A Survival Guide to

ADVANCED CALCULUS

Revised 2016 World Scientific Edition

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Partial scrutiny,

Comments, Suggestions and Errata

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Comment, p. 37, **Fig. 1.4**
The coordinate axes have different units of measurement:

$$|OQ_i| \neq |OQ_j| \text{ for } i \neq j.$$

This usually happens in sciences.
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EX. 2.4, p. 41
See **Fig. 1.5**, p. 38.
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EX. 4.1, p. 55
4) Consider $A = N_1 + \alpha_1$ and $B = N_2 + \alpha_2$ where N_i is a subspace of V and $\alpha_i \in V$, $i = 1, 2$. Then $\alpha_1 + \alpha_2 = \alpha \in V$, $N_1 + N_2 = N$ is a subspace of V and $A + B = N + \alpha$.
5) Consider $A = N + \alpha$ where N is a subspace of V and $\alpha \in V$. Then $T(\alpha) \in W$, $T[N]$ is a subspace of W and $T[A] = T[N] + T(\alpha)$.
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EX. 4.9, p. 55
As a matter of fact, **Theo. 4.4** follows directly from **Theo. 4.3**. Indeed, consider $\overline{T} = \pi \circ T$ with $\pi : V \to V/N$ and T as in **Theo. 4.4**. Furthermore, denote V/N by W such that $\overline{T} \in \mathrm{Hom}(V, W)$. On the other hand, since the null space of π equals N, that is, $N(\pi) = N$, and from the assumption that $T[N] \subset N$, the null space of $\overline{T}$ includes N, that is, $N \subset N(\overline{T})$.[1] Now consider T and M as in **Theo. 4.3**. Then replace them by $\overline{T}$ and N.
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Comment, p. 59, *Proof* of **Lemma 5.5**
At the very end, 'corollary' refers to '**Lemma 5.4**'.
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EX. 5.1, p. 60
Consider $\boldsymbol{\alpha} = \prec\alpha_1, \ldots, \alpha_n\succ$ satisfies the hypotheses of the **Corollary** and $\boldsymbol{\beta} = \prec\beta_1, \ldots, \beta_n\succ$ with $\beta_j = \sum_1^n \alpha_i$ and $\beta_i = 0$ for $i \neq j$. Then $\pi(\boldsymbol{\alpha}) = \pi(\boldsymbol{\beta})$. Therefore $\boldsymbol{\alpha} = \boldsymbol{\beta}$ since π is injective.
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Comments, p. 63, **Theo. 5.5**

- 1st sentence
  Hom V is used in place of $\mathrm{Hom}(V)$;
- *Proof*, 1st paragraph
  Hom N is used in place of $\mathrm{Hom}(N)$.

This minor notational variant, with no parentheses, appears several times in the text.[2]
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Erratum, p. 67, l. -4
$\boxed{\prec x, y\succ \to x + y}$ should be $\boxed{\prec x, y\succ \mapsto x + y}$.
==
Comment, p. 68, ll. (-15)–(-11)
The bilinearity of

$$\omega : \mathrm{Hom}(W, X) \times \mathrm{Hom}(V, W) \ni \prec S, T\succ \mapsto \omega(S, T) = S \circ T \in \mathrm{Hom}(V, X)$$

[1] In fact,

$$\begin{aligned} \xi \in N &\Longrightarrow T(\xi) \in N = N(\pi) \\ &\Longrightarrow \pi(T(\xi)) = \overline{0} \\ &\Longrightarrow \overline{T}(\xi) = \overline{0} \\ &\Longrightarrow \xi \in N(\overline{T}). \end{aligned}$$

[2] See, for example, ***Block decompositions of linear maps**, pp. 65–7.

follows from **Theo. 3.3**, whereas the linearity property concerning its **Corollary** (i.e., the fact that

$$\mathrm{Hom}(V,W) \ni T \mapsto \omega_T \in \mathrm{Hom}\left(\mathrm{Hom}(W,X),\mathrm{Hom}(V,X)\right)$$

is linear) is also a direct consequence of **Theo. 6.1**.

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Erratum, p. 71, l. -8
$\boldsymbol{\alpha}_i$ should not be written in bold: $x_i\alpha_i$ is the ith summand!
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Comment, p. 74, *Proof* of **Theo. 1.3**
The skeleton of a linear mapping from $\mathbb{R}^n$ to W is defined right before **Theo. 1.2′**, p. 32. Now, by the converse part of **Theo. 1.2**, p. 31, $T \circ L_{\boldsymbol{\alpha}}$ is the linear combination mapping $\mathbf{x} \mapsto \sum x_i \left(T \circ L_{\boldsymbol{\alpha}}\right)\left(\delta^i\right)$.
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Comment, p. 79, l. 1
The existence of such a V follows from **Theo. 2.2** and its **Corollary**: U is finite-dimensional and $U \cap W$ is a subspace of U.
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Comment, p. 85, *Proof* of **Theo. 3.4**

- 1st paragraph, 1st sentence
  The bilinearity of $\omega : W^* \times \mathrm{Hom}(V, W) \to V^*$ is a consequence of **Theo. 3.3**, p. 45.

- 2nd paragraph, 2nd sentence
  If $\{\beta_i\}_1^n$ is an ordered basis for W and $T(\alpha) = \sum_1^n x_i\beta_i$ with $x_j \neq 0$, then consider l is the corresponding jth coordinate functional on W.[3]

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EX. 3.4, p. 87

- 3) Since $A \subset L(A)$, $L(A)^\circ \subset A^\circ$ by 2). The other inclusion also holds since

$$\begin{aligned} f \in A^\circ &\Longrightarrow A \subset N(f) \\ &\Longrightarrow L(A) \subset N(f) \\ &\Longrightarrow f \in L(A)^\circ. \end{aligned}$$

- 5) By definition,

$$A^\circ = \{f \in V^* \;:\; f(\alpha) = 0 \text{ for all } \alpha \in A\}.$$

  Then, since $A^\circ \subset V^*$,

$$A^{\circ\circ} = \{\alpha \in V \;:\; f(\alpha) = 0 \text{ for all } f \in A^\circ\}$$

  by (the second) definition. Therefore

$$\begin{aligned} \alpha \in A &\Longrightarrow f(\alpha) = 0 \text{ for all } f \in A^\circ \\ &\Longrightarrow \alpha \in A^{\circ\circ}. \end{aligned}$$

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Comment, p. 88, ll. (-7)–(-6)
The doubly-indexed sequence $\{t_{ij}\}$ can also be written as a single-indexed sequence $\left\{t_{k(i,j)}\right\}$ where k is a bijection between $\overline{m} \times \overline{n}$ and $\overline{mn}$.
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Comments/Erratum, p. 89

- ll. 9-12 (1st paragraph after **Theo. 4.1**)
  By **Theo. 6.2**, p. 68, if $V = \mathbb{R}^m$, then $\boldsymbol{\alpha} \mapsto L_{\boldsymbol{\alpha}}$ is an isomorphism between $\left(\mathbb{R}^m\right)^n$ and $\mathrm{Hom}\left(\mathbb{R}^n, \mathbb{R}^m\right)$.

- l. 14
  $\mathbf{x}_j$ should not be written in bold: $x_j\mathbf{t}^j$ is the jth summand!

[3] $l(T(\alpha)) = x_j$.

- *Proof* of **Theo. 4.2**
  $\varphi = L_{\boldsymbol{\alpha}}$ and $\psi = L_{\boldsymbol{\beta}}$.

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Erratum, p. 91, l. -6
For notational consistency (Section 0.7), $\boxed{\mathbf{x} \to \mathbf{z}}$ should be $\boxed{\mathbf{x} \mapsto \mathbf{z}}$.

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Erratum, p. 94, right before **Change of basis**
$\boxed{t^*}$ should be $\boxed{\mathbf{t}^*}$.

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Erratum, p. 117, l. 1
Concerning $G(\xi\eta)$, a comma is missing between the greek letters.
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Comment, p. 121, **Definition**
$p(0) = 0$. In fact, concerning **n2**, consider $x = 0$.
==
Comment, p. 122, sentence immediately followed by **Lemma 2.1**
If V is n-dimensional, in the lemma, consider that p is one of the norms on $W = \mathbb{R}^n$ and T is a coordinate isomorphism (p. 72).
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Erratum, p. 125

- **EX. 2.3**
  $\boxed{|x|}$ should be $\boxed{|\mathbf{x}|}$;

- **EX. 2.10**
  For notational consistency (Section 0.7), $\boxed{t \to t\beta + (1-t)\alpha}$ should be $\boxed{t \mapsto t\beta + (1-t)\alpha}$.

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Comment, p. 126, **Definition**
The displayed implication can be rewritten as

$$f\left[A \cap B_\delta(\alpha)\right] \subset B_\epsilon(\beta).$$

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Comment, p. 127, 3rd sentence right before **Theo. 3.1**
Suppose there is a constant c such that

$$||T(\zeta)|| \leq c$$

for all $\zeta \in V$. Then

$$||T(x\alpha)|| \leq c$$

$\forall x \in \mathbb{R}$ and $\forall \alpha \in V$. In particular,

$$||T(\alpha)|| \leq \frac{c}{|x|}$$

$\forall x \neq 0$ and $\forall \alpha \in V$. So, for all $\alpha \in V$,

$$||T(\alpha)|| = 0, \text{ that is, } T(\alpha) = 0.$$

Therefore $T = 0$!
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Erratum, p. 128, l. -10
$\boxed{F}$ should be $\boxed{T}$.
==
Comments, p. 129

- 1st sentence
  $N(T) = \{0\}$ since $||\xi|| \leq \frac{||T(\xi)||}{b}$.[4]

- 2nd sentence
  $\mathrm{Hom}(V, W)$ now also denotes the subspace of $\mathcal{B}(V, W)$ consisting of all linear maps from V to W.[5]

[4] See p. 79.
[5] See p. 123.

