A Survival Guide to

Dynamical Systems

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Partial scrutiny,

Comments, Suggestions and Errata

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Comment, p. 11
The tangent to the curve $G = \{(x, y) \in \mathbb{R}^2 \mid y = P(x)\}$ at $(x, y) = (x_n, P(x_n))$ is given by

$$T_n = \left\{(x, y) \in \mathbb{R}^2 \mid P'(x_n)(x - x_n) + (-1)(y - P(x_n)) = 0\right\}.$$

Therefore, for $(x, y) \in T_n$ with $y = 0$, we have $x = x_{n+1}$ from Newton's method.
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Comment, p. 12, l. -1
As a matter of fact, the inequality is strict, that is,

$$|x_{new} - z| < \frac{\mu}{2}.$$

Hence, since

$$x_{new} \in \left(z - \frac{\mu}{2}, z + \frac{\mu}{2}\right),$$

if we denote the next iterate by x_{newer}, that is,

$$x_{newer} := x_{new} - \frac{f(x_{new})}{f'(x_{new})},$$

it follows that

$$|x_{newer} - z| < \frac{\mu}{4}.$$

This iteration goes on and each *Newton step* more than halves the distance to z.
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Erratum, p. 13, 1st sentence of **1.2.3**
Change to "Now let f **be** a function ...".
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Comment, p. 21, ll. -2 and -1
To start the sequence of Newton steps, **Proposition 1.2.1** considers $x_0 = 0$ as the initial guess (starting point). Then, in case $P(z) = 0$ holds for some z with $x_n \to z$, (1.14) means

$$|P(x_0) - P(z)| \leq K^{-5},$$

that is, $P(0)$ within K^{-5} of $P(z)$.
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Comments, p. 22

- (1.26)
  $x_0 = 0$ is **the** initial guess after a shifting of coordinates and it is most likely that $P(x_0) \neq 0$ (when considering P as a single function of x). $u = \mathbf{0}$ is supposed to be a zero of $P(u, 0)$.

- 1st sentences right after (1.26)
  The inequality with $x_0 = 0$ guarantees that (1.14) holds. Hence **Proposition 1.2.1** guarantees the existence of a zero: there is a point z which is the limit of the sequence of Newton steps with initial guess $x_0 = 0$:

  $$P(u, z) = 0, \quad x_n \to z \quad \text{and} \quad x_0 = 0.$$

  Therefore **1.2.3** implies that the convergence to z also takes place for the sequence of Newton steps with initial guess $x_0 \neq 0$ provided that $|x_0 - z|$ is sufficiently small. (Furthermore, since $|z| \leq |x_0 - z| + |x_0|$, notice that $|z|$ is also small enough. So both $x_0 \neq 0$ and z must be sufficiently close to $x_0 = 0$!)

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Errata, p. 23

- 1st line after (1.29)
  "... $u = 0$, ..." should be "... $u = \mathbf{0}$, ...".

- l. -9
  "..., say $\in \mathbb{R}^n$, ..." should be "..., say $u \in \mathbb{R}^n$, ...".

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Erratum, p. 34
2.1.1 does not deal with the $\mu = 1$ case.
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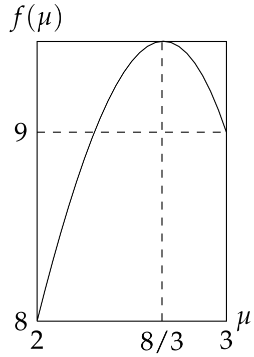
Comment, p. 38, ll. -6 and -5
The value of $f(\mu) = \mu^2(4-\mu)$, since $f'(\mu) = \mu(8-3\mu)$ and $f''(8/3) < 0$, increases from $f(2) = 8$ to $f(8/3) > 9$ and decreases from $f(8/3)$ to $f(3) = 9$.



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Erratum, p. 40, Figure 2.5
$L_\mu^{(2)}$ should be $L_\mu^{\circ 2}$.
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Comments, p. 40
As a matter of fact, (2.3) refers to the discriminant Δ of the quadratic function (which precedes it) with zeros

$$p_{2\pm} = \frac{-\left(\mu^2+\mu\right) \pm \sqrt{\Delta}}{-2\mu^2}.$$

2.1.7 The double root comes from the fact that $\Delta = 0$ for $\mu = 3$.

2.1.8 l. -1
For $\mu > 3$, since $\sqrt{(\mu+1)(\mu-3)} = \sqrt{\mu^2-2\mu-3} < \sqrt{\mu^2-2\mu+1} = \mu - 1$,

$$\begin{aligned} 0 < \frac{1}{\mu} &= \frac{1}{2} + \frac{1}{2\mu} - \frac{\mu-1}{2\mu} \\ &< p_{2-} \\ &< p_{2+} \\ &< \frac{1}{2} + \frac{1}{2\mu} + \frac{\mu-1}{2\mu} = 1. \end{aligned}$$

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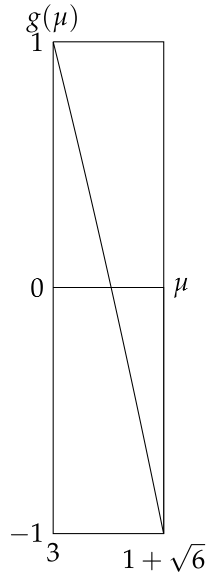
Erratum, p. 41
$L_\mu^{(2)}$ should be $L_\mu^{\circ 2}$.
==
Comments, p. 41, last two sentences of **2.1.8**

$$\begin{aligned} \left(L_3^{\circ 2}\right)'(p_{2\pm}) &= \left(L_3^{\circ 2}\right)'\left(\frac{2}{3}\right) \\ &= L_3'(2/3)L_3'(2/3) \\ &= (-1)(-1) \\ &= 1. \end{aligned}$$

Furthermore, consider the graph of $g(\mu) = -\mu^2 + 2\mu + 4$ for $\mu \in \left[3, 1+\sqrt{6}\right]$.



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Erratum, p. 43, **2.1.10**

Shouldn't p_{2+} be p_{2-}?

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Erratum, p. 45, Figure 2.9

Change μ to r or vice versa.

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Comment, p. 49, $\mu''(0)$

If $\gamma(x) = (x, \mu(x))$ and $\Gamma(x, \mu)$ denotes $-\frac{P_x}{P_\mu}$, then $\mu'(x) = \Gamma(\gamma(x))$. Hence

$$\begin{aligned}
\mu''(0) &= \gamma'(0) \cdot \nabla\Gamma(\gamma(0)) \\
&= 1 \cdot \Gamma_x(0, \mu(0)) + \mu'(0) \cdot \Gamma_\mu(0, \mu(0)) \\
&= \Gamma_x(0, 0) + 0 \\
&= -\frac{P_{xx}(0,0) \cdot P_\mu(0,0) - P_x(0,0) \cdot P_{x\mu}(0,0)}{\left(P_\mu(0,0)\right)^2} \\
&= -\frac{P_{xx}(0,0)}{P_\mu(0,0)} + \frac{0 \cdot P_{x\mu}(0,0)}{\left(P_\mu(0,0)\right)^2} \\
&= -\frac{\partial^2 P/\partial x^2}{\partial P/\partial \mu}(0,0).
\end{aligned}$$

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Comment, p. 52, first paragraph of **2.3.2**

See the paragraph starting with "Before embarking ..." on p. 47.

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Comment, p. 54, **Step I**

(2.7) holds since $P(x, \mu)$ is well-defined via the first paragraph of **2.3.2**.

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Comment, p. 56

l. 6

The chain rule implies that

$$\frac{d}{dx}\left(\frac{\partial F^{\circ 2}}{\partial x}(x, \nu(x))\right)\bigg|_{x=0} = \frac{\partial^2 F^{\circ 2}}{\partial x^2}(0,0) \cdot 1 + \frac{\partial^2 F^{\circ 2}}{\partial \mu \partial x}(0,0) \cdot \nu'(0).^{1}$$

[1] Also, see $\phi'(x)$, **Step VI**.

l. 7
By considering l. 6, p. 53, and l. -3, p. 52, it follows that

$$\frac{\partial F^{\circ 2}}{\partial x}(0,0) = \left(\frac{\partial F}{\partial x}(0,0)\right)^2 = 1 = (\lambda(0))^2.$$

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Comment, pp. 56–7, 1st sentence of **2.4**
Suppose (2.10) holds. Therefore, if $'$ denotes differentiation with respect to x and $p = \frac{1}{2}$, then

$$\left(L_\mu^{\circ 2^{n-1}}\right)'(p) = L_\mu'(p) L_\mu'\left(L_\mu(p)\right)\cdots L_\mu'\left(L_\mu^{\circ(2^{n-1}-1)}(p)\right) = 0$$

since $L_\mu'(p) = 0$, which implies that p is a superattractive periodic point of period 2^{n-1}.[2]
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Comment, p. 58, ll. 9–12
Consider s_n is a solution of (2.10) for which $\frac{1}{2}$ has period $2^{i-1} < 2^{n-1}$. Therefore $L_{s_n}^{\circ 2^{i-1}}(1/2) = 1/2$ and, since $\left(L_{s_n}^{\circ 2^{i-1}}\right)'(1/2) = 0$, it follows that $s_n = s_i$ with $i < n$!

