A SURVIVAL GUIDE TO

MATHEMATICS & CLIMATE

2013 SIAM EDITION

Hans Kaper and Hans Engler

PARTIAL SCRUTINY,

COMMENTS, SUGGESTIONS AND ERRATA

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Exercises, pp. 23–7

1.

$$\frac{hc}{\lambda kT} = \frac{(h \text{ in Js})\left(c \text{ in } \frac{\text{m}}{\text{s}}\right)}{(\lambda \text{ in m})\left(k \text{ in } \frac{\text{J}}{\text{K}}\right)(T \text{ in K})} \text{ in } \frac{\text{Jm}}{\text{mJ}} \text{ is dimensionless;}$$

$$\frac{hc^2}{\lambda^5} = \frac{(h \text{ in Js})\left(c^2 \text{ in } \frac{\text{m}^2}{\text{s}^2}\right)}{\lambda^5 \text{ in m}^5} \text{ in } \frac{\text{Js}^{-1}\text{m}^2}{\text{m}^5} = \text{Wm}^{-3} \Longrightarrow B(\lambda, T) \text{ has the dimension of radiance.}$$

3.

$$\begin{aligned} F(T) &= \pi \int_0^\infty B(\lambda, T) d\lambda \\ &= 2\pi hc^2 \int_0^\infty \frac{1}{\lambda^5\left(e^{hc/\lambda kT} - 1\right)} d\lambda. \end{aligned}$$

Therefore

$$\begin{aligned} x = \frac{hc}{\lambda kT}, \text{ i.e., } \lambda = \frac{hc}{xkT} \Longrightarrow \frac{d\lambda}{dx} = -\frac{hc}{x^2 kT} \text{ and } \frac{1}{\lambda^5} &= \left(\frac{kT}{hc}\right)^5 x^5 \\ \Longrightarrow F(T) &= 2\pi hc^2 \left(\frac{hc}{kT}\right)\left(\frac{k^5T^5}{h^5c^5}\right)\left(-\int_\infty^0 \frac{x^5}{x^2\left(e^x - 1\right)} dx\right) \\ \Longrightarrow F(T) &= \frac{2\pi k^4 T^4}{h^3 c^2}\left(\frac{1}{15}\pi^4\right) \\ \Longrightarrow F(T) &= \frac{2\pi^5 k^4}{15 h^3 c^2} T^4. \end{aligned}$$

8. ($Q = \frac{S_0}{4}$ varies approximately between 341.375 Wm^{-2} and 341.75 Wm^{-2}.)
(i) Via (2.9), T^* varies approximately between

$$\left(\frac{(0.7)(341.375)}{(0.6)(5.67 \cdot 10^{-8})}\right)^{1/4} \approx 289.5002\,\text{K}$$

and

$$\left(\frac{(0.7)(341.75)}{(0.6)(5.67 \cdot 10^{-8})}\right)^{1/4} \approx 289.5797\,\text{K},$$

whose difference is 0.0795 K.
(ii) $T^* = ((1-\alpha)Q - A)/B$ varies approximately between

$$\begin{aligned} \frac{(0.7)(341.375) - (203.3)}{2.09} &\approx 17.0634 \text{ degrees Celsius} \\ &\approx 290.2134\,\text{K} \end{aligned}$$

and

$$\begin{aligned} \frac{(0.7)(341.75) - (203.3)}{2.09} &\approx 17.1890 \text{ degrees Celsius} \\ &\approx 290.3390\,\text{K}, \end{aligned}$$

whose difference is 0.1256 degrees Celsius or Kelvin.
(iii) The *heat capacity* of the Earth's climate system quantifies the amount of incoming solar energy (heat) required to increase $T(t)$ by 1 degree Celsius and its actual value (assumed to be constant over the entire globe)

depends on the medium under consideration.[1] For example, land heats up faster than water, which has to absorb a great deal of energy before its temperature rises.[2] For this reason, the ocean takes a long time to change temperature significantly, whereas land can heat up very quickly.

10.
(i) Based on $\alpha(T)$ of section **2.5**, let us consider

$$f(x) = a + \frac{b}{2} \cdot \tanh(x) \tag{1}$$

as a function that connects the value $a - \frac{1}{2}b$ smoothly with the value $a + \frac{1}{2}b$.
(ii) In (1), for $\varepsilon > 0$ sufficiently small, replace b and x by $b - \varepsilon$ and εx respectively.
(iii) $\tanh(x)$ is a rescaled $g(x)$. In fact, since

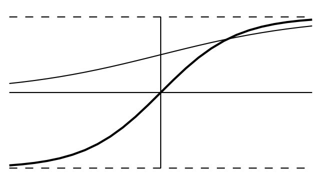
$$\begin{aligned}\tanh(x) &= \frac{e^x - e^{-x}}{e^x + e^{-x}} \\ &= \frac{e^x - \frac{1}{e^x}}{e^x + \frac{1}{e^x}} \\ &= \frac{e^{2x} - 1}{e^{2x} + 1}\end{aligned}$$

and

$$\begin{aligned}g(x) &= \frac{1}{1 + e^{-x}} \\ &= \frac{1}{1 + \frac{1}{e^x}} \\ &= \frac{e^x}{e^x + 1},\end{aligned}$$

$$\tanh(x) = 2g(2x) - 1.$$

Furthermore, there is a linear function from $\tanh(\mathbb{R}) = (-1, 1)$ to $g(\mathbb{R}) = (0, 1)$,[3] $\tanh(x)$ and $g(x)$ are infinitely differentiable, the inflection points of $\tanh(x)$ and $g(x)$ occur at $(0, 0)$ and $(0, 0.5)$ respectively, and the graphs of $\tanh(x)$ and $g(x)$ are symmetric with respect to the inflection points.[4]



[1]See p. 15.
[2]Heat capacity can also be defined as resistance to temperature change.
[3]Note that,

$$\begin{aligned}\lim_{x\to-\infty} \tanh(x) &= \frac{0-1}{0+1} \\ &= -1; \\ \lim_{x\to\infty} \tanh(x) &= \lim_{x\to\infty} \frac{2e^{2x}}{2e^{2x}} \text{ (L'Hôpital's Rule)} \\ &= 1; \\ \lim_{x\to-\infty} g(x) &= \frac{0}{0+1} \\ &= 0; \\ \lim_{x\to\infty} g(x) &= \lim_{x\to\infty} \frac{e^x}{e^x} \text{ (L'Hôpital's Rule)} \\ &= 1.\end{aligned}$$

[4]In fact, $f(x)$ is an odd function!

12. Since $x = T - T^*$ and $x \to 0$ as $T \to T^*$,

$$\begin{aligned} C\dot{x} &= C\dot{T} \\ &= (1 - \alpha(x + T^*))Q - \varepsilon\sigma(x + T^*)^4 \\ &= \left(1 - \alpha(T^*) - \alpha'(T^*)x - \mathcal{O}\left(x^2\right)\right)Q - \varepsilon\sigma\left(x^4 + 4x^3T^* + 6x^2(T^*)^2 + 4x(T^*)^3 + (T^*)^4\right) \\ &= (1 - \alpha(T^*))Q - \varepsilon\sigma(T^*)^4 - \left(\alpha'(T^*)Q + 4\varepsilon\sigma(T^*)^3\right)x \end{aligned}$$

where $(1 - \alpha(T^*))Q - \varepsilon\sigma(T^*)^4 = 0$.
Without loss of generality, the general solution of $\dot{x} = (-D/C)x$ is $x = e^{(-D/C)t}$, which converges to 0 as $t \to \infty$ if $D > 0$.[5]

[5] $C > 0$ is defined on page 15!

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Comment, p. 36, (3.7)
The general solution of

$$\frac{dT_0}{dt} = -cT_0$$

is

$$T_0 = e^{-ct}.$$

A particular solution of the first equation of (3.7) is

$$T_0 = T_0^*,$$

which implies that its general solution is

$$T_0 = e^{-ct} + T_0^*.$$

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Comment, p. 37, (3.13)
Multiply both sides of (3.12) by $\frac{\beta}{\alpha\Delta T}$ and rewrite the expression within the absolute value bars of (3.12) as the product of $\alpha\Delta T$ and another expression. Therefore

$$\frac{d}{dt}\left(\frac{\beta\Delta S}{\alpha\Delta T}\right) = \frac{2\beta H}{\alpha\Delta T} - 2k\left|\alpha\Delta T\left(1 - \frac{\beta\Delta S}{\alpha\Delta T}\right)\right|\frac{\beta\Delta S}{\alpha\Delta T} \Longrightarrow 2\alpha k|\Delta T|\frac{dx}{dt'} = \frac{2\beta H}{\alpha\Delta T} - 2\alpha k|\Delta T|\,|1-x|x.$$

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Comment, p. 38, (3.15)
For $x < 1$,

$$\begin{aligned}\dot{x} &= \lambda - (1-x)x \\ &= \lambda - x + x^2\end{aligned}$$

by (3.13). So

$$\begin{aligned}\dot{y} &= \frac{d}{dt}\left(x - x^*\right) \\ &= \dot{x} \\ &= \lambda - (x^* + y) + (x^* + y)^2 \\ &= \lambda - x^* - y + (x^*)^2 + 2x^*y + y^2 \\ &= \lambda - (1 - x^*)\,x^* + (2x^* - 1)\,y + y^2.\end{aligned}$$

Now let y be small enough and note that $x^* < 1$ satisfies (3.14).[6]
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Comment, p. 38, ultimate paragraph of **3.5.2**
Since $\Delta T = 2T^*$ by the first sentence of section **3.5**,

$$\begin{aligned}x &= \frac{\beta\Delta S}{\alpha\Delta T} \\ &= \frac{\beta\Delta S}{2\alpha T^*}\end{aligned}$$

[6]A similar reasoning can be applied for $x > 1$. In any case,

$$y = e^{\pm(2x^*-1)t}, \quad x^* \lesseqgtr 1,$$

is the solution of (3.15). Now analize $x = x^* + y$ as $t \to \infty$.

and (3.5) can be rewritten as

$$\begin{aligned} q &= k(\alpha\Delta T - \beta\Delta S) \\ &= k\alpha\Delta T\left(1 - \frac{\beta\Delta S}{\alpha\Delta T}\right) \\ &= 2k\alpha T^*(1-x). \end{aligned}$$

On the other hand, by (3.9), $2T^* = T_2^* - T_1^*$ is positive since the average temperature near the equator is higher than the average temperature near the poles. Therefore $q(1-x) > 0$.
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Exercises, pp. 39–40

3–4.

$$\begin{aligned} \frac{d}{dt}(\Delta T) &= \dot{T}_2 - \dot{T}_1 \\ &= c\left(T^* - T_2\right) - |q|\Delta T - c\left(-T^* - T_1\right) - |q|\Delta T \\ &= c\left(2T^* - \Delta T\right) - 2|q|\Delta T \\ &= -(c+2|q|)\Delta T + 2cT^*, \\ \frac{d}{dt}(\Delta S) &= \dot{S}_2 - \dot{S}_1 \\ &= H + d\left(S^* - S_2\right) - |q|\Delta S + H - d\left(-S^* - S_1\right) - |q|\Delta S \\ &= 2H + d\left(2S^* - \Delta S\right) - 2|q|\Delta S \\ &= -(d+2|q|)\Delta S + 2\left(H + dS^*\right). \end{aligned}$$

Now suppose that H, T^* and S^* become zero.[7] So the flow q ceases to exist and the equations above become

$$\begin{aligned} \frac{d}{dt}(\Delta T) &= -c\Delta T, \\ \frac{d}{dt}(\Delta S) &= -d\Delta S. \end{aligned}$$

Therefore, for each $t \in \mathbb{R}$,

$$\begin{aligned} \Delta T &= c_1 e^{-ct}, \\ \Delta S &= c_2 e^{-dt}, \end{aligned}$$

where c_i is constant, $i = 1, 2$.

[7] The authors (Kaper and Engler) provided an errata where, concerning this exercise, it is also assumed that both T^* and S^* vanish!

