10), the same integral.

- Ultimate sentence
  $-f, f+g \in L^1(\mathbb{R}) \implies -f + (f+g) \in L^1(\mathbb{R})$.

==

Comment, p. **48**, sentence right before **Theo. 2.4.1**
If $z=0$ is a simple pole of an analytic function $g(z)$, then

$$\lim_{\epsilon\to 0} \int_{\gamma(\epsilon)} g(z)\, dz = \pi i \operatorname{Res}(g, 0),$$

where $\gamma(\epsilon)$ is a semicircle of small radius ϵ, centered at the origin, situated in the upper half-plane and described in the direction of increasing argument, and the residue $\operatorname{Res}(g, z_0)$ is the coefficient of $(z-z_0)^{-1}$ in the Laurent series expansion of g at $z_0 = 0$.[16] Hence, since $\frac{\sin x}{x} = \frac{e^{ix} - \cos x}{ix}$ and $\frac{\cos x}{x}$ is an odd function,

$$\begin{aligned}
\int_{-\infty}^{\infty} \frac{\sin x}{x}\, dx &= \frac{1}{i} \lim_{\epsilon\to 0} \left(\int_{-\infty}^{-\epsilon} \frac{e^{ix}}{x}\, dx + \int_{\epsilon}^{\infty} \frac{e^{ix}}{x}\, dx \right) \\
&= \frac{1}{i} \lim_{\epsilon\to 0} \int_{\gamma(\epsilon)} \frac{e^{iz}}{z}\, dz \\
&= \pi \operatorname{Res}\left(\frac{e^{iz}}{z}, 0 \right) \\
&= \pi
\end{aligned}$$

where $\operatorname{Res}\left(\frac{e^{iz}}{z}, 0\right)$ is the coeficient of z^{-1} in the Laurent series

$$\frac{1}{z} + i - \frac{z}{2} - \frac{iz^2}{6} + \frac{z^3}{24} + \mathcal{O}\left(z^4\right).$$

On the other side, $\frac{\sin x}{x}$ is not absolutely integrable over $[0, \infty)$ since

$$\int_0^{\infty} \left| \frac{\sin x}{x} \right| dx = \sum_{k=0}^{\infty} \int_{k\pi}^{(k+1)\pi} \frac{|\sin x|}{x}\, dx \geq \sum_{k=0}^{\infty} \frac{1}{(k+1)\pi} \int_{k\pi}^{(k+1)\pi} |\sin x| dx = s$$

with

$$\begin{aligned}
s &= \frac{1}{\pi} \int_0^{\pi} \sin x\, dx + \frac{1}{2\pi} \int_{\pi}^{2\pi} (-\sin x) dx + \frac{1}{3\pi} \int_{2\pi}^{3\pi} \sin x\, dx + \frac{1}{4\pi} \int_{3\pi}^{4\pi} (-\sin x) dx + \cdots \\
&= \frac{1}{\pi} \overbrace{\cos x\big|_{\pi}^{0}}^{2} + \frac{1}{2\pi} \overbrace{\cos x\big|_{\pi}^{2\pi}}^{2} + \frac{1}{3\pi} \overbrace{\cos x\big|_{3\pi}^{2\pi}}^{2} + \frac{1}{4\pi} \overbrace{\cos x\big|_{3\pi}^{4\pi}}^{2} + \cdots \\
&= \frac{2}{\pi} \left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots \right) \\
&= \infty.
\end{aligned}$$

[16] Refer to Elementary Theory of Analytic Functions of One or Several Complex Variables by Henri Cartan, p. 104.

(Note that, since

$$\begin{aligned}\int_{-\infty}^{0}\left|\frac{\sin x}{x}\right|dx &= \int_{-\infty}^{0}\left|\frac{-\sin(-x)}{x}\right|dx\\ &= -\int_{\infty}^{0}\left|\frac{\sin u}{u}\right|du\\ &= \int_{0}^{\infty}\left|\frac{\sin u}{u}\right|du,\end{aligned}$$

integration on $(-\infty,\infty)$ was not necessary.)

==

Comment, p. **50**, sentence right before **Theo. 2.4.3** and 1st sentence of its *Proof*
See **Exs.** 7–8, p. **85**.

==

Comment, p. **51**, **Lemma 2.5.2**, *Proof*, 5th sentence, right before the comma
See **Ex.** 11, p. **85**.

==

Comment/Erratum, p. **52**, **Theo. 2.5.3**, *Proof*

- $f \simeq \sum_{n=1}^{\infty}\sum_{k=1}^{\infty} f_{n,k}$ since:

  (a) $\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\int |f_{n,k}| \leq \sum_{n=1}^{\infty}\int |f_n| + \sum_{n=1}^{\infty} 2^{-n} < \infty$;

  (b) $f(x) = \sum_{n=1}^{\infty} f_n(x) = \sum_{n=1}^{\infty}\sum_{k=1}^{\infty} f_{n,k}(x)$ for each $x \in \mathbb{R}$ such that $\overbrace{\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\left|f_{n,k}(x)\right|}^{(*)} < \infty$.
  In fact,

  $$\sum_{n=1}^{\infty}|f_n(x)| = \sum_{n=1}^{\infty}\left|\sum_{k=1}^{\infty} f_{n,k}(x)\right| \leq (*).$$

- Change '$g_{n,k}$' to '$f_{n,k}$'.

==

Comments, p. **53**

- ll. 7–9
  The restriction of $f = \chi_{\{0\}}$ to each $[a,b]$ containing $\{0\}$ is Riemann integrable and its Riemann integral is 0. Now use **Theo. 2.10.1**, p. **64**.[17]

- Sentence right before **Theo. 2.6.3**
  Use (2.14) with f in place of g.

==

Comment, p. **54**, **Theo. 2.6.6**, *Proof*, 3rd sentence, right before the first comma
See **Ex.** 11, p. **85**.

==

Comment, p. **55**, paragraph right before **Theo. 2.7.2**
See **Ex.** 19, p. **86**.

==

Comments, pp. **55–6**, last 3 sentences before **Theo. 2.7.5**
If $f, g \in L^1(\mathbb{R})$ with $f = g$ a.e., then $\left|\int(f-g)\right| \leq \int |f-g| = 0$. Thus $\int f = \int g$.

