UNIVERSIDADE FEDERAL DO PARANÁ

Department of Mathematics

1st List of Exercises - Functional Analysis

1. Consider $1 \leq p < \infty$ and the function $\|\cdot\|_p : \mathcal{C}[0,1] \longrightarrow \mathbb{R}$ given by $\|f\|_p = \left(\int_0^1 |f(t)|^p dt\right)^{\frac{1}{p}}$, for all $f \in \mathcal{C}[0,1]$. Show that $||\cdot||_p$ is a norm in $\mathcal{C}[0,1]$.

2. Show that $\left(\mathcal{C}[0,1], \|\cdot\|_p\right)$ is not complete.

3. For $f \in \mathcal{C}(\mathbb{R})$, consider the set $A(f) = \{x \in \mathbb{R} \,|\, f(x) \neq 0\}$ and define the *support of* f (denoted by $supp(f)$) as $supp(f) = \overline{A(f)}$. Let $\mathcal{C}_c(\mathbb{R})$ be the space of all continuous real valued functions on $\mathbb{R}$ whose support is a compact subset of $\mathbb{R}$. Show that is a normed linear space with the *sup-norm* and that it is not complete.

4. Let $\mathcal{C}_0(\mathbb{R})$ be the space of all continuous real valued functions on $\mathbb{R}$ which *vanish at infinity, i e* if $f \in \mathcal{C}_0(\mathbb{R})$ then for all $\varepsilon > 0$ there exists a compact set $K_\varepsilon \subset \mathbb{R}$ such that $|f(x)| < \varepsilon$, for all $x \in K_\varepsilon^c$. Show that $\mathcal{C}_0(\mathbb{R})$ is a Banach space with the *sup-norm*. Also, show that $\mathcal{C}_c(\mathbb{R})$ is dense in $\mathcal{C}_0(\mathbb{R})$.

5. Let $\mathcal{C}^1[0,1]$ be the space of all continuous real valued functions on $[0,1]$ which are continuously differentiable on $(0,1)$ and whose derivatives can be continuously extended to [0,1]. For $f \in \mathcal{C}^1[0,1]$, define $\|f\|_* = \max\limits_{x\in[0,1]} \{|f(x)|, |f'(x)|\}$. Show that $\left(\mathcal{C}^1[0,1], \|\cdot\|_*\right)$ is a Banach space. State and prove an analogous result for $\mathcal{C}^k[0,1]$.

6. For $f \in \mathcal{C}^1[0,1]$, define $\|f\|_1 = \left(\int_0^1 (|f(x)|^2 + |f'(x)|^2) dx\right)^{\frac{1}{2}}$. Show that $\|\cdot\|_1$ defines a norm on $\mathcal{C}^1[0,1]$. The expression $|f|_1 = \left(\int_0^1 |f'(x)|^2 dx\right)^{\frac{1}{2}}$ defines a norm on $\mathcal{C}^1[0,1]$?

7. Let $V = \{f \in \mathcal{C}^1[0,1] | f(0) = 0\}$. Show that $|\cdot|_1$ defines a norm on V.

8. Let V be a Banach space with norm $\|\cdot\|_V$ and $X = \mathcal{C}([0,1]; V)$ the space of all continuous functions from $[0,1]$ into V. For $f \in X$, define $\|f\|_X = \max\limits_{x\in[0,1]} \|f(x)\|_V$. Show that $\|\cdot\|_X$ is well defined and it is a norm on X. Also, show that $(X, \|\cdot\|_X)$ is a Banach space.

9. Let $\mathcal{C}^1[0,1]$ be endowed with the norm $\|\cdot\|_*$ and $f \in \mathcal{C}[0,1]$ be endowed with the usual *sup-norm*. Show that $T : \mathcal{C}^1[0,1] \longrightarrow \mathcal{C}[0,1]$ given by $T(f) = f'$, is a continuous linear transformation and $\|T\| = 1$.

10. Let $\mathcal{C}[0,1]$ be endowed with its usual norm. For $f \in \mathcal{C}[0,1]$, define $T(f(t)) = \int_0^t f(s)\, ds, \quad t \in [0,1]$. For every $n \in \mathbb{N}$, evaluate $\|T^n\|$.