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Comment, 2nd paragraph of section **4.1**, pp. 41–42

$$\begin{aligned}(\dot{x}_1, \dot{x}_2, \ldots, \dot{x}_{n-1}, \dot{x}_n) &= \left(x^{(1)}, x^{(2)}, \ldots, x^{(n-1)}, x^{(n)}\right) \\ &= (x_2, x_3, \ldots, x_n, g(x_1, \ldots, x_n)).\end{aligned}$$

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Comment, p. 43, (ii) and (iii)
Concerning the solutions,

$$\begin{aligned}\frac{dx}{dt} = x^2 &\Longrightarrow \int x^{-2} dx = \int dt \\ &\Longrightarrow -\frac{1}{x} = t + \text{constant with constant} = -\frac{1}{x_0} - t_0 \text{ if } x(t_0) = x_0 \\ &\Longrightarrow x = -\frac{1}{t - \frac{1+x_0t_0}{x_0}}.\end{aligned}$$

and

$$\begin{aligned}\frac{dx}{dt} = \sqrt{x} &\Longrightarrow \int x^{-1/2} dx = \int dt \\ &\Longrightarrow 2\sqrt{x} = t + \text{constant with constant} = 2\sqrt{x_0} - t_0 \text{ if } x(t_0) = x_0 \\ &\Longrightarrow 4x = (t - t_0 + 2\sqrt{x_0})^2.\end{aligned}$$

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Comments, pp. 44–5

- 3rd paragraph, 1st sentence

$$\begin{aligned}f \text{ is Lipschitz} &\Longrightarrow f \text{ is continuous} \\ &\Longrightarrow \text{there exists a solution for the IVP } \begin{cases} \dot{x} = f(x), \\ x(t_0) = x_0 \end{cases} \text{ (by **Theo. 4.1**).}\end{aligned}$$

  Concerning the first implication above, for any $x_i \in D$, $i = 1, 2$, and $\varepsilon > 0$, consider $\delta < \frac{\varepsilon}{k}$. Therefore

$$\begin{aligned}\|x_1 - x_2\| < \delta \Longrightarrow \|f(x_1) - f(x_2)\| &\leq k\|x_1 - x_2\| \\ &< k\delta \\ &< \varepsilon.\end{aligned}$$

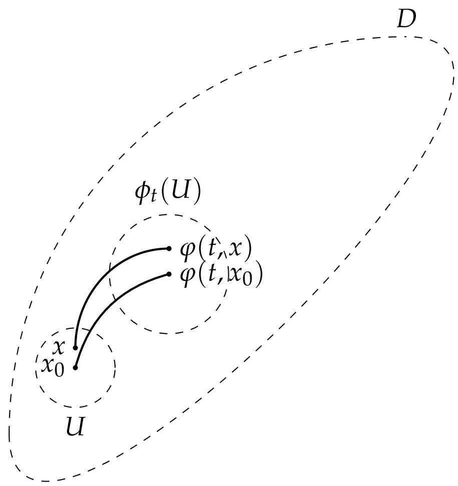
- **Theo. 4.3** can be rewritten as

  Let f be C^k on D.[8] Fix $t \in I(x_0)$.[9] So there is a neighborhood U of x_0 such that $x \overset{\phi_t}{\mapsto} \phi_t(x) := \varphi(t, x)$ is C^k on U.

  U could represent a very small open ball centered at x_0, consisting of initial conditions arbitrarily close to x_0. $\phi_t(U)$ represents the result of allowing U to evolve through t units of time (forward for $t > 0$ or backward for $t < 0$). The transition from U to $\phi_t(U)$ is as smooth as f.

[8] In particular, f is Lipschitz on D if $k \geq 1$.
[9] By **Lemma 4.1**, $I(x_0)$ represents the domain of the solution $\varphi(t, x_0) = \varphi(t; 0, x_0)$ for the IVP

$$\begin{cases} \dot{x} = f(x), \\ x(0) = x_0. \end{cases}$$



- 2nd paragraph of section **4.2**
  Let f be C^k on D, $k = 1, 2, \ldots$. A *dynamical system* associated with $\dot{x} = f(x)$ is the set consisting of the maps ϕ_t, obtained as described above, for each initial condition $x_0 \in D$ and each $t \in I(x_0)$.

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Comment, p. 46, **Def. 4.4**

$$\omega(x) = \{y \in D : \phi_{t_n}(x) = \varphi(t_n, x) \to y \text{ for some sequence } t_n \to \infty\} \text{ and}$$
$$\alpha(x) = \{y \in D : \phi_{t_n}(x) = \varphi(t_n, x) \to y \text{ for some sequence } t_n \to -\infty\}.$$

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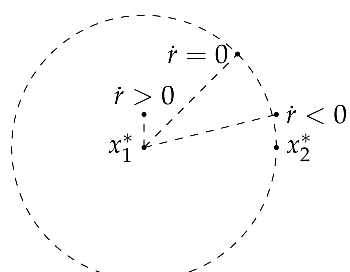
Comment, p. 49, (4.10)
The figure



illustrates an initial condition x_0 which is either in the interior ($r < 1$), boundary ($r = 1$) or exterior ($r > 1$) of the open ball centered at x_1^*. For $r \geq 0$, since $\dot{\theta} \geq 0$, θ is a increasing function. So, for the $r < 1$ case, since $\dot{r} > 0$, r is strictly increasing, which implies that solutions $\varphi(t, x_0)$ that start near x_1^* will spiral away from the origin.[10] For the $r = 1$ case, since $\dot{r} = 0$, solutions $\varphi(t, x_0)$ move along the boundary $r = 1$ and will converge to x_2^* as time goes by. For the $r > 1$ case, since $\dot{r} < 0$, r is strictly decreasing, which implies that solutions $\varphi(t, x_0)$ will eventually converge to x_2^*.

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Comments, p. 51

- (4.13)
  The fact that the only critical point is the origin is a direct consequence of assuming the existence of A^{-1}:

$$\begin{aligned} Ax = 0 &\Longrightarrow A^{-1}Ax = A^{-1}0 \\ &\Longrightarrow x = 0. \end{aligned}$$

[10] x_1^* is called an *unstable spiral point*.

- Last paragraph
  $\mathbb{R}^{n^2}$ is isomorphic to the space $\mathbb{R}^{n\times n}$ of matrices of order n.[11] For example, consider the isomorphism

$$\mathbb{R}^{n\times n} \ni \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \mapsto (a_{11}, a_{12}, \ldots, a_{1n}, a_{21}, a_{22}, \ldots, a_{2n}, \ldots, a_{n1}, a_{n2}, \ldots, a_{nn}) \in \mathbb{R}^{n^2}.$$

  Since all norms in $\mathbb{R}^{n^2}$ are equivalent, we might also consider

$$\lim_{N\to\infty} \sum_{k=0}^{N} \frac{M^k}{k!} = e^M$$

  with respect to the Euclidean norm.

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Comments, p. 52

- (4.15)
  Differentiate (4.14) with respect to t term by term!
- 1st paragraph after **Theo. 4.4**
  Let J and P be real matrices with P invertible and $A = PJP^{-1}$. So $A^k = PJ^kP^{-1}$ for $k = 0, 1, 2, \ldots$. Therefore

$$\begin{aligned} e^{tA} &= P\left(\sum_{k=0}^{\infty} \frac{t^k}{k!} J^k\right) P^{-1} \\ &= Pe^{tJ}P^{-1} \end{aligned}$$

  by (4.14). For example, if J is the diagonal matrix with diagonal entries $\lambda_1, \ldots, \lambda_n$, then e^{tJ} is the diagonal matrix with diagonal entries $e^{\lambda_1 t}, \ldots, e^{\lambda_n t}$.
- 2nd paragraph after **Theo. 4.4**
  E^s and E^u are invariants under e^{tA}. In fact, for simplicity, let A be diagonalizable and consider an initial condition $x_0 \in E^s$. So

$$x_0 = \sum_{j=1}^{r} \alpha_j v_{i_j} \tag{2}$$

  is a linear combination of eigenvectors $v_{i_1}, \ldots, v_{i_r}$ associated with eingenvalues $\lambda_{i_1}, \ldots, \lambda_{i_r}$ of A which are in the left half of the complex plane, i.e.,

$$Av_{i_j} = \lambda_{i_j} v_{i_j} \tag{3}$$

  where the real part of λ_{i_j} is negative for $j = 1, \ldots, r$. Therefore

$$\begin{aligned} e^{tA}x_0 &= \sum_{k=0}^{\infty} \frac{t^k}{k!} A^k x_0 \\ &= \sum_{j=1}^{r} \alpha_j \sum_{k=0}^{\infty} \frac{t^k \lambda_{i_j}^k}{k!} v_{i_j} \\ &= \sum_{j=1}^{r} \alpha_j e^{\lambda_{i_j} t} v_{i_j} \in E^s \end{aligned}$$

  by (2) and (3).

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Comment/Erratum, p. 53, (i)

[11] $L(\mathbb{R}^n)$ often denotes the space of linear operators on $\mathbb{R}^n$, which is also isomorphic to $\mathbb{R}^{n\times n}$.

- 1st paragraph
  Consider $e^{tJ}y_0$ with nonzero $y_0 = (y_{0,1}, y_{0,2})$. Without loss of generality, suppose $y_{0,2} \neq 0$. Therefore

$$
\begin{aligned}
|y_1|^{|\lambda_2|} &= \left|y_{0,1}e^{\lambda_1 t}\right|^{|\lambda_2|} \\
&= \left|y_{0,1}e^{-|\lambda_1|t}\right|^{|\lambda_2|} \\
&= |y_{0,1}|^{|\lambda_2|}\left|e^{-|\lambda_2|t}\right|^{|\lambda_1|} \\
&= \frac{|y_{0,1}|^{|\lambda_2|}}{|y_{0,2}|^{|\lambda_1|}}\left|y_{0,2}e^{\lambda_2 t}\right|^{|\lambda_1|} \\
&= C\,|y_2|^{|\lambda_1|}.
\end{aligned}
$$

- Antepenultimate sentence
  '$0 < \lambda_2 < \lambda_2$' should be '$0 < \lambda_1 < \lambda_2$'.

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Comments/Erratum, p. 54

- Sentence that precedes (i)
  - Consider $J = \alpha I + \beta B$ with

$$
I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \text{ and } B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.
$$

    Note that the sequence with k-th term B^k begins with the terms

$$
B^0 = I, B^1 = B, B^2 = -I, B^3 = -B, B^4 = I, B^5 = B, B^6 = -I, B^7 = -B, \ldots
$$

    and continues to repeat indefinitely. Therefore

$$
\begin{aligned}
e^{tJ} &= e^{t(\alpha I + \beta B)} \\
&= e^{\alpha t I}e^{\beta t B} \\
&= \left(\sum_{k=0}^{\infty}\frac{(\alpha t)^k}{k!}I^k\right)\left(\sum_{k=0}^{\infty}\frac{(\beta t)^k}{k!}B^k\right) \\
&= e^{\alpha t}I\left(I + \beta t B + \frac{(\beta t)^2}{2!}B^2 + \frac{(\beta t)^3}{3!}B^3 + \frac{(\beta t)^4}{4!}B^4 + \frac{(\beta t)^5}{5!}B^5 + \cdots\right) \\
&= e^{\alpha t}\left(I + \beta t B - \frac{(\beta t)^2}{2!}I - \frac{(\beta t)^3}{3!}B + \frac{(\beta t)^4}{4!}I + \frac{(\beta t)^5}{5!}B - \cdots\right) \\
&= e^{\alpha t}\left(\left(1 - \frac{(\beta t)^2}{2!} + \frac{(\beta t)^4}{4!} - \cdots\right)I + \left(\beta t - \frac{(\beta t)^3}{3!} + \frac{(\beta t)^5}{5!} - \cdots\right)B\right) \\
&= e^{\alpha t}\big((\cos\beta t)I + (\sin\beta t)B\big).
\end{aligned}
$$

  - The diagonal entry at the bottom right corner, $-\cos\beta t$, should be $\cos\beta t$.

- (ii)
  The orbits are circles. In fact, for a nonzero initial condition

$$
y_0 = \begin{pmatrix} y_{0,1} \\ y_{0,2} \end{pmatrix},
$$

$$
\begin{aligned}
\begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix} &= e^{tJ}y_0 \\
&= \begin{pmatrix} y_{0,1}\cos\beta t + y_{0,2}\sin\beta t \\ -y_{0,1}\sin\beta t + y_{0,2}\cos\beta t \end{pmatrix}
\end{aligned}
$$

with

$$\begin{aligned} y_1^2 + y_2^2 &= (y_{0,1})^2 \left(\cos^2 \beta t + \sin^2 \beta t\right) + (y_{0,2})^2 \left(\sin^2 \beta t + \cos^2 \beta t\right) \\ &= \left(\sqrt{(y_{0,1})^2 + (y_{0,2})^2}\right)^2. \end{aligned}$$

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Comments, **4.6.3**, pp. 55–6

- (i) means that A is non-diagonalizable, that is, $\mathbb{R}^2$ does not have a basis consisting of eigenvectors of A. So the λ-eigenspace of A, which is either E^s or E^u, has dimension 1. On the other hand, since $J^0 = I$ and

$$J^k = \begin{pmatrix} \lambda^k & 0 \\ k\lambda^{k-1} & \lambda^k \end{pmatrix} \text{ for } k = 1, 2, 3, \ldots,$$

$$\begin{aligned} e^{tJ} &= \sum_{k=0}^{\infty} \frac{t^k}{k!} J^k \\ &= I + \sum_{k=1}^{\infty} \begin{pmatrix} \frac{(t\lambda)^k}{k!} & 0 \\ \frac{t(t\lambda)^{k-1}}{(k-1)!} & \frac{(t\lambda)^k}{k!} \end{pmatrix} \\ &= \begin{pmatrix} e^{\lambda t} & 0 \\ te^{\lambda t} & e^{\lambda t} \end{pmatrix} \end{aligned}$$

and, if y_0 is an initial condition,

$$\lim_{t \to \pm\infty} e^{tJ} y_0 = (0, 0)$$

if $\lambda \lesseqgtr 0$.