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Erratum, p. 58, l. -1
As a matter of fact, since the superattractive value s_r is the solution of the equation (for μ) $f_\mu^{\circ 2^{r-1}}(X_m) = X_m$,[3] then

$$f_{s_r}^{\circ 2^{r-1}}(X_m) - X_m = 0.$$

Since d_r denotes the difference between X_m and the next nearest point on the superstable 2^r orbit, which is the orbit $\left\{X_m, f_{s_r}(X_m), \ldots, f_{s_r}^{\circ 2^{r-1}-1}(X_m)\right\}$, then

$$d_r = f_{s_r}^{\circ j}(X_m) - X_m \text{ for } j \in \left\{1, \ldots, 2^{r-1}-1\right\}.$$

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Erratum, p. 61, l. 12
It seems the insertion of 2^{r+1} is unnecessary! It should be just

$$\mathcal{R}(g_k)(y) = \lim(-\alpha)^{r+1} g_{s_{k+r}}^{2^{r+1}}(y/(-\alpha)^{r+1}) = g_{k-1}(y)$$

since $g_k(y) = \lim(-\alpha)^r g_{s_{k+r}}^{2^r}(y/(-\alpha)^r)$ implies that

$$g_{k-1}(y) = \lim(-\alpha)^r g_{s_{k-1+r}}^{2^r}(y/(-\alpha)^r) \underbrace{=}_{\mathbf{r} = r-1} \lim(-\alpha)^{\mathbf{r}+1} g_{s_{k+\mathbf{r}}}^{2^{\mathbf{r}+1}}(y/(-\alpha)^{\mathbf{r}+1}).$$

[2] See p. 26, **1.4.6**.
[3] See the discussion around (2.10), p. 57.

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Comments, p. 63, *Proof* of **Lemma 3.1.2**

- Negating the existence of the greatest point $c \in J$ with $f(c) = a$ leads us to an increasing sequence $\{c_n\}$ in J, which converges to $\sup \{c_n \mid n \in \mathbb{N}\} = s \in J$ since J is compact, such that $f(c_n) = a$ for all n. Then $f(s) = a$ since the sequence $\{f(c_n)\}$ converges to both $f(s)$ and a. Therefore s is the greatest point $c \in J$ with $f(c) = a$!

- "Then we may take $L = [c, d]$."
  If $f([c, d]) = [a', b']$, then The Intermediate Value Theorem guarantees that $[a', b'] \supseteq [a, b]$. Hence, if $a' < a$, there exists $c' > c$ with $f(c') = a$,[4] which is a contradiction. Therefore, $a' = a$. Similarly, $b' = b$.

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Comments/Errata, p. 64

- **Notation**
  Where do we use $< a, b >$ from this point on?

- *Proof* of **Theo. 3.1.1**
  "$f(I_0) \supset I_1, \quad f(I_1) \supset I_0 \cup I_1$."
  Apply The Intermediate Value Theorem.[5]
  "Finally, since $f(I_1) \supset I_0$**,** there is a compact interval $A_n \subset I_1$ with $f(A_n) = A_{n-1}$."

  (Note the insertion of the second comma - the one in bold.)

  In fact, $I_0 \supset A_{n-1}$.[6]
  "By Lemma 3.1.1, $f^{\circ n}$ has a fixed point, x, in A_n."

  (Note the use of the adopted notation in place of f^n.)

  In fact

$$\begin{aligned} A_n \subset I_1 &= f^{\circ(n-2)}(A_{n-2}) \\ &= f^{\circ(n-2)}(f(A_{n-1})) \\ &= f^{\circ(n-1)}(A_{n-1}) \\ &= f^{\circ(n-1)}(f(A_n)) \\ &= f^{\circ n}(A_n). \end{aligned}$$

  "But $f(x)$ lies in I_0 and all the higher iterates up to n lie in I_1 so the period can not be smaller than n."
  In fact, on the one hand $f(x) \in I_0$ since $x \in A_n$ and $f(A_n) = A_{n-1} \subset I_0$. On the other hand, if $f^{\circ i}(x) = x \in A_n \subset I_1$ for some index $1 < i < n$, then $f^{\circ(i+1)}(x) = f(x) \in I_0$, which is absurd since $f^{\circ 2}(x), \ldots, f^{\circ n}(x) \in I_1$.

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Erratum, p. 67, l. -3
The denominator $[f'(g(x))g'(x)$ should be $[f'(g(x))]g'(x)$ or $f'(g(x))g'(x)$.

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Errata, p. 68

[4] Draw a graph illustrating the situation!
[5] Note that $I_0 \cup I_1 = [a, c]$.
[6] See p. 64, l. -10.

- *Proof* of **Lemma 3.2.1**, last sentence
  Write | The reverse **occurs** at a relative maximum. |.

- *Proof* of **Lemma 3.2.3**, first sentence
  'fixed' should be 'critical'.

==

Comment, p. 69

- *Proof* of **Lemma 3.2.3**
  | "By the mean value theorem, there is a point s with $r < s < t$ such that $g'(s) < 1$." |
  In fact, if $f(x) = g(x) - x$, there is a point s in the open interval (r, t) at which

$$\begin{aligned} g'(s) - 1 &= f'(s) \\ &= \frac{f(t) - f(r)}{t - r} \\ &= \frac{g(t) - t - (g(r) - r)}{t - r} \\ &< 0. \end{aligned}$$

- *Proof* of **Lemma 3.2.4**
  Suppose first that the set $C\left(f^{\circ 2}\right)$ of all critical points of $f^{\circ 2}$ is infinite. On the other hand, the set $C(f)$ consisting of all critical points of f is finite by hypothesis. Therefore there can be only finitely many elements of $C\left(f^{\circ 2}\right)$ in $C(f)$ and

$$\left\{x \in C\left(f^{\circ 2}\right) \mid f(x) \in C(f)\right\}$$

  is an infinite set whose image under f cannot be infinite, which is a contradiction by the second sentence of the proof.

- last sentence, | "So there are three possibilities: ..." |
  On the one hand, neither $g(L)$ nor $g(R)$ is in (L, R). In fact, suppose otherwise. So one or both of them are attracted to p by g. Then one or both of the endpoints of (L, R) are attracted to p by g, which contradicts the maximality of (L, R). On the other hand, $g(\{L, R\}) \subset \{L, R\}$.[7] In fact, L and R are accumulation points of (L, R). Hence there are sequences $L_1, L_2, \ldots$ and $R_1, R_2, \ldots$ in (L, R) with $L_n \to L$ and $R_n \to R$ as $n \to \infty$. Then, since g is continuous, $g(L_n) \to g(L)$ and $g(R_n) \to g(R)$ as $n \to \infty$. Now suppose $|g(L) - L| > 0$ and $|g(L) - R| > 0$. So there is some index n_0 such that $g(L_{n_0})$ is not in (L, R).[8] Therefore, since L_{n_0} is attracted to p by g, it follows that $g(L_{n_0})$ is attracted to p by g, which contradicts the maximality of (L, R). Similarly, the supposition that $g(R) \notin \{L, R\}$ contradicts the maximal choice of (L, R).

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Erratum, p. 70, l. 2

| "... $z, f(z), \ldots f^{\circ m-1}(z)$ ..." | should be | "... $z, f(z), \ldots, f^{\circ(m-1)}(z)$ ..." |.

[7] Notice that, if that inclusion holds, then the last line of the p. 69 means that $g(L) = g(R)$ equals either L or R - one point is fixed and the other is eventually fixed!

[8] As a matter of fact, almost all the terms $g(L_n)$ are not in (L, R).

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Erratum, p. 79, **Prop. 4.1.1**

"Let $f = ax^2 + bx + d$." should be "Let $f(x) = ax^2 + bx + d$.".

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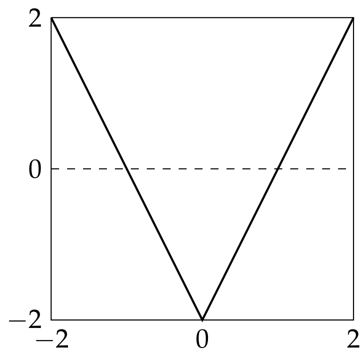
Erratum, p. 82, Figure 4.3

The graph of $V(x)$ with the proper scale for the y-axis should be:



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Comment, p. 84, **The tent map is chaotic**, "..., there is a point $x \in \left[\frac{k}{2^n}, \frac{k+1}{2^n}\right]$ which is mapped into itself."

The diagonal $D = \{(x, y) \in [0,1] \times [0,1] \,|\, y = x\}$ traverses the narrow rectangle $\left[\frac{k}{2^n}, \frac{k+1}{2^n}\right] \times [0,1]$, which contains the graph of the surjective restriction of $T^{\circ n}$ to $\left[\frac{k}{2^n}, \frac{k+1}{2^n}\right]$. Since the image of this restriction is the unit interval $[0,1]$, its graph intersects D.

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Erratum, p. 85

In the verification of $T \circ S = T \circ T$, seven commas (at the very end of each line) and a period (at the very end of the 8th line) are missing!

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Erratum, p. 86

Change S^k to $S^{\circ k}$ twice.

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Comment, p. 88, the sentence right before **Going to the unit circle**

Since S is chaotic on $[0,1]$, the commutative diagram at the bottom of the page 84 gives us an alternative proof that T is chaotic.

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Comment/Errata, p. 90, **Proof of Prop. 4.4.1 with** $\delta = c/4$

- l. 6
  $1 \leq nj - k \leq n$ since $n(j-1) \leq k \leq nj - 1$.
- l. 11
  $\left(f^{-(nj-k)}\left(W_{nj-k}\right)\right.$ should be $\left(f^{-(nj-k)}\left(W_{nj-k}\right)\right)$.
- ll. 13-16 and 20
  Replace f^{nj} by $f^{\circ nj}$ eight times and, at the very beginning, replace f^{nj-k} by $f^{\circ(nj-k)}$ once.
- last sentence, "So ... $m = nj$."
  n should be m in the enunciation of **Prop. 4.4.1**, p. 89.