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EXERCISES, p. 130

$$\begin{aligned}||L_{\mathbf{a}}|| &= ||L_{\mathbf{a}}|B_1||_\infty \\ &= \text{lub}\ \{|L_{\mathbf{a}}(\mathbf{x})|\ :\ ||\mathbf{x}|| \le 1\}\,,\end{aligned}$$

$$L_{\mathbf{a}}(\mathbf{x}) = \sum_1^n a_i x_i,\ ||\mathbf{x}||_1 = \sum_1^n |x_i| \ \text{ and }\ ||\mathbf{x}||_\infty = \max\ \{|x_i|\ :\ i = 1, \ldots, n\}\,.$$

3.7 Here, $||\mathbf{x}|| = ||\mathbf{x}||_1$. So, on the one hand, since

$$|a_i| = \left|L_{\mathbf{a}}\left(\delta^i\right)\right|,\ i = 1, \ldots, n,^{6}$$

it follows that

$$||\mathbf{a}||_\infty \le ||L_{\mathbf{a}}||\,.$$

On the other hand, if $||\mathbf{x}||_1 \le 1$, then

$$\begin{aligned}|L_{\mathbf{a}}(\mathbf{x})| &\le \sum_1^n |a_i|\,|x_i| \\ &\le ||\mathbf{a}||_\infty \sum_1^n |x_i| \\ &\le ||\mathbf{a}||_\infty.\end{aligned}$$

Therefore

$$||L_{\mathbf{a}}|| \le ||\mathbf{a}||_\infty.$$

3.8 Here, $||\mathbf{x}|| = ||\mathbf{x}||_\infty$. So, on the one hand, since

$$\begin{aligned}||x||_\infty \le 1 &\Longrightarrow |x_i| \le 1 \text{ for } i = 1, \ldots, n \\ &\Longrightarrow |a_i x_i| \le |a_i| \ \text{ for } i = 1, \ldots, n \\ &\Longrightarrow \left|\sum_1^n a_i x_i\right| \le \sum_1^n |a_i|\,,\end{aligned}$$

it follows that

$$||L_{\mathbf{a}}|| \le ||\mathbf{a}||_1.$$

On the other hand, consider $\boldsymbol{\sigma} \in \mathbb{R}^n$ such that

$$\sigma_i = \begin{cases} 1 & \text{if } a_i \ge 0, \\ -1 & \text{if } a_i < 0. \end{cases}$$

Then $||\boldsymbol{\sigma}||_\infty = 1$ and

$$\begin{aligned}|L_{\mathbf{a}}(\boldsymbol{\sigma})| &= \left|\sum_1^n a_i \sigma_i\right| \\ &= \left|\sum_1^n |a_i|\right| \\ &= \sum_1^n |a_i|\,.\end{aligned}$$

Therefore

$$||\mathbf{a}||_1 \le ||L_{\mathbf{a}}||\,.$$

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Comments, p. 132, **EQUIVALENT NORMS**

It's not necessary to include the following extra underlined words:

[6] See pp. 73–4.

- 1st sentence: "... are *norm isomorphic* to one another if ...";
- **Definition**: "... are *equivalent* to one another if there exist positive constants ...".

In fact, on the one hand, we are dealing with two equivalence relations. On the other hand,

$$a = 0 \text{ or } b = 0 \Longrightarrow p = q = 0.$$

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EX. 4.10, p. 135
Consider $p_i(\alpha_i + \beta_i) = x_i$, $p_i(\alpha_i) = y_i$ and $p_i(\beta_i) = z_i$, $i = 1, \ldots, n$. So $x_i \leq y_i + z_i$, $i = 1, \ldots, n$. Therefore, provided that $|| \; ||$ is increasing,

$$\begin{aligned} |||\boldsymbol{\alpha} + \boldsymbol{\beta}||| &= ||\mathbf{x}|| \\ &\leq ||\mathbf{y} + \mathbf{z}|| \\ &\leq ||\mathbf{y}|| + ||\mathbf{z}|| = |||\boldsymbol{\alpha}||| + |||\boldsymbol{\beta}|||. \end{aligned}$$

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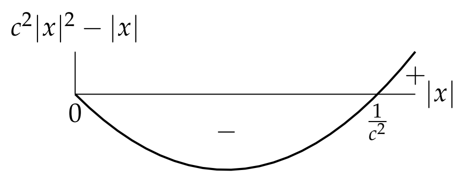
Comment, p. 137, paragraph right before **Fig. 3.7**
f is not in $\mathcal{O}$. In fact, suppose otherwise. So there are positive constants r and c such that

$$\begin{aligned} |x| < r &\Longrightarrow |x|^{1/2} \leq c|x| \\ &\Longrightarrow |x| \leq c^2|x|^2 \\ &\Longrightarrow c^2|x|^2 - |x| = |x|\left(c^2|x| - 1\right) \geq 0 \\ &\Longrightarrow c^2|x| - 1 \geq 0 \\ &\Longrightarrow |x| \geq \frac{1}{c^2}, \end{aligned}$$



which is a contradiction since r can be chosen small enough so that $r < \frac{1}{c^2}$.

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Comment, p. 138, **Theo. 5.1**, *Proof*, 1st sentence
$\mathcal{L}_\varepsilon \neq \emptyset$. In fact, consider the zero function.

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Comment, p. 139, **Lemma 5.1**, *Proof*
The last inequality can be rewritten as

$$2||\eta|| - ||\eta|| \leq 2b||\xi|| + ||\xi||.$$

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EXERCISES, p. 140

5.1 Consider two equivalent norms on V and two equivalent norms on W. Then, in particular, there are constants a and b such that

$$| \; |_V \leq a|| \; ||_V \text{ and } || \; ||_W \leq b| \; |_W.$$

Consider $f \in \mathcal{I}((V, | \; |_V), (W, | \; |_W))$. Then $f \in \mathcal{I}((V, || \; ||_V), (W, || \; ||_W))$. In fact, consider $\epsilon > 0$. So, for $\varepsilon = \frac{\epsilon}{b}$, there exists $r > 0$ such that

$$|\xi|_V < r \Longrightarrow |f(\xi)|_W < \varepsilon.$$

Now consider $\delta = \frac{r}{a}$. Therefore

$$\begin{aligned} ||\xi||_V < \delta &\Longrightarrow |\xi|_V \leq a||\xi||_V < r \\ &\Longrightarrow |f(\xi)|_W < \varepsilon \\ &\Longrightarrow ||f(\xi)||_W \leq b|f(\xi)|_W < \epsilon. \end{aligned}$$

5.5 There are constants c_i, r_i, $i = 1, 2$, such that

$$||f(\xi)|| \leq c_1||\xi|| \text{ on } B_{r_1}(0) \text{ and } ||g(\xi)|| \leq c_2||\xi|| \text{ on } B_{r_2}(0).$$

Therefore, if $r = \min\{r_1, r_2\}$ and $|\xi| < r$,

$$\begin{aligned} ||\prec f(\xi), g(\xi)\succ|| &= ||f(\xi)|| + ||g(\xi)|| \\ &\leq (c_1 + c_2)\,||\xi||. \end{aligned}$$

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Comment, p. 141, sentence right after "... $\Delta f_a - l \in \mathcal{o}$."

$$\begin{aligned} l_1 + f_1 = l_2 + f_2 \text{ such that } l_1, l_2 \text{ are linear maps and } f_1, f_2 \in \mathcal{o} &\Longrightarrow l_1 - l_2 = f_2 - f_1 \in \text{Hom}(\mathbb{R}, \mathbb{R}) \cap \mathcal{o}(\mathbb{R}, \mathbb{R}) \\ &\Longrightarrow l_1 = l_2 \text{ and } f_1 = f_2. \end{aligned}$$

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Comment, p. 142, **Definition**

As before, $\beta = \Delta F_\alpha(\xi)$ can be viewed as $\eta = F(\chi)$ under the translation

$$\prec\chi, \eta\succ \mapsto \prec\xi, \beta\succ = \prec\chi - \alpha, \eta - F(\alpha)\succ$$

that maps $\prec\alpha, F(\alpha)\succ$ to $\prec 0, 0\succ$ and translates A to the neighborhood

$$\text{dom}\,\Delta F_\alpha = \{\xi \,:\, \alpha + \xi \in A\}$$

of 0 in V.

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Comment, p. 143, **Theo. 6.1**, *Proof*

The conclusion of 2) takes into account the uniqueness of $d(F+G)_\alpha$.

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EX. 6.18, p. 146

First note that

$$\Delta F_\alpha - dF_\alpha \in \mathcal{o} \Longrightarrow dF_\alpha - \Delta F_a \in \mathcal{o}.$$

So, for each $\xi \in \text{dom}\,\Delta F_\alpha$,[7]

$$\begin{aligned} ||dF_\alpha(\xi)|| &= ||\Delta F_\alpha(\xi) + \mathcal{o}(\xi)|| \\ &\leq ||\Delta F_\alpha(\xi)|| + ||\mathcal{o}(\xi)|| \\ &\leq C||\xi|| + ||\mathcal{o}(\xi)|| \end{aligned}$$

(since F is Lipschitzian with Lipschitz constant C). Then, for each such vector $\xi \neq 0$,

$$\left|\left|dF_\alpha\left(\frac{\xi}{||\xi||}\right)\right|\right| \leq C + \frac{||\mathcal{o}(\xi)||}{||\xi||}.$$

Now, if $t \neq 0$ is small enough, then $t\xi \in \text{dom}\,\Delta F_\alpha$ and

$$\begin{aligned} \left|\left|dF_\alpha\left(\frac{\xi}{||\xi||}\right)\right|\right| &= \frac{|t|}{|t|}\left|\left|dF_\alpha\left(\frac{\xi}{||\xi||}\right)\right|\right| \\ &= \left|\left|\frac{t}{|t|}dF_\alpha\left(\frac{\xi}{||\xi||}\right)\right|\right| \\ &= \left|\left|dF_\alpha\left(\frac{t\xi}{||t\xi||}\right)\right|\right| \\ &\leq C + \frac{||\mathcal{o}(t\xi)||}{||t\xi||} \\ &\leq C \end{aligned}$$

[7] Cf. **Def.**, p. 142.

as $t \to 0$ for each unit vector $\xi/||\xi||$. Therefore, since

$$||T|| = \sup\{||T(\beta)|| \,:\, ||\beta|| = 1\}$$

for each bounded linear transformation T,[8]

$$||dF_\alpha|| \leq C$$

holds.

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Comment, p. 147, paragraph that defines $D_\xi F(\alpha)$

A more precise 2nd sentence is "The restriction of F to the intersection of this line and A is ...".

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Comments, pp. 148–9, **Theo. 7.3**, *Proof*

- 1st sentence
  $A \neq \emptyset$ since $a \in A$;
- 2nd sentence
  In fact, suppose otherwise. So there is a sequence $\{x_n\} \subset (a, b]$ such that $x_n \to a$ and

  $$||f(x_n) - f(a)|| > (m + \epsilon)(x_n - a) + \epsilon$$

  for each index n. Therefore the continuity of f at a gives us $0 > \epsilon$, which is a contradiction.

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EX. 7.10, p. 151

If $|\mathbf{y}| = 1$, then

$$\begin{aligned} \left|D_{\mathbf{y}} f(\mathbf{a})\right| &= |df_{\mathbf{a}}(\mathbf{y})| \\ &= |(\mathbf{L}, \mathbf{y})| \\ &\leq |\mathbf{L}| \end{aligned}$$

by the Schwarz inequality.[9] In particular, if

$$\mathbf{y} = \frac{\mathbf{L}}{|\mathbf{L}|},$$

$$\begin{aligned} D_{\mathbf{y}} f(\mathbf{a}) &= \left(\mathbf{L}, \frac{\mathbf{L}}{|\mathbf{L}|}\right) \\ &= |\mathbf{L}|. \end{aligned}$$

(It is worth noting that $D_{\mathbf{y}} f(\mathbf{a})$ is minimum when $\mathbf{y}$ points in the opposite direction of the gradient of f at $\mathbf{a}$.)