- (ii) means that A is diagonalizable, that is, $\mathbb{R}^2$ has a basis consisting of eigenvectors of A. So either $E^s = \mathbb{R}^2$ or $E^u = \mathbb{R}^2$. Furthermore, for each initial condition $x_0 \in \mathbb{R}^2$, since $J = \lambda I$ and

$$\begin{aligned} A &= F^{-1} J F \\ &= F^{-1} (\lambda I) F \\ &= \lambda F^{-1} I F \\ &= \lambda F^{-1} F \\ &= \lambda I, \end{aligned}$$

$$\begin{aligned} e^{tA} x_0 &= e^{tJ} x_0 \\ &= e^{\lambda t I} x_0 \\ &= e^{\lambda t} I x_0 \\ &= e^{\lambda t} x_0 \end{aligned}$$

is a scalar multiple of x_0 for every $t \in \mathbb{R}$.

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Exercises, pp. 58–62

1. Firstly,

$$\begin{aligned} X_1 = x \text{ and } X_2 = \dot{x} &\Longrightarrow \begin{bmatrix} \dot{X}_1 \\ \dot{X}_2 \end{bmatrix} = \begin{bmatrix} X_2 \\ X_1 \end{bmatrix} \\ &\Longrightarrow \dot{X} = AX \text{ with } X = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} \text{ and } A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}. \end{aligned}$$

So, by subsection **4.6.1**, $\lambda_1 = -1$ and $\lambda_2 = 1$,

$$J = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix},$$

$$e^{tJ} = \begin{bmatrix} e^{-t} & 0 \\ 0 & e^{t} \end{bmatrix}$$

and the orbits are similar to the ones of **Figure 4.8** with a saddle point at the origin as the only fixed point. Furthermore, concerning the trajectory

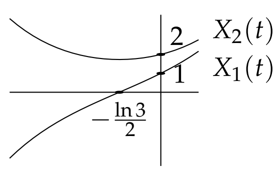
$$\left\{ \left(t, e^{tA}X_0\right) \,:\, t \in I(X_0) \right\}$$

of an initial condition X_0,[12] it is worth noting that, since $A^{2k} = I$ (2×2 identity matrix) and $A^{2k+1} = A$ for $k = 0, 1, 2, \ldots$,

$$\begin{aligned} e^{tA} &= \sum_{k=0}^{\infty} \frac{t^k}{k!} A^k \\ &= \sum_{k=0}^{\infty} \frac{t^{2k}}{(2k)!} I + \sum_{k=0}^{\infty} \frac{t^{2k+1}}{(2k+1)!} A \\ &= (\cosh t) I + (\sinh t) A \\ &= \begin{bmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{bmatrix}. \end{aligned}$$

As an illustration, let us consider the solution with $X_0 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$:

$$\begin{aligned} X_1(t) &= \cosh t + 2 \sinh t \\ &= \frac{1}{2}\left(3e^{t} - e^{-t}\right), \\ X_2(t) &= \sinh t + 2 \cosh t \\ &= \frac{1}{2}\left(3e^{t} + e^{-t}\right). \end{aligned}$$



3. Consider (4.5), p. 42. Therefore:

- $x_2 = 0$ and $\sin x_1 = 0$ give us the fixed points

$$(x_1^*, x_2^*) = (k\pi, 0) \text{ for } k \in \mathbb{Z};$$

  In **Figure 4.3**, p. 46, $\boldsymbol{A} = (0,0)$ and $\boldsymbol{B} = (\pm\pi, 0)$.

- $\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ is the linearization of

$$\begin{cases} \dot{x}_1 = x_2, \\ \dot{x}_2 = -x_1 + \mathcal{O}\left(x_1^2\right). \end{cases}$$

  Furthermore, by subsection **4.6.2**, $\lambda_1 = i$ and $\lambda_2 = -i$, $A = J$,

$$e^{tJ} = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}$$

  and the orbits are similar to the ones of **Figure 4.10**, p. 55, with a center at the origin as the only fixed point.

[12] Cf. **Def. 4.2**, p. 45.

So the phase portrait around $x^* \in \{\boldsymbol{A}, \boldsymbol{B}\}$ and the phase portrait of **Figure 4.10** (left) are locally similar.

5.

(i) On the one hand, a first integral is any function that is constant along the solutions of an ODE. So, if $F(x, y)$ is constant on a solution curve, $\frac{dF}{dt} = \dot{x}F_x + \dot{y}F_y$ equals zero by the chain rule. Then

$$\frac{\dot{y}}{\dot{x}} = -\frac{F_x}{F_y} \tag{4}$$

provided that $\dot{x}F_y \neq 0$. On the other hand, by considering $y = \dot{x}$, the equations of the exercise can be written as $(\dot{x}, \dot{y}) = f(x, y)$ with $\dot{x} = y$ and

$$\begin{array}{ll} \dot{y} = -x - x^2, & \dot{y} = -x + x^2, \\ \dot{y} = -x - x^3, & \dot{y} = -x + x^3, \end{array}$$

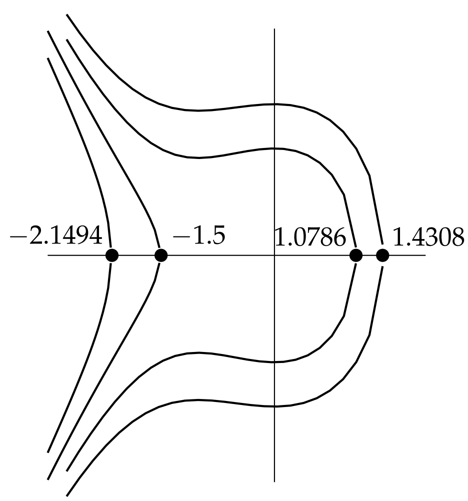
respectively. So, firstly, consider

$$\begin{cases} \dot{x} = y, \\ \dot{y} = -x - x^2. \end{cases} \tag{5}$$

Therefore, by (4) and due to fact that

$$\begin{aligned} \frac{dy}{dx} = \frac{\dot{y}}{\dot{x}} &\Longrightarrow \frac{dy}{dx} = -\frac{x + x^2}{y} \\ &\Longrightarrow \int y\,dy = -\int \left(x + x^2\right) dx \\ &\Longrightarrow \frac{y^2}{2} + \frac{x^2}{2} + \frac{x^3}{3} = \text{constant}, \end{aligned}$$

$F(x, y) = \frac{y^2}{2} + \frac{x^2}{2} + \frac{x^3}{3}$ is the first integral of (5).[13] (The next figure depicts level curves $F(x, y) = c$, $c \in \{-1, 0, 1, 2\}$.)



The mirror image of those curves in respect to the y-axis are level curves of the first integral of the system

$$\begin{cases} \dot{x} = y, \\ \dot{y} = -x + x^2. \end{cases} \tag{6}$$

Now, consider

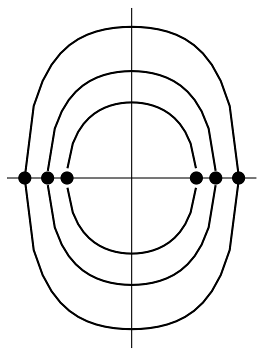
$$\begin{cases} \dot{x} = y, \\ \dot{y} = -x - x^3. \end{cases} \tag{7}$$

[13] In fact, $F_x = x + x^2$ and $F_y = y$ confirm (4).

Therefore, by (4) and due to fact that

$$\begin{aligned}\frac{dy}{dx} = \frac{\dot{y}}{\dot{x}} \Longrightarrow \frac{dy}{dx} &= -\frac{x+x^3}{y}\\ \Longrightarrow \int y\,dy &= -\int \left(x+x^3\right)dx\\ \Longrightarrow \frac{y^2}{2}+\frac{x^2}{2}+\frac{x^4}{4} &= \text{constant},\end{aligned}$$

$F(x,y) = \frac{y^2}{2}+\frac{x^2}{2}+\frac{x^4}{4}$ is the first integral of (7).[14] (The next figure depicts level curves $F(x,y) = c$, $c \in \{0.5, 1, 2\}$.)



x-intercepts: $\mp\left(\sqrt{2},0\right), \mp\left(\sqrt{\sqrt{5}-1},0\right), \mp\left(\sqrt{\sqrt{3}-1},0\right)$.

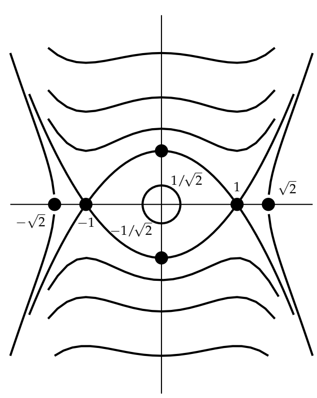
Finally, consider

$$\begin{cases}\dot{x} = y,\\ \dot{y} = -x + x^3.\end{cases} \tag{8}$$

Therefore, by (4) and due to fact that

$$\begin{aligned}\frac{dy}{dx} = \frac{\dot{y}}{\dot{x}} \Longrightarrow \frac{dy}{dx} &= -\frac{x-x^3}{y}\\ \Longrightarrow \int y\,dy &= -\int \left(x-x^3\right)dx\\ \Longrightarrow \frac{y^2}{2}+\frac{x^2}{2}-\frac{x^4}{4} &= \text{constant},\end{aligned}$$

$F(x,y) = \frac{y^2}{2}+\frac{x^2}{2}-\frac{x^4}{4}$ is the first integral of (8).[15] (The next figure depicts some level curves of the first integral.)



(ii) By considering $f(x,y) = (0,0)$, the critical points of (5), (6), (7) and (8) are obtained, respectively, via:

- $-x(1+x) = 0$ and $y = 0 \Longrightarrow (x^*, y^*) \in \{(0,0), (-1,0)\}$;
- $-x(1-x) = 0$ and $y = 0 \Longrightarrow (x^*, y^*) \in \{(0,0), (1,0)\}$;

[14]In fact, $F_x = x + x^3$ and $F_y = y$ confirm (4).
[15]In fact, $F_x = x - x^3$ and $F_y = y$ confirm (4).

- $-x(1+x^2) = 0$ and $y = 0 \Longrightarrow (x^*, y^*) = (0,0)$;
- $-x(1-x^2) = 0$ and $y = 0 \Longrightarrow (x^*, y^*) \in \{(0,0), (\mp 1, 0)\}$.