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Comments/Erratum, *Proof* of **Prop. 4.5.1**, pp. 91–2

- 2nd sentence, "If (x_0, y_0) ... $y_1 = g(y_0)$."
  (x_1, y_1) satisfies $y = h(x)$ since

$$\begin{aligned} y_1 &= g(y_0) \\ &= g(h(x_0)) \\ &= h(f(x_0)) \\ &= h(x_1). \end{aligned}$$

- 4th sentence, "By hypothesis ... zero."
  By continuity, if $\lim_{n\to+\infty} x_n = p$, then

$$\begin{aligned} f(p) &= \lim_{n\to+\infty} f(x_n) \\ &= \lim_{n\to+\infty} x_{n+1} \\ &= p. \end{aligned}$$

  Hence $p = 0$.

- 9th sentence, "Extend ... $x_2 \leq x \leq x_1$."
  The extension of h to $[x_2, x_1]$ is well-defined since $f^{-1}(x) \in [x_1, x_0]$ for $x \in [x_2, x_1]$.[9]

- 12th sentence, "Continuing ... $h = g^n \circ h \circ f^{-n}$."
  At the very end, $h = g^{\circ n} \circ h \circ f^{-n}$ is more consistent with the adopted notation.

- last sentence
  Let us verify for $[x_3, x_2]$. It follows from:

$$h = g^{\circ 2} \circ h \circ f^{-2} \text{ implies that } h \circ f = g \circ h.$$

  In fact, on the one hand, $f^{-1}(x) \in [x_2, x_1]$ for $x \in [x_3, x_2]$. Furthermore, concerning $[x_2, x_1]$, $g \circ h = h \circ f$ holds. On the other hand, if $x \in [x_3, x_2]$, then

$$\begin{aligned} (h \circ f)(x) &= \left(g^{\circ 2} \circ h \circ f^{-2} \circ f\right)(x) \\ &= \left(g \circ (g \circ h) \circ f^{-1} \circ f^{-1} \circ f\right)(x) \\ &= \left(g \circ (g \circ h) \circ f^{-1}\right)(x) \\ &= g\left((g \circ h)\left(f^{-1}(x)\right)\right) \\ &= g\left((h \circ f)\left(f^{-1}(x)\right)\right) \\ &= \left(g \circ h \circ f \circ f^{-1}\right)(x) \\ &= (g \circ h)(x). \end{aligned}$$

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Comments/Erratum, p. 94

- 5th sentence,

  "For $-2 \leq c \leq 1/4$, the iterate of any point in $[-p_+, p_+]$ remains in the interval $[-p_+, p_+]$."

  Let $|x| \leq p_+$. On the one hand,

$$x^2 + c \leq p_+^2 + c = \frac{1}{4} + \frac{\sqrt{1-4c}}{2} + \frac{1}{4} - c + c = p_+.$$

[9] $f^{-1}(x_2) = x_1$, $f^{-1}(x_1) = x_0$ and f^{-1} is a continuous strictly increasing function on $[x_2, x_1]$.

On the other hand, since $x^2 + c \geq -p_+$ for $c \geq 0$ (because $-p_+$ is negative) and $x^2 + c \geq c$, it suffices to prove that $c \geq -p_+$ for $-2 \leq c < 0$. In fact, suppose otherwise. Hence

$$c < -\frac{1+\sqrt{1-4c}}{2} \Rightarrow -(2c+1) > \sqrt{1-4c},$$

which does not hold if $2c + 1 \geq 0$, that is, if $c \geq -\frac{1}{2}$. So suppose that $-2 \leq c < -\frac{1}{2}$. Therefore

$$\begin{aligned} -(2c+1) > \sqrt{1-4c} &\Rightarrow 4c^2 + 4c + 1 > 1 - 4c \\ &\Rightarrow 4c(c+2) > 0, \end{aligned}$$

which is a contradiction since $4c < 0$ and $c + 2 \geq 0$ for $-2 \leq c < -\frac{1}{2}$.

- 7th sentence, "To visualize **the** what is going on, ..."
  It should be just "To visualize what is going on, ...".

- 8th sentence, "The bottom of the graph will protrude below the bottom of the square."
  It follows since $c < -p_+$ for $c < -2$. In fact, suppose otherwise.

- last sentence
  A_2 is open since, under a continuous function, the inverse image of an open set (in the codomain) is always an open set (in the domain).

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Erratum, p. 95, l. -3
In order to be consistent with a previous notation (p. 90, l. 1), $Q_c^{-\circ n}(A_1)$ should be just $Q_c^{-n}(A_1)$.
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Errata/Comment, p. 100

- l. 1
  Replace **itenerary** by **itinerary**.

- l. 4
  As a matter of fact, by abuse of notation, S may used in place of **Sh** as can be seen on p. 87.

- ll. -1, -3 and -4
  In order to be consistent with a previous notation (p. 90, l. 1), $Q_c^{-\circ n}$ should be just Q_c^{-n}. Similarly, $Q_c^{-\circ(n-1)}$ should be changed.

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Erratum, p. 101, **Theo. 4.6.2**
Change Q_c^n to $Q_c^{\circ n}$.

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Suggestion, p. 103
The 4th sentence should end as in

$$\boxed{\text{"... in } I_k, k = 1, \ldots, N.\text{"}}$$

since the 3rd sentence considers I_k for $k = 1, \ldots, N-1$ separetely from I_N.
==
Comment, p. 105, **The push forward of a discrete measure**, last sentence
Using (5.2), we have

$$(F_*\mu)(I) = \sum_{y_\ell \in F^{-1}(I)} n\left(y_\ell\right).$$

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Comment, p. 106

- (5.3) and (5.4)
  - x_k is used, twice, instead of just x, to make clear that the sum is over k;
  - If (5.4) holds, then we can put each of the two densities, ρ and σ, in the other's place by (5.3).
- **Back to the histogram**
  Suppose that $\rho \approx$ constant on I_k. Therefore

$$\begin{aligned} p\left(I_k\right) &= \int_{I_k} \rho(x) dx \\ &\approx \rho(x) \int_{\frac{k-1}{N}}^{\frac{k}{N}} dt = \rho(x) \cdot \frac{1}{N}. \end{aligned}$$

  For N large enough, the previous supposition is a possible one.

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Comments, p. 108

- **First proof of (i)**
  Note that

$$\begin{aligned} \rho(F(x)) &= \frac{1}{\pi\sqrt{4x(1-x)(1-4x(1-x))}} \\ &= \frac{2}{\pi 4|1-2x|\sqrt{x(1-x)}} \\ &= \frac{1+1}{\pi 4|1-2x|\sqrt{x(1-x)}} \\ &= \frac{\frac{1}{\pi\sqrt{x(1-x)}}}{4|1-2x|} + \frac{\frac{1}{\pi\sqrt{(1-x)x}}}{4|-1+2x|} \\ &= \frac{\rho(x)}{|F'(x)|} + \frac{\rho(1-x)}{|F'(1-x)|}. \end{aligned}$$

- **Second proof of (i)**, 4th sentence, $\boxed{\text{"In other words ... } \nu \text{ has density } \rho(x) \equiv 1.\text{"}}$
  In fact, the measure of I is just the length of I, which means that $\nu(I) = \int_I dx$.

==

Comment, p. 109, **Proof of (ii)**
At the very end, we are dealing with a constant sequence of Riemann sums which converges to both $\int f(t)dt$, whose convergence follows from the continuity of f, and the constant $f(x)$. Then

$$f(x) = \int f(t)dt$$

is just a consequence of the uniqueness of the limit of any sequence.
==
Comment, p. 110, **What about (iii)?**
To fix ideas, let $(X, \mathcal{B}, \mu)$ be a probability space. Therefore:

- X is a set of possible outcomes;
- $\mathcal{B}$ is a set of events, which are sets of outcomes;
- μ assigns probabilities to each event in $\mathcal{B}$.

Furthermore, let $T : X \to X$ be an ergodic transformation with respect to an invariant measure, μ. Then:

- $T_* \mu = \mu$;
- $\forall A \in \mathcal{B}$ such that $T^{-1}(A) = A$, either $\mu(A) = 0$ or $\mu(A) = 1$.

An ergodic theorem describes the limiting behavior of a sequence

$$\frac{1}{n} \sum_{i=0}^{n-1} f \circ T^i$$

(as $n \to \infty$) and depends on the function f (for example, it may be assumed that f is integrable, or square integrable (L^2), or continuous) and the type of convergence used in the theorem (for example, pointwise, L^2 or uniform convergence).
The **Birkhoff Ergodic Theorem** deals with pointwise convergence of the previous sequence of partial sums for integrable functions f if $\mu(X)$ is finite.[10] It says that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} f\left(T^i(x)\right) = \hat{f}(x) \text{ for almost all } x$$

with

$$\hat{f} = \frac{1}{\mu(X)} \int f\mu.$$

==
Comment, pp. 110–111, "Indeed, applying ... as $n \to \infty$"
Since U is an isometry,

$$\frac{||U^n f||}{n} = \frac{||f||}{n} \to 0$$

as $n \to \infty$.
==
Comments, p. 112

- **Lemma 5.3.2**
  The denseness of $D \cup I$ is a consequence of a well-known result (in Functional Analysis) which says that the orthogonal complement of the orthogonal complement of $S \subset H$ is the closure of S. Therefore

$$\begin{aligned} \overline{D \cup I} &= (D \cup I)^{\perp\perp} \\ &= 0^{\perp} \\ &= H. \end{aligned}$$

[10] For a probability space, $\mu(X) = 1$.