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Comment, p. 153

- *general chain rule*
  By **Theo. 6.2**, p. 143, if $\zeta \in A$, then

$$\begin{aligned} d(F \circ G)_\gamma(\zeta) &= \left(dF_{G(\gamma)} \circ dG_\gamma\right)(\zeta) \\ &= dF_{G(\gamma)}\left(dG_\gamma(\zeta)\right) \\ &= dF_{G(\gamma)}\left(dg^1_\gamma(\zeta), \ldots, dg^n_\gamma(\zeta)\right) \\ &= \sum_1^n dF^i_{G(\gamma)}\left(dg^i_\gamma(\zeta)\right) \\ &= \sum_1^n \left(dF^i_{G(\gamma)} \circ dg^i_\gamma\right)(\zeta) \\ &= \left(\sum_1^n dF^i_{G(\gamma)} \circ dg^i_\gamma\right)(\zeta). \end{aligned}$$

[8] Cf. p. 128

[9] Cf. p. 125.

- **Lemma 8.2**
  Consider that
  $$\Delta F_{\boldsymbol{\alpha}} = dF_{\boldsymbol{\alpha}} + o.$$
  Therefore
  $$\begin{aligned}\Delta F_{\boldsymbol{\alpha}} \circ \theta_i &= (dF_{\boldsymbol{\alpha}} + o) \circ \theta_i \\ &= dF_{\boldsymbol{\alpha}} \circ \theta_i + o \circ \theta_i \\ &= dF_{\boldsymbol{\alpha}} \circ \theta_i + o,\end{aligned}$$
  which implies that
  $$dF^i_{\boldsymbol{\alpha}} = dF_{\boldsymbol{\alpha}} \circ \theta_i$$
  by uniqueness of the differential.[10]

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Erratum, p. 154, **Theo. 8.3**, l. 1
$\boxed{F \to W}$ should be $\boxed{F : A \to W}$.

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Comment, p. 155, last sentence of the paragraph that follows the first $\square$[11]
The Lipschitz condition for a linear map $T : V \to W$ is equivalent to
$$\left|\left| T\left(\frac{\zeta}{||\zeta||}\right)\right|\right| \leq c$$
for all $\zeta \in V$. Such condition for ω must be equivalent to
$$\left|\left| \omega\left(\frac{\xi}{||\xi||}, \frac{\eta}{||\eta||}\right)\right|\right| \leq b$$
for all $\xi \in X$ e $\eta \in Y$.

==

EX. 8.10, p. 156
Since $F = \omega \circ \prec g, h \succ$, by the *general chain rule*, p. 153, it follows that
$$dF_\beta(\zeta) = d\omega^1_{\prec g(\beta), h(\beta) \succ}\left(dg_\beta(\zeta)\right) + d\omega^2_{\prec g(\beta), h(\beta) \succ}\left(dh_\beta(\zeta)\right).$$
Now consider the *Proof* of **Lemma 8.3**, concerning the partial differentials of ω.

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Comment, p. 157, 1st *Proof*, 2nd sentence
Check the paragraph where $D_\xi F(\alpha)$ is defined on p. 147 and the (real) definition of a partial differential, p. 153, l. -10, along with **Theo. 7.1**.

==

Comment/Errata, p. 158

- 2nd paragraph and **Theo. 9.4**
  The domain of F is an open subset of $\mathbb{R}^n$, not an interval of real numbers if $n > 1$. So, rather than invoking **Lemma 8.1**, **Theo. 8.1** should be used as the justification for the m-tuple $\frac{\partial F}{\partial x_j}(\mathbf{a})$. Furthermore, for notational consistency, f^i should be used, not f_i, $i = 1, \ldots, m$, as component functions of F. (In fact, check the *Jacobian* of F, p. 159.) Therefore, by Theorems **9.1**, **7.2** and **8.1**,
  $$\begin{aligned}\frac{\partial F}{\partial x_j}(\mathbf{a}) &= D_{\delta^j} F(\mathbf{a}) \\ &= dF_{\mathbf{a}}\left(\delta^j\right) \\ &= \prec df^1_{\mathbf{a}}\left(\delta^j\right), \ldots, df^m_{\mathbf{a}}\left(\delta^j\right) \succ \\ &= \prec D_{\delta^j} f^1(\mathbf{a}), \ldots, D_{\delta^j} f^m(\mathbf{a}) \succ \\ &= \prec \frac{\partial f^1}{\partial x_j}(\mathbf{a}), \ldots, \frac{\partial f^m}{\partial x_j}(\mathbf{a}) \succ.\end{aligned}$$

[10] Cf. p. 142.
[11] As a matter of fact, the book uses an elongated rectangle as a Q.E.D. symbol!

- l. -6
  "... Section 8 ..." should be "... Section 7 ...".

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EXERCISES/Errata, p. 160

9.7-8 Since the determinant of a block diagonal matrix is the product of the determinants of the individual blocks, it follows that

$$\begin{aligned}\frac{\partial(x,y,z)}{\partial(r,\theta,z)} &= \begin{vmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} \\ &= \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} \cdot 1 \\ &= r \\ &= \frac{\partial(x,y)}{\partial(r,\theta)}.\end{aligned}$$

9.9 First, note that $\boxed{z = r\cos\theta}$ should be $\boxed{z = r\cos\varphi}$. So

$$\begin{aligned}\frac{\partial(x,y,z)}{\partial(r,\varphi,\theta)} &= \begin{vmatrix} \sin\varphi\cos\theta & r\cos\varphi\cos\theta & -r\sin\varphi\sin\theta \\ \sin\varphi\sin\theta & r\cos\varphi\sin\theta & r\sin\varphi\cos\theta \\ \cos\varphi & -r\sin\varphi & 0 \end{vmatrix} \\ &= r^2\sin\varphi.\end{aligned}$$

9.11 First, note that the LHS should be

$$\left(\frac{\partial w}{\partial r}\right)^2 + \left(\frac{1}{r}\frac{\partial w}{\partial \theta}\right)^2.$$

Then add

$$\begin{aligned}\left(\frac{\partial w}{\partial r}\right)^2 &= \left(\frac{\partial w}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial r}\right)^2 \\ &= \left(w_x\cos\theta + w_y\sin\theta\right)^2 \\ &= w_x^2\cos^2\theta + 2w_xw_y\cos\theta\sin\theta + w_y^2\sin^2\theta\end{aligned}$$

to

$$\begin{aligned}\left(\frac{1}{r}\frac{\partial w}{\partial \theta}\right)^2 &= \frac{1}{r^2}\left(\frac{\partial w}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial \theta}\right)^2 \\ &= \frac{1}{r^2}\left(w_xr(-\sin\theta) + w_yr\cos\theta\right)^2 \\ &= w_x^2\sin^2\theta - 2w_xw_y\sin\theta\cos\theta + w_y^2\cos^2\theta\end{aligned}$$

to get the RHS.

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Comment, p. 162

- l. 19 (or -19)

$$\begin{aligned}F(\xi) - G(\xi) &= F(\xi) - dF_\alpha(\zeta) - F(\alpha) \\ &= F(\xi) - \Delta F_\alpha(\zeta) + o(\zeta) - F(\alpha) \\ &= F(\xi) - F(\alpha+\zeta) + F(\alpha) + o(\zeta) - F(\alpha) \\ &= F(\xi) - F(\alpha+\xi-\alpha) + o(\xi-\alpha) \\ &= o(\xi-\alpha).\end{aligned}$$

- **Theo. 10.2**, *Proof*, 2nd sentence
  By **Lemma 8.1**, the definition of $D_\xi F(\alpha)$ on p. 147 and **Theo. 7.2**, if $\gamma = \prec\gamma_1, \gamma_2\succ$, then

$$\begin{aligned}\gamma'(0) &= \prec\gamma_1'(0), \gamma_2'(0)\succ \\ &= \prec\xi, D_\xi F(\alpha)\succ \\ &= \prec\xi, dF_\alpha(\xi)\succ.\end{aligned}$$

==
Erratum, p. 165, l. -10
'F' should be 'F'.
==
Comment, p. 167, **Theo. 11.3**, *Proof*, 2nd sentence

$$\begin{aligned}\Delta G_{\prec\alpha,\beta\succ}(\xi,\eta) &= G(\alpha+\xi,\beta+\eta) - G(\alpha,\beta) \\ &= \alpha+\xi - H(\beta+\eta) - \alpha + H(\beta) \\ &= \xi - \Delta H_\beta(\eta) \\ &= \xi - dH_\beta(\eta) - o(\eta)\end{aligned}$$

implies that

$$\begin{aligned}\left(\Delta G_{\prec\alpha,\beta\succ} \circ \theta_2\right)(\eta) &= \Delta G_{\prec\alpha,\beta\succ}(0,\eta) \\ &= -dH_\beta(\eta) - o(\eta).\end{aligned}$$

Now consider l. -10 on p. 153.
==
Erratum, p. 168, l. 4
Delete 'for $i = 1, \ldots, n$'.
==
EX. 11.1, p. 169
For the Jacobian matrix, see **EX. 11.24**.
==
EX. 11.24, p. 171
For a), consider $G(\xi,\eta) = \xi - H(\eta)$. Therefore

$$\begin{aligned}dF_\alpha(\xi) &= \left(-\left(dG^2_{\prec\alpha,\beta\succ}\right)^{-1} \circ dG^1_{\prec\alpha,\beta\succ}\right)(\xi) \\ &= -\left(-dH_\beta\right)^{-1}\left(dG^1_{\prec\alpha,\beta\succ}(\xi)\right) \\ &= \left(dH_\beta\right)^{-1}(\xi).\end{aligned}$$

For b), consider **Theo. 6.2**.[12]
==
Comment, p. 173, l. 3
See **Cor.**, p. 79.
==
Comment, p. 174, **Theo. 12.2**, *Proof*
See the *Proof* of **Theo. 12.1**.
==

[12]**Theo. 6.1.**5) has been applied to the identity map.

==
==
4
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==
Comment, p. 199, 2nd paragraph after **Cor.**, last sentence
Since $\overline{A}$ is the smallest closed set including A,

$$A \text{ is closed iff } \overline{A} = A.$$

On the other hand,

$$0 \in \overline{f\left[\mathbb{Z}^{+}\right]} - f\left[\mathbb{Z}^{+}\right].$$

==
Comment, p. 200, *Proof*, last sentence
$N = kN$ for each scalar k and $-\eta \in N$. Therefore

$$\begin{aligned}\rho\left(\{\alpha\}, N\right) &= \rho\left(\frac{1}{||\beta-\eta||}\{\beta-\eta\}, \frac{1}{||\beta-\eta||}N\right)\\ &= \frac{1}{||\beta-\eta||}\rho\left(\{\beta\}+(-\eta), N\right)\\ &= \frac{1}{||\beta-\eta||}\rho\left(\{\beta\}, N\right).\end{aligned}$$

==
Comment, p. 206

- **Def.**
  Compactness and sequential compactness are equivalent for metric spaces.[13]
- **Lemma 4.2**, *Proof*
  $\rho\left(x_{n(i)}, b\right) \to \rho(a,b)$ since $\rho\left(x, \{b\}\right)$ is a continuous function of x.[14] For $n(i) \geq i$, use **EX. 4.1**, p. 209.