(iii) By definition, $P \in \mathbb{R}^n$ is a critical point of a real valued function F of several variables if $\nabla F(P) = 0$. So, since the critical points of the first integral $F(x,y)$ are obtained via $(F_x, F_y) = (0,0)$, we have to solve

$$\begin{array}{ll} (x+x^2, y) = (0,0), & (x-x^2, y) = (0,0), \\ (x+x^3, y) = (0,0), & (x-x^3, y) = (0,0). \end{array}$$

By (ii), $P = (x^*, y^*)$ in each case.
(iv) Concerning (5), (6), (7) and (8), $Df(x^*, y^*)$ equals

$$\begin{array}{cc} \begin{pmatrix} 0 & 1 \\ -1-2x^* & 0 \end{pmatrix}, & \begin{pmatrix} 0 & 1 \\ -1+2x^* & 0 \end{pmatrix}, \\ \begin{pmatrix} 0 & 1 \\ -1-3(x^*)^2 & 0 \end{pmatrix} \text{ and } & \begin{pmatrix} 0 & 1 \\ -1+3(x^*)^2 & 0 \end{pmatrix}, \end{array} \tag{9}$$

respectively. So, first, if $x^* = 0$, then $\pm i$ are the eigenvalues of each matrix of (9) and the origin is a center. Now, if $x^* = -1$ (respectively, $x^* = 1$), then ± 1 are the eigenvalues of the first (respectively, second) matrix of (9), implying that (x^*, y^*) is a saddle point. Finally, if $x^* = \mp 1$, then $\pm\sqrt{2}$ are de eigenvalues of the fourth matrix of (9), implying that (x^*, y^*) is a saddle point.
(v) Let us consider a tabular presentation of the 2nd derivative test for real-valued functions $F(x,y)$ with F_{xx}, F_{xy}, F_{yx} and F_{yy} continuous around a critical point $X^* = (x^*, y^*)$ of F:

$F_{xx}(X^*)\,F_{yy}(X^*) - \left(F_{xy}(X^*)\right)^2$	$F_{xx}(X^*)$	X^*
positive	positive	local minimum
positive	negative	local maximum
negative	positive/negative	saddle
zero	whatever	no information

Therefore:

- For the 1st integral of (5), $X^* \in \{(0,0), (-1,0)\}$, $F_{xx} = 1+2x$, $F_{xy} = 0$, $F_{yy} = 1$ and $F_{xx}F_{yy} - (F_{xy})^2 = 2x$. Then $X^* = (-1,0)$ is a saddle point, confirming the nomenclature of (iv), but there is no information about the origin.

- For the 1st integral of (6), $X^* \in \{(0,0), (1,0)\}$, $F_{xx} = 1-2x$, $F_{xy} = 0$, $F_{yy} = 1$ and $F_{xx}F_{yy} - (F_{xy})^2 = -2x$. Then $X^* = (1,0)$ is a saddle point, confirming the nomenclature of (iv), but there is no information about the origin.

- For the 1st integral of (7), $X^* = (0,0)$, $F_{xx} = 1+3x^2$, $F_{xy} = 0$, $F_{yy} = 1$ and $F_{xx}F_{yy} - (F_{xy})^2 = 3x^2$. Then there is no information about the origin.

- For the 1st integral of (8), $X^* \in \{(0,0), (\mp 1, 0)\}$, $F_{xx} = 1-3x^2$, $F_{xy} = 0$, $F_{yy} = 1$ and $F_{xx}F_{yy} - (F_{xy})^2 = -3x^2$. Then $X^* = (\mp 1, 0)$ are saddle points, confirming the nomenclature of (iv), but there is no information about the origin.

18.
(i) The method of variation of parameters for a non-homogeneous 1st order linear equation $\dot{x} + p(t) = f(t)$ gives us the general solution

$$x(t) = Ae^{P(t)} + v(t)e^{P(t)}$$

of the equation where A is a constant, $P(t)$ is an antiderivative of $-p(t)$ and $v(t)$ is an antiderivative of $f(t)e^{-P(t)}$. So, since $p(t) = 1$ and $f(t) = \cos t$ here, $P(t) = -t$ and

$$\begin{aligned} v(t) &= \int \cos t\, e^t dt \\ &= \frac{\sin t + \cos t}{2} e^t. \end{aligned}$$

Therefore
$$x(t) = Ae^{-t} + \frac{\sin t + \cos t}{2}$$
and, for $x(0) = x_0$,
$$x(t) = \left(x_0 - \frac{1}{2}\right) e^{-t} + \frac{\sin t + \cos t}{2}. \tag{10}$$
(ii) Take $x_0 = \frac{1}{2}$ in (10). Otherwise, (10) is not periodic.
(iii) For arbitrarily large t, the first summand of (10) becomes arbitrarily small and the second one becomes bounded.

19.
(i) $x + 2\beta\dot{x} + \ddot{x}$ equals
$$a\cos\omega t + b\sin\omega t + e^{-\beta t}(c_1\cos\lambda t + c_2\sin\lambda t)$$
$$+$$
$$2\beta\left(\omega(-a\sin\omega t + b\cos\omega t) + e^{-\beta t}((-\beta)(c_1\cos\lambda t + c_2\sin\lambda t) + \lambda\left(-c_1\sin\lambda t + c_2\cos\lambda t\right))\right)$$
$$+$$
$$\left(-\omega^2\right)(a\cos\omega t + b\sin\omega t)$$
$$+$$
$$e^{-\beta t}\left(\left(\beta^2 - \lambda^2\right)(c_1\cos\lambda t + c_2\sin\lambda t) + (-2\beta\lambda)(-c_1\sin\lambda t + c_2\cos\lambda t)\right),$$
which equals
$$a\left(\cos\omega t - 2\beta\omega\sin\omega t - \omega^2\cos\omega t\right)$$
$$+$$
$$b\left(\sin\omega t + 2\beta\omega\cos\omega t - \omega^2\sin\omega t\right)$$
$$+$$
$$e^{-\beta t}\left(\left(1 - 2\beta^2 + \beta^2 - \lambda^2\right)(c_1\cos\lambda t + c_2\sin\lambda t) + (2\beta - 2\beta)\lambda(-c_1\sin\lambda t + c_2\cos\lambda t)\right),$$
which equals
$$\gamma\cos\omega t$$
for λ, a and b given in the exercise.
(ii) Note that $x(t)$ (given in (i)) is also a solution of (4.19) for $\beta = 0$. Therefore, if $x_\beta(t) := x(t)$ for $\beta \in [0, 1)$, $x_0(t)$ is periodic, $x_\beta(t)$ is not periodic and
$$\lim_{t\to\infty} x_\beta(t) = x_0(t)$$
for each $\beta \neq 0$.
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5
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Comments, pp. 64–8
Firstly, consider

$$\frac{d\varphi}{dt} = f(\lambda, \varphi(t)). \tag{11}$$

- **5.2.1**
  $f(\lambda, x) = x(\lambda - x)$. So $f(\lambda, x) = 0$ implies that $x^* \in \{0, \lambda\}$ for each $\lambda \in \mathbb{R}$. Now, consider $\lambda < 0$. Therefore, by **Fig. 5.1** (left) and (11),

$$\begin{aligned}\varphi(t) < \lambda &\Longrightarrow f(\lambda, \varphi(t)) < 0\\ &\Longrightarrow \frac{d\varphi}{dt} < 0\\ &\Longrightarrow \varphi(t) \text{ is decreasing,}\end{aligned}$$

$$\begin{aligned}\varphi(t) \in (\lambda, 0) &\Longrightarrow f(\lambda, \varphi(t)) > 0\\ &\Longrightarrow \frac{d\varphi}{dt} > 0\\ &\Longrightarrow \varphi(t) \text{ is increasing}\end{aligned}$$

  and

$$\begin{aligned}\varphi(t) > 0 &\Longrightarrow f(\lambda, \varphi(t)) < 0\\ &\Longrightarrow \frac{d\varphi}{dt} < 0\\ &\Longrightarrow \varphi(t) \text{ is decreasing.}\end{aligned}$$

  Analogously, for $\lambda > 0$,

$$\begin{aligned}\varphi(t) < 0 &\Longrightarrow \frac{d\varphi}{dt} < 0\\ &\Longrightarrow \varphi(t) \text{ is decreasing,}\end{aligned}$$

$$\begin{aligned}\varphi(t) \in (0, \lambda) &\Longrightarrow \frac{d\varphi}{dt} > 0\\ &\Longrightarrow \varphi(t) \text{ is increasing}\end{aligned}$$

  and

$$\begin{aligned}\varphi(t) > \lambda &\Longrightarrow \frac{d\varphi}{dt} < 0\\ &\Longrightarrow \varphi(t) \text{ is decreasing,}\end{aligned}$$

  and, for $\lambda = 0$,

$$\begin{aligned}\varphi(t) < 0 &\Longrightarrow \frac{d\varphi}{dt} < 0\\ &\Longrightarrow \varphi(t) \text{ is decreasing.}\end{aligned}$$

- **5.2.2–4**
  Use the same reason as above and consider the following points:
  - Concerning (5.4), if

$$\begin{aligned}f(\lambda, x) &= \lambda - x^2\\ &= 0,\end{aligned}$$

    then

* $\lambda < 0 \Longrightarrow \nexists x^*$,
* $\lambda = 0 \Longrightarrow x^* = 0$,
* $\lambda > 0 \Longrightarrow x^* = \pm\sqrt{\lambda}$;

– Concerning (5.6), if

$$\begin{aligned} f(\lambda, x) &= x\left(\mu - x^2\right) \\ &= 0, \end{aligned}$$

then

* $\mu \leq 0 \Longrightarrow x^* = 0$,
* $\mu > 0 \Longrightarrow x^* \in \{0, \pm\sqrt{\mu}\}$;

– Concerning (5.7), if

$$\begin{aligned} f(\lambda, x) &= x\left(\mu + x^2\right) \\ &= 0, \end{aligned}$$

then

* $\mu < 0 \Longrightarrow x^* \in \{0, \pm\sqrt{-\mu}\}$,
* $\mu \geq 0 \Longrightarrow x^* = 0$;

– Concerning (5.5), λ^* can be checked by solving

$$\lambda = x^3 - x \text{ for } x = \pm\frac{1}{\sqrt{3}}$$

from the system of page 67. So

$$\begin{aligned} \lambda &= \frac{1}{3\sqrt{3}} - \frac{1}{\sqrt{3}} \\ &= \frac{1-3}{3\sqrt{3}} \\ &= -\frac{2}{\sqrt{27}} \\ &= -\sqrt{\frac{4}{27}} \end{aligned}$$

or

$$\begin{aligned} \lambda &= -\frac{1}{3\sqrt{3}} + \frac{1}{\sqrt{3}} \\ &= -\left(\frac{1}{3\sqrt{3}} - \frac{1}{\sqrt{3}}\right) \\ &= \sqrt{\frac{4}{27}}. \end{aligned}$$

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Errata/Comments, p. 71, **5.2.6**

- The authors (Kapler and Engler) provided an errata correcting the first equation of (5.9):

$$\lambda x_1 \text{ should be } \lambda.^{16}$$

[16] The manner the equation is presented in the book give us $x^* = (2\lambda, \lambda)$. In fact, consider

$$\begin{cases} \lambda x_1 - x_1^2 + x_1 x_2 = 0, \\ \qquad\;\; x_1^2 - 2x_1 x_2 = 0. \end{cases}$$

Then, if you add the two equations,

$$\lambda x_1 - x_1 x_2 = 0.$$

With that correction, consider

$$\begin{cases} \lambda & - & x_1^2 & + & x_1 x_2 & = & 0, \\ & & x_1^2 & - & 2x_1 x_2 & = & 0. \end{cases} \tag{12}$$

Then, if you add the two equations of (12),

$$\lambda - x_1 x_2 = 0.$$

Now, substitute $x_1 x_2 = \lambda$ into the first equation of (12) to obtain

$$x_1^2 - 2\lambda = 0.$$

Therefore

$$x_1 = \pm\sqrt{2\lambda}$$

for $\lambda > 0$ and, since $x_1 x_2 = \lambda$,

$$\begin{aligned} x_2 &= \pm\frac{\lambda}{\sqrt{2\lambda}} \\ &= \pm\frac{\sqrt{2\lambda}}{2}. \end{aligned}$$

- As discussed in the preceding subsections, where $f(\lambda, x)$ was scalar, solution branches were expected to meet at points (λ, x^*) where

$$\begin{cases} f(\lambda, x^*) = 0, \\ \frac{\partial f}{\partial x}(\lambda, x^*) = 0. \end{cases}$$

  Such points were candidates for bifurcation points. Here, the candidates for bifurcation points of planar vector fields are obtained by solving

$$\begin{cases} f(\lambda, x^*) = 0, \\ \det(Df(\lambda, x^*)) = 0. \end{cases}$$

- Consider T and D as in section **4.6**. Then the discriminant

$$T^2 - 4D = \left(\frac{49}{2} - 16\right)\lambda$$

  is positive. Therefore, since $D > 0$, the eigenvalues of $Df(\lambda, x_\pm^*)$ are real with the same sign and the critical points $x_\pm^*$ are nodes: $T \lesseqgtr 0$ imply that the branch of x_+^*-solutions consists of stable nodes but, contrary to what is affirmed in the book, the branch of x_-^*-solutions consists of unstable nodes.