==

Comments, pp. **57–8**, paragraph right before **Theo. 2.7.10**

[17] See also **Ex.** 9, p. **85**.

- 1st sentence
  A sequence of functions $f_1, f_2, \ldots$ defined on $X \subset \mathbb{R}$ *converges uniformly to* f if

$$\sup_{x \in X} |f_n(x) - f(x)| \to 0 \quad \text{as} \quad n \to \infty.$$

  So, concerning **Ex. 2.7.8**, $f_n \to 0$ uniformly since

$$\sup_{x \in \mathbb{R}} |f_n(x)| = \frac{1}{\sqrt{n}} \quad \forall n \in \mathbb{N}.$$

- 2nd and 3rd sentences
  The inequality follows from

$$|f_n - f| \leq \sup_{x \in \mathbb{R}} |f_n(x) - f(x)| \, \chi_{[a,b]} \quad \forall n \in \mathbb{N}.$$

==
Comments, p. **58**

- **Theo. 2.7.10**, *Proof*
  In place of the ultimate sentence, consider p. **47**, paragraph that follows □, last three sentences.

- **Theo. 2.7.12**
  $f = f_1 + f_2 + \cdots$ i.n. signifies $f_1 + \cdots + f_n \to f$ i.n..

- **Theo. 2.8.1**, *Proof*, 2nd sentence
  Recall that $\int |f_n| = \|f_n\|$, where $\|\cdot\|$ is the L^1-norm.[18]

==
Comments, p. **59**

- **Theo. 2.8.2**, *Proof*
  - 3rd sentence, right before the second comma
    See **Cor. 2.5.4**, p. **52**.
  - 4th sentence

$$\begin{aligned} f_{p_n} &= f_{p_1} + (f_{p_2} - f_{p_1}) + \cdots + (f_{p_n} - f_{p_{n-1}}) \\ &\to g \text{ a.e.} \end{aligned}$$

    is another way to write the equality that ends the 3rd sentence.
  - 5th sentence, right before the first comma
    The equality that ends the 3rd sentence and **Theo. 2.7.12**, p. **58**, imply that

$$g = f_{p_1} + (f_{p_2} - f_{p_1}) + (f_{p_3} - f_{p_2}) + \cdots \text{ i.n.},$$

    which can also be written as

$$\begin{aligned} f_{p_n} &= f_{p_1} + (f_{p_2} - f_{p_1}) + \cdots + (f_{p_n} - f_{p_{n-1}}) \\ &\to g \text{ i.n..} \end{aligned}$$

    On the other hand,

$$f_n \to f \text{ i.n.} \Longrightarrow f_{p_n} \to f \text{ i.n..}$$

- Penultimate sentence
  See **Theo. 2.6.5**, p. **54**.

- Ultimate sentence
  The equality is known as *passage to the limit under the integral sign*.

[18]See **Def. 2.6.1**, p. **52**.

==
Comment/Erratum, p. **60**

- 1st sentence
  See **Ex. 2.7.8**, pp. **56–7**.

- **Theo. 2.8.3**, *Proof*, last equality
  Change the last '$-$' to '$+$'.

==
Comments/Errata, p. **61**

- 1st sentence
  $\int h < \infty$ by **Def. 2.3.1**, p. **45**, and **Theo. 2.4.1**, p. **48**.

- 2nd sentence

  - For a fixed $m \in \mathbb{N}$, define

    $$u_n = g_{m,n+1} = \max\{|f_m|, |f_{m+1}|, \ldots, |f_{m+n+1}|\} \text{ and } v_n = g_{m+1,n} = \max\{|f_{m+1}|, \ldots, |f_{m+n+1}|\}.$$

    Then $u_n \geq v_n$ for every $n \in \mathbb{N}$. Therefore $g_m = \lim_{n\to\infty} u_n \geq \lim_{n\to\infty} v_n = g_{m+1}$.

  - Change '$|f_1|$' to 'h'.

- Case 1, 3rd sentence
  Change 'f_n' to 'g_n'.

- Case 2, 3rd sentence
  See **Theo. 1.4.2**, pp. **19–20**.

==
Erratum/Comments, p. **62**

- **2.9**, 1st sentence
  Change '$\int_{\mathbb{R}}$' to '$\int_{\mathbb{R}} f$'.

- **Theo. 2.9.2**, *Proof*
  Note that $f\chi_{[a.b]} \simeq \sum_{n=1}^{\infty} f_n\chi_{[a.b]} = \sum_{n=1}^{\infty} g_n$.

- Ultimate paragraph, right before **Def. 2.9.3**
  By **Ex.** 25, p. **87**, the constant function $f = 1$ does not belong to $L^1(\mathbb{R})$. By **Theo. 2.10.1**, p. **64**, $\int_a^b f$ exists for every $-\infty < a < b < \infty$.

==
Comment/Erratum, p. **63**

- 2nd sentence
  Consider an arbitrary $[a, b]$. Let N be a positive integer such that $[a, b] \subset [-N, N]$ and consider the *Proof* of **Theo. 2.9.2** with $f\chi_{[-N,N]}$ in place of f.[19] Therefore

  $$f\chi_{[a,b]} = f\chi_{[-N,N]}\chi_{[a,b]} \simeq g_1 + g_2 + \cdots.$$

- Penultimate paragraph
  "In applications it often ..." should be "In applications it is often ...".

==
Comments, pp. **64–5**

[19] N is used here since n is used in the above-mentioned *Proof*.

- 1st sentence
  See **Ex. 28**, p. **87**.