==
Comment, p. 209, **Cor.**, *Proof*, 3rd sentence
There exists N such that

$$\rho\left(\xi_n, \alpha\right) < 1$$

for each $n > N$. On the other hand, if 0 is the zero vector, then

$$\rho\left(\xi_n, 0\right) \leq \rho\left(\xi_n, \alpha\right) + \rho(\alpha, 0)$$

for each positive integer n. So, for

$$r = \max\left\{1+\rho(\alpha,0), \rho\left(\xi_1,\alpha\right)+\rho(\alpha,0), \ldots, \rho\left(\xi_N,\alpha\right)+\rho(\alpha,0)\right\},$$

$$\rho\left(\xi_n, 0\right) \leq r$$

for each index n. Therefore each term ξ_n lies in some closed ball about 0,[15] which is a closed set that contains the closure of $\{\xi_n\}$.
==
EXERCISES, p. 209

4.2 Consider $\epsilon > 0$. Then there exists an N such that

$$n > N \Longrightarrow \rho\left(x_n, a\right) < \epsilon.$$

Now,

$$\ldots > n(N+2) > n(N+1) > n(N) > N$$

by **EX. 4.1**. Therefore, for each positive integer i,

$$n(N+i) > n(N) \Longrightarrow \rho\left(x_{n(N+i)}, a\right) < \epsilon.$$

[13]See p. 214.
[14]See **EX. 1.9**, p. 200.
[15]See the definition of boundedness on p. 206, last paragraph.

4.9 Consider $\{y_n\} \subset f[A]$ and $\{x_n\} \subset A$ such that $y_n = f(x_n)$. Then there exists a subsequence $\left\{x_{n(i)}\right\}_i$ that converges to a point $a \in A$. Therefore, due to the fact that f is continuous, $\left\{y_{n(i)}\right\}_i$ converges to $f(a)$.

==

Comment, p. 212

- l. -15
  The r-denseness of S signifies that $\{B_r(a) : a \in A\}$ is an open covering of S.[16]

- l. -4
  The total boundedness of B signifies that, for each positive r, there is a finite set of open balls of radius r that covers B.

==

Erratum, p. 213, **Lemma 5.1**, *Proof*, last sentence
'ξ_1' should be 'ξ_n'.

==

EXERCISES, pp. 214–215

5.1 Consider an arbitrary positive integer n. Then

$$\begin{aligned} ||f_n||_\infty &= \sup\{x^n : x \in (0,1)\} \\ &= 1. \end{aligned}$$

(As a matter of fact,

$$\lim_{x \to 1} x^n = 1.)$$

Therefore

$$||f_n||_\infty \not\to 0.$$

5.2 Suppose f is uniformly continuous on $(0,1)$ and consider $\epsilon = 1$. So there is a positive δ such that (for all $x, y \in (0,1)$)

$$|x - y| < \delta \Longrightarrow \left|\frac{1}{x} - \frac{1}{y}\right| < 1.$$

Now pick $x \in (0,1)$ with $x < \delta$. Then set $y = \frac{x}{2}$. Therefore, on the one hand,

$$\begin{aligned} |x - y| &= \frac{x}{2} \\ &< \frac{\delta}{2} \\ &< \delta. \end{aligned}$$

On the other hand,

$$\begin{aligned} \left|\frac{1}{x} - \frac{1}{y}\right| &= \left|\frac{1}{x} - \frac{2}{x}\right| \\ &= \frac{1}{x} \\ &> 1! \end{aligned}$$

5.6 Let C be compact. So, by **EX. 1.9**, p. 200, and the **Cor.** on p. 207, there exists a point $p \in C$ such that

$$\rho(p, A) = \rho(C, A).$$

Since A is closed, $p \notin \overline{A}$.[17] Now apply **EX. 1.6**, p. 200.

[16] See **Lemma 5.3**, p. 213.
[17] Otherwise, $p \in A$, which contradicts the fact that $A \cap C = \emptyset$.

==
Erratum, p. 217, **Lemma 7.3**, *Proof*
'F' should be 'T'.
==
Comment, p. 218, ll. 5–6
If $T \in \text{Hom}(V, W)$, then $T : V \to W$ is a bounded linear map.[18] Clearly, if T is invertible, then $T^{-1} : W \to V$ is linear. However, in order to prove that $T^{-1} \in \text{Hom}(W, V)$, it is necessary to check that T^{-1} is bounded. For example, T^{-1} is necessarily bounded if V and W are finite-dimensional.[19] What if one of them is infinite-dimensional?
==
Comment, p. 219, **Fig. 4.3**
In the *Proof*, f is denoted by g.[20]
==
EXERCISES, p. 222

7.7 If i is a positive integer, let d_i be the ith digit of π after the decimal point. Consider the sequence whose nth term is given by

$$x_n = 3 + \sum_{i=1}^{n} d_i 10^{-i}.$$

Clearly $x_n \to \pi$ as $n \to \infty$.[21]

7.8 Given $\epsilon > 0$, produce the δ that is guaranteed by uniform continuity. Then pick N so that $\rho(x_m, x_n) < \delta$ for $m, n > N$. Therefore $\rho(F(x_m), F(x_n)) < \epsilon$ for $m, n > N$.

7.10 (All notations here follow those defined in the *Proof* of **Theo. 7.5** taking into account the boundedness meaning related to **Theo. 7.6**.) Since each f_n is linear and $g(a) = \lim f_n(a)$ for each $a \in V$, g is also linear. Therefore $f_m - g$ is linear and, since

$$\left|\left|(f_m - g)\left(\frac{a}{||a||}\right)\right|\right| \le ||f_m - g||_\infty \text{ for each nonzero } a \in V,$$

$f_m - g$ is bounded, that is, $f_m - g \in \text{Hom}(V, W)$.

7.11 Let A be a metric space and $B \subset A$.

- Suppose A is complete and B is closed. Consider a Cauchy sequence in B converging to a limit a in A. Therefore $a \in B$ due to the closedness of B.[22]
- Suppose B is complete. Consider a convergent sequence lying in B. Then this sequence is Cauchy in B.[23] Therefore the convergence to a limit in B is guaranteed by the completeness of B.

==
Comment, p. 223, §**8**, 2nd paragraph, 3rd sentence
An 'algebra' is a vector space equipped with a bilinear product.
==
Comment/Erratum, p. 224

- **Theo. 8.1**, *Proof*, l. 4, 2nd equal sign
  Use that $e = x^0$.
- **Cor.**, *Proof*, l. -2
  'S' should be 'S'.

==
Comments, p. 225

[18] See p. 129.
[19] See **Theo. 4.2**, p. 132.
[20] See also the *Remark* following the *Proof*.
[21] $\{x_n\}$ is a Cauchy sequence in $\mathbb{R}$ which consists only of rational terms!
[22] See p. 203, right after **Theo. 3.1**.
[23] See **Lemma 7.1**, p. 216.

- 1st paragraph after the *Proof* of **Theo. 8.3**
  - The first e, which is the identity of A, has nothing to do with the second one, which is related to the exponential function.
  - By **Theo. 8.3**, $\sum x^n/n!$ converges for x in $B_1(0)$, and if $0 < s < 1$, then the series converges uniformly on $B_s(0)$. In fact,

$$\left\{\left|\left|\frac{1}{n!}e\right|\right|1^n\right\} = \left\{\frac{1}{n!}||e||\right\}$$
$$= \left\{\frac{1}{n!}\right\}$$

    is bounded, where e denotes the identity of A.

- **Theo. 8.4**
  - F^n is not the n-th iterate of F. This n is just a superscript, not a power!
  - Concerning the 2nd sentence of the *Proof*, take $\alpha = \beta$ and an arbitrary index $n \geq N$. So, by applying the **Cor.** of the mean-value theorem, p. 149, twice,

$$\left|\left|\Delta F^n_\beta(\xi) - dF^N_\beta(\xi)\right|\right| \leq \epsilon||\xi|| \quad \text{and} \quad \left|\left|\Delta F^N_\beta(\xi) - dF^n_\beta(\xi)\right|\right| \leq \epsilon||\xi||$$

    for all ξ such that $\beta + \xi \in B$. Then

$$\left|\left|\left(\Delta F^n_\beta(\xi) - dF^n_\beta(\xi)\right) - \left(dF^N_\beta(\xi) - \Delta F^N_\beta(\xi)\right)\right|\right| \leq 2\epsilon||\xi||$$

    for all such ξ, which is not exactly the inequality of the 3rd sentence of the *Proof*.[24] However, letting $n \to \infty$, we may consider

$$\left|\left|(\Delta F_\beta(\xi) - T(\xi)) - \left(dF^N_\beta(\xi) - \Delta F^N_\beta(\xi)\right)\right|\right| \leq 2\epsilon||\xi||$$

    in place of the inequality of the 4th sentence of the *Proof* and the remaining sentences remain unchanged since

$$\left|\left|dF^N_\beta(\xi) - \Delta F^N_\beta(\xi)\right|\right| = \left|\left|\Delta F^N_\beta(\xi) - dF^N_\beta(\xi)\right|\right|.$$

==

EXERCISES, p. 227

8.4 Concerning **Lemma 8.3**, p. 155, consider $X = Y = W = A$, $\omega = p$, $\alpha = a$, $\beta = b$, $\xi = x$ and $\eta = y$. The boundedness inequality here is $||xy|| \leq ||x||\,||y||$.[25]

8.14 First, from **Lemma 8.2**, p. 226, it follows that

$$F^n(x) = \sum_{i=0}^{n} a_i x^i \Longrightarrow dF^n_y(x) = \left(\sum_{i=1}^{n} i a_i y^{i-1}\right) \cdot x.$$

Second, from **Theo. 8.3**, p. 225, it follows that

$\{F^n\}$ converges pointwise to F on $B_r(0)$.

Third, from **Lemma 8.3**, p. 226, it follows that

$\left\{dF^n_y\right\}$ converges uniformly over y in any ball smaller than $B_r(0)$.

Now apply **Theo. 8.4**.

[24] The signs of $\Delta F^N_\beta(\xi)$ and $dF^N_\beta(\xi)$ are changed!

[25] See p. 155, right before **Lemma 8.3**.