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Comments, p. 72

- 1st paragraph

  The positivity of the amplitude is used for discarding the minus sign in

$$\lambda - r^2 = 0 \Longrightarrow r = \pm\sqrt{\lambda}.$$

Now, substitute $x_1 x_2 = \lambda x_1$ into the first equation of the system. So

$$\lambda x_1 - x_1^2 + \lambda x_1 = 0,$$

which implies that

$$\begin{aligned} x_1^2 - 2\lambda x_1 = 0 &\Longrightarrow (x_1 - 2\lambda)\, x_1 = 0 \\ &\Longrightarrow \underline{x_1 = 2\lambda} \text{ or } x_1 = 0. \end{aligned}$$

By substituting $x_1 = 2\lambda$ into the second equation of the system, it follows that

$$\begin{aligned} 4\lambda^2 - 4\lambda x_2 = 0 &\Longrightarrow \lambda\,(\lambda - x_2) = 0 \\ &\Longrightarrow \lambda = 0 \text{ or } \underline{x_2 = \lambda}. \end{aligned}$$

Furthermore, we must add λ to the $1,1$ entry of $Df(\lambda, x)$, which implies that $\det(Df(\lambda, x)) = -2\lambda x_1 + 2x_1^2$.

- 2nd paragraph
  If I represents the 2×2 identity matrix, consider

$$\begin{aligned} p(\ell) &= \det(A(\lambda) - \ell I) \\ &= \ell^2 - 2\lambda\ell + \lambda^2 + 1. \end{aligned}$$

  So, due to the fact that the discriminant of $p(\ell) = 0$ is equal to -4, $A(\lambda)$ has a pair of complex conjugate eigenvalues:

$$\begin{aligned} \ell &= \frac{2\lambda \pm 2i}{2} \\ &= \lambda \pm i. \end{aligned}$$

- (5.12)

$$\begin{aligned} \dot{r} &= \frac{d}{dt}\left(\left(x_1^2 + x_2^2\right)^{1/2}\right) \\ &= \frac{1}{2}\left(x_1^2 + x_2^2\right)^{-1/2}(2x_1\dot{x}_1 + 2x_2\dot{x}_2) \\ &= \frac{\left(x_1^2 + x_2^2\right)\left(\lambda - x_1^2 - x_2^2\right)}{\left(x_1^2 + x_2^2\right)^{1/2}} \\ &= \frac{r^2\left(\lambda - r^2\right)}{r}, \end{aligned}$$

$$\begin{aligned} \dot{\theta} &= \frac{d}{dt}(\arctan(x_2/x_1)) \\ &= \frac{1}{1 + (x_2/x_1)^2} \cdot \frac{\dot{x}_2 x_1 - x_2\dot{x}_1}{x_1^2} \\ &= \frac{x_1^2}{x_1^2 + x_2^2} \cdot \frac{-x_1^2 - x_2^2}{x_1^2} \\ &= -\frac{x_1^2 + x_2^2}{x_1^2 + x_2^2}. \end{aligned}$$

- 1st sentence after (5.14)
  See the solid line in the first quadrant in **Figure 5.4** (right), p. 68.

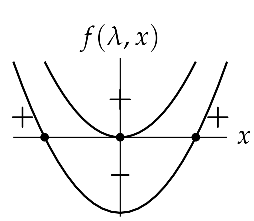
==

Exercises, pp. 75–6

1. Consider $f(\lambda, x) = \lambda + x^2$. So $f(\lambda, x) = 0$ implies that $x^* \in \left\{0, \pm\sqrt{-\lambda}\right\}$ exists only for $\lambda \le 0$:

$$\begin{aligned} \lambda + x^2 = 0 &\Longrightarrow x^2 = -\lambda \\ &\Longrightarrow x = \pm\sqrt{-\lambda}. \end{aligned}$$

So the phase portraits for $\lambda \in \{-1, 0\}$



and equation (11), p. 17 of this text, tell us that

$$\begin{aligned}\varphi(t) < -\sqrt{-\lambda} &\Longrightarrow f(\lambda,\varphi(t)) > 0\\ &\Longrightarrow \frac{d\varphi}{dt} > 0\\ &\Longrightarrow \varphi(t) \text{ is increasing}\end{aligned}$$

and

$$\begin{aligned}\varphi(t) > -\sqrt{-\lambda} &\Longrightarrow f(\lambda,\varphi(t)) < 0\\ &\Longrightarrow \frac{d\varphi}{dt} < 0\\ &\Longrightarrow \varphi(t) \text{ is decreasing}\end{aligned}$$

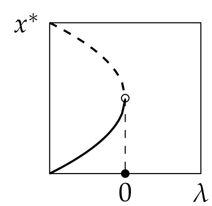
(meaning $x^* = -\sqrt{-\lambda}$ is stable), whereas

$$\begin{aligned}\varphi(t) < \sqrt{-\lambda} &\Longrightarrow f(\lambda,\varphi(t)) < 0\\ &\Longrightarrow \frac{d\varphi}{dt} < 0\\ &\Longrightarrow \varphi(t) \text{ is decreasing}\end{aligned}$$

and

$$\begin{aligned}\varphi(t) > \sqrt{-\lambda} &\Longrightarrow f(\lambda,\varphi(t)) > 0\\ &\Longrightarrow \frac{d\varphi}{dt} > 0\\ &\Longrightarrow \varphi(t) \text{ is increasing}\end{aligned}$$

(meaning $x^* = \sqrt{-\lambda}$ is unstable). Furthermore, $x^* = 0$ is clearly unstable. The previous reasoning along with $f_x(\lambda,x) = 2x = 0$ (meaning $x^* = 0$ is the candidate for bifurcation point) imply that there are two fixed points for $\lambda < 0$: $x^*_- = -\sqrt{-\lambda}$ (stable) and $x^*_+ = \sqrt{-\lambda}$ (unstable). They merge with each other at $\lambda = 0$ and, from this unstable point on, there are no fixed points as depicted in the following bifurcation diagram:



2. Consider $f(\lambda,x) = \sin x - \lambda$. So $f(\lambda,x) = 0$ implies that $x^* = \arcsin\lambda$ exists only for

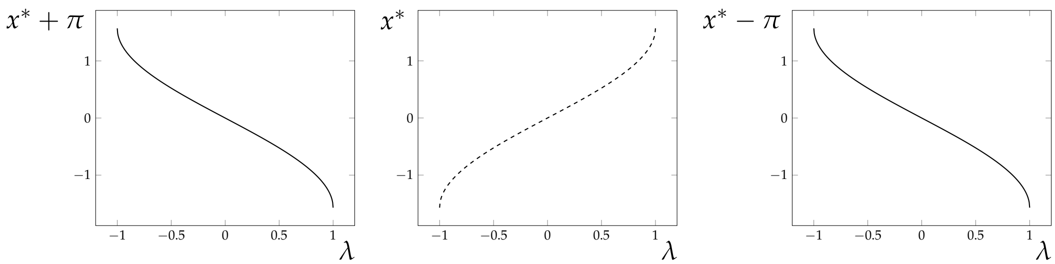
$$\lambda = \sin x \in [-1,1].$$

So there are no fixed points for $\lambda \in [-2,-1) \cup (1,2]$ and the phase portraits can be analyzed by horizontally translating the graph of $f(0,x) = \sin x$ (meaning shifting the graph of $f(0,x)$ left or right in the direction of the x-axis) in order to obtain the graph of $f(\lambda,x) = \sin x - \lambda$ (as a function of x) with $(\lambda,x) \in [-1,1] \times [-4\pi,4\pi]$. Note that $f(\lambda,x)$ changes sign at $(x^*,0)$:

- x^* is stable where $f(\lambda,x)$ changes sign from positive to negative;[17]
- x^* is unstable where $f(\lambda,x)$ changes sign from negative to positive.[18]

Now, concerning the bifurcation points, consider $f_x(\lambda,x) = \cos x = 0$. Then, due to the fact that $x \in [-4\pi,4\pi]$, $x^* \in \left\{\pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \pm\frac{5\pi}{2}, \pm\frac{7\pi}{2}\right\}$, which implies that $\lambda = \pm 1$ are the candidates for bifurcation points. This fact and the previous reasonig allow us to depict bifurcation diagrams as follows:

[17] This takes place in an interval where $f(\lambda,x)$ is decreasing.
[18] This takes place in an iterval where $f(\lambda,x)$ is increasing.



where the values displayed on the vertical axes measure $x^* \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. Hence, the vertical axis of the first diagram represents $x^* + \pi \in \left[\frac{\pi}{2}, \frac{3\pi}{2}\right]$, whereas the vertical axis of the third diagram represents $x^* - \pi \in \left[-\frac{3\pi}{2}, -\frac{\pi}{2}\right]$. Furthermore, compared to the three previous diagrams, the bifurcation diagrams for:

- $x^* \in \left[\frac{3\pi}{2}, \frac{5\pi}{2}\right]$ and $x^* \in \left[-\frac{5\pi}{2}, -\frac{3\pi}{2}\right]$ are identical to the second one;
- $x^* \in \left[\frac{5\pi}{2}, \frac{7\pi}{2}\right]$ and $x^* \in \left[-\frac{7\pi}{2}, -\frac{5\pi}{2}\right]$ are identical to the first/third one;
- $x^* \in \left[\frac{7\pi}{2}, 4\pi\right]$ (respectively, $x^* \in \left[-4\pi, -\frac{7\pi}{2}\right]$) is identical to the first (respectively, second) half of the second diagram.

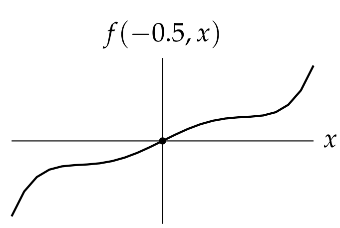
3. Since $f(\lambda, x) = x\left(\lambda + x^2 - x^4\right)$,

$$x^* \in \left\{0, \pm\sqrt{\frac{1 \pm \sqrt{1+4\lambda}}{2}}\right\}$$

where $x^* = 0$ exists for $-1 < \lambda < 1$, whereas the other fixed points exist for $-\frac{1}{4} \leq \lambda < 1$,[19] provided that $-2 < x^* < 2$.[20] Then:

- $-1 < \lambda < -\frac{1}{4} \Longrightarrow$ there is only one fixed point: $x^* = 0$;
- $\lambda = -\frac{1}{4} \Longrightarrow$ there are three fixed points: $x^* \in \left\{0, \pm\sqrt{\frac{1}{2}}\right\}$;
- $-\frac{1}{4} < \lambda < 0 \Longrightarrow$ there are five fixed points for each such λ;
- $\lambda = 0 \Longrightarrow$ there are three fixed points: $x^* \in \{0, \pm 1\}$;
- $0 < \lambda < 1 \Longrightarrow$ there are three fixed points for each such λ: $x^* \in \left\{0, \pm\sqrt{\frac{1+\sqrt{1+4\lambda}}{2}}\right\}$.

(So the number of fixed points changes three times as λ varies between -1 and 1.) Now, in order to analyze the stability of such fixed points via sign diagrams, consider the phase portraits for $\lambda \in \{-0.5, -0.25, -0.2, 0.5\}$:



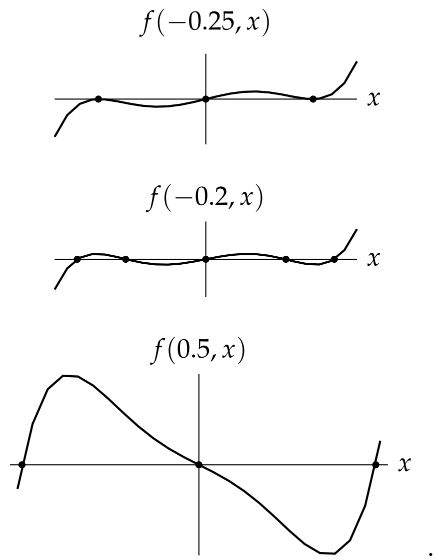
[19] In fact, consider the biquadratic equation $x^4 - x^2 - \lambda = 0$ and the change of variable $x^2 = t$. So

$$t^2 - t - \lambda = 0 \Longrightarrow t = \frac{1 \pm \sqrt{1+4\lambda}}{2}$$

with $1 + 4\lambda \geq 0$ and $-1 < \lambda < 1$.

[20] As a matter of fact, $x^* \in (-2, 2)$ for

$$\begin{aligned} -\frac{1}{4} \leq \lambda < 1 &\Longleftrightarrow 0 \leq 1 + 4\lambda < 5 \\ &\Longleftrightarrow 0 \leq \sqrt{1+4\lambda} < \sqrt{5}. \end{aligned}$$



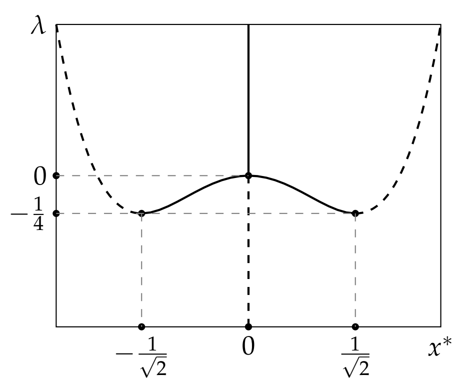
Therefore:

- $x^* = 0$ is unstable (respectively, stable) for $\lambda < 0$ (respectively, $\lambda > 0$);
- each x^* is unstable for $\lambda = -0.25$;
- the fixed points farthest from (respectively, closest to) $x^* = 0$ are unstable (respectively, stable) for $-0.25 < \lambda \leq 0$;
- the nonzero fixed points are unstable for $0 < \lambda < 1$.