11. Let $T : \mathcal{C}_c(\mathbb{R}) \longrightarrow \mathbb{R}$ given by $T(f(t)) = \displaystyle\int_{-\infty}^{\infty} f(t)\,dt$. Show that T is well defined and that it is a linear functional on $\mathcal{C}_c(\mathbb{R})$. Is T continuous?

12. Let $\{t_i\}_{i=1}^n$ be given points in the closed interval $[0,1]$ and let $\{\alpha_i\}_{i=1}^n$ be given real numbers. For $f \in \mathcal{C}[0,1]$ define $T(f) = \displaystyle\sum_{i=1}^n \alpha_i f(t_i)$. Show that T is a continuous linear functional on $\mathcal{C}[0,1]$ and evaluate $\|T\|$.

13. Let $M_{n\times n}(\mathbb{C})$ be the linear space of the $n \times n$ complex matrices and let $\|\cdot\|_{p,n}$ denote the matrix norm induced by the vector norm $\|\cdot\|_p$ on $\mathbb{C}^n$, for $1 \leq p \leq \infty$. If $\mathbf{A} = (a_{ij}) \in M_{n\times n}(\mathbb{C})$ show that $\|\mathbf{A}\|_{1,n} = \displaystyle\max_{1\leq j\leq n}\left\{\sum_{i=1}^n |a_{ij}|\right\}$. State and prove an analogous result for $\|\mathbf{A}\|_{\infty,n}$.

14. Show that, for any matrix $\mathbf{A} \in M_{n\times n}(\mathbb{C})$, it holds $\|\mathbf{A}\|_{2,n} \leq \|\mathbf{A}\|_E \leq \sqrt{n}\|\mathbf{A}\|_{2,n}$, where $\|\mathbf{A}\|_E = \left\{\displaystyle\sum_{i,j=1}^n |a_{ij}|^2\right\}^{\frac{1}{2}}$.

15. Let $1 \leq p < q \leq \infty$. Show that $\ell^p \subset \ell^q$, and that, for all $x \in \ell^p$, $\|x\|_q \leq \|x\|_p$.

16. Let V be a Banach space and let $\{T_n\}$ be a sequence of continuous linear operators on V. Define $S_n = \displaystyle\sum_{k=1}^n T_k$. If $\{S_n\}$ is a convergent sequence in $\mathcal{B}(V)$, we say that the series $\displaystyle\sum_{k=1}^\infty T_k$ is *convergent* and the limit of the sequence $\{S_n\}$ is called the *sum of the series*. If $\displaystyle\sum_{k=1}^\infty \|T_k\| < \infty$, we say that the series $\displaystyle\sum_{k=1}^\infty T_k$ is *absolutely convergent*. Show that any absolutely convergent series is convergent.

17. Let V be a Banach space. If $T \in \mathcal{B}(V)$ is such that $\|T\| < 1$, show that the series $I + \displaystyle\sum_{k=1}^\infty T^k$ is convergent and that its sum is $(I - T)^{-1}$.

18. (a) Let V be a Banach space and let $T \in \mathcal{B}(V)$. Show that the series $I + \displaystyle\sum_{k=1}^\infty \frac{T^k}{k!}$ is convergent. The sum is denoted $\exp(T)$.

(b) If $T, S \in \mathcal{B}(V)$ are such that $TS = ST$, show that $\exp(T+S) = \exp(T)\exp(S)$.

(c) Deduce that $\exp(T)$ is invertible for any $T \in \mathcal{B}(V)$.

(d) Let $A = \begin{bmatrix} \alpha & -\beta \\ \beta & \alpha \end{bmatrix}$, where α and β are real numbers. Show that, for any $t \in \mathbb{R}$,

$$\exp(tA) = e^{\alpha t}\begin{bmatrix} \cos\beta t & -\sin\beta t \\ \sin\beta t & \cos\beta t \end{bmatrix}.$$

19. Let V be a Banach space. Show that $\mathcal{G}$, the set of invertible linear operators in $\mathcal{B}(V)$ is an open subset of $\mathcal{B}(V)$ (endowed with its usual norm topology).