==
Comment, p. 229, **Theo. 9.1**, *Proof*
$\sum_{n}^{m-1} C^i\delta < C^n\delta/(1-C)$ follows from

$$\begin{aligned}\left(C^n + \cdots + C^{m-1}\right)(1-C) &= C^n - C^m \\ &< C^n.\end{aligned}$$

==
Comment, p. 231, $*$, 1st paragraph, last sentence
The geometric convergence follows from the *Proof* of **Theo. 9.1**, l. -4.
==
EXERCISES, pp. 234–236

9.1 B is complete by **Theo. 7.9**. Furthermore,

$$rF : B \to rB \subset B$$

is a contraction since

$$\begin{aligned}||rF(\xi) - rF(\eta)|| &= ||r(F(\xi) - F(\eta))|| \\ &\le r||\xi - \eta||\end{aligned}$$

for all $\xi, \eta \in B$. So rF has a unique fixed point by **Theo. 9.1**. Hence there is a unique point $p_r \in B$ such that $rF(p_r) = p_r$. Now, one way to prove $F(p) = p$ for some point $p \in B$ is to show that

$$\inf\{||F(\xi) - \xi|| : \xi \in B\} = 0$$

since the infimum must be attained by compactness. In fact,

$$\begin{aligned}||F(p_r) - p_r|| &= ||(1-r)F(p_r)|| \\ &\le (1-r)\,||F(p_r)|| \\ &\le (1-r)\sup\{||F(\xi)|| : \xi \in B\}\end{aligned}$$

where the supremum is attained by compactness.

9.2 Let F be an identity map.

9.8 Comment

There is another discrepancy between the two statements. Concerning **Theo. 9.3**, G is continuous and has a continuous second partial differential. Also, F is continuous. Concerning **Theo. 11.2** in Chapter 3, both G and F are continuously differentiable. So if G in **Theo. 9.3** is (continuously) differentiable,[26] so is K in the *Proof* of that theorem. Therefore F in **Theo. 9.3** is (continuously) differentiable by **Theo. 9.4** and its **Cor.**.

==
Comment, p. 236, **Theo. 10.1**, *Proof*, 5th sentence
Consider $\alpha_n = \alpha \in U$ for each index n. This assumption clearly gives us a sequence $\{\alpha_n\} \subset U$ so that $\alpha_n \to \alpha$. Therefore

$$\begin{aligned}T(\alpha) &= \lim_{n\to\infty} T(\alpha_n) \\ &= \beta\end{aligned}$$

by the 3rd sentence of the *Proof*.
==
Comment, p. 238, **Theo. 10.2**
For the well-definiteness of $J(f)$, see also **EX. 10.9**, p. 240.
==

[26]Notice that the above-mentioned hypothesis of **Theo. 9.3** related to G follows from this supposition!

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EXERCISES, p. 252

1.9 First note that $V \ni \eta \mapsto (\beta, \eta) \in \mathbb{R}$ is continuous. In fact, for an arbitrary $\epsilon > 0$, consider

$$\delta < \frac{\epsilon}{||\beta||},$$

where the norm arises from the scalar product. Hence $||\xi - \eta|| < \delta$ implies that

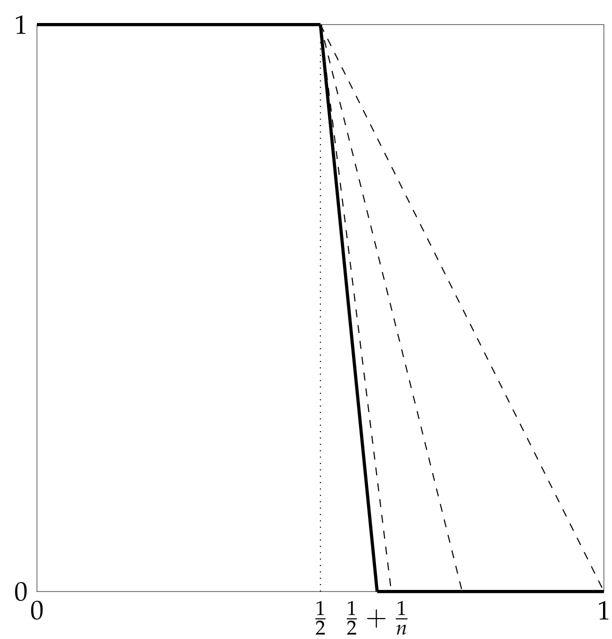
$$\begin{aligned} |(\beta,\xi) - (\beta,\eta)| &= |(\beta, \xi - \eta)| \text{ (by linearity)} \\ &< ||\beta||\,||\xi - \eta|| \text{ (by the Schwarz inequality)} \\ &< \epsilon. \end{aligned}$$

Now consider $\xi \in \overline{L(A)}$. So there is a sequence $\{\xi_n\}$ in $L(A)$ such that $\xi_n \to \xi$. Furthermore, since $\beta \in L(A)^{\perp}$, $(\beta, \xi_n) = 0$ for each index n. Therefore

$$\begin{aligned} (\beta, \xi) &= \left(\beta, \lim_{n\to\infty} \xi_n\right) \\ &= \lim_{n\to\infty} (\beta, \xi_n) \ \text{(by continuity)} \\ &= 0. \end{aligned}$$

1.10 Consider an arbitrary positive integer $n \geq 2$. Define

$$f_n(x) := \begin{cases} 1 & \text{if } x \in \left[0, \frac{1}{2}\right]; \\ -nx + \frac{n+2}{2} & \text{if } x \in \left[\frac{1}{2}, \frac{1}{2} + \frac{1}{n}\right]; \\ 0 & \text{if } x \in \left[\frac{1}{2} + \frac{1}{n}, 1\right]. \end{cases}$$



Then

$$\left(f_n - k\right)(x) := \begin{cases} 0 & \text{if } x \in \left[0, \frac{1}{2}\right] \cup \left[\frac{1}{2} + \frac{1}{n}, 1\right]; \\ -nx + \frac{n+2}{2} & \text{if } x \in \left[\frac{1}{2}, \frac{1}{2} + \frac{1}{n}\right]. \end{cases}$$

Therefore

$$\begin{aligned}
||f_n - k||_2 &= \sqrt{\int_{\frac{1}{2}}^{\frac{1}{2}+\frac{1}{n}} \left(n^2x^2 - n(n+2)x + \left(\frac{n+2}{2}\right)^2 \right) dx} \\
&= \sqrt{\left. \frac{n^2}{3}x^3 - \frac{n}{2}(n+2)x^2 + \left(\frac{n+2}{2}\right)^2 x \right|_{\frac{1}{2}}^{\frac{n+2}{2n}}} \\
&= \sqrt{\frac{(n+2)^3}{24n} - \frac{(n+2)^3}{8n} + \frac{(n+2)^3}{8n} - \frac{n^2}{24} + \frac{n(n+2)}{8} - \frac{(n+2)^2}{8}} \\
&= \sqrt{\frac{1}{3n}}.
\end{aligned}$$

So $||f_n - f||_2 \to 0$ for some $f \in \mathcal{C}([0,1])$ does not hold. Otherwise, by the uniqueness of limits, $f = k$ in $\overline{\mathcal{E}}$. This is absurd since k is not a continuous function.

Comment, p. 253, **Lemma 2.2**, *Proof*

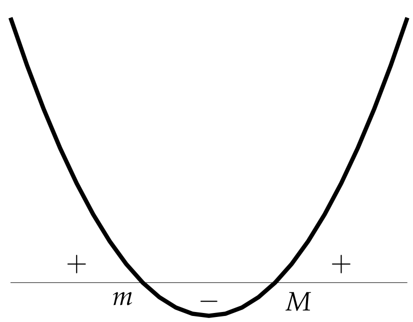
$$\frac{1}{2}(\mu_n + \mu_m) \in M \Longrightarrow \left|\left| \alpha - \frac{1}{2}(\mu_n + \mu_m) \right|\right| \le \rho(\alpha, M).$$

Comment, p. 254, **Theo. 2.3**, *Proof*

$$\begin{aligned}
\left|\left| \xi - \sum_1^m x_i\varphi_i \right|\right| &\le \left|\left| \xi - \sum_1^n x_i\varphi_i \right|\right| \quad \left(\sum_1^n x_i\varphi_i \text{ is in the span of } \{\varphi_i\}_1^m\right) \\
&\le \left|\left| \xi - \sum_1^n y_i\varphi_i \right|\right| \quad \left(\sum_1^n y_i\varphi_i \text{ is in the span of } \{\varphi_i\}_1^n\right) \\
&\le \epsilon.
\end{aligned}$$

EXERCISES, p. 256

2.1 Consider $a = ||\xi||^2$, $b = 2(\alpha - \mu, \xi)$ and $f(t) = at^2 + bt$.[27] Let m and M be the minimum and maximum of $\{0, -b/a\}$, respectively. Now, if $b \neq 0$, then $f(t) < 0$ for each $t \in (m, M)$.



2.7 (and **2.4**)

First note that if $f(t)$ is an even (respectively odd) function, then

$$\int_{-L}^{L} f(t)dt = 2\int_0^L f(t)dt$$

$$\text{(respectively } \int_{-L}^{L} f(t)dt = 0).$$

[27] $a > 0$ since $\xi \neq 0$.

Now, for each integer n, $\cos\frac{\pi nt}{L}$ and $\sin\frac{\pi nt}{L}$ are $2L$-periodic elements in the pre-Hilbert space $\mathcal{C}[-L,L]$ with respect to the scalar product

$$(f,g) = \int_{-L}^{L} f(t)\overline{g(t)}dt.$$

Therefore:

- $\{\sin\frac{\pi nt}{L}\}_1^\infty$ is orthogonal on $[-L,L]$ (and on $[0,L]$).
  In fact, since the product of two odd functions is an even function and

$$\sin A \sin B = \frac{1}{2}[\cos(A-B) - \cos(A+B)],$$

$$\begin{aligned}
\left(\sin\frac{\pi mt}{L}, \sin\frac{\pi nt}{L}\right) &= \int_{-L}^{L} \sin\frac{\pi mt}{L}\ \sin\frac{\pi nt}{L}dt \\
&= 2\int_0^L \sin\frac{\pi mt}{L}\ \sin\frac{\pi nt}{L}dt \\
&= \int_0^L \cos\frac{\pi(m-n)t}{L}dt - \int_0^L \cos\frac{\pi(m+n)t}{L}dt \\
&= \begin{cases} \frac{L}{\pi(m-n)}\int_0^{\pi(m-n)}\cos u du - \frac{L}{\pi(m+n)}\int_0^{\pi(m+n)}\cos v dv & \text{if } m \neq n, \\ \int_0^L \cos 0 dt - \int_0^L \cos\frac{2\pi mt}{L}dt & \text{if } m = n \end{cases} \\
&= \begin{cases} \left[\frac{L}{\pi(m-n)}\sin u\right]_0^{\pi(m-n)} - \left[\frac{L}{\pi(m+n)}\sin v\right]_0^{\pi(m+n)} & \text{if } m \neq n, \\ L - \left[\frac{L}{2\pi m}\sin\frac{2\pi mt}{L}\right]_0^L & \text{if } m = n \end{cases} \\
&= \begin{cases} 0 & \text{if } m \neq n, \\ L & \text{if } m = n. \end{cases}
\end{aligned}$$