- **Lemma 5.3.3** and **Lemma 5.3.4**
  Let S be the set of elements for which the limit in (5.8) exists. Consider that $f \in H$. So the denseness of S guarantees the existence of a sequence in S which converges to f. Then the closeness of S implies that $f \in S$. Therefore $S = H$.

==

Erratum, p. 113, (5.9)
A dx is missing right after the integrand.

==

Erratum, p. 121
The 2nd **Proof o Prop. 5.4.3** is, in fact, the **Proof o Prop. 5.4.4**.

==
==
6
==
==
Suggestion, p. 129, **6.1**, 1st sentence
Either change $\mathbb{R}$ to $\mathbb{R}^+ \cup \{0\}$, twice, or put the word 'non-negative', which is placed right before the word 'real', right before the word 'function'.
==
Suggestion, p. 130, **Open balls, the topology on a pseudo-metric space** X**.**, 1st sentence
Concerning "..., the **(open) ball of radius** r about x ...", the word 'about' should be written in boldface.
==
Comment, p. 131, the sentence that precedes **An example.**,

"Clearly a Lipschitz map is uniformly continuous."

In fact, for $\epsilon > 0$, consider $\delta \leq \frac{\epsilon}{C+1}$.
==
Comments, p. 132

- **Identifying points at zero distance.**, "It is also open, that is, it maps open sets to open sets."
  If A is open and $\overline{\{x\}} \in A/R$, then $B_r\left(\overline{\{x\}}\right) \subset A/R$ for each $B_r(x) \subset A$.[11] Therefore A/R is open.
- **Prop. 6.1.1**
  $F = p \circ f$ where $p : f(Y) \ni x \mapsto \overline{\{x\}} \in f(Y)/R$.

==
Comment, p. 133, **Complete metric spaces.**, $d\left(\{x_n\}, \{y_n\}\right) := \lim_{n\to\infty} d\left(x_n, y_n\right)$
Since $\mathbb{R}$ is complete, the Cauchy sequence $\{d\left(x_n, y_n\right)\}$ converges.[12] Therefore $d_{X_{seq}}$ is well-defined.
==
Comment, p. 134, last sentence before the definition of a **contraction** map,

"The inequality will continue to hold for this value of C which is known as the Lipschitz constant of f and denoted by $\text{Lip}(f)$."

In fact, on the one hand,

$$d_Y(f(x), f(x)) = 0 = \text{Lip}(f) d_X(x, x) \quad \forall x \in X.$$

On the other hand, as a lower bound of $\{C \,|\, d_Y(f(x_1), f(x_2)) \leq C d_X(x_1, x_2)\, \forall x_1, x_2 \in X\}$,

$$\frac{d_Y(f(x_1), f(x_2))}{d_X(x_1, x_2)} \leq \text{Lip}(f) \quad \forall x_1, x_2 \in X \text{ with } x_1 \neq x_2.$$

==
Erratum/Comment, p. 136

- **Cor. 6.3.1**, *Proof*
  Prop. should be Theo., twice.

[11] In fact, since $d\left(\overline{\{x\}}, \overline{\{y\}}\right) = d(x, y)$, $d\left(\overline{\{x\}}, \overline{\{y\}}\right) < r$ whenever $d(x, y) < r$.

[12] The basic idea to prove that the sequence is Cauchy is based on

$$\begin{aligned} |d\left(x_n, y_n\right) - d\left(x_m, y_m\right)| &= |d\left(x_n, y_n\right) - d\left(y_n, x_m\right) + d\left(y_n, x_m\right) - d\left(x_m, y_m\right)| \\ &\leq |d\left(x_n, y_n\right) - d\left(y_n, x_m\right)| + |d\left(y_n, x_m\right) - d\left(x_m, y_m\right)| \\ &\leq d\left(x_n, x_m\right) + d\left(y_n, y_m\right), \end{aligned}$$

which follows from

$$|d(x,z) - d(z,y)| \leq d(x,y) \iff d(x,z) - d(y,z) \leq d(x,y),\; d(y,z) - d(x,z) \leq d(y,x).$$

for all x, y, z in X.

- **6.4**
Since (S, d_S) and (X, d_X) are metric spaces, then $(S \times X, d_{S\times X})$ is a metric space with

$$d_{S\times X}((s_1, x_1), (s_2, x_2)) = d_S(s_1, s_2) + d_X(x_1, x_2)$$

for all points (s_1, x_1) and (s_2, x_2) in $S \times X$.

==

Errata/Comment, p. 137

- *Proof* of **Theo. 6.4.1**
  - In the middle of the proof, a) is missing right before $\leq$;
  - At the very end of the proof, change Prop. to Cor..

- "..., wish to conclude the existence of an inverse to F, ...", l. -10
Isn't a 1st person (subject) pronoun missing before the word wish?

- "... the mean value theorem ...",[13,14] ll. -5 and -4

If V and W are normed linear spaces, and if F is from $B_r(\alpha) \subset V$ to W with F differentiable and $||dF_\beta|| \leq \epsilon$ for every $\beta \in B_r(\alpha)$, then $||F(\beta + \xi) - F(\beta)|| \leq \epsilon||\xi||$ whenever $\beta, \beta + \xi \in B_r(\alpha)$.

==

Erratum/Comments/Suggestion, **Theo. 6.5.1** and its *Proof*, p. 138

- "..., $\text{Lip}[v] < \lambda < 1$.", 1st sentence of the proof
Either the first $<$ should be the equal sign $=$ or the $=$ of (6.2) on p. 137 should be the inequality sign $<$.

- 3rd sentence of the proof
In fact,

$$\begin{aligned}(\text{id} + v) \circ (\text{id} + w) = \text{id} &\Leftrightarrow \text{id} \circ (\text{id} + w) + v \circ (\text{id} + w) = \text{id}\\ &\Leftrightarrow \text{id} + w + v \circ (\text{id} + w) = \text{id}\\ &\Leftrightarrow w = -v \circ (\text{id} + w).\end{aligned}$$

- "Let X be the space of continuous maps of $\overline{B_s(0)} \to E$ ...", 4th sentence of the proof
Use either

"Let X be the space of continuous maps u from $\overline{B_s(0)}$ to E ..."

or

"Let X be the space of continuous maps $u : \overline{B_s(0)} \to E$ ...".

- "Then X is a complete metric space relative to the sup norm, ...", 5th sentence of the proof
The completeness of X follows from a classical result of Functional Analysis provided that E is complete. In fact, by hypothesis, E is a Banach space.

==

Errata/Suggestions/Comment, p. 139

- **The setup.**, 3rd sentence
"... and y ranges over <u>and</u> open ball ..." should be "... and y ranges over <u>an</u> open ball ...".

[13]See **ADVANCED CALCULUS** (revised edition) by Lynn H. Loomis and Shlomo Sternberg, **Theo. 7.4**, p. 149.
[14]For later use in **Chapter 8**, call it MVT.

- **The method.**, 2nd, 3rd and last sentences
  - "... map from $A \times B \to Y$ ..." should be "... map from $A \times B$ to Y ...";
  - $K(x, y)y = y$ should be $K(x, y) = y$;
  - "The mean value theorem ...", l. -1
    See the previous footnote.

==

Suggestion/Errata, p. 140
Concerning the underlined text:

- "... $||K(x, y_1) - K(x, y_2)|| \leq ||y_1 - y_2||\underline{C}, \quad 0 \leq C < 1, \ \forall \underline{s \in S}$, ..." should be

$$\text{“... } ||K(x, y_1) - K(x, y_2)|| \leq \underline{C}||y_1 - y_2||, \quad 0 \leq C < 1, \ \forall \underline{y_1, y_2 \in B}, \text{ ...”};$$

- "... $y_x \in B$ *such that* $K(x, \underline{y}) = y_x$, ..." should be "... $y_x \in B$ *such that* $K(x, \underline{y_x}) = y_x$, ...".

==

Erratum, p. 141, l. 4
Concerning "..., and the the right ...", delete the extra 'the'.

==

Comment/Errata, p. 142

- 1[st] sentence
  - Notice that F is bounded on $\overline{L \times U}$, which contains $L \times U$;
  - Based on the rest of the proof from here, $L = J$.
- At the very end of the proof, $\delta(m + cr) \leq 1$ should be $\delta(m + cr) \leq r$.

==
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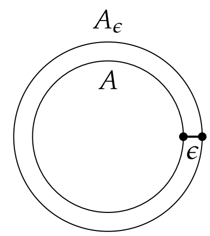
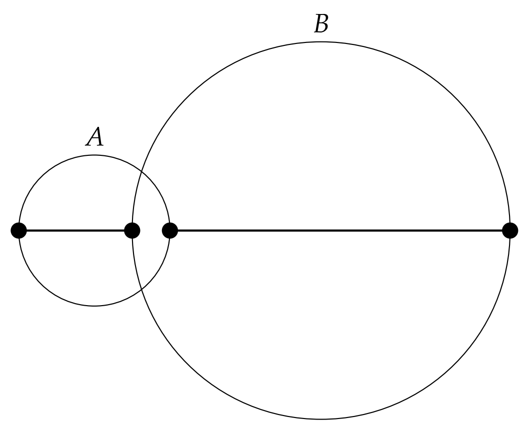
7
==
==
Comments, pp. 143–4

- Illustration for A_ϵ (where A is a closed ball):



- $d(A,B)$ and $h(A,B)$ are well-defined. In fact, the maxima are achieved since both A and B are bounded.
- Illustration for $d(A,B) \neq d(B,A)$ (where A and B are closed balls):



- **Proof of (7.3).**
  Using set-builder notation and propositional logic, we have
  $$\{\epsilon \mid A \subset C_\epsilon, B \subset D_\epsilon, C \subset A_\epsilon \text{ and } D \subset B_\epsilon\} \subset \{\epsilon \mid A \cup B \subset (C \cup D)_\epsilon \text{ and } C \cup D \subset (A \cup B)_\epsilon\}.$$
  Therefore, the infimum of the latter is less than or equal to the infimum of the former.