- **Theo. 2.10.1**, *Proof*

  - 1st paragraph
    Denote the inf (resp. sup) of $f([a+(k-1)c, a+kc))$ by m_k (resp. M_k) and the characteristic function of $[a+(k-1)c, a+kc)$ by f_k, $k = 1, \ldots, n$. Therefore
    $$g_n = m_1 f_1 + \cdots + m_n f_n \ \text{(resp.}\ h_n = M_1 f_1 + \cdots + M_n f_n\text{)}.$$[20]

  - 2nd paragraph
    * 1st sentence
      As finer partitions of $[a, b)$ are considered, (g_n) (resp. (h_n)) keeps nondecreasing (resp. nonincreasing).
    * 3rd and 4th sentences
      Consider $n \in \mathbb{N}$. Then, since $f(\mathbb{R}) \subset [-M, M]$,
      $$-M \leq g_n \leq f \leq h_n \leq M,$$
      that is,
      $$-M \leq -h_n \leq -f \leq -g_n \leq M.$$
      So, if
      $$\varphi(x) := \begin{cases} M & \text{if } x \in [a,b), \\ 0 & \text{otherwise,} \end{cases}$$
      then $|g_n| \leq \varphi$ and $|h_n| \leq \varphi$. Therefore we can use **Theo. 2.8.4** properly. Now, one the one hand, note that $\int g_n$ and $\int h_n$ are Riemann sums.[21] One the other hand, note that the *passage to the limit under the integral sign* was used, twice.[22]
    * Antepenultimate sentence
      $g = h$ a.e. by **Theo. 2.7.4**, p. **55**.
    * Penultimate sentence
      By **Theo. 2.7.4**, p. **55**, $\int |f - g| = 0$. Then $f - g \in L^1(\mathbb{R})$.[23] So, since $g \in L^1(\mathbb{R})$, $f = f - g + g \in L^1(\mathbb{R})$.

- **Theo. 2.10.2** and **Theo. 2.10.3**
  To be Lebesgue integrable is to be Lebesgue integrable on $\mathbb{R}$. Then f is Lebesgue integrable on (a, b) if $f\chi_{(a,b)}$ is Lebesgue integrable, that is, f is integrable over (a, b).[24]

==

Comments, pp. **68–9**

- **Def. 2.11.1**
  S is measurable if $\chi_S \chi_{[a,b)}$ is integrable for every $-\infty < a < b < \infty$.[25]

- Sentence that comes right after **Def. 2.11.2**
  See **Def. 2.7.1**, p. **55**, and **Def. 2.6.2**, p. **53**.

- **Theo. 2.11.3**, *Proof*

  - 3rd sentence
    Note that
    $$\begin{aligned} \int |f| &= \int \chi_S \\ &= \mu(S) \\ &= 0 \end{aligned}$$
    due to the sentence that comes right after **Def. 2.11.2**.

---

[20] See p. **40**, (2.1).
[21] See **Def. 2.2.1**, p. **41**
[22] See p. **59**, ultimate paragraph.
[23] See p. **53**, right after **Def. 2.6.2**.
[24] See p. **62**, **2.9**, everything before **Theo. 2.9.2**.
[25] See **2.9**.

- 4th sentence
  Since $f_1 + f_2 + \cdots \simeq \chi_S$ and $f_1 + \cdots + f_n \leq |f_1| + \cdots + |f_n|$ for each $n \in \mathbb{N}$, there exists an index n_0 such that $A_{n_0} \neq \emptyset$.[26]
- 7th sentence

$$\begin{aligned}\sum_{k=1}^{k_n} (b_{n,k} - a_{n,k}) &= \int \chi_{A_n} \\ &\leq \int \left(2 \sum_{i=1}^{n} |f_i| \right) = 2 \sum_{i=1}^{n} \int |f_i| \\ &\leq 2 \sum_{n=1}^{\infty} \int |f_n| \\ &< \frac{2\varepsilon}{3},\end{aligned}$$

  where:
  - the first equality comes from the fact that

$$\chi_{A_n} = \sum_{k=1}^{k_n} \chi_{[a_{n,k}, b_{n,k})};^{27}$$

  - the first inequality comes from the fact that

$$\left(2 \sum_{i=1}^{n} |f_i| \right) \geq \chi_{A_n}.$$

- Penultimate paragraph
  - l. -6
    Use **Cor. 2.5.4**, p. **52**.
  - Ultimate sentence
    Since $h_n \to h$ i.n., use the *passage to the limit under the integral sign*.[28]

- **Theo. 2.11.4**, *Proof*, 1st part
  On the one hand, since $S = \bigcup_{n=1}^{\infty} S_n$ is a disjoint union,

$$\chi_S(x) = \big(\chi_{S_1} + \chi_{S_2} + \cdots\big)(x) \text{ for every } x \in \mathbb{R}. \tag{4}$$

  On the other hand, since each S_n is measurable, each χ_{S_n} is a locally integrable function. So, since $\chi_{S_n} \leq \chi_{[a,b]}$ for every $n \in \mathbb{N}$, every χ_{S_n} is an integrable function by **Theo. 2.9.5**, p. **63**. Then, since $\big(\chi_{S_1} + \cdots + \chi_{S_n}\big)(x) \to \chi_S(x)$ for every $x \in \mathbb{R}$ (by (4)) and $\chi_{S_1} + \cdots + \chi_{S_n} \leq \chi_{[a,b]}$ for every $n \in \mathbb{N}$, χ_S is integrable and $\chi_{S_1} + \cdots + \chi_{S_n} \to \chi_S$ i.n. by **Theo. 2.8.4**, p. **60**. So

$$\begin{aligned}\sum_{n=1}^{\infty} \int \chi_{S_n} &= \lim_{n\to\infty} \sum_{k=1}^{n} \int \chi_{S_k} \\ &= \lim_{n\to\infty} \int \sum_{k=1}^{n} \chi_{S_k} \\ &= \int \left(\lim_{n\to\infty} \sum_{k=1}^{n} \chi_{S_k} \right) \\ &= \int \chi_S \\ &< \infty\end{aligned}$$

  (where the penultimate equality comes from the *passage to the limit under the integral sign*).
  Therefore $\chi_S \simeq \chi_{S_1} + \chi_{S_2} + \cdots$.

[26] In fact, suppose otherwise to obtain a contradiction.
[27] See **Def. 2.2.1**, p. **41**.
[28] p. **59**, ultimate paragraph.

