DISCRETE AND CONTINUOUS
DYNAMICAL SYSTEMS SERIES B
Volume **16**, Number **1**, July **2011**

doi:10.3934/dcdsb.2011.16.225

pp. **225–238**

ON A CLASS OF THREE DIMENSIONAL NAVIER-STOKES EQUATIONS WITH BOUNDED DELAY

Sandro M. Guzzo

Universidade Estadual do Oeste do Paraná - UNIOESTE
Colegiado do curso de Matemática
Rua Universitária, 2069. Cx.P. 711
85819-110 Cascavel, PR, Brazil

Gabriela Planas

Departamento de Matemática
IMECC - UNICAMP
Rua Sergio Buarque de Holanda, 651
13083-859 Campinas, SP, Brazil

(Communicated by Peter E. Kloeden)

Abstract. In this paper we consider a three dimensional Navier-Stokes type equations with delay terms. We discuss the existence of weak and strong solutions and we study the asymptotic behavior of the strong solutions. Moreover, under suitable assumptions, we show the exponential stability of stationary solutions.

1. **Introduction and preliminaries.** In this work we study the existence and qualitative properties of solutions for a