==
Comment/Suggestion, p. 145

- **A sketch of the proof of completeness.**
  $A \neq \emptyset$. In fact, since $\{A_n\}$ is Cauchy with respect to h, by selecting a subsequence if necessary, we may assume that, for each index n,
  $$h(A_n, A_{n+1}) < 2^{-n},$$
  that is,
  $$A_n \subset (A_{n+1})_{2^{-n}} \quad \text{and} \quad A_{n+1} \subset (A_n)_{2^{-n}}.$$
  Then, for any natural number N, there is a sequence $\{x_n\}_{n \geq N}$ in (X,d) such that $x_n \in A_n$ and $d(x_n, x_{n+1}) < 2^{-n}$ for each index $n \geq N$. Any such sequence is Cauchy with respect to d and thus converges to some $x \in X$ (due to the completeness of (X,d)).
- **7.1.1**, at the very end of the 2nd line
  'K' should be 'K'.

==
Suggestion/Errata, p. 146, **Theorem 7.2.1**

- Put 'X' right after '*space*';
- '*Lifschitz*' should be '*Lipschitz*';
- '$K_n(a)$' should be '$K_n(A)$';
- At the very end, an '*i*' is missing in '*Lipschtz*'.

==
Errata, p. 148, **7.3.2**

- 1st line after $A_1 = TA_0$
  "... be deleting ..." should be "... by deleting ...";
- 4th line after $A_1 = TA_0$
  'sussive' should be 'successive';
- 7th line after $A_1 = TA_0$
  Put '=' right after 'B_0'.

==
Comment, p. 153, ll. -3 and -2, "..., $M_{t,\epsilon}$ is non-decreasing as $\epsilon \to 0$, ..."
In fact, if $\epsilon \leq \delta$, then a countable cover $\mathcal{B}_\epsilon$ by balls of radius at most ϵ is also a countable cover $\mathcal{B}_\delta$ by balls of radius at most δ. Thus the set of all $m_t(\mathcal{B}_\epsilon)$ is a subset of the set of all $m_t(\mathcal{B}_\delta)$. Therefore

$$M_{t,\epsilon} \geq M_{t,\delta}.$$

==
==
8
==
==

Comment, p. 159, 1st sentence

The theory presented here is built on complete metric spaces, which were defined in chapter 6. So, it is worth recalling that, a Banach space is a normed linear space which is complete with respect to the metric derived from its norm.

==

Erratum/Comment, p. 160

- The equation

$$L(u) := Au - u \circ (A + \phi)$$

is displayed twice. So (8.6) should be the equation number to

$$L(u) = \phi - \psi(\mathrm{id} + u)$$

and the 2nd sentence that starts with

'So we wish ...'

should be deleted.

- l. -1

$$\begin{aligned} \left|\left|L^{-1}\right|\right| \cdot \mathrm{Lip}[\psi] &< \frac{||A^{-1}||}{(1-a)} \cdot \epsilon \\ &< \frac{||A^{-1}||}{1-a} \cdot \frac{1-a}{||A^{-1}||} \\ &< 1 \end{aligned}$$

$$\Downarrow$$

$$2\left|\left|L^{-1}\right|\right| \cdot \mathrm{Lip}[\psi] - \left|\left|L^{-1}\right|\right| \cdot \mathrm{Lip}[\psi] < 1$$

$$\Downarrow$$

$$\begin{aligned} \left|\left|L^{-1}\right|\right| \cdot \mathrm{Lip}[\psi] &< \frac{||L^{-1}|| \cdot \mathrm{Lip}[\psi] + 1}{2} := c \\ &< 1. \end{aligned}$$

==

Errata/Comment, p. 161

- At the very begining, there is a subjacent hypothesis for the conclusion of the existence and uniqueness of the solution to (8.5): X is complete.

- $\left|\left|A^{-1}\right|\right|$, right before the sentence which ends in (8.8), should be $\left|\left|M^{-1}A^{-1}\right|\right|$.

- **The map $A + \phi$ is injective.**

  - The 1st '$\geq$', l. -5, comes from

$$\begin{aligned} ||x|| &= \left|\left|A^{-1}Ax\right|\right| \\ &\leq \left|\left|A^{-1}\right|\right| ||Ax||; \end{aligned}$$

– The 2nd '$\geq$', l. -3, comes from

$$\begin{aligned}||Ax + \phi(x) - Ay - \phi(y)|| + \text{Lip}[\phi]||x-y|| &\geq ||A(x-y) + \phi(x) - \phi(y)|| + ||\phi(x) - \phi(y)|| \\ &\geq ||A(x-y) + \phi(x) - \phi(y) + \phi(y) - \phi(x)|| \\ &\geq ||A(x-y)|| \\ &\geq \frac{1}{||A^{-1}||}||x-y||;\end{aligned}$$

– The 3rd '$\geq$', l. -2, comes from

$$\begin{aligned}\text{Lip}[\phi] < \frac{1-a}{||A^{-1}||} &\Rightarrow \frac{1-a}{||A^{-1}||} - \text{Lip}[\phi] > 0 \\ &\Rightarrow \frac{a}{||A^{-1}||} + \frac{1-a}{||A^{-1}||} - \text{Lip}[\phi] > \frac{a}{||A^{-1}||};\end{aligned}$$

– '(8.4.', l. -1
Put a parentheses right after '4'.

==

Errata/Comments, p. 162

- **The map $A + \phi$ is surjective with continuous inverse.**
  – Remove the period for estimate (8.4).
  – The contraction argument follows from

$$\begin{aligned}\left|\left|A^{-1}(y - \phi(x_1)) - A^{-1}(y - \phi(x_2))\right|\right| &= \left|\left|A^{-1}(\phi(x_2) - \phi(x_1))\right|\right| \\ &\leq \left|\left|A^{-1}\right|\right| \, ||\phi(x_2) - \phi(x_1)|| \\ &\leq \left|\left|A^{-1}\right|\right| \text{Lip}[\phi]||x_2 - x_1|| \\ &\leq \left|\left|A^{-1}\right|\right| \epsilon||x_2 - x_1|| \\ &\leq (1-a)||x_1 - x_2||.\end{aligned}$$

  – 3rd line after estimate (8.4)
  Change "... $A + \phi$ a ..." to "... $A + \phi$ is a ...".

- Concerning the item that completes the proof of the lemma, notice that:
  – $NN^{-1}f = A^{-1}A_s f \circ (A+\phi)^{-1} \circ (A+\phi) = f$ and $N^{-1}Nf = A_s A^{-1} f \circ (A+\phi) \circ (A+\phi)^{-1} = f$;
  – Via the sup norm on Y,

$$\left|\left|N^{-1}f\right|\right| = \sup\left\{\left|\left|N^{-1}fx\right|\right| \,:\, x \in E\right\} \text{ and } ||f|| = \sup\left\{||fy|| \,:\, y \in E\right\}.$$

  Hence, since

$$\begin{aligned}\left|\left|N^{-1}fx\right|\right| &= \left|\left|A_s f \circ (A+\phi)^{-1}x\right|\right| \\ &\leq ||A_s|| \cdot \left|\left|f \circ (A+\phi)^{-1}x\right|\right| \\ &\leq a||f||,\end{aligned}$$

  it follows that $\left|\left|N^{-1}f\right|\right| \leq a||f||$.

==

Comment, pp. 162–3

Concerning the convergence, the geometric series corresponding to $\left(I - N^{-1}\right)^{-1}$ used $\left|\left|N^{-1}\right|\right|| \leq a$, which was obtained by

$$N^{-1}f = A_s f \circ (A+\phi)^{-1},\ ||A_s|| \leq a \Rightarrow \left|\left|N^{-1}f\right|\right| \leq a||f||.$$

Similarly, the geometric series corresponding to $(I-Q)^{-1}$uses $||Q||| \leq a$, which is obtained by

$$Qg = A_u^{-1} g \circ (A+\phi),\ \left|\left|A_u^{-1}\right|\right| \leq a \Rightarrow ||Qg|| \leq a||g||.$$

==

Errata/Comment, p. 163

- Before the completion of the proof of the 1st part of **Theorem 8.1.1**:
  - $||M_u||$ and $||M||$ should be $\left|\left|M_u^{-1}\right|\right|$ and $\left|\left|M^{-1}\right|\right|$, respectively;
  - $M^{-1} = M_s^{-1} \oplus M_u^{-1}$ has the maximum property, that is,

$$\left|\left|M^{-1}\right|\right| = \max\left\{\left|\left|M_s^{-1}\right|\right|, \left|\left|M_u^{-1}\right|\right|\right\}.$$

- Right after the completion of the proof of the 1st part of **Theorem 8.1.1**:
  Since $X \ni u \mapsto \mathcal{F}_0(u) := u(0) \in E$ is a continuous linear functional, if $Ku = u$, then

$$K^n u_0 \to u \Rightarrow \mathcal{F}_0(K^n u_0) \to \mathcal{F}_0(u) = u(0)$$

  for any point $u_0 \in X$.[15] In particular, if $u_0(0) = 0$, then $K^n u_0(0) = 0$ for each index n. So $u(0) = 0$.