- $\{\sin\frac{\pi nt}{L}\}_1^\infty \cup \{\cos\frac{\pi nt}{L}\}_0^\infty$ is orthogonal on $[-L,L]$.
  In fact, since

$$\sin A \cos B = \frac{1}{2}[\sin(A-B) + \sin(A+B)]$$

  and sine is an odd function,

$$\begin{aligned}
\left(\sin\frac{\pi mt}{L}, \cos\frac{\pi nt}{L}\right) &= \int_{-L}^{L} \sin\frac{\pi mt}{L}\ \cos\frac{\pi nt}{L}dt \\
&= \frac{1}{2}\left(\int_{-L}^{L}\sin\frac{\pi(m-n)t}{L}dt + \int_{-L}^{L}\sin\frac{\pi(m+n)t}{L}dt\right) \\
&= \begin{cases} \frac{1}{2}\left(\frac{L}{\pi(m-n)}\int_{-\pi(m-n)}^{\pi(m-n)}\sin u du + \frac{L}{\pi(m+n)}\int_{-\pi(m+n)}^{\pi(m+n)}\sin v dv\right) & \text{if } m \neq n, \\ \frac{1}{2}\left(\int_{-L}^{L}\sin 0 dt + \int_{-L}^{L}\sin\frac{2\pi mt}{L}dt\right) & \text{if } m = n \end{cases} \\
&= \frac{1}{2}(0+0) \\
&= 0.
\end{aligned}$$

- $\{\cos\frac{\pi nt}{L}\}_0^\infty$ is orthogonal on $[-L,L]$.
  The proof is similar to the previous ones. Now use

$$\cos A \cos B = \frac{1}{2}[\cos(A-B) + \cos(A+B)].$$

==

Comment, p. 257, paragraph preceding **Lemma 3.1**
See p. 85, paragraph preceding **Theo. 3.4**.

==

Erratum/Comments, p. 258

- l. 6
  '$|(T\xi,\eta)| = [\xi,\eta]$' should be '$|(T\xi,\eta)| = |[\xi,\eta]|$'.

- **Theo. 3.1**, *Proof*

  - 2nd and 3rd sentences
    The continuity follows from the composition

    $$\xi \mapsto \prec T\xi,\xi \succ \mapsto (T\xi,\xi)$$

    since T, I and the scalar product are continuous.[28]
    $m \leq ||T||$ since

    $$\begin{aligned}(T\xi,\xi) &\leq |(T\xi,\xi)| \\ &\leq ||T\xi||\,||\xi|| \quad \text{(by Schwarz)} \\ &\leq ||T||\end{aligned}$$

    if $||\xi|| = 1$.[29]
  - 5th sentence
    The self-adjointness follows from

    $$\begin{aligned}((mI-T)\xi,\eta) &= (m\xi,\eta) - (T\xi,\eta) \\ &= (\xi,m\eta) - (\xi,T\eta) \\ &= (\xi,(mI-T)\eta)\end{aligned}$$

    and the nonnegativeness follows from

    $$\begin{aligned}((mI-T)\xi,\xi) &= ||\xi||^2\left((mI-T)\frac{\xi}{||\xi||},\frac{\xi}{||\xi||}\right) \\ &= ||\xi||^2\left[m\left(\frac{\xi}{||\xi||},\frac{\xi}{||\xi||}\right) - \left(T\frac{\xi}{||\xi||},\frac{\xi}{||\xi||}\right)\right] \\ &= ||\xi||^2\left[m - \left(T\frac{\xi}{||\xi||},\frac{\xi}{||\xi||}\right)\right] \\ &\geq 0\end{aligned}$$

    provided that $\xi \neq 0$.

==
Comments/Erratum, p. 259

- *Proof* of **Lemma 3.3**

  - 3rd sentence
    Since $\{\xi_i\}_1^r$ is independent,[30] none of its vectors can be written as a linear combination of the others.
  - 5th sentence
    Suppose $\xi_i \neq 0$ for some $i \neq j$. Then $x = \lambda_i$ as before. Therefore $\lambda_i = \lambda_j$ for some $i \neq j$![31]

- 2nd paragraph after **Theo. 3.2**, 4th sentence
  '$T^2\alpha$' should be '$T^2\xi$'.

---

[28]Consider $\xi_n \to \xi$ and $\eta_n \to \eta$. Therefore $(\xi_n,\eta_n) \to (\xi,\eta)$. In fact,

$$\begin{aligned}|(\xi_n,\eta_n) - (\xi,\eta)| &= |(\xi_n,\eta_n) - (\xi_n,\eta) + (\xi_n,\eta) - (\xi,\eta)| \\ &\leq |(\xi_n,\eta_n) - (\xi_n,\eta)| + |(\xi_n,\eta) - (\xi,\eta)| \\ &\leq ||\xi_n||\,||\eta_n - \eta|| + ||\xi_n - \xi||\,||\eta|| \quad \text{(by Schwarz)}\end{aligned}$$

and $||\xi_n||$ is bounded by convergence.

[29]See p. 128, l. -12.

[30]See **Cor.**, p. 251.

[31]See p. 258, l. -11.

==
EXERCISES, pp. 260–261

3.3 Since

$$\begin{aligned} p(t) &= t^2 + 2 \cdot t \cdot \frac{b}{2} + \frac{b^2}{4} - \frac{b^2}{4} + c \\ &= \left(t + \frac{b}{2}\right)^2 + c - \frac{b^2}{4}, \end{aligned}$$

$$T + \frac{b}{2} I \text{ is self-adjoint (by \textbf{EX. 3.1})}$$

and

$$c - \frac{b^2}{4} > 0,$$

$p(T) \neq 0$ (by **EX. 3.2**).

3.4 The case $n = 2$ follows from

$$\begin{aligned} ||T(\xi)||^2 &= (T(\xi), T(\xi)) \\ &= \left(\xi, T^2(\xi)\right) \quad (T \in SA) \\ &= (\xi, 0) \\ &= 0 \end{aligned}$$

for each $\xi \in V$.
Now suppose $T^{2^{m+1}} = 0$. Then $T^{2^m} = 0$ due to the fact that

$$\begin{aligned} \left|\left|T^{2^m}(\xi)\right|\right|^2 &= \left(T^{2^m}(\xi), T^{2^m}(\xi)\right) \\ &= \left(T^{2^{m-1}}(\xi), T^{2^{m+1}}(\xi)\right) \quad (T \in SA) \\ &= \left(T^{2^{m-1}}(\xi), 0\right) \\ &= 0 \end{aligned}$$

for each $\xi \in V$. Therefore $T = 0$ by the induction hypothesis.

3.5 Suppose p and q are minimal polynomials of T. Without loss of generality, assume both polynomials are monic. Then $(p - q)(T) = 0$ and the degree of $p - q$ is less than the degree of p, which is a contradiction unless $p = q$.

==
Comment/Erratum, pp. 262–263

- 1st paragraph and **EX. 4.1**
  If $\alpha, \beta \in V$, then

$$\begin{aligned} (T\alpha, \beta) &= \theta_\beta(T\alpha) \\ &= \left(T^*(\theta_\beta)\right)(\alpha) \quad (V^* \ni l \overset{T^*}{\mapsto} l \circ T \in V^*, \text{ p. 85}) \\ &= \left(\left(\theta \circ \theta^{-1} \circ T^* \circ \theta\right)(\beta)\right)(\alpha) \\ &= \theta_{\left(\theta^{-1} \circ T^* \circ \theta\right)(\beta)}(\alpha) \\ &= \left(\alpha, \left(\theta^{-1} \circ T^* \circ \theta\right)(\beta)\right). \end{aligned}$$

  Now consider $S \in \operatorname{Hom} V$ such that

$$(T\alpha, \beta) = (\alpha, S\beta) \text{ for all } \alpha, \beta \in V.$$

  Therefore $S = \theta^{-1} \circ T^* \circ \theta$ by **EX. 4.3**, p. 264.

- paragraph preceding **Theo. 4.1**
  "... columns of t ..." should be "... columns of $\mathbf{t}$ ...".

==

Erratum, p. 265, l. 13
'$T\varphi_2$' should be '$T\varphi_1$'.

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Comment, p. 295, ll. 3–5
See pp. 61–62, **On solving a linear equation**.
==
Comment, p. 296, paragraph right after **Lemma 6.2**
See **Theo. 4.1**, p. 282.
==
Erratum/Comments, p. 297, **Lemma 6.3**, *Proof*

- '$f \to \prec l_1(f), l_2(f) \succ$' should be '$f \mapsto \prec l_1(f), l_2(f) \succ$';
- Denote the map of the previous bullet by L. Therefore:
  - $L|N$ is injective since

$$\begin{aligned} f \in N \text{ and } l_i(f) = 0,\ i = 1, 2 &\Longrightarrow f \in M \cap N \\ &\Longrightarrow f = 0; \end{aligned}$$

  - L is in place of T of **Theo. 5.3**, p. 61, whereas M and N have changed places with each other: M is the null space of L!

==
Comment, p. 299, **Theo. 6.2**
Concerning the uniform convergence of the Fourier expansion, if

$$f_n = \sum_{i=1}^{n} (f, \varphi_i)\, \varphi_i$$

for each positive integer n,[32]

$$f_n \to f \text{ uniformly} \Longleftrightarrow ||f_n - f||_\infty \to 0.$$

==
Comment, p. 301, **FOURIER SERIES**, 3rd paragraph, 1st sentence
λ here is an eigenvalue of $T|M$. It depends on the "λ" of **Theo. 6.1**, p. 298.[33]
==
Comment, p. 302, **Theo. 7.2** and its *Remaining Proof*
The orthogonality of the eigenvectors also follows from **EX. 2.7**, p. 256. The uniform convergence of the Fourier series follows from **Theo. 6.2**, p. 299.
==

[32]See p. 211, ll. 5–10, and p. 254.
[33]See the *Proof* of **Theo. 6.2**, p. 299.