==
Comments, p. **70**

- Continuation of **Theo. 2.11.4**, *Proof*
  - 2nd sentence
    In fact, for each $[a,b]$,
    $$\chi_S = \chi_{S_1} + \chi_{S_2} + \cdots \text{ pointwise} \Longrightarrow \chi_S\chi_{[a,b]} = \chi_{S_1}\chi_{[a,b]} + \chi_{S_2}\chi_{[a,b]} + \cdots \text{ pointwise}$$
    and, by **Theo. 2.9.4**, p. **63**, each $\chi_{S_n}\chi_{[a,b]}$ is locally integrable. Therefore, by using a similar argument as in the 1st part of the proof,
    $$\chi_S\chi_{[a,b]} \simeq \chi_{S_1}\chi_{[a,b]} + \chi_{S_2}\chi_{[a,b]} + \cdots \text{ for each } [a,b].$$
    Therefore χ_S is locally integrable.
  - Case 1
    Each χ_{S_n} is integrable by **Theo. 2.9.5**, p. **63**. Now, note that since χ_S is integrable, (4) holds and $\chi_{S_1} + \cdots + \chi_{S_n} \leq \chi_S$ for every $n \in \mathbb{N}$, we have $\chi_{S_1} + \cdots + \chi_{S_n} \to \chi_S$ i.n. by **Theo. 2.8.4**, p. **60**. Then, by using a similar argument as in the 1st part of the proof,
    $$\sum_{n=1}^{\infty} \int \chi_{S_n} < \infty.$$
    Therefore $\chi_S \simeq \chi_{S_1} + \chi_{S_2} + \cdots$.
  - Case 2
    * $\sum_{n=1}^{\infty} \int \chi_{S_n} < \infty \Longrightarrow \int \chi_{S_n} < \infty$ for every $n \in \mathbb{N} \Longrightarrow \chi_{S_1}, \chi_{S_2}, \ldots \in L^1(\mathbb{R})$;
    * $\sum_{n=1}^{\infty} \chi_{S_n} = f$ a.e. by **Theo. 2.7.10**, p. **58**;
    * Since $|f - \chi_S|$ is integrable (by **Theo. 2.7.4**, p. **55**) and f is integrable, $\chi_S = f - (f - \chi_S)$ is integrable.[29]

- 1st paragraph after □, ultimate sentence
  Follow the *Proof* of **Theo. 2.8.1**, p. **58**, but now with each f_n in $L^1(\Omega)$. So $f \in L^1(\mathbb{R})$. As a matter of fact, $f \in L^1(\Omega)$ since $f = \sum_{n=1}^{\infty} f_n$ a.e..[30]

- 2nd paragraph after □, ultimate sentence
  Let $g = |f|$ and consider **Theo. 2.11.7**, p. **71**, and **Theo. 2.9.5**, p. **63**.

- Ultimate paragraph
  - 1st sentence
    Let $f \simeq f_1 + f_2 + \cdots$ be as in **Def. 2.3.1**, p. **45**. Then $f_1 + \cdots + f_n \to f$ a.e. by **Cor. 2.7.11**, p. **58**.
  - 4th (last) sentence
    Suppose that $f\chi_{[0,1]} \in L^1(\mathbb{R})$ to obtain a contradiction. In fact, on the one hand, for each positive integer n,
    $$\begin{aligned} \ln n &= \int_{[\frac{1}{n},1]} f \text{ (by \textbf{Theo. 2.10.1}, p. \textbf{64})} \\ &\leq \int_{[0,1]} f. \end{aligned}$$
    On the other hand, $\int f\chi_{(0,1]} < \infty$ by **Def. 2.3.1**, p. **45**, and **Theo. 2.4.1**, p. **48**.

==
Comments/Erratum, p. **71**

[29] See p. **53**, right after **Def. 2.6.2**.
[30] See p. **55**, **Def. 2.7.3**, and p. **47**, antepenultimate and penultimate paragraphs, starting from the italicized sentence.

- **Theo. 2.11.6**, *Proof*
  Consider **Exercises** 21 and 19.(b), pp. **86-7**.

- Sentence that precedes **Theo. 2.11.7**
  See p. **49**, **Cor. 2.4.2**, *Proof*, l. 2.

- (2.26)
  '$f(x)$' should be 'f'. '$b_n]$' should be '$b_n)$'.

==
Comment/Erratum, p. **73**, **Theo. 2.12.2**, *Proof*

- *only if* part
  Consider

$$C := \left\{ x \in \mathbb{R} : \sum_{n=1}^{\infty} |f_n(x)| < \infty \right\},$$

$$C_r := \left\{ x \in \mathbb{R} : \sum_{n=1}^{\infty} |\mathrm{Re} f_n(x)| < \infty \right\} \text{ and}$$

$$C_i := \left\{ x \in \mathbb{R} : \sum_{n=1}^{\infty} |\mathrm{Im} f_n(x)| < \infty \right\}.$$

  So, since $C \subset C_r \cap C_i$ (by the triangle inequality) and $\mathbb{R} \setminus C$ is a null set,[31] $\mathbb{R} \setminus (C_r \cap C_i)$ is a null set.[32] For this reason, both $\mathrm{Re} f$ and $\mathrm{Im} f$ have representations where both $\mathrm{Re} f_n$ and $\mathrm{Im} f_n$ are used!

- *if* part
  It seems that the representations of $\mathrm{Re} f$ and $\mathrm{Im} f$ were switched.

==
Comments, pp. **74–5**

- Hölder's inequality, *Proof*
  - 1st sentence
    If $\|f\|_p = 0$, then $f^p = 0$ a.e. by **Theo. 2.7.4**, p. **55**. Hence $f = 0$ a.e.. Thus $fg = 0$ a.e.. Therefore $\|fg\|_1 = 0$ by **Theo. 2.7.4**.
  - Last sentence
    Since f and g are measurable,[33] $|fg|$ is measurable by **Theo. 2.11.6**, p. **71**. So, by the first inequality of p. **75** and **Theo. 2.11.7**, p. **71**, fg is locally integrable. Then, by the first inequality of p. **75** and **Theo. 2.9.5**, p. **63**, $fg \in L^1(\mathbb{R})$. Therefore $|fg| \in L^1(\mathbb{R})$ by **Theo. 2.4.1**, p. **48**.

- Minkowski's inequality, *Proof*, 3rd sentence
  Use an argument similar to the one presented in the previous item to prove that $|f+g|^p \in L^1(\mathbb{R})$.

==
Erratum, p. **76**, l. 5
Remove the preposition 'in'.
==
Erratum, p. **81**, (2.38),
'$\int_c^d F$' should be '$\int_a^b F$'.
==
Exercises, pp. 84–91

5. $\mathrm{supp}|f| = \mathrm{supp} f$ is a finite union of semiopen intervals, which is contained in $\cup_{k=1}^n [a_k, b_k)$. On the other hand, consider the step function $g = Mg_1 + \cdots + Mg_n$ where g_k is the characteristic function of $[a_k, b_k)$, $k = 1, \ldots, n$. So $|f| \le g$. Now use **Theo. 2.2.2.**(c).