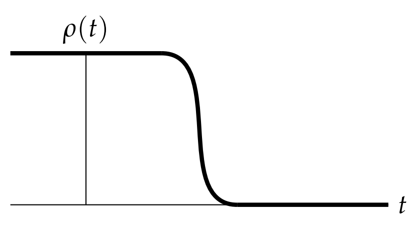
- 1st line of **8.1.2**
  Remove 'differentiable,'.

==

Errata/Suggestions/Comment, p. 164

- **Plan of the proof.**, 1st line after '$K > 2$.'
  Remove the first 'the'.

- **Checking the Lipschitz constant.**
  - The next figure illustrates the graph of a possible $\rho(t)$ and has three parts:
    1. a horizontal semi-line with endpoint $(0.5, 1)$;
    2. a curve from $(0.5, 1)$ to $(1, 0)$ which leaves at an angle of 0° and arrives at an angle of 180°;
    3. a horinzontal semi-line with endpoint $(1, 0)$.



    Notice that $\rho(t) \leq 1$ for all $t \in \mathbb{R}$.
  - l. -2 follows from the MVT (see my footnote 14) since

$$\left|\rho'(t)\right| < K\ \forall t \in \mathbb{R},\ \left|\ ||x_1|| - ||x_2||\ \right| \leq ||x_1 - x_2||,\ ||d\phi_x|| < \frac{\epsilon}{2K}\ \forall x \in B(0,r) \text{ and } \rho\left(\frac{||x_2||}{r}\right) \leq 1.$$

  - l. -1
    Change '$||x_1 - x_2|$' to '$||x_1 - x_2||$'.

[15] See the *Proof* of **Theorem 6.3.1**, p. 135.

==
Erratum, p. 165, last line before (8.10)
Change "... are ..." to "... be ...".
==
Comments/Suggestion/Errata, p. 166

- **More details in the linear case.**
  - As far as $||A^n x|| = ||A^n x_s|| + ||A^n x_u||$ is concerned, if E is a Banach space with a norm $||\cdot||_E$, then

    $$x_s + x_u \mapsto ||x_s + x_u||_\oplus := ||x_s||_E + ||x_u||_E$$

    defines a norm on E.
  - Consider $c = \frac{1}{a}$ since

    $$||x_u|| = \left|\left|A_u^{-1}A_u x_u\right|\right| \leq \left|\left|A_u^{-1}\right|\right| \cdot ||A_u x_u|| \leq a\,||A_u x_u|| \Rightarrow ||Ax_u|| \geq \frac{1}{a}\,||x_u||\,.$$

    Furthermore, observe that $c > 1$.

- **Back to the general case.**
  - 3rd sentence
    "... and homeomorphism ..." should be "... and **a** homeomorphism ...".
  - 4th sentence
    Firstly, change:
    * 'U' to 'B_r';
    * '$h(x) \in S(A)$' to '$h(x) \in S \cap V$'.

    Now, notice that r and V can be taken such that V is bounded.[16] Then, as B_r in the linear case,

    $$A^n y \in V \ \ \forall n \geq 0 \Leftrightarrow y \in S \cap V.$$

    Therefore the 4th sentence can be rewritten as

    $$\begin{aligned} f^n(x) \in B_r \ \forall n \geq 0 &\Leftrightarrow A^n h(x) \in V \ \forall n \geq 0 \\ &\Leftrightarrow h(x) \in S \cap V \\ &\Leftrightarrow A^n h(x) \to 0. \end{aligned}$$

    (Concerning the last $\Leftrightarrow$, the proof of $\Rightarrow$ follows from (8.11), p. 165, since $S = W^s(0, A)$. On the other hand, since $h(x) \in V$ by definition, it remains to show that $h(x) \in S$ in order to prove $\Leftarrow$. In fact, if $h(x) = y_s \oplus y_u$, then $||A^n y_s|| + ||A^n y_u|| \to 0$. Hence $y_u = 0$, which means that $h(x) = y_s \in S$.)
  - l. -2
    In fact, H is the restriction of h^{-1} to $S \cap V$.

==
Errata/Comment, p. 167

- **An important remark.**
  - 2nd line before (8.14)
    "... $p \in f^{-n}[B_r^s(p)]$ ..." should be "... $x \in f^{-n}[B_r^s(p)]$ ...".
  - 1st line after (8.14)
    At the end of p. 166, it was proved that $B_r^s(p)$ is a topological submanifold via h, which is a homeomorphism. Then

    $W^s(p)$ is a topological submanifold

    by (8.14). **Theorem 8.2.1** confirms it with a map which is Lipschitz, a stronger condition than continuity.

[16] Consider $r' < r$, $V' := h\left(\overline{B_{r'}}\right)$ and the appropriate restriction of h.

- 4th line after (8.14)
  At the very end, change 'Her' to 'Here'.
- 6th line after (8.14), "... in[**?**,Shub]"
  * a space between 'in' and '[' is missing;
  * the bibliografical reference related to '**?**' is missing;
  * a period is missing right after ']'.
- l. -1
  At the very end, a 'c' is missing in 'charater'.

• **The Lipschitzian case.**, 5 lines before (8.15)
Change 's' to 'r'.

===

Errata/Suggestions/Comment, p. 168

• **Theo. 8.2.1.**

- l. 1
  Concerning $\epsilon(a)$ and $\delta(a, \epsilon, r)$, change 'a' to 'c' (since the bound a, p. 167, is replaced by c from now on).
- l. 3
  '$g : E_u(r) \to E_s(r)$' should be '$g : U(r) \to S(r)$'.
- l. 6
  Change 'f^{-1}' to 'f'.
- l. 9
  * Change '$U(p)$' to '$U(r)$';
  * It is worth noting that, as a "cograph",

$$\text{graph}(g) \subset S(r) \times U(r),$$

which is identified with

$$S(r) \oplus U(r) \subset S \oplus U.$$

Furthermore, graph(g) has the subspace topology inherited from E and it has a single coordinate chart given by

$$(\text{graph}(g), p_u)$$

where p_u is the projection onto U. On the other hand, $p_u^{-1}(x) = (g(x), x)$ is continuous if and only if g is continuous, which is the case here since g is Lipschitz by *(i)*.
To summarize: graph(g) is a topological submanifold.

• **The idea of the proof.**

- l. 4
  A comma is used to represent 'such that', which is usually represented by a vertical bar (|) or a slash (/) or a colon (:). So, it should be, for example,

$$\text{graph}(v) = \{(v(x), x) \mid x \in U(r)\}.$$

- l. 6
  The shorthand notation represents

$$f[\text{graph}(v)] = \{f(v(x), x) \mid x \in U(r)\} = \{(f_s(v(x), x), f_u(v(x), x)) \mid x \in U(r)\}.$$

- l. -9
  Change "... map of $U(r) \to E$ ..." to "... map $U(r) \to E$ ..." or "... map of $U(r)$ to E ...".
- l. -7
  Change "... map of $U(r) \to U$." to "... map $U(r) \to U$." or "... map of $U(r)$ to U."

- l. -6
  It should be

$$f[\text{graph}(v)] = \left\{ \left(f_s \circ (v, id)([f_u \circ (v, id)]^{-1}(y)), y \right) \,\middle|\, y \in U(r) \right\} = \text{graph} \left[G_f(v) \right].$$

==

Errata/Suggestion, p. 169

- 2 lines before **Lemma 8.2.1**
  "... direc ..." should be "... direct ...".
- **Lemma 8.2.2**
  - 2 lines before (8.18)
    '$f_u \circ (v, id) : E_u(r) \to E_u$' should be '$f_u \circ (v, id) : U(r) \to U$'.
  - 2 lines after (8.18)
    - '$f_u - A_u$' should be '$f_u \circ (v, id) - A_u$';
    - The last '$<$' should be '$\leq$'.
  - 3 lines after (8.18)
    "By the Lipschitz implicit function theorem ..." should be

    "By the Lipschitz inverse function theorem, Theorem 6.5.1, ...".

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Erratum/Comment, p. 176, **9.1.1**

- 1st sentence
  "... matrix square ..." should be "... square matrix ...".
- last two sentences
  Since $T \geq 0$ by definition, $T^l \geq 0$ for each non-negative integer l. Therefore, once K is big enough, the irreducibility hypothesis guarantees that, for any i, j, there is a $k = k(i, j)$ such that

$$\left[(I+T)^K\right]_{ij} = \left[\sum_{\substack{l=0 \\ l \neq k}}^{K} \binom{K}{l} \left(T^l\right)_{ij}\right] + \binom{K}{k} \left(T^k\right)_{ij}$$

  is positive.

==
Comment, p. 176, **Theorem 9.1.1.***5*
As usual, '$S \leq T$' means that $T - S$ is a non-negative matrix.
==
Comments, p. 177

- (9.1)
  Consider $j \in \{1, \ldots, n\}$ and

$$\frac{(Tz)_j}{z_j} = \min\left\{\frac{(Tz)_i}{z_i} \,:\, 1 \leq i \leq n \text{ with } z_i \neq 0\right\}.$$

  Since $z \geq 0$ and, by definition, $T \geq 0$, it follows that $(Tz)_i \geq 0$ for each $i \in \{1, \ldots n\}$. So, whatever i we choose,

$$\frac{(Tz)_j}{z_j} \cdot z_i \leq (Tz)_i$$

  holds, even if $z_i = 0$. Thus

$$\frac{(Tz)_j}{z_j} z \leq Tz.$$

  Then $\{s \,:\, sz \leq Tz\} \neq \emptyset$ and

$$\frac{(Tz)_j}{z_j} = \max\{s \,:\, sz \leq Tz\}.$$

  In fact, suppose otherwise. Hence, there is a number s such that $Tz \geq sz$ and $s > \frac{(Tz)_j}{z_j}$. Therefore

$$\begin{aligned}(Tz)_j &\geq sz_j \\ &> (Tz)_j\,!\end{aligned}$$

- 1st sentence after (9.1)
  Whenever convenient, as far as $L(z)$ is concerned, we may restrict our attention to the case where $z \in C$. As an example, a maximum value of L on C is, in fact, the maximum value of L on all of Q.