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Comment, p. 323, $\boldsymbol{\mu}$**5**
An elementary consequence of $\boldsymbol{\mu}$**1** and $\boldsymbol{\mu}$**2** is the monotonicity of μ:

$$\boldsymbol{\mu}\mathbf{m}.\quad A, B \in \mathscr{D} \text{ with } A \subset B \Longrightarrow \mu(A) \leq \mu(B).$$

In fact, since $B = A \cup (B - A)$ is a disjoint union of elements of $\mathscr{D}$,[34]

$$\begin{aligned} \mu(B) &= \mu(A) + \mu(B - A) \\ &\geq \mu(A). \end{aligned}$$

Now define $B_1 = A_1$ and

$$B_k = A_k - \bigcup_{j=1}^{k-1} A_j$$

for each $k \in \{2, \ldots, n\}$. So $B_1, \ldots, B_n$ are pairwise disjoint elements of $\mathscr{D}$,[35] whose union equals $\bigcup_1^n A_i$, and $B_i \subset A_i$ for each $i \in \{1, \ldots, n\}$. Therefore

$$\begin{aligned} \mu\left(\bigcup_1^n A_i\right) &= \mu\left(\bigcup_1^n B_i\right) \\ &\leq \sum_1^n \mu(B_i) \quad (\text{by } \boldsymbol{\mu}\mathbf{2}) \\ &\leq \sum_1^n \mu(A_i) \quad (\text{by } \boldsymbol{\mu}\mathbf{m}). \end{aligned}$$

==
Erratum, p. 325, l. 4
'a^k', the last component of $\boldsymbol{a}$, should be 'a^n'.
==
Erratum, p. 328, l. -9
"... to use Axiom $\boldsymbol{\mu}$**1** ..." should be "... to use Axiom $\boldsymbol{\mu}$**2** ..."
==
Erratum/Comments, pp. 331–332, **CONTENTED SETS**

- 3rd sentence
  '(iii)' should be '(iv)';

- (6.1) and *Proof* of **Prop. 6.1**

  – Note a slight abuse of notation: p is a paving $\Longrightarrow \mu(|p|) = \mu(p)$;
  – Since $|p|, |z| \in \mathscr{D}_{\min}$, note that (6.1) follows from $\boldsymbol{\mu}\mathbf{m}$;[36]
  – Now consider the final arguments of the "only if" part of the *Proof*. Since $v \prec p_\eta$, every rectangle of p_η is a union of rectangles of v.[37] However this does not mean that every $\square \in v$ must be one of some rectangles of v whose union is in p_η. In fact, we also have $v \prec z_\eta$. On the other hand, since $|p_n| \subset \operatorname{int} A$, every rectangle of p_η is in $\operatorname{int} A$. Therefore

  $$\emptyset \neq s \subsetneqq v.$$

  Note also that both ∂A *is contented* and $\mu(\partial A) = 0$ follow from the last sentence of those final arguments together with (6.4) and **Def. 6.3**.

[34] See $\mathscr{D}$**1**.
[35] Idem.
[36] See my previous **Comment** on $\boldsymbol{\mu}$**5**.
[37] See p. 329.

==
Comments, pp. 332-333, *Proof* of **Theo. 6.1**

- $\mathscr{D}$**1**
  For the sake of completeness, note that, since $A - B \subset A$, $A - B$ is contented by **Prop. 6.2**.

- $\boldsymbol{\mu}$**2**, 2nd sentence
  On the one hand, $p_1 \cup p_2$ is an inner paving of $A_1 \cup A_2$ by (6.3). So

$$\mu\left(A_1 \cup A_2\right) \geq \mu\left(|p_1 \cup p_2|\right).$$

  On the other hand, since we are dealing with disjoint unions,

$$\begin{aligned}\mu\left(|p_1 \cup p_2|\right) &= \mu\left(|p_1| \cup |p_2|\right)\\ &= \mu\left(|p_1|\right) + \mu\left(|p_2|\right)\\ &> \mu\left(A_1\right) + \mu\left(A_2\right) - \epsilon.\end{aligned}$$

  Therefore

$$\mu\left(A_1 \cup A_2\right) + \epsilon > \mu\left(A_1\right) + \mu\left(A_2\right)$$

  for each $\epsilon > 0$.

==
Comment, p. 333, (7.2) and **Fig. 8.9**
Since $B_{\mathbf{x}}^r$ is a closed ball, ϵ cannot be zero. In fact, see last two paragraphs of p. 324.
==
Comments/Erratum, p. 334

- 1st sentence, 1st inequality
  From (7.2) and (7.3),

$$\square_{\mathbf{x}-r\mathbf{1}}^{\mathbf{x}+r\mathbf{1}} \subset B_{\mathbf{x}}^{\sqrt{n}r} \subset \square_{\mathbf{x}-\sqrt{n}r\mathbf{1}}^{\mathbf{x}+\left(\sqrt{n}r+\epsilon\right)\mathbf{1}}.$$

  On the other hand,

$$\begin{aligned}\mu\left(\square_{\mathbf{x}-\sqrt{n}r\mathbf{1}}^{\mathbf{x}+\left(\sqrt{n}r+\epsilon\right)\mathbf{1}}\right) &= \prod_{i=1}^{n}\left(x^i + \sqrt{n}r + \epsilon - \left(x^i - \sqrt{n}r\right)\right) \quad \text{(by (4.1), p. 327)}\\ &= \prod_{1}^{n}\left(2\sqrt{n}r + \epsilon\right)\\ &= \left(2\sqrt{n}r + \epsilon\right)^n\\ &= 2^n\left(\sqrt{n}\right)^n r^n + b(\epsilon) \quad \text{with} \quad b(\epsilon) = \sum_{i=1}^{n}\binom{n}{i}\left(2\sqrt{n}r\right)^{n-i}\epsilon^i\\ &\leq 2^n\left(\sqrt{n}\right)^n 2^n r^n + b(\epsilon) = 2^n\left(\sqrt{n}\right)^n \prod_{i=1}^{n}\left(x^i + r - \left(x^i - r\right)\right) + b(\epsilon).\end{aligned}$$

  Therefore, by **Def. 6.2**, p. 331, and (4.1), p. 327,

$$\overline{\mu}\left(B_{\mathbf{x}}^{\sqrt{n}r}\right) \leq 2^n\left(\sqrt{n}\right)^n \mu\left(\square_{\mathbf{x}-r\mathbf{1}}^{\mathbf{x}+r\mathbf{1}}\right) + b(\epsilon).$$

- *Proof* of **Prop. 7.1**
  Let us make a slight change by considering a finite number of balls covering A with

$$\sum \overline{\mu}\left(B_{\mathbf{x}_i}^{r_i}\right) < \frac{\epsilon}{K^n\left(\sqrt{n}\right)^n}.$$

  Now, on the one hand, by (7.3), p. 333,

$$\square_{\mathbf{x}_i-\frac{r_i}{\sqrt{n}}\mathbf{1}}^{\mathbf{x}_i+\frac{r_i}{\sqrt{n}}\mathbf{1}} \subset B_{\mathbf{x}_i}^{r_i}.$$

Then

$$\mu^*\left(B_{\mathbf{x}_i}^{r_i}\right) \geq \left(\frac{2r_i}{\sqrt{n}}\right)^n.$$

On the other hand, by (7.2), p. 333,

$$B_{\varphi(\mathbf{x}_i)}^{Kr_i} \subset \square_{\varphi(\mathbf{x}_i)-Kr_i\mathbf{1}}^{\varphi(\mathbf{x}_i)+(Kr_i+\epsilon)\mathbf{1}},$$

which implies that

$$\overline{\mu}\left(B_{\varphi(\mathbf{x}_i)}^{Kr_i}\right) \leq (2Kr_i+\epsilon)^n$$

for each $\epsilon > 0$. Therefore

$$\begin{aligned}\overline{\mu}\left(B_{\varphi(\mathbf{x}_i)}^{Kr_i}\right) &\leq K^n\,(2r_i)^n \\ &\leq K^n\left(\sqrt{n}\right)^n \mu^*\left(B_{\mathbf{x}_i}^{r_i}\right) \\ &\leq K^n\left(\sqrt{n}\right)^n \overline{\mu}\left(B_{\mathbf{x}_i}^{r_i}\right) \quad \text{by (6.4), p. 331.}\end{aligned}$$

(It is worth noting that the original supposition

$$\sum \overline{\mu}\left(B_{\mathbf{x}_i}^{r_i}\right) < \frac{\epsilon}{K^n}$$

is clearly related to

$$\overline{\mu}\left(B_{\varphi(\mathbf{x}_i)}^{Kr_i}\right) \leq K^n \overline{\mu}\left(B_{\mathbf{x}_i}^{r_i}\right),$$

which is a more natural setting. For example, consider areas of disks in $\mathbb{E}^2$ and volumes of spheres in $\mathbb{E}^3$.)

- Final paragraph
  - Consider $\mathbf{x} = \prec x^1, \ldots, x^{n-1}, 0 \succ$ such that $||\mathbf{x}||_\infty < r$.
  - '0)', l. -2, should be just '0'.

==
Erratum/Comments, p. 335

- 1st sentence
  For notational consistency,[38] the 2nd '$\to$' should be '$\mapsto$'.

- **Exercise**
  For $\mathbb{E}^n$, $n \in \{2,3\}$, consider **Prop. 7.4** and suitable parameterizations of circles and spheres.

- **Prop. 8.1**, *Proof*, 2nd sentence
  As a matter of fact, homeomorphisms preserve not only boundaries, but also interiors and closures.

==
Erratum, p. 336, l. 6
'Theorem 5.2' should be 'Theorem 6.1'.
==
Comments, p. 337, last paragraph

- 1st sentence
  See $\mathscr{F}$**3** and $\mathscr{F}$**1**.

- 2nd sentence
  See (9.7).

==
Comments/Erratum, p. 338, **Prop. 9.1**, *Proof*

- 1st sentence
  For '$e_A \in \mathscr{F}$ for every $A \in \mathscr{D}_{\min}$', see the last sentence of p. 337 along with Theo. 5.1, p. 331.

[38] See p. 11.

- 2nd sentence
  'Proposition 4.1' should be 'Theorem 5.1'.

- 3rd sentence
  Use $\int \mathbf{3'}$.

==
Errata/Comments, p. 339

- **Prop. 10.1**, *Proof*

  - 1st sentence
    '$|f_1(\mathbf{x})$' should be '$|f_1(\mathbf{x})|$'.
  - 3rd sentence

$$\begin{aligned}|g_2(\mathbf{x})| &= |f_2(\mathbf{x}) + g_2(\mathbf{x}) - f_2(\mathbf{x})| \\ &\leq |f_2(\mathbf{x})| + |g_2(\mathbf{x}) - f_2(\mathbf{x})| \\ &< M + \epsilon.\end{aligned}$$

- **Prop. 10.2**, *Proof*

  - l. -2
    Consider $\boldsymbol{\mu}\mathbf{2}$ and the fact that

$$B = (B - |p|) \cup |p|$$

    is a disjoint union.
  - l. -1
    Just to make the notation uniform, '$e_{|p|}, |p|$' should be '$\prec e_{|p|}, |p| \succ$'.