[31] See p. **55**, **Def. 2.7.1**, and p. **47**, antepenultimate and penultimate paragraphs, starting from the italicized sentence. The arguments are similar for complex-valued functions.

[32] See **Theo. 2.7.2**, p. **55**.

[33] See p. **70**, last paragraph, second sentence.

7–8.

- Let f and (f_n) be as in **Def. 2.3.1**, p. **45**. Therefore:
(a) Since $\tau_z|f_n|(x) = |f_n|(x-z) = |f_n(x-z)| = |\tau_z f_n(x)| = |\tau_z f_n|(x)$ for every $x \in \mathbb{R}$,

$$\sum_{n=1}^{\infty} \int |\tau_z f_n| = \sum_{n=1}^{\infty} \int \tau_z |f_n| \underbrace{=}_{\textbf{Theo. 2.2.2.}\text{(e), p. }\textbf{41}} \sum_{n=1}^{\infty} \int |f_n| < \infty;$$

(b) $\tau_z f(x) = f(x-z) = \sum_{n=1}^{\infty} f_n(x-z) = \sum_{n=1}^{\infty} \tau_z f_n(x)$ for every $x \in \mathbb{R}$ such that $\sum_{n=1}^{\infty} |\tau_z f_n(x)| = \sum_{n=1}^{\infty} |f_n(x-z)| < \infty$.
So $\tau_z f \simeq \tau_z f_1 + \tau_z f_2 + \cdots$ and

$$\int \tau_z f = \sum_{n=1}^{\infty} \int \tau_z f_n \underbrace{=}_{\textbf{Theo. 2.2.2.}\text{(e), p. }\textbf{41}} \sum_{n=1}^{\infty} \int f_n = \int f.$$

- Without loss of generality, suppose f is the characteristic function of $[a,b)$ and $z > 0$ is sufficiently small with $[a,b) \cap [a+z, b+z) \neq \emptyset$.[34] Therefore, since

$$(\tau_z f - f)(x) = \begin{cases} -1 & \text{if } x \in [a, a+z), \\ 0 & \text{if } x \in [a+z, b), \\ 1 & \text{if } x \in [b, b+z), \end{cases}$$

if $z \to 0$, then

$$\int |\tau_z f - f| = 2z \to 0.$$

10. Let (f_n) be as in **Def. 2.3.1**, p. **45**. Then $f_n = \sum_{m=1}^{m(n)} \lambda_{m,n} \chi_{[a_{m,n}, b_{m,n})}$ and $|f_n| = \sum_{m=1}^{m(n)} |\lambda_{m,n}| \chi_{[a_{m,n}, b_{m,n})}$ for every $n \in \mathbb{N}$.[35] Therefore

(a) $\sum_{n=1}^{\infty} \sum_{m=1}^{m(n)} \int \left| \lambda_{m,n} \chi_{[a_{m,n}, b_{m,n})} \right| = \sum_{n=1}^{\infty} \int |f_n| < \infty$;[36]

(b) $f(x) = \sum_{n=1}^{\infty} f_n(x) = \sum_{n=1}^{\infty} \sum_{m=1}^{m(n)} \lambda_{m,n} \chi_{[a_{m,n}, b_{m,n})}(x)$ whenever $\sum_{n=1}^{\infty} \sum_{m=1}^{m(n)} \left| \lambda_{m,n} \chi_{[a_{m,n}, b_{m,n})}(x) \right| = \sum_{n=1}^{\infty} |f_n(x)| < \infty$.

Now arrange the family of all intervals $[a_{m,n}, b_{m,n})$ and the family of all scalars $\lambda_{m,n}$ into sequences

$$[a_1, b_1), [a_2, b_2), \ldots \quad \text{and} \quad \lambda_1, \lambda_2, \ldots,$$

respectively, so that none of them are missed. Thus $f \simeq \lambda_1 \chi_{[a_1, b_1)} + \lambda_2 \chi_{[a_2, b_2)} + \cdots$.[37]

19.

(a) Let X be a countable subset of $\mathbb{R}$. If X is finite, use **Comments**, p. **53**, ll. 7–9, p. 12 of this material, with X in place of $\{0\}$. Now let $X = \{x_n \mid n \in \mathbb{N}\}$ be infinite and consider $\chi_{\{x_n\}}$ for each $n \in \mathbb{N}$. So $\chi_{\{x_n\}} \in L^1(\mathbb{R})$ and $\int \chi_{\{x_n\}} = 0$ for each $n \in \mathbb{N}$.[38] Therefore, due to the fact that

$$\chi_X = \chi_{\{x_1\}} + \chi_{\{x_2\}} + \cdots,$$

$\chi_X \simeq \chi_{\{x_1\}} + \chi_{\{x_2\}} + \cdots$ and, by **Theo. 2.5.3**, p. **52**, $\int \chi_X = 0$.

[34] Note that $\tau_z f$ is the characteristic function of $[a+z, b+z)$.

[35] See p. **40**.

[36] See **Theo. 2.2.2.**(a), p. **41**.

[37] For the converse, the proof is obvious!

[38] Use **Comments**, p. **53**, ll. 7–9, p. 12 of this material, with $\{x_n\}$ in place of $\{0\}$.

(b) Consider $\varepsilon > 0$ is sufficiently small, $S_n \subset \mathbb{R}$ is a null set for each $n \in \mathbb{N}$ and $S = \bigcup_{n=1}^{\infty} S_n$.[39] By **Theo. 2.11.3**, p. **68**, there exist intervals $I_{n,k} = [a_{n,k}, b_{n,k})$ such that

$$S_n \subset \bigcup_{k=1}^{\infty} I_{n,k} \qquad \text{and} \qquad \sum_{k=1}^{\infty} l(I_{n,k}) < \frac{\varepsilon}{2^n} \qquad \text{for each } n \in \mathbb{N}.^{40}$$

Now arrange the doubly-indexed family of intervals $I_{n,k}$ into a sequence $I_1, I_2, \ldots$ (where none of the $I_{n,k}$ are missed).[41] Therefore

$$S \subset \bigcup_{n=1}^{\infty} \left(\bigcup_{k=1}^{\infty} I_{n,k} \right) = \bigcup_{i=1}^{\infty} I_i \qquad \text{and} \qquad \sum_{i=1}^{\infty} l(I_i) = \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} l(I_{n,k}) < \varepsilon.$$

28. See **Ex. 37**, p. **89**.

33.

(a) The constant function $\chi_{\mathbb{R}} = 1$ is locally integrable,[42] that is $\mathbb{R} \in \mathcal{M}$. Then $\emptyset = \mathbb{R} \backslash \mathbb{R} \in \mathcal{M}$ by (d).

(e) Let I be an arbitrary interval and consider arbitrary numbers a and b with $a < b$. Thus χ_I is locally integrable since $\chi_I \chi_{I_{[a,b)}}$ is Lebesgue integrable by **Theo. 2.10.1**, p. **64**.

(f) Any open subset of $\mathbb{R}$ is a countable union of disjoint open intervals. Now use (e) and (b).

(g) Consider $A = \mathbb{R}$ and let B be an arbitrary open subset of A. So $A \backslash B \in \mathcal{M}$ by (a), (f) and (d).

34.(b) Consider $S = B$, $S_1 = A$, $S_2 = B \backslash A$ and $S_n = \emptyset$ for $n = 3, 4, \ldots$. Now use **Theo. 2.11.4**, p. **69**.

35. By **Ex.** 33.(a,d), (b)$\iff$(e) and (c)$\iff$(d).

37. See **Comments**, p. **70**, 1st paragraph after □, ultimate sentence, p. 17 of this material.

39. **Comment**
Neither g nor g^2 is defined for $x \in (-1, 0)$!

[39]Notice that $\{n \in \mathbb{N} \mid S_n \neq \emptyset\}$ can be finite or infinite.
[40]If $I = [a, b)$, then $l(I) = b - a$.
[41]This is possible since $I_{n,k} \mapsto n/k$ is a bijection between that doubly-indexed family of intervals and $\{x \in \mathbb{Q} : x > 0\}$.
[42]See p. **62**, right after the second □.

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Comment, p. **95**, **Exs. 3.2.4–7**
Concerning the conjugate simmetry condition (**Def. 3.2.1.**(a), p. **94**), note that, since

$$|\overline{z} - \overline{z_0}| = |\overline{z - z_0}| = |z - z_0|$$

for all $z, z_0 \in \mathbb{C}$, $z \mapsto \overline{z}$ is continuous at each $z_0 \in \mathbb{C}$. So complex conjugation is a continuous mapping.
==
Erratum, p. **98**, l. -10
Change '(b)' to '(a)'.
==
Erratum, p. **100**, l. 1
Remove the comma.
==
Comment, p. **100**, continuation of **Ex. 3.3.5**, $\boxed{\|f_n - f_m\| \le \left(\frac{1}{n} + \frac{1}{m}\right)^{1/2}}$

Suppose $n > m$. Concerning **Figure 3.1** on p. **99**, visualize the graphs of f_n and f_m simultaneously and denote the points where the oblique line segments intersect the x-axis by $x_1 = \frac{1}{2} + \frac{1}{2n}$ and $x_2 = \frac{1}{2} + \frac{1}{2m}$.[43] Hence

$$f_n(x) - f_m(x) = \begin{cases} 0 & \text{if } 0 \le x \le \frac{1}{2}, \\ 2(m-n)\left(x - \frac{1}{2}\right) & \text{if } \frac{1}{2} \le x \le x_1, \\ 2m\left(x - \frac{1}{2}\right) - 1 & \text{if } x_1 < x \le x_2, \\ 0 & \text{if } x_2 \le x \le 1. \end{cases}$$

Then $\|f_n - f_m\| = \sqrt{I_{m,n}}$ with

$$\begin{aligned} I_{m,n} &= \int_0^1 (f_n(x) - f_m(x))^2 \, dx \\ &= 4(m-n)^2 \int_{1/2}^{x_1} \left(x - \frac{1}{2}\right)^2 dx + 4m^2 \int_{x_1}^{x_2} \left(x - \frac{1}{2}\right)^2 dx - 4m \int_{x_1}^{x_2} \left(x - \frac{1}{2}\right) dx + \int_{x_1}^{x_2} dx \\ &= 4(m-n)^2 \int_0^{1/2n} t^2 dt + 4m^2 \int_{1/2n}^{1/2m} t^2 dt - 4m \int_{1/2n}^{1/2m} t dt + \frac{1}{2m} - \frac{1}{2n} \\ &= \frac{4(m-n)^2}{3} \left(\frac{1}{2n}\right)^3 + \frac{4m^2}{3} \left[\left(\frac{1}{2m}\right)^3 - \left(\frac{1}{2n}\right)^3\right] - 2m \left[\left(\frac{1}{2m}\right)^2 - \left(\frac{1}{2n}\right)^2\right] + \frac{1}{2m} - \frac{1}{2n} \\ &= \frac{1}{2} \left\{\frac{(m-n)^2}{3} \left(\frac{1}{n}\right)^3 + \frac{m^2}{3} \left[\left(\frac{1}{m}\right)^3 - \left(\frac{1}{n}\right)^3\right] - m \left[\left(\frac{1}{m}\right)^2 - \left(\frac{1}{n}\right)^2\right] + \frac{1}{m} - \frac{1}{n}\right\} \\ &= \frac{1}{2} \left[-\frac{2m}{3n^2} + \frac{1}{3n} + \frac{1}{3m} + \frac{m}{n^2} - \frac{1}{n}\right] \\ &= \frac{1}{2} \left[\frac{m}{3n^2} - \frac{2}{3n} + \frac{1}{3m}\right] \\ &= \frac{1}{2} \cdot \frac{m^2 - 2mn + n^2}{3mn^2} \\ &= \frac{(m-n)^2}{6mn^2}. \end{aligned}$$

Therefore, if $\frac{(m-n)^2}{6mn^2} > \frac{m+n}{mn}$, then $m > n + \sqrt{6n(m+n)} > n$, which is a contradiction.
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Comments, p. **103**

[43] Note that $x_1 < x_2$.