==
Erratum/Comment, p. 178

- l. 4
  '$|\lambda|y_i|$' should be '$|\lambda||y_i|$'.

- **Showing that $\lambda_{\max}(T^\dagger) = \lambda_{\max}(T)$.**
  This result is clearly correct since taking transpose does not change the eigenvalues of a matrix. However, in order to show that the result holds, the book's argument is made in such a way that it is also used for **Proving the first two assertions in item 4 of the theorem.**, p. 179.

==
Comments/Errata/Suggestion, p. 179

- **Proving the first two assertions in item 4 of the theorem.**
  - 1st paragraph
    * The subjacent hypothesis is that w is as described on p. 178;
    * At the very end, remove the extra 'then' in "... then then ...".
  - 2nd paragraph
    Change '$n-1$' to 'k'.
  - last paragraph
    * Concerning "If $\mu = \lambda_{\max}$ then $w^\dagger(Ty - \lambda_{\max}y) = 0$ but $Ty - \lambda_{\max}y \le 0$ ...", only '$Ty - \lambda_{\max}y \le 0$' comes from the hypothesis '$\mu = \lambda_{\max}$', whereas '$w^\dagger(Ty - \lambda_{\max}y) = 0$' has to do with the first displayed formula of p. 179;
    * '4)' and '2)' should be '4.'and '2.'.

- In order to maintain the notation compatible with **Theorem 9.1.1.**_6_ and p. 180, in the last two paragraphs,

  'T_i' and 'Λ_i' should be '$T_{(i)}$' and '$\Lambda_{(i)}$'.

==
Comments/Erratum, p. 180

- 1st displayed formula
  See $\lambda I - T$ as a parametric curve with parameter λ, det as a real function of n^2 real variables and $\det(\lambda I - T)$ as the composite function $(\det \circ \text{curve})(\lambda)$. Therefore

$$\begin{aligned}\frac{d}{d\lambda}\det(\lambda I - T) &= \nabla \det(\lambda I - T) \cdot \frac{d}{d\lambda}\text{curve}(\lambda) \\ &= \nabla \det(\lambda I - T) \cdot I,\end{aligned}$$

  where both factors can be seen as vectors in $\mathbb{R}^{n^2}$. Furthermore, notice that

$$(\nabla \det(\lambda I - T))_{ii} = \frac{\partial}{\partial \lambda_i}\det(\lambda I - T)$$

  for each $i = 1, \ldots, n$.

- **Showing that $\lambda_{\max}$ has algebraic (and hence geometric) multiplicity one.**
  - 1st sentence
    Since T is an $n \times n$ matrix, if $\lambda_{i,1}, \ldots, \lambda_{i,n-1}$ are the eigenvalues of $T_{(i)}$ and $j \in \{1, \ldots, n-1\}$, then:
    * $\left|\lambda_{i,j}\right| < \lambda_{\max}$ by **Theorem 9.1.1.**_6_;
    * $\det\left(\lambda I - T_{(i)}\right) = \prod_{j=1}^{n-1}\left(\lambda - \lambda_{i,j}\right)^{n_j}$, where n_j is a positive integer which depends on $\lambda_{i,j}$.

    Therefore, since the roots of a polynomial with real coefficients occur in conjugate pairs,

$$\det\left(\lambda_{\max} I - T_{(i)}\right) > 0.$$

– '2)' should be '2.'.[17]

- **Proof of the last assertion of the theorem.**, 1st sentence
  Clearly, $T^k y = \lambda^k y$ if $Ty = \lambda y$.

==
Errata, p. 181

- l. 2
  A period is missing right after 'x';
- l. -6
  A double apostrophe is missing right after 'v_j'.

==
Comment/Erratum, p. 182

- **Paths and powers.**, 2nd paragraph
  (The result follows by induction on ℓ.) The case $\ell = 1$ is trivial. Now suppose that the result holds for some $\ell > 1$, so that the entries of A^ℓ are as claimed. Consider any path of lenght $\ell + 1$ from v_j to v_i. Hence there is an adjacent vertex, v_k, to v_i on this path. Delete v_i. So the remaining path is a path of lenght ℓ from v_k to v_j. By induction, the number of such paths is given by $\left(A^\ell\right)_{kj}$. On the other hand, each such v_k corresponds to a 1 for A_{ik}. Now consider

$$\begin{aligned}\left(A^{\ell+1}\right)_{ij} &= \left(AA^\ell\right)_{ij} \\ &= \sum_{k=1}^{n} A_{ik} \left(A^\ell\right)_{kj}\end{aligned}$$

  with A of order n.

- **9.2.2**, l. 6
  'postive' should be 'positive'.

==
Comments/Erratum, p. 183

- 2nd paragraph, 1st sentence, "The paths ... have total lengths at most $3(n-1)$."
  In fact, $V = \{v_1, \ldots, v_n\}$.

- **Proof of the lemma.**

  – Concerning the equivalence '$\Longleftrightarrow$', it may seem that the '$\Longrightarrow$' part does not hold for $i \geq b$. However, since j is an arbitrary non-negative integer, we may consider $j = \alpha a + \beta$ with α and β non-negative integers. Therefore, for $i = r$ with $0 \leq r < b$,

$$\begin{aligned}n &= ra + (\alpha a + \beta)b \\ &= (\alpha b + r)a + \beta b.\end{aligned}$$

  – Before the last sentence, in "... mod b, So ...", the comma should be a period.

- **The Frobenius form of an irreducible non-primitive matrix.**

  – $C_i \neq \emptyset \; \forall i$.
  In fact, consider $u_0 = v$. Since irreducibility is equivalent to strong connectedness, there is a path joining u_0 to u_0. Call it u_0-cycle. By the definition of p, the length of u_0-cycle is an integer multiple of p. Then $u_0 \in C_0$. Now consider $u_1 \neq u_0$ on u_0-cycle such that u_1 is adjacent to u_0. Specifically, suppose that u_1 goes to u_0. Clearly $u_1 \in C_1$. Next consider $u_2 \notin \{u_0, u_1\}$ on u_0-cycle such that u_2 is adjacent to u_1. (Notice that u_2 goes to u_1.) Clearly $u_2 \in C_2$. Similar reasoning shows that $u_3 \in C_3, \ldots, u_{p-1} \in C_{p-1}$.

[17] See **Theorem 9.1.1**.2.

- Concerning the first sentence after the definition of C_i, suppose there is a vertex $u \in C_{i_1} \cap C_{i_2}$ with $i_1 \neq i_2$. Hence there are two paths from u to v of lengths n_1 and n_2 with
$$n_j \equiv i_j \mod p, \quad j = 1, 2. \tag{*}$$
On the other hand, since A is irreducible, there is a path of length n joining v to u. Now, if we combine each one of the paths joining u to v with the path joining v to u, by the definition of p, we get two cycles of lengths
$$n_j + n \equiv 0 \mod p, \quad j = 1, 2.$$
Therefore
$$n_1 \equiv n_2 \mod p,$$
which is a contradiction since the remainders on dividing n_1 and n_2 by p are i_1 and i_2, respectively, by (*).
- If $u \in C_0$, there is no path of length 0 from u to v.
In fact, there is no cycle of length 0 by the definition of p!

==

Comment, pp. 183–4, **The Frobenius form of an irreducible non-primitive matrix.**, from "This means ..." on
Let (u, w) be the first edge in a path from u to v. By the convention for edges, p. 181, the edge goes from $u = v_j$ to $w = v_i$ with $A_{ij} \neq 0$. Then u is related to the j-th column of A, whereas w is related to the i-th row of A. Therefore, after relabeling the vertices as required on p. 183, the vertices in C_k are related to consecutive columns of PAP^{-1}, $k = 0, \ldots, p-1$. So let us analyze the $p = 4$ example (since it explains the general case) where C_k is related to A_k, $k = 1, 2, 3$, and
$$C_0 = \left\{v_1, \ldots, v_{\#(C_0)}\right\}$$
is related to A_4.

- First, $\left(PAP^{-1}\right)_{1j} = 0$, $j = 1, \ldots, \#(C_0)$. In fact, suppose otherwise. Hence there exists such an index j for which $\left(v_j, v_1\right)$ is an edge. However, since $v_1 \in C_0$, there is a path of length
$$n \equiv 0 \mod p$$
from v_1 to v. So there is a path of length
$$n + 1 \equiv 1 \mod p$$
from v_j to v. Then $v_j \in C_0 \cap C_1$, which is a contradiction since $C_0 \cap C_1 = \emptyset$.
- Second, as in the previous case, if $i \in \{2, \ldots, \#(C_0)\}$, then $\left(PAP^{-1}\right)_{ij} = 0$, $j = 1, 2, \ldots, \#(C_0)$.

So the first $\#(C_0)$ rows of PAP^{-1} rule out the elements of C_0 as ending vertices of edges whose starting vertices are such elements. Therefore A_4 is the bottom left block of PAP^{-1}.

- Finally, similar arguments show that A_k is where it should be, $k = 1, 2, 3$, once that the $[n - (\#(C_0))] \times [n - (\#(C_0))]$ bottom right block of PAP^{-1} is null. In fact, suppose otherwise. Therefore we get to $C_k \cap C_\ell \neq \emptyset$ for some $\ell \neq k$!