==
Comments, p. 340

- **Cor.**, *Proof*
  This **Cor.** is a consequence of **Theo. 10.1**. As a matter of fact, since the *Proof* of that theorem does not depend on this **Cor.**, we can use it here. So,[39] for $\epsilon > 0$, there are paved functions $h \leq f_1$ and $k \geq f_2$ such that

$$\int f_1 - \epsilon < \int h \text{ and } \int f_2 + \epsilon > \int k.$$

- **Theo. 10.1**, *Proof*, 1st sentence
  $\forall h, k \in \mathscr{F}_P$ such that $h \leq f \leq k$,

$$\int h \leq \int f \leq \int k \tag{1}$$

  by $\int \mathbf{3'}$. So $\int f$ is both an upper bound of

$$H = \left\{ \int h \,:\, h \in \mathscr{F}_P, h \leq f \right\}$$

  and an lower bound of

$$K = \left\{ \int k \,:\, k \in \mathscr{F}_P, f \leq k \right\}.$$

  Take now an arbitrary positive ϵ and let h and k be as in **Prop. 10.3**. Therefore, by (1),

$$\begin{aligned}0 \leq \int (f - h) \leq \int (k - h) < \epsilon &\Longrightarrow \left(\int f \right) - \epsilon < \int h \\ &\Longrightarrow \int f = \sup H\end{aligned}$$

[39] See l. -2 of the *Proof* of **Theo. 10.1**.

and

$$-\int k \le -\int f \le -\int h \Longrightarrow 0 \le \int (k-f) \le \int (k-h) < \epsilon$$
$$\Longrightarrow \int k < \left(\int f\right) + \epsilon$$
$$\Longrightarrow \int f = \inf K.$$

- (10.5)
  See **Prop. 10.2** and **Prop. 10.1**, p. 339.

==

Comment/Erratum, p. 342

- ll. (-16)–(-13)
  First, note that if $\varphi \in \mathrm{Hom}(U, V)$, then $U = V = \mathbb{R}^n$ since every proper subspace of a normed linear space is not open. (In fact, proper subspaces do not contain open balls centered at $\mathbf{0}$.[40]) Now, by considering (8.1), (9.10) and (10.5),

$$\begin{aligned}\int_V e_A &= \int e_V e_A \\ &= \int e_A \\ &= \mu(A) \\ &= |\det\varphi| \left|\det\varphi^{-1}\right| \mu(A) \\ &= |\det\varphi|\, \mu\left(\varphi^{-1}A\right) \\ &= \int e_{\varphi^{-1}A} |\det\varphi| \\ &= \int (e_A \circ \varphi)\, |\det\varphi| \\ &= \int e_U\, (e_A \circ \varphi)\, |\det\varphi| \\ &= \int_U (e_A \circ \varphi) \left|\det J_\varphi\right|.\end{aligned}$$

- l. -11
  The first 'φ' should be 'η'.[41]

==

EXERCISES, p. 345

11.3–5 See **EXERCISES 9.7–9**, p. 160.

==

[40]Consider a vector $\mathbf{v}$ which does not belong to a proper subspace V. So $r\mathbf{v} \notin V$ for each $r > 0$.

[41]See **Prop. 10.4**, p. 341.

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Comment, p. 365, **Fig. 9.2**
If the "south pole" is the initial point of two vectors whose end points are shown in the figure, then the two vectors are multiples of one another and the ratio of their magnitudes is in $\left\{1+x^{n+1},\left(1+x^{n+1}\right)^{-1}\right\}$.
==
Comment, p. 367, last sentence

$$\begin{aligned}\frac{2}{1+\|y\|^2}-1 &= \frac{2}{1+\|\varphi_1(x)\|^2}-1\\ &= \frac{2}{1+\frac{1-x^{n+1}}{1+x^{n+1}}}-1 \qquad \text{(See p. 365)}\\ &= x^{n+1}.\end{aligned}$$

==
Errata, p. 368, 2nd i), 2nd ii) and l. -9

- 'N' should be 'N_i';
- 'φ' should be 'φ_i';
- '$\varphi_j(x_k)\in U_j$' should be '$\varphi_j(x_k)\in\varphi_j(U_j)$'.

==
Comment, p. 369, last paragraph

- 2nd sentence
  Use that
  $$f\circ\psi_j^{-1}=\left(f\circ\varphi_i^{-1}\right)\circ\varphi_i\circ\psi_j^{-1}.$$
- 3rd (last) sentence
  Consider two charts as in the penultimate paragraph, p. 369. Suppose there is a convergent sequence with respect to the atlas containing the first chart. Now, on p. 368, penultimate paragraph, use W_j and ψ_j in place of U_j and φ_j.

==
Errata/Comments, pp. 371–2

- **Prop. 3.1**, *Proof*
  - l. 3
    'β' should be 'β_1'.
  - l. 4
    - 'x' should be 'x_1';
    - the 2nd 'α_1' should be 'α_2'.
  - The existence of W_1 containing x_1 is guaranteed by **A2**, p. 364.
  - For the differentiability of $\beta_2\circ\alpha_2^{-1}$ and $\alpha_1\circ\beta_1^{-1}$, see the paragraph that follows the introduction of the equivalence relation between atlases, p. 369.
- 1st sentence after **Ex. 3.1**
  If W is a Banach space, then 'differentiable as a function (in the sense of Section 2)' means that each of the functions f_i defined by (2.1) is differentiable for $f: M\to W$.

==
Comments, p. 374

- See Section 3.7, pp. 146–150;[42]
- (4.3) implies that

$$\begin{aligned} a\xi_\beta + b\eta_\beta &= aJ_{\beta\circ\alpha^{-1}}(\alpha(x))\xi_\alpha + bJ_{\beta\circ\alpha^{-1}}(\alpha(x))\eta_\alpha \\ &= J_{\beta\circ\alpha^{-1}}(\alpha(x))\,(a\xi_\alpha + b\eta_\alpha) \\ &= J_{\beta\circ\alpha^{-1}}(\alpha(x))\zeta_\alpha \\ &= \zeta_\beta. \end{aligned}$$

- Concerning the definition of ψ_{*x}, suppose g is a differentiable real-valued function on M_2 defined in a neighborhood of $\psi(x)$. So $f = g\circ\psi$ is a differentiable real-valued function on M_1 defined in a neighborhood of x. Then

$$\begin{aligned} (g\circ(\psi\circ\varphi))'(0) &= (f\circ\varphi)'(0) \\ &= (f\circ\overline{\varphi})'(0) \\ &= (g\circ(\psi\circ\overline{\varphi}))'(0) \end{aligned}$$

  provided that $\varphi \sim \overline{\varphi}$. Therefore $\psi\circ\varphi \sim \psi\circ\overline{\varphi}$.

==
Comment, p. 375, last paragraph
Writing 'ψ_*' in place of 'ψ_{*x}' is an abuse of notation. In fact, see the last paragraph of Section 4, p. 376.
==
Errata/Comments, p. 376

- ll. 3–4
  Delete the extra 'of' before '$x \in M$'.
- l. 6
  '$T_{*x}(M)$' should be '$T_x(M)$'.
- (4.5)
  See penultimate paragraph on p. 373.
- (4.6)
  For φ^*, see p. 372.

==
Comments, p. 379

- (5.2)

$$\begin{aligned} Y_\beta\left(\beta\circ\alpha^{-1}(v)\right) &= Y\left(\beta^{-1}\left(\beta\circ\alpha^{-1}(v)\right)\right)_\beta && \text{(via (5.1))} \\ &= Y\left(\alpha^{-1}(v)\right)_\beta \\ &= J_{\beta\circ\alpha^{-1}}\left(\alpha\left(\alpha^{-1}(v)\right)\right)Y\left(\alpha^{-1}(v)\right)_\alpha && \text{(via (4.3))} \\ &= J_{\beta\circ\alpha^{-1}}(v)Y_\alpha(v). \end{aligned}$$

- The sentence right after (5.2) means that the "local expression" is unique up to an automorphism.

==
Comment/Errata, p. 380, ll. 6-7 and 10

- See the definition of ξ_α on p. 374;
- 'Φ' should be 'Φ_α', twice.[43]

[42] In particular, see **Theo. 7.2**, p. 148, and l. 10, p. 150.
[43] In fact, see l. 8, (5.3) and pp. 381–4.

===
Comments/Erratum, pp. 384–5

- (6.1)
  See pp. 372 and 376.
- (6.2–3)
  See pp. 373 and 378–9. Furthermore, it is worth noting that

  $$X(x)f = D_\varphi f$$

  provided that $\varphi \in X(x)$.
- '$D_X f$ is linear in X'
  See p. 380, ll. 18–20.
- (6.4)
  - For ψ_*, see pp. 374–6;
  - 'M' should be 'M_1'.

===
Comment, p. 386, l. -1
See (4.4) and (6.4).
===
Comment, p. 387, (6.8)
Due to the fact that φ_t is a diffeomorphism,[44] $\varphi_t^*[Y]$ is a smooth vector field if Y is.[45] If so,

$$\frac{1}{t}\left\{\varphi_t^*[Y] - Y\right\}$$

is a smooth vector field.[46]
===
Comments/Errata, p. 391

- Concerning the identification of $\xi \in T_x(M)$ with $\xi_\alpha \in V$, see p. 374.
- (7.1)
  l_α is defined via a scalar product using angled brackets, not the round bracket notation from the fifth chapter.
- l. 5

  $$\begin{aligned}\langle \xi_\alpha, l_\alpha \rangle &= \left\langle J_{\alpha\circ\beta^{-1}}(\beta(x))\xi_\beta, l_\alpha \right\rangle \qquad \text{(via (4.3))} \\ &= \left\langle \xi_\beta, \left(J_{\alpha\circ\beta^{-1}}(\beta(x))\right)^* l_\alpha \right\rangle\end{aligned}$$

  by the definition of T^* on p. 262. So, since

  $$\left(J_{\alpha\circ\beta^{-1}}(\beta(x))\right)^{-1} = J_{\beta\circ\alpha^{-1}}(\alpha(x)), \tag{2}$$

  the result holds.[47]
- l. 15
  'df' should be '$df(x)$'.
- l. -14
  $(U, \alpha) \notin M$![48]

---

[44] See p. 376.
[45] See p. 385.
[46] See p. 380.
[47] See p. 262.
[48] In fact, it belongs to an atlas of M.

- (7.4)
  Consider that $v = \alpha(x)$. Then see equation (2).

==

Errata/Comment, p. 392

- l. 5
  '$\xi \to \langle \varphi_*(\xi), l \rangle$' should be '$\xi \mapsto \langle \varphi_{*x}(\xi), l \rangle$'.
- l. 12
  'map' should be 'linear differential form'.
- (7.10)
  On the one hand,

$$\begin{aligned} \langle \xi, d(\psi^*[f])(x) \rangle &= \langle \xi, d(f \circ \psi)(x) \rangle \\ &= \xi(f \circ \psi) \\ &= \xi\left(\psi^*[f]\right). \end{aligned}$$

  On the other hand,

$$\begin{aligned} \langle \xi, (\psi^* df)(x) \rangle &= \left\langle \xi, (\psi_{*x})^* \, df(\psi(x)) \right\rangle \\ &= \langle \psi_{*x}(\xi), df(\psi(x)) \rangle \\ &= \psi_{*x}(\xi) f \\ &= \xi\left(\psi^*[f]\right). \end{aligned}$$

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