20. Define $T, S : \ell^2 \longrightarrow \ell^2$ by $T(x) = (0, x_1, x_2, \cdots)$ and $S(x) = (x_2, x_3, \cdots)$, for all $x = (x_1, x_2, \cdots) \in \ell^2$. Show that T and S define continuous linear operators on ℓ^2 and that $ST = I$ while $TS \neq I$ (Thus, T and S, which are called the *right* and *left shift operators* respectively, are not invertible.)

21. Let $\mathcal{P}$ be the space of all polynomials in one variable with real coefficients. For $p(x) = \sum_{i=1}^{n} a_i x \in \mathcal{P}$, define $\|p\|_1 = \sum_{i=1}^{n} |a_i|$ and $\|p\|_\infty = \max_{1 \leq i \leq n} |a_i|$. Show that $\|\cdot\|_1$ and $\|\cdot\|_\infty$ define norms on $\mathcal{P}$ and that they are not equivalent.

22. Let V be a normed linear space and let W be a finite dimensional subspace of V. Show that, for all $v \in V$, there exists $w \in W$ such that $\|v + W\| = \|v + w\|$.

23. Let V and W be normed linear spaces and let $U \subset V$ be an open subset. Let $J : U \longrightarrow W$ be a mapping. We say that J is *(Fréchet) differentiable* at $u \in U$ if there exists $T \in \mathcal{B}(V, W)$ such that

$$\lim_{h \to 0} \frac{\|J(u+h) - J(u) - T(h)\|}{\|h\|} = 0$$

(Equivalently, $J(u+h) - J(u) - T(h) = \varepsilon(h)$, with $\lim_{h \to 0} \frac{\|\varepsilon(h)\|}{\|h\|} = 0$.)

(a) If such a T exists, show that it is unique. (We say that T is the *(Fréchet) derivative* of J at $u \in U$ and write $T = J'(u)$.)

(b) If J is differentiable at $u \in U$, show that J is continuous at $u \in U$.

24. Let V and W be normed linear spaces and let $U \subset V$ be an open subset. Let $J : U \longrightarrow W$ be a mapping. We say that J is *Gâteau differentiable* at $u \in U$ along a vector $w \in V$ if $\lim_{t \to 0} \frac{J(u+tw) - J(u)}{t}$ exists. (We call the limit the *Gâteau derivative* of J at u along w.) Show that if J is Fréchet differentiable at $u \in U$ then J is Gâteau differentiable at u along any vector $w \in V$ and the corresponding Gâteau derivative is given by $J'(u)w$.

25. Let V and W be normed linear spaces and $T \in \mathcal{B}(V, W)$ and $w_0 \in W$ be given. Define $J : V \longrightarrow W$ by $J(u) = T(u) + w_0$. Show that J is differentiable at every $u \in V$ and $J'(u) = T$.

26. (a) Let V be a real normed linear space and let $J : V \longrightarrow \mathbb{R}$ be a given mapping. A subset $K \subset V$ is said to be *convex* ir, for every u and $v \in K$ and for all $t \in [0, 1]$ we have that $tu + (1-t)v \in K$. Let $K \subset V$ be a closed convex set. Assume that J attains its minimum over K at $u \in K$. If J is differentiable at u, show that $J'(u)(v - u) \geq 0$, for all $v \in K$.

(b) Let $K = V$. If J attains its minimum at $u \in V$ and if J is differentiable at u, show that $J'(u) = 0$.

27. Let V be a real normed linear space. A mapping $J : V \longrightarrow \mathbb{R}$ is said to be *convex* if, for every $u, v \in V$ and for every $t \in [0, 1]$, we have $J(tu + (1-t)v) \leq tJ(u) + (1-t)J(v)$.

(a) If $J : V \longrightarrow \mathbb{R}$ is convex and differentiable at every point, show that $J(v) - J(u) \geq J'(u)(v - u)$, for every $u, v \in V$.

(b) Let $J : V \longrightarrow \mathbb{R}$ be convex and differentiable at every point of V. Let $K \subset V$ be a closed convex set. Let $u \in K$ be such that $J'(u)(v - u) \geq 0$, for every $v \in K$. Show that $J(u) = \min_{v \in K} J(v)$.

(c) If $J : V \longrightarrow \mathbb{R}$ is convex and differentiable at every point of V and if $u \in V$ is such that $J'(u) = \mathbf{0}$, show that J attains its minimum (over all of V) at u.