> Right from the beginning of the previous analysis, the use of PAP^{-1} is an abuse of notation since the block decomposition, which can be written as PAP^{-1}, is obtained as the conclusion of the preceding arguments.

Now let s be the size of a matrix A. If $\wp$ is a permutation of $S = \{1, \ldots, s\}$, there is a permutation matrix P associated to $\wp$. In fact,
$$P = \sum_{i=1}^{s} e_{\wp(i)i}$$
where the i-th term of the sum is the unit matrix which has a 1 in the $\wp(i), i$ position as its only nonzero entry, $i = 1, \ldots, s$. Left multiplication by P permutes the entries of a column vector X using $\wp$. For example, if $s = 3$,

i	1	2	3
$\wp(i)$	2	3	1

and

$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix},$$

then

$$\begin{aligned} P &= e_{\wp(1)1} + e_{\wp(2)2} + e_{\wp(3)3} \\ &= e_{21} + e_{32} + e_{13} \\ &= \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \end{aligned}$$

and

$$\begin{aligned} PX &= \begin{bmatrix} x_3 \\ x_1 \\ x_2 \end{bmatrix} \\ &= x_3 e_1 + x_1 e_2 + x_2 e_3 \\ &= x_3 e_{\wp(3)} + x_1 e_{\wp(1)} + x_2 e_{\wp(2)}. \end{aligned}$$

As a matter of fact, whatever s and $\wp$ we consider,

$$\begin{aligned} PX &= \left(\sum_{i=1}^{s} e_{\wp(i)i} \right) \left(\sum_{j=1}^{s} x_j e_j \right) \\ &= \sum_{i,j=1}^{s} x_j e_{\wp(i)i} e_j \\ &= \sum_{i=1}^{s} x_i e_{\wp(i)i} e_i \\ &= \sum_{i=1}^{s} x_i e_{\wp(i)} \end{aligned}$$

since, whenever possible,

$$e_{ij} e_k = \begin{cases} e_i & \text{if } j = k; \\ 0 & \text{if } j \neq k. \end{cases}$$

On the other hand, it is a well-known fact that $P^{-1} = P^t$. Hence, concerning PAP^{-1}, not only P acts on each column of A in the same way as $\wp$ acts on S, but also P^t acts on each row of PA as if it were a permutation of indices. In fact, right multiplication by P^t permutes the entries of an arbitrary $1 \times s$ row vector Y since

$$\begin{aligned} YP^t &= X^t P^t \\ &= (PX)^t \end{aligned}$$

if $X = Y^t$. The relabeling related to

$$\bigcup_{k=0}^{p-1} C_k$$

turns out to be a permutation $\wp$ with associated permutation matrix P such that, by construction, PAP^{-1} is in the block form as previously described.

===

Comment, p. 184, last paragraph before **Proposition 9.2.1**, 1st sentence
Consider $D = D(i)^k$, $E = D(i+1)^k$ and $T = (RS)^{k-1}R$. Hence

$$D = ST \text{ and } E = TS.$$

Furthermore, $d_{ij} = \sum_k s_{ik} t_{kj}$ is the (i, j)-entry of D, where s_{ik} and t_{kj} are entries of S and T, respectively, and a similar representation holds for an arbitrary entry of E. Therefore,

$$\begin{aligned}
\operatorname{trace}(D) &= \sum_i d_{ii} \\
&= \sum_i \sum_k s_{ik} t_{ki} \\
&= \sum_i \sum_k t_{ki} s_{ik} \\
&= \sum_k \sum_i t_{ki} s_{ik} \\
&= \sum_k e_{kk} \\
&= \operatorname{trace}(E).
\end{aligned}$$

==

Comments/Errata, pp. 185–6, **9.3**

- 1st paragraph
  - For the existence of y, see **Showing that** $\lambda_{\max}\left(T^{\dagger}\right) = \lambda_{\max}(T)$**.**, p. 178. Hence, if $A = T$, then $r = \eta$. Now consider $y = w^{\dagger}$.
  - A ')' is missing right after the word 'number'.
  - $y \cdot x = \alpha > 0$. If $\alpha \neq 1$, then $\left(\alpha^{-1} y\right) \cdot x = 1$.
- 2nd paragraph
  - 1st sentence
    - For A $n \times n$, R is the column space of

$$\begin{aligned}
x \otimes y^{\dagger} &:= \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \begin{bmatrix} y_1 & \cdots & y_n \end{bmatrix} \\
&= \begin{bmatrix} x_1 y_1 & \cdots & x_1 y_n \\ \vdots & \cdots & \vdots \\ x_n y_1 & \cdots & x_n y_n \end{bmatrix} \\
&= \begin{bmatrix} y_1 x & \cdots & y_n x \end{bmatrix}.
\end{aligned}$$

    - $H^2 = H$ follows immediately from the definition of outer product, along with the assumption that $y \cdot x = 1$.[18]
  - 2nd sentence
    In fact, $H(I - H) = 0$.
  - 3rd sentence
    Change y to $y^{\dagger}$.
  - 4th and 5th sentences
    - They are, in fact, the same sentence. Delete one of them!

---

[18] In fact,

$$\begin{aligned}
\left(x \otimes y^{\dagger}\right)\left(x \otimes y^{\dagger}\right) &= xyxy \\
&= xy \\
&= x \otimes y^{\dagger}.
\end{aligned}$$

* R and N are invariant under A. In fact, on the one hand, let $u \in R$. Hence $u = sx$ where s is a scalar. Then

$$\begin{aligned} Au &= sAx \\ &= (sr)x \in R. \end{aligned}$$

On the other hand, let $v \in N$. Thus $Hv = 0$. So

$$\begin{aligned} HAv &= AHv \\ &= 0. \end{aligned}$$

Therefore $Av \in N$.

- 3rd paragraph, 1st sentence, 1st clause
  In fact, consider the Perron-Frobenius theorem.[19]

- **Theorem 9.3.1.**
  It means that, eventually, the k-iterate of a typical starting vector concerning the iteration matrix P, say $P^k v_0$ for k large enough, will lie in the direction of the positive eigenvector x associated with $\lambda_{\max} = r$.

==

Comment, p. 188, **An imprimitive Leslie matrix**
L is not primitive since its natural powers have one of the following forms:

$$\begin{pmatrix} 0 & 0 & * \\ * & 0 & 0 \\ 0 & * & 0 \end{pmatrix}, \begin{pmatrix} 0 & * & 0 \\ 0 & 0 & * \\ * & 0 & 0 \end{pmatrix} \text{ and } \begin{pmatrix} * & 0 & 0 \\ 0 & * & 0 \\ 0 & 0 & * \end{pmatrix}.$$

==

Comment, p. 189, **9.5**

- 1st paragraph
  Let T, $\lambda_{\max}$ and x be as in **Theorem 9.1.1**. Therefore, if $M = T$, since $\mathbf{1}$ is a multiple of x,[20] it follows that

$$\begin{aligned} Mx = \lambda_{\max} x &\Longrightarrow M(\alpha \mathbf{1}) = \lambda_{\max}(\alpha \mathbf{1}) \\ &\Longrightarrow M\mathbf{1} = \lambda_{\max} \mathbf{1} \\ &\Longrightarrow \lambda_{\max} = 1. \end{aligned}$$

- 2nd paragraph
  Consider

$$\mathbf{p}^\dagger = \begin{bmatrix} \pi_1 \\ \vdots \\ \pi_n \end{bmatrix}$$

  such that

$$\mathbf{p}M = \mathbf{p}.$$

  Without loss of generality, suppose

$$\sum_{i=1}^{n} \pi_i = 1.$$

  (If $\sum_1^n \pi_i = \pi$, replace $\mathbf{p}$ by $\mathbf{q} = \frac{1}{\pi}\mathbf{p}$.) Therefore, by **Theorem 9.3.1**,

$$\begin{aligned} \lim_{k \to \infty} \left(\frac{1}{1} M^k \right) &= \mathbf{1} \otimes \mathbf{p} \\ &= \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} \begin{bmatrix} \pi_1 & \cdots & \pi_n \end{bmatrix}. \end{aligned}$$

[19] Specifically, **Theorem 9.1.1**.*7*.
[20] See **Theorem 9.1.1**.*3*.

==

Errata, p. 190, **9.6.2**

- 2nd paragraph, 2nd sentence
  "... is dangling row ..." should be "... is a dangling row ...";
- Last sentence
  "... dangling mode ..." should be "... dangling node ...".

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Erratum, p. 200, **Non-real eigenvalues**
Concerning the diagonal of A, 0 should be a.
==
Comment, p. 202, **variation of constants formula**

$$\begin{aligned}\frac{d}{dt}\left(e^{-tH}x(t)\right) &= -He^{-tH}x(t) + e^{-tH}\frac{d}{dt}x(t) \\ &= -He^{-tH}x(t) + e^{-tH}\big(Hx(t) + f(t)\big) \\ &= e^{-tH}f(t)\end{aligned}$$

since $He^{-tH} = e^{-tH}H$. Therefore

$$e^{-tH}x(t) - x_0 = \int_0^t e^{-sH}f(s)ds.$$

Now multiply both sides of this equation by e^{tH}.
==
Comment, p. 211, (10.6)

$$\begin{aligned}\dot{y} &= \ddot{x} + \frac{d}{dt}F(x) \\ &= -f(x)\dot{x} - x + \dot{x}\frac{d}{dx}F(x) \\ &= -x\end{aligned}$$

since $\frac{d}{dx}F(x) = f(x)$ by property a. of f.
==