On the other hand, concerning the candidates for bifurcation points, consider $f_x(\lambda, x) = 5x^4 - 3x^2 - \lambda$ and note that

$$f_x(0,0) = 0 \text{ and } f_x\left(\mp\frac{1}{\sqrt{2}}, -\frac{1}{4}\right) = 0.$$

The previous reasoning, along with the equations $x = 0$ and $\lambda + x^2 - x^4 = 0$, give us the bifurcation diagram



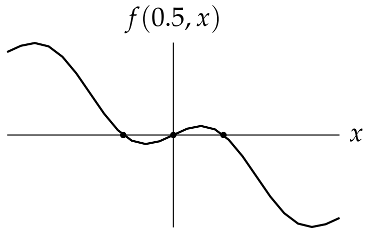
for $(x^*, \lambda) \in \left(-\frac{\sqrt{1+\sqrt{5}}}{2}, \frac{\sqrt{1+\sqrt{5}}}{2}\right) \times (-1, 1)$.[21]

4.
Let $f(\lambda, x) = \sin x - \lambda x$ with $\lambda \in [0.5, 2]$. So $x^* = 0$ is a fixed point for each λ and, since

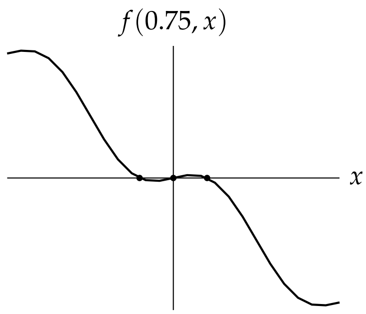
$$\begin{aligned} f(\lambda, x) &= (1-\lambda)x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots \\ &= x\left(1 - \lambda - \frac{x^2}{3!} + \frac{x^4}{5!} - \cdots\right), \end{aligned}$$

[21]Note that the bifurcation diagram is rotated about the origin at $\pi/2$ radians CCW.

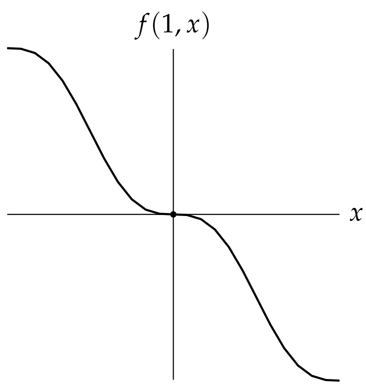
there are two more fixed points if and only if $1-\lambda>0$ (due to the fact that $-\frac{x^2}{3!}+\frac{x^4}{5!}-\cdots$ is an even function with a concave down graph). Now, in order to analyze the stability of the fixed points via sign diagrams, consider the phase portraits for $\lambda \in \{0.5, 0.75, 1, 1.5, 2\}$:



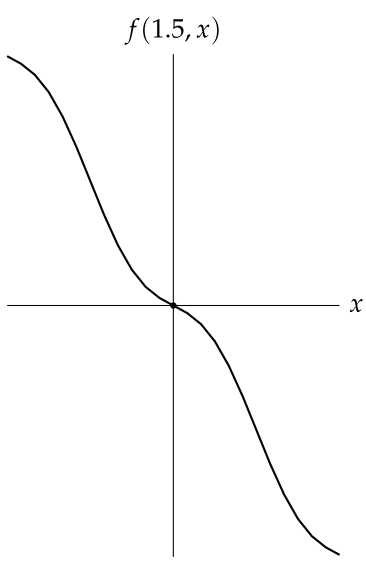
with $x^* = 0$ unstable and $x^* \approx \pm 1.8955$ stable,



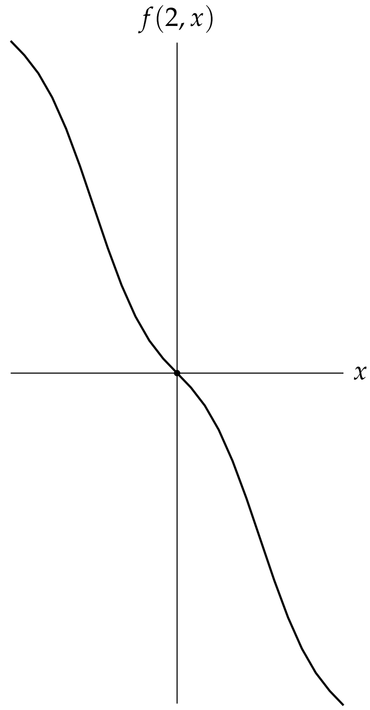
with $x^* = 0$ unstable and $x^* \approx \pm 1.2757$ stable,



with $x^* = 0$ stable,



with $x^* = 0$ stable, and

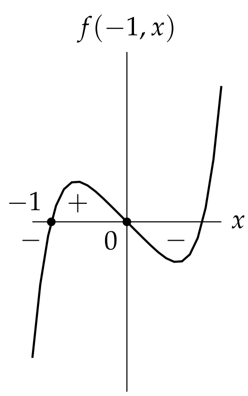


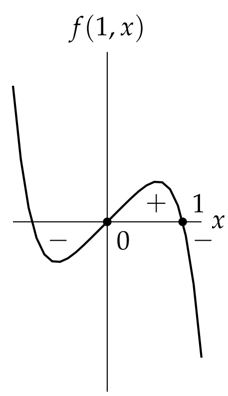
with $x^* = 0$ stable. Therefore everything indicates that there is a pitchfork bifurcation at $\lambda = 1$ which is similar to the mirror image of the subcritical pitchfork bifurcation of **Figure 5.5**, p. 69, with respect to the $y-$axis and with $\lambda - 1$ in place of μ.[22]

5.
(i) Consider $f(\lambda, x) = \lambda x\left(\lambda - x^2\right)\left(\lambda + x^2\right)$.[23] So

$$x^* = \begin{cases} 0 \text{ for each } \lambda; \\ \sqrt{\lambda} \text{ for } \lambda > 0; \\ -\sqrt{-\lambda} \text{ for } \lambda < 0. \end{cases}$$

The following two graphs represent the phase portraits for $\lambda \lesseqgtr 0$ ($\lambda = \mp 1$ in the figures). The figures show us that $x^* = 0$ changes from stable to unstable at $\lambda = 0$, $x^* = -\sqrt{-\lambda}$ is unstable and nonexistent for $\lambda > 0$, whereas $x^* = \sqrt{\lambda}$ is stable and nonexistent for $\lambda < 0$.





[22] As a matter of fact, compare

$$f(\lambda, x) = -\left((\lambda - 1)x + \frac{x^3}{3!}\right) + \mathcal{O}\left(x^5\right)$$

to (5.7), p. 68.

[23] $f(\lambda, x) = \lambda^3 x - \lambda x^5$.

On the other hand, the candidates for bifurcation points are obtained by

$$\begin{aligned} f_x(\lambda, x) = 0 &\Longrightarrow \lambda^3 - 5\lambda x^4 = 0 \\ &\Longrightarrow \lambda\left(\lambda^2 - 5x^4\right) = 0 \\ &\Longrightarrow \lambda \in \left\{0, \pm\sqrt{5}x^2\right\}. \end{aligned}$$

Therefore the previous analysis confirms **Figure 5.9** (left) with $\lambda = 0$ as the bifurcation point.
(ii) It looks like that the bifurcation diagram of **Figure 5.9** (right) is a rescaled version of the bifurcation diagram of **Figure 5.4** (right), which is given rise by the vector field $f(\lambda, x) = x\left(\lambda - x^2\right)$, after being rotated through an angle of $\pi/4$ radians in anti-clockwise direction about the origin. So let us analyze the equations $x = 0$ and $\lambda - x^2 = 0$, which are building blocks of the bifurcation diagram, after being subjected to such rotation. Clearly, $x = 0$ becomes $x = \lambda$, whereas $\lambda - x^2 = 0$ becomes $x^2 + \lambda^2 - 2x\lambda - \sqrt{2}x - \sqrt{2}\lambda = 0$.[24] Therefore, concerning **Figure 5.9**, the vector field which gives rise to the bifurcation diagram (on the right) is

$$f(\lambda, x) = (x - \lambda)\left(x^2 + \lambda^2 - 2x\lambda - \sqrt{2}x - \sqrt{2}\lambda\right).$$

===

[24]Consider the parabola $\lambda' = x'^2$ and the rotation

$$\begin{bmatrix} \cos\frac{\pi}{4} & -\sin\frac{\pi}{4} \\ \sin\frac{\pi}{4} & \cos\frac{\pi}{4} \end{bmatrix} \begin{bmatrix} x \\ \lambda \end{bmatrix} = \begin{bmatrix} x' \\ \lambda' \end{bmatrix}.$$

==
==
6
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==
Comment, p. 77, penultimate sentence
Consider p. 36, 1st sentence along with (3.8) and (3.9). Therefore

$$\begin{aligned}\bar{T}_2^* &= T_2^* - T_0^* \\ &= T_2^* - \frac{1}{2}(T_1^* + T_2^*) \\ &= \frac{1}{2}(T_2^* - T_1^*) \\ &:= T^*\end{aligned}$$

and

$$\begin{aligned}\bar{T}_1^* &= T_1^* - T_0^* \\ &= T_1^* - \frac{1}{2}(T_1^* + T_2^*) \\ &= \frac{1}{2}(T_1^* - T_2^*) \\ &= -T^*.\end{aligned}$$

Similarly,

$$\bar{S}_2^* = S^* \text{ and } \bar{S}_1^* = -S^*.$$

==
Comments, p. 78

- Sentence that follows (6.3)
  Since

$$\begin{aligned}\frac{d}{dt}\left(\frac{1}{2}(T_1+T_2)\right) &= -c\left(\frac{1}{2}(T_1+T_2)\right) \text{ and} \\ \frac{d}{dt}\left(\frac{1}{2}(S_1+S_2)\right) &= -d\left(\frac{1}{2}(S_1+S_2)\right),\end{aligned}$$

$$\begin{aligned}\frac{1}{2}(T_1+T_2) &= \text{constant} \cdot e^{-ct} \longrightarrow 0 \text{ and} \\ \frac{1}{2}(S_1+S_2) &= \text{constant} \cdot e^{-dt} \longrightarrow 0\end{aligned}$$

  when $t \longrightarrow \infty$.

- (6.6)

$$\begin{aligned}\dot{x} &= \frac{dx}{dt'} \\ &= \frac{1}{c\Delta S^*}\left(\frac{d\Delta S}{dt}\right) \\ &= \frac{d}{c}(1-x) - \left|\frac{2q}{c}\right| x, \\ \dot{y} &= \frac{dy}{dt'} \\ &= \frac{1}{c\Delta T^*}\left(\frac{d\Delta T}{dt}\right) \\ &= 1 - y - \left|\frac{2q}{c}\right| y.\end{aligned}$$

==

Erratum, p. 79, right after (6.8)
λf^* should be equal to $Rx^* - y^*$.