- 2nd paragraph following 1st □
  - 2nd sentence
    See **Def. 1.5.2**, p. **26**.
  - 3rd sentence
    In fact, since $\langle \cdot, x \rangle$ and complex conjugation are continuous,[44] $\langle x, \cdot \rangle = \overline{\langle \cdot, x \rangle}$ is continuous.
- l. -1

$$\begin{aligned} \mathrm{Re}\langle x_n, x \rangle &\leq |\langle x_n, x \rangle| \\ &\leq \|x_n\| \, \|x\| \to \|x\|^2. \end{aligned}$$

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Comment, p. **105**, 1st sentence
See p. **103** (2nd paragraph following 1st □, 2nd sentence) and p. **27** (**Theo. 1.5.7**).

==

Erratum, p. **106**, l. 9
'$\mathbb{N}$' should be '$\{1, \ldots, n\}$'.

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Comments, p. **111**, last sentence

- 2nd equality, last summand

$$\begin{aligned} \left\langle \sum_{j=1}^{n} \alpha_j x_j, \sum_{k=1}^{n} \alpha_k x_k \right\rangle &= \sum_{k=1}^{n} \left\langle \sum_{j=1}^{n} \alpha_j x_j, \alpha_k x_k \right\rangle \\ &= \sum_{k=1}^{n} \sum_{j=1}^{n} \left\langle \alpha_j x_j, \alpha_k x_k \right\rangle \\ &= \sum_{k=1}^{n} \left\langle \alpha_k x_k, \alpha_k x_k \right\rangle . \end{aligned}$$

- 4th equality, last summand

$$\begin{aligned} \sum_{k=1}^{n} |\langle x, x_k \rangle - \alpha_k|^2 &= \sum_{k=1}^{n} \left(\langle x, x_k \rangle - \alpha_k \right) \overline{\left(\langle x, x_k \rangle - \alpha_k \right)} \\ &= \sum_{k=1}^{n} \left(\langle x, x_k \rangle \, \overline{\langle x, x_k \rangle} - \overline{\alpha_k} \, \langle x, x_k \rangle - \alpha_k \overline{\langle x, x_k \rangle} + \alpha_k \overline{\alpha_k} \right) . \end{aligned}$$

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Comments, p. **112**, 2nd paragraph after □

- The sequence
$$\left(\sum_{k=1}^{n} |\langle x, x_k \rangle|^2 \right)$$
is increasing and bounded above. Then the series in (3.26) is convergent. Therefore
$$\lim_{n \to \infty} |\langle x, x_n \rangle|^2 = 0.$$
- *zero* (in the 2nd sentence) is the zero vector.[45]

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Comments, p. **113**, **Ex. 3.4.11**

[44] See the first comment of the previous page!
[45] See **Def. 3.3.10**, p. **102**.

- 2nd sentence
  See **Ex. 3.4.17**.

- 3rd sentence
  $\cos t \sin nt$ is an odd function.

==
Comment, p. **120**, 2nd sentence
See **Theo. 3.4.14**, p. **115**.
==
Comments, p. **122**

- 1st paragraph, 4th sentence
  Concerning the 1st equality, $f = f\chi_{[-\pi,\pi]}$ and

$$\int f = \int \tau_x f$$

  by section **2.9**, pp. **62–4**, and **Ex.** 7, p. **85**. The 2nd equality follows from **Theo. 2.10.4**, p. **66**.

- (3.34), 1st equality
  Let us prove

$$f_0 + f_1 + \cdots + f_n = \sum_{k=-n}^{n} (n+1-|k|)\langle f, \varphi_k \rangle \varphi_k \tag{5}$$

  by induction on n. In fact, since (5) holds trivially for $n \in \{0, 1\}$ and

$$\begin{aligned} f_0 + f_1 + \cdots + f_n + f_{n+1} &= \sum_{k=-n}^{n} (n+1-|k|)\langle f, \varphi_k \rangle \varphi_k + \sum_{k=-(n+1)}^{n+1} \langle f, \varphi_k \rangle \varphi_k \\ &= \sum_{k=-n}^{n} (n+1-|k|+1)\langle f, \varphi_k \rangle \varphi_k + \left\langle f, \varphi_{-(n+1)} \right\rangle \varphi_{-(n+1)} + \left\langle f, \varphi_{n+1} \right\rangle \varphi_{n+1} \\ &= \sum_{k=-(n+1)}^{n+1} ((n+1)+1-|k|)\langle f, \varphi_k \rangle \varphi_k, \end{aligned}$$

  (5) also holds true for $n = 2, 3, \ldots$.

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Erratum, p. **124**, *Proof* of **Lemma 3.5.3**
'x' should be 't'.
==
Comment, p. **126**, *Proof* of **Theo. 3.5.6**, 1st sentence
See **Ex.** 41, p. **89**.
==
Erratum, p. **127**, 2nd paragraph after **Def. 3.6.1**
'H' should be 'E'.
==
Comment, p. **128**, *Proof* of **Theo. 3.6.2**, 4th sentence
'$(x_n) \in S^\perp$' is an abuse of notation.
==
Comments, pp. **128–9**, *Proof* of **Theo. 3.6.4**

- 4th sentence
  It is straightforward to prove the first two equalities. (3.5) is used to prove the third equality.

- penultimate sentence

$$\begin{aligned}4\left\|x-\frac{y+y_1}{2}\right\|^2+\|y-y_1\|^2&=\|2x-(y+y_1)\|^2+\|y-x+x-y_1\|^2\\&=\|x-y_1+x-y\|^2+\|x-y_1-(x-y)\|^2\\&=2\left(\|x-y_1\|^2+\|x-y\|^2\right)\\&=2\left(d^2+d^2\right).\end{aligned}$$

(Note that (3.5) was used in the penultimate equality.)

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Erratum, p. **132**, 1st sentence of Section **3.7**
'3.5' should be '3.3'. In fact, cf. p. **103**, 3rd and 4th sentences after the *Proof* of **Theo. 3.3.11**.[46]

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Exercises, pp. 135–143

10.

$$\begin{aligned}4\times\mathrm{RHS}&=\langle x+y,x+y\rangle-\langle x-y,x-y\rangle+i(\langle x+iy,x+iy\rangle-\langle x-iy,x-iy\rangle)\\&=2(\langle x,y\rangle+\overline{\langle x,y\rangle})+2i(\langle x,iy\rangle+\overline{\langle x,iy\rangle})\\&=2(\langle x,y\rangle+\overline{\langle x,y\rangle}+i(\bar{i}\langle x,y\rangle+i\overline{\langle x,y\rangle}))\\&=2(2\langle x,y\rangle).\end{aligned}$$

15.

$$\begin{aligned}4\left\|z-\frac{x+y}{2}\right\|^2+\|x-y\|^2&=\|2z-(x+y)\|^2+\|x-z+z-y\|^2\\&=\|z-y+z-x\|^2+\|z-y-(z-x)\|^2\\&=2\left(\|z-y\|^2+\|z-x\|^2\right).\end{aligned}$$

(Note that (3.5) was used in the ultimate equality.)

34. Consider $p\in H=\operatorname{span}\{p_1,p_2,p_3\}$ where $p_1(x)=1$, $p_2(x)=x$ and $p_3(x)=x^2$.[47] Note that

$$\|x^3-p(x)\|^2=\int_{-1}^{1}|x^3-p(x)|^2dx$$

reaches its minimum where $p(x)=P_H(x^3)$. So calculate

$$p=\langle x^3,q_1\rangle q_1+\langle x^3,q_2\rangle q_2+\langle x^3,q_3\rangle q_3$$

where $B=\{q_1,q_2,q_2\}$ is an orthonormal basis of H. To obtain B, apply Gram-Schmidt to $\{p_1,p_2,p_3\}$.

43. See **Ex. 3.4.17**, pp. **116–7**.

44–5. Concerning the orthonormality, see **Ex. 3.4.17**, pp. **116–7**.

[46] See p. **27**, **Theo. 1.5.7**.
[47] Clearly, p_1, p_2 and p_3 are linearly independent.

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Comment, pp. **146–7**, **Ex. 4.2.2**, penultimate sentence
As in **Ex. 3.2.3**, pp. **94–5**, consider the standard inner product. Then, since

$$Ax = \sum_{i=1}^{N} \langle Ax, e_i \rangle e_i,$$

$$\begin{aligned}\|Ax\|_2 &= \sqrt{\sum_{i=1}^{N} \left| \sum_{j=1}^{N} \alpha_{ij}\lambda_j \right|^2} \\ &\le \sqrt{\sum_{i=1}^{N} \left(\sqrt{\sum_{j=1}^{N} |\alpha_{ij}|^2} \underbrace{\sqrt{\sum_{j=1}^{N} |\lambda_j|^2}}_{\|x\|_2} \right)^2}\end{aligned}$$

by (4.1) and the Cauchy-Schwarz inequality. Therefore

$$\sqrt{\sum_{i=1}^{N} \sum_{j=1}^{N} |\alpha_{ij}|^2}$$

is an upper bound of $\{\|Ax\|_2 \,:\, \|x\|_2 = 1\}$.

==

Erratum, pp. **150–1**, *Proof* of **Theo. 4.2.9**, ultimate sentence
'a_{ij}' should be 'α_{ij}'.

==

Erratum, p. **155**, *Proof* of **Theo. 4.3.12**, 3rd sentence
'$\|\varphi\|\|Ax\|\|Ax\|$' should be '$\|\varphi\|\|x\|\|Ax\|$'.

==

Comment, p. **161**, **Cor. 4.4.12**
Note that the product (**Theo. 4.4.11**) and the sum (first consequence of **Def. 4.4.1**, p. **158**) of self-adjoint operators are self-adjoint.

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Comments, p. **162**, *Proof* of **Theo. 4.4.14**

- Note that T is bounded by **Def. 4.4.1** and **Def. 4.4.3**, pp. **158–9**.
- (4.6)
  Consider $\varphi(x,z) = \langle Tx, z\rangle$ with $\varphi = \varphi_1$ and $T = A$ as in **Ex. 4.3.3**, p. **151**, and let Φ be the quadratic form of φ as in p. **152**. Therefore
  $$4\,\mathrm{Re}\langle Tx, z\rangle = \Phi(x+z) - \Phi(x-z),$$
  $\|\Phi\| = M$ and the inequality follows from the sentence presented after **Def. 4.3.6**, p. **152**. Furthermore, the equality holds by the Parallelogram law, p. **97**.

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Comment, p. **165**, **Ex. 4.5.9**
For all $x \in H$, if $Lx = -ix$, then

$$\begin{aligned}\langle T^* x, x\rangle &= \langle x, Tx\rangle \\ &= \langle x, ix\rangle \\ &= -i\langle x, x\rangle \\ &= \langle -ix, x\rangle \\ &= \langle Lx, x\rangle.\end{aligned}$$

So $T^* = L$ by **Cor. 4.3.8**.

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Comment, p. **166**, (4.11)

There is no need to use **Theo. 4.4.14**. In fact,

$$\begin{aligned} \left\|T^2x\right\| &= \|TTx\| \\ &= \|T^*Tx\| \qquad (\textbf{Theo. 4.5.8}) \\ &= \|T^*Tx\|\|x\| \\ &\geq |\langle T^*Tx, x\rangle| \text{ (Schwarz's inequality, p. } \textbf{96}\text{).} \end{aligned}$$

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Comment, p. **167**

On the one hand,

$$T \text{ is unitary} \Rightarrow T \text{ is isometric}$$

by **Def. 4.5.16** and **Theo. 4.5.15**. On the other hand,

$$T \text{ is isometric} \not\Rightarrow T \text{ is unitary.}$$

In fact, the operator A in **Ex. 4.5.3**, **p. 164**, is isometric by **Def. 4.5.13**. However, since A is not surjective, A is not invertible. Therefore, A is not unitary by **Theo. 4.5.17**.[48]

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Exercises, pp. **211–6**

11. Let C and D be operators with $T = C + iD$ and $T^* = C - iD$. Therefore

$$\begin{aligned} C &= \frac{1}{2}(T + T^*) \\ &= A, \\ D &= \frac{1}{2i}(T - T^*) \\ &= B. \end{aligned}$$

28. Check my **Comment** in regard to p. **167**.

[48]Concerning **Exercise** 28, **p. 213**, the answer is NO!