==

Comments, p. 80

- (6.11)
  The 1,1 entry of A is obtained by

$$\begin{aligned}\frac{\partial}{\partial x}(\delta(1-x)-|f|x) &= \frac{\partial}{\partial x}\left(\delta-\delta x\mp\frac{1}{\lambda}\left(Rx^2-xy\right)\right)\\ &= -\delta\mp\frac{1}{\lambda}(2Rx-y)\\ &= -\delta\mp\frac{1}{\lambda}((Rx-y)+Rx)\\ &= -\delta-\left(\pm\frac{Rx-y}{\lambda}\right)\mp\frac{Rx}{\lambda}\\ &= -(\delta+|f|)\mp\frac{Rx}{\lambda}.\end{aligned}$$

  Computing the 1,2 and 2,1 entries of A is straightforward. Finally, the 2,2 entry of A is obtained by

$$\begin{aligned}\frac{\partial}{\partial y}(1-y-|f|y) &= \frac{\partial}{\partial y}\left(1-y\mp\frac{1}{\lambda}\left(Rxy-y^2\right)\right)\\ &= -1\mp\frac{1}{\lambda}(Rx-2y)\\ &= -1\mp\frac{1}{\lambda}((Rx-y)-y)\\ &= -1-\left(\pm\frac{Rx-y}{\lambda}\right)\pm\frac{y}{\lambda}\\ &= -(1+|f|)\pm\frac{y}{\lambda}.\end{aligned}$$

- (6.12)

$$\begin{aligned}D &= \delta+\delta\,|f^*|+|f^*|+|f^*|^2\pm\left(\frac{Rx^*}{\lambda}-\frac{\delta y^*}{\lambda}\right)+|f^*|\left(\pm\frac{Rx^*-y^*}{\lambda}\right)\\ &= \delta+\delta\,|f^*|+|f^*|+2\,|f^*|^2\pm\left(\frac{Rx^*}{\lambda}-\frac{\delta y^*}{\lambda}\right)\pm\left(-\frac{y^*}{\lambda}+\frac{y^*}{\lambda}\right)\\ &= \delta+\delta\,|f^*|+2\,|f^*|+2\,|f^*|^2\pm(1-\delta)\frac{y^*}{\lambda}.\end{aligned}$$

- Penultimate and ultimate sentences
  Since $f^* > 0$ and $\delta > 0$,

$$(\delta+2\,|f^*|)(1+|f^*|)>0$$

  and, since $\delta\in(0,1]$, $y^*>0$ and $\lambda>0$,[25]

$$\frac{(1-\delta)y^*}{\lambda}\geq 0.$$

  So $D > 0$. Furthermore, since

$$\begin{aligned}T^2 &= (1+\delta)^2+6(1+\delta)f^*+9(f^*)^2\\ &= 1+2\delta+\delta^2+6f^*+6\delta f^*+9(f^*)^2\end{aligned}$$

[25] $\delta\in(0,1]$ and $y^*>0 \Longleftarrow$ l. 5 and (6.8), p. 79;
$\lambda>0 \Longleftarrow \lambda=\frac{c}{2\alpha k(2T^*)}$, p. 78, and T^* is the temperature anomaly in the basin surrounding Box 2, p. 77.

and

$$-4D = -4\delta - 4\delta f^* - 8f^* - 8(f^*)^2 - 4(1-\delta)\frac{y^*}{\lambda},$$

$$\begin{aligned} T^2 - 4D &= 1 - 2\delta + \delta^2 - 2f^* + 2\delta f^* + (f^*)^2 - 4(1-\delta)\frac{y^*}{\lambda} \\ &= (1-\delta)^2 - 2(1-\delta)f^* + (f^*)^2 - 4(1-\delta)\left(\frac{\frac{1}{1+f^*}}{\lambda}\right) \\ &= ((1-\delta) - f^*)^2 - \frac{4(1-\delta)}{\lambda\,(1+f^*)}. \end{aligned}$$

Now, note that $T^2 - 4D > 0$ for $\delta = 1$. So, here,

$$\delta, 1-\delta \in (0,1). \tag{13}$$

Let us prove that $T^2 - 4D < 0$ holds with some simple heuristics. So, on the one hand,

$$\begin{aligned} ((1-\delta) - f^*)^2 < \frac{4(1-\delta)}{\lambda(1+f^*)} &\iff \lambda(1+f^*) < 4\left(\frac{1-\delta}{((1-\delta)-f^*)^2}\right) \\ &\iff \lambda f^* < 4\left(\frac{1-\delta}{((1-\delta)-f^*)^2}\right) - \lambda. \end{aligned}$$

On the other hand, by subsection **6.2.1** along with **Figure 6.1** ($f > 0$),[26]

$$\lambda f^* = \phi(f^*) \Longrightarrow 0 < \lambda f^* < 1.$$

Then $T^2 - 4D < 0$ if

$$1 < 4\left(\frac{1-\delta}{((1-\delta)-f^*)^2}\right) - \lambda, \tag{14}$$

which is equivalent to

$$((1-\delta) - f^*)^2 < \frac{4}{\lambda+1}(1-\delta) \iff (1-\delta)^2 - \left(2f^* + \frac{4}{\lambda+1}\right)(1-\delta) + (f^*)^2 < 0$$

with positive roots

$$1 - \delta_\pm = \frac{2f^* + \frac{4}{\lambda+1} \pm \sqrt{\left(2f^* + \frac{4}{\lambda+1}\right)^2 - 4(f^*)^2}}{2}. \tag{15}$$

So (14) holds for each $1-\delta \in (1-\delta_-, 1-\delta_+)$. Then (14) holds for each $\delta \in (\delta_+, \delta_-) \subset (0,1)$.[27] Therefore $T^2 - 4D < 0$ for each $f^* > 0$.

==

Comment, p. 81, 3rd sentence

$$\frac{dD}{df^*} = -\delta - 2 + 4f^* - \frac{1-\delta}{\lambda\,(1-f^*)^2}$$

is negative for $f^* \in (-\infty, 0)$,

$$\lim_{f^* \to -\infty} D = +\infty \ \text{ and } \ \lim_{f^* \to 0} D = \delta - \frac{1-\delta}{\lambda},$$

which is negative if $\lambda \in (0,1)$ is small enough.[28]

==

Erratum/Comments, p. 82, 3rd paragraph

[26]Since $\delta \neq 1$, points like e and g are not considered here!

[27]On the one hand, if $\delta_- > 1$, then $1 - \delta_- < 0$, which contradicts (15). On the other hand, if $\delta_+ < 0$, then δ can take nonpositive values, which is a contradiction because $\delta \in (0,1)$.

[28]See (13)!

- 4th sentence[29]
  Interchange 'S-mode' and 'T-mode'.

- Last four sentences

  - "..., a reversal of the flow, ..."
    f depends on q.[30]
  - "... an increase of the temperature anomaly."
    See **Figure 6.3**.[31]
  - "... the salinity anomaly will also increase."
    Here, x^* depends on y^*.[32]
  - "... salinity anomaly on the vertical axis, ..."
    See the previous comment.

==

Exercises, pp. 83–6

1.

$$\begin{aligned}
\phi_+(0) &= \frac{\delta R}{\delta} - 1 \\
&= R - 1 \\
&> 0 \ \text{ if } R > 1; \\
\left.\frac{d\phi_+}{df}\right|_{f=0} &= -\frac{\delta R}{(\delta + f)^2} + \left.\frac{1}{(1+f)^2}\right|_{f=0} \\
&= -\frac{\delta R}{\delta^2} + 1 \\
&= -\frac{R}{\delta} + 1 \\
&< 0 \ \text{ if } R > \delta; \\
\lim_{f \longrightarrow \infty} \phi_+(f) &= \lim_{f \longrightarrow \infty} \frac{\delta R(1+f) - (\delta + f)}{(\delta + f)(1+f)} \\
&= \lim_{f \longrightarrow \infty} \frac{(\delta R - 1)f + \delta R - \delta}{f^2 + (\delta + 1)f + \delta} \\
&= \lim_{f \longrightarrow \infty} \frac{\frac{\delta R - 1}{f}}{1} \\
&= 0^- \ \text{ if } \delta R < 1.
\end{aligned}$$

The critical point 'c' satisfying (6.9) (for $\lambda = \frac{1}{5}$, $R = 2$ and $\delta = \frac{1}{6}$) is shown in figures 6.1 and 6.2, which are consistent with the properties above. In fact, the graph of ϕ curves up as it moves toward 'c',[33] crosses the f-axis, keeps curving up a little bit more and approaches the f-axis asymptotically.[34] Since $\lambda \in (0, \infty)$, the graphs of ϕ and λf have exactly one point of intersection, which is 'c'. Furthermore, concerning $f \in [0, \infty)$, 'c' is close to the equiflow line $f = 0$ and the phase portrait does not have another steady state close to any equiflow line.

2.
If $\delta = \frac{1}{6}$ and $R = \frac{3}{2}$, $\lambda f = \phi(f)$ has exactly two (respectively, one) negative solutions (respectively, solution)

[29] The one after "..., $y^* = \frac{4}{5}$.".
[30] See p. 78.
[31] See the anomaly component y^* on the right.
[32] See (6.7), p. 78.
[33] Which is consistent with the first two properties.
[34] Which is consistent with the ultimate property.

$f = f^*$ for each $\lambda \in (0, \frac{4}{5})$ (respectively, for $\lambda = \frac{4}{5}$). In fact,

$$\begin{aligned}
\lambda f &= \phi(f) \\
&= \frac{\frac{1}{4}}{\frac{1}{6} - f} - \frac{1}{1-f} \\
&= \frac{\frac{1}{2}}{\frac{1-6f}{3}} - \frac{1}{1-f} \\
&= \frac{3(1-f) - 2(1-6f)}{2(1-6f)(1-f)} \\
&= \frac{9f+1}{2(6f^2 - 7f + 1)}.
\end{aligned}$$

So, let us find the negative roots f^* of

$$p(\lambda, f) = 12\lambda f^3 - 14\lambda f^2 + (2\lambda - 9)f - 1$$

for

$$\lambda \in \left\{ \frac{1}{20}, \frac{1}{10}, \frac{1}{5}, \frac{3}{10}, \frac{2}{5}, \frac{1}{2}, \frac{3}{5}, \frac{7}{10}, \frac{4}{5} \right\},$$

that is, the cubic polynomials

$$\begin{aligned}
0.6f^3 - 0.7f^2 - 8.9f - 1 &= 0, \\
1.2f^3 - 1.4f^2 - 8.8f - 1 &= 0, \\
12f^3 - 14f^2 - 43f - 5 &= 0, \\
3.6f^3 - 4.2f^2 - 8.4f - 1 &= 0, \\
24f^3 - 28f^2 - 41f - 5 &= 0, \\
6f^3 - 7f^2 - 8f - 1 &= 0, \\
36f^3 - 42f^2 - 39f - 5 &= 0, \\
8.4f^3 - 9.8f^2 - 7.6f - 1 &= 0 \text{ and} \\
48f^3 - 56f^2 - 37f - 5 &= 0.
\end{aligned}$$

The negative roots of these polynomials are given by

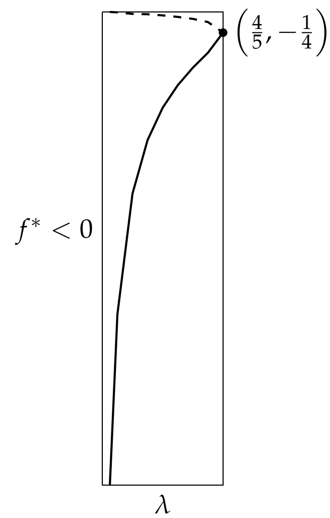
$$\begin{aligned}
f^* &\approx -3.246, -0.113, \\
f^* &\approx -2.115, -0.116, \\
f^* &\approx -1.316, -0.126, \\
f^* &\approx -0.961, -0.128, \\
f^* &\approx -0.747, -0.136, \\
f^* &\approx -0.598, -0.146, \\
f^* &\approx -0.482, -0.159, \\
f^* &\approx -0.383, -0.180 \text{ and} \\
f^* &= -\frac{1}{4},
\end{aligned}$$

respectively. Furthermore, note that $\lambda = \frac{4}{5}$ is a candidate for bifurcation point since, at $(\lambda, f) = \left(\frac{4}{5}, -\frac{1}{4}\right)$,

$$\begin{cases} p(\lambda, f) = 0; \\ \frac{\partial p}{\partial f} = 0. \end{cases}$$

This analysis and the comments on page 81 allow us to consider the following bifurcation diagram:[35]

[35]Note the consistency with the bifurcation diagram of figure 6.3 (left), p. 82.



3.
Firstly, by (6.8), p. 79, note that x_i^* and y_i^* are positive for each $i \in \{1,2,3\}$. Secondly, since $f_1^* < f_2^* < 0$, that is, $-f_1^* > -f_2^* > 0$, it follows that, on the one hand,

$$\begin{aligned} 1 - f_1^* > 1 - f_2^* > 1 &\Longrightarrow 0 < \frac{1}{1-f_1^*} < \frac{1}{1-f_2^*} < 1 \\ &\Longrightarrow 0 < y_1^* < y_2^* < 1, \end{aligned}$$

and, on the other hand, since $\delta > 0$,

$$\begin{aligned} -f_1^* + \delta > -f_2^* + \delta > \delta &\Longrightarrow \frac{\delta - f_1^*}{\delta} > \frac{\delta - f_2^*}{\delta} > 1 \\ &\Longrightarrow \frac{1}{x_1^*} > \frac{1}{x_2^*} > 1 \\ &\Longrightarrow 0 < x_1^* < x_2^* < 1. \end{aligned}$$

Now, note that

$$\begin{aligned} y_2^* < y_3^* &\Longleftrightarrow \frac{1}{1-f_2^*} < \frac{1}{1+f_3^*} \\ &\Longleftrightarrow 1 + f_3^* < 1 - f_2^* \\ &\Longleftrightarrow f_3^* < -f_2^* \\ &\Longleftrightarrow f_3^* + \delta < -f_2^* + \delta \\ &\Longleftrightarrow \frac{1}{\delta - f_2^*} < \frac{1}{\delta + f_3^*} \\ &\Longleftrightarrow \frac{\delta}{\delta - f_2^*} < \frac{\delta}{\delta + f_3^*} \\ &\Longleftrightarrow x_2^* < x_3^*. \end{aligned}$$

Similarly, $y_2^* = y_3^* \Longleftrightarrow x_2^* = x_3^*$ and $y_2^* > y_3^* \Longleftrightarrow x_2^* > x_3^*$. However, if $x_2^* = x_3^*$ and $y_2^* = y_3^*$, then $f_2^* = f_3^*$, which is a contradiction. In the same vein, $x_2^* > x_3^*$ and $y_2^* > y_3^*$ also contradicts the hypothesis $f_2^* < 0 < f_3^*$.

4.
Since $\delta > 1$ and $R > 1$, $\delta R > 1$. So

$$\delta R - 1 > 0, \ \delta > 0 \ \text{and} \ 1 - R < 0. \tag{16}$$

Now, concerning (6.9), a necessary condition for finding three points of intersection is that the graph of $\phi(f)$

dips below the horizontal axis, $\phi(f) = 0$ for some $f > 0$.[36] However,

$$\begin{aligned}\phi(f^*) = 0 &\Longrightarrow \frac{\delta R}{\delta + |f^*|} = \frac{1}{1 + |f^*|} \\ &\Longrightarrow (\delta R - 1)\,|f^*| = \delta(1 - R),\end{aligned}$$

which contradicts (16) for $f^* > 0$. Therefore (6.6) has only one equilibrium solution with $f^* > 0$. Furthermore, f^* is a stable node. In fact, $T < 0$ and, since $1 - \delta < 0$ and $\lambda > 0$, $D > 0$ and $T^2 - 4D > 0$.[37]
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[36] See p. 79.

[37] See p. 80, last paragraph.

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Comments, pp. 88–90

- **7.2**, 2nd bullet, 1st paragraph
  By the existence and uniqueness theorems,[38]

$$\varphi(t) = \left(0, 0, e^{\beta t}\right), \, t \in \mathbb{R},$$

  is the unique solution of (7.1) passing through the point $(0, 0, z_0)$.

- 2nd sentence after (7.2)
  Being a subset of $\mathbb{R}^n$,

$$\mathscr{D} \text{ is closed and bounded} \Longleftrightarrow \mathscr{D} \text{ is compact},$$

  which implies that $\phi_t(\mathscr{D})$ is compact.[39] Furthermore, since the intersection of a decreasing family of compact sets is compact,[40] $\mathscr{A}$ is compact by (7.2).[41]

- (7.3)
  For each $c \in \mathbb{R}$, the level surface of value c for V, that is,

$$V^{-1}(\{c\}) = \left\{P \in \mathbb{R}^3 \,:\, V(P) = c\right\},$$

  is an ellipsoid.

- (7.4)
  Note that

$$\begin{aligned}\frac{d}{dt}\left(V(\phi_t(P)) = \nabla V((\phi_t(P)) \cdot \frac{d}{dt}(\phi_t(P))\right. \\ &= \|\nabla V(\phi_t(P))\| \left\|\frac{d}{dt}(\phi_t(P))\right\| \cos\theta,\end{aligned}$$

  where θ is the smallest angle between the gradient $\nabla V(\phi_t((P))$ and the velocity vector $\frac{d}{dt}(\phi_t(P))$. Therefore, since $\nabla V(\phi_t(P))$ is perpendicular to the level surface of value $V(\phi_t(P))$ at $\phi_t(P)$, that is, the ellipsoid $V^{-1}(\{\phi_t(P)\})$ at $\phi_t(P)$, if $\frac{d}{dt}\left(V(\phi_t(P)) < 0\right.$, the vector field is directed inward at $\phi_t(P)$.

- $\mathscr{E}$ and m
  $\mathscr{E}$ being open, m may not exist. So, concerning the definition of $\mathscr{E}$, change $<$ to $\leq$.

- 1st sentence after (7.5)
  Suppose $\mathscr{E} \not\subset \mathscr{D}$. So, there exists some $P \in \mathscr{E}$ such that $V(P) > m$, which contradicts the definition of m.

- **7.3**

  - 1st sentence and '$C_\pm$'
    Let the right-hand sides of Eq. (7.1) be zero. So, from the first equation, $x = y$. Then, the second equation becomes $x(\rho - 1 - z) = 0$, which implies that $z = \rho - 1$ for $x \neq 0$, and the third equation becomes $-\beta z + x^2 = 0$. Therefore,

$$x^2 = \beta(\rho - 1).$$

  - 1st sentence after (7.7)
    $(1 + \sigma)^2 > 4(1 - \rho)\sigma$ must hold for $0 < \rho < 1 < 1 + \beta < \sigma$.
  - (7.8)
    For example, $A_{21} = 1$ since $\frac{\partial}{\partial x}(\rho x - y - xz)$ at C_+ is equal to $\rho - (\rho - 1)$.

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[38] Cf. pp. 43–4.
[39] Because ϕ_t is continuous.
[40] By the Cantor Intersection Theorem.
[41] See **7.5**, 1.(ii).

– (ii) for (7.9)
  On the one hand,

$$\rho < \rho_H \Longrightarrow \rho < \rho_H \sigma.$$

  On the other hand

$$\begin{aligned}(1+\beta+\sigma)\beta(\rho+\sigma) > 2\beta(\rho-1)\sigma &\Longleftrightarrow (1+\beta+\sigma)(\rho+\sigma) > 2(\rho-1)\sigma\\ &\Longleftrightarrow (1+\beta+\sigma-2\sigma)\rho > -(1+\beta+\sigma+2)\sigma\\ &\Longleftrightarrow -(\sigma-\beta-1)\rho > -(\sigma+\beta+3)\sigma\\ &\Longleftrightarrow \rho < \frac{\sigma+\beta+3}{\sigma-\beta-1}\sigma.\end{aligned}$$

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Exercises, pp. 92–94

1.
(i) By **Definition 7.1**, p. 88, a trapping set $\mathscr{D}$ is a closed connected set in $\mathbb{R}^n$. Besides being closed, let $\mathscr{D}$ be bounded. So, since $\mathscr{D}$ is compact and ϕ_t is continuous, $\phi_t(\mathscr{D})$ is compact. Now, consider $t_0 \in \mathbb{R}$ and let T be as in **Definition 7.1**. Therefore,

$$\phi_t(\phi_{t_0}(\mathscr{D})) \subset \phi_{t_0}(\mathscr{D}) \text{ for all } t \geq T.$$

In fact, consider $z \in \phi_t(\phi_{t_0}(\mathscr{D}))$. Then, there is a point $y \in \phi_{t_0}(\mathscr{D})$ such that $z = \phi_t(y)$. Therefore, since there is a point $x \in \mathscr{D}$ such that $y = \phi_{t_0}(x)$,

$$\begin{aligned}z &= \phi_t(\phi_{t_0}(x))\\ &= \phi_{t+t_0}(x)\\ &= \phi_{t_0+t}(x)\\ &= \phi_{t_0}(\phi_t(x)) \in \phi_{t_0}(\mathscr{D})\end{aligned}$$

because, by **Definition 7.1**,

$$x \in \mathscr{D} \Longrightarrow \phi_t(x) \in \mathscr{D}.$$

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Comments, p. 109

- (9.8)
  The criterion is to minimize (9.2) with

$$f(\mathbf{x}_i;\boldsymbol{\alpha}) = \mathbf{x}_i^T\boldsymbol{\alpha},\, i = 1,\ldots,n,$$

  which can be written as

$$\begin{aligned}\sum_{i=1}^{n}\varepsilon_i^2 &= \boldsymbol{\varepsilon}^T\boldsymbol{\varepsilon} \\ &= (\mathbf{y}-\mathbf{X}\boldsymbol{\alpha})^T(\mathbf{y}-\mathbf{X}\boldsymbol{\alpha}) \\ &= \mathbf{y}^T\mathbf{y} - \mathbf{y}^T\mathbf{X}\boldsymbol{\alpha} - \boldsymbol{\alpha}^T\mathbf{X}^T\mathbf{y} + \boldsymbol{\alpha}^T\mathbf{X}^T\mathbf{X}\boldsymbol{\alpha} \\ &= \mathbf{y}^T\mathbf{y} - 2\mathbf{y}^T\mathbf{X}\boldsymbol{\alpha} + \boldsymbol{\alpha}^T\mathbf{X}^T\mathbf{X}\boldsymbol{\alpha}.\end{aligned}$$

  Therefore

$$\begin{aligned}\nabla_{\boldsymbol{\alpha}}\left(\sum_{i=1}^{n}\varepsilon_i^2\right) = \mathbf{0} &\Longrightarrow -2\mathbf{y}^T\mathbf{X} + 2\boldsymbol{\alpha}^T\mathbf{X}^T\mathbf{X} = \mathbf{0} \\ &\Longrightarrow \mathbf{X}^T\mathbf{y} = \mathbf{X}^T\mathbf{X}\boldsymbol{\alpha}.\end{aligned}$$

- (9.9)
  The invertibility of $\mathbf{X}^T\mathbf{X}$ means that $\mathbf{X}$ should have rank p.[42] This requires in particular that $n \geq p$.[43]

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Comments, p. 110, **9.3**
Suppose that $\nabla_{\boldsymbol{\alpha}} Q_2 = \mathbf{0}$. Therefore

$$\begin{aligned}\frac{\partial Q_2}{\partial \alpha_1} = 0 &\Longrightarrow -2\sum_{i=1}^{n}(y_i - \alpha_1 - \alpha_2 x_i) = 0 \\ &\Longrightarrow \sum_{i=1}^{n} y_i = n\alpha_1 + \alpha_2\sum_{i=1}^{n} x_i \\ &\Longrightarrow \overline{y} = \alpha_1 + \alpha_2\overline{x},\end{aligned}$$

which confirms (9.13), and, furthermore,

$$\begin{aligned}\frac{\partial Q_2}{\partial \alpha_2} = 0 &\Longrightarrow -2\sum_{i=1}^{n} x_i(y_i - \alpha_1 - \alpha_2 x_i) = 0 \\ &\Longrightarrow \sum_{i=1}^{n}\left(x_i y_i - x_i(\overline{y} - \alpha_2\overline{x}) - \alpha_2 x_i^2\right) = 0 \\ &\Longrightarrow \sum_{i=1}^{n}(x_i y_i - x_i\overline{y}) - \alpha_2\sum_{i=1}^{n}\left(x_i^2 - x_i\overline{x}\right) = 0 \\ &\Longrightarrow \alpha_2 = \frac{\left(\sum_{i=1}^{n} x_i y_i\right) - n\overline{x}\,\overline{y}}{\left(\sum_{i=1}^{n} x_i^2\right) - n\overline{x}^2}.\end{aligned}$$

Concerning (9.12), it is worth recalling that the correlation coefficient can be defined as

$$r_{xy} = \frac{\sum_{i=1}^{n}(x_i - \overline{x})(y_i - \overline{y})}{\sqrt{\sum_{i=1}^{n}(x_i - \overline{x})^2}\sqrt{\sum_{i=1}^{n}(y_i - \overline{y})^2}}.$$

[42] So, in that case, the nullity of $\mathbf{X} \in \mathbb{R}^{n\times p}$ is zero. Therefore, due to the fact that the kernel of $\mathbf{X}^T\mathbf{X}$ is contained in the kernel of $\mathbf{X}$, the rank of $\mathbf{X}^T\mathbf{X} \in \mathbb{R}^{p\times p}$ is also p.

[43] That is, the number of parameters is smaller than or equal to the number of observations.

Now, consider the last paragraph. Note that

$$\begin{aligned} \hat{y} - \overline{y} &= \hat{\alpha}_2 \left(x - \overline{x} \right) \\ &= r_{xy} \frac{s_y}{s_x} \left(x - \overline{x} \right) \end{aligned}$$

is (9.14) with $\hat{y}$ in place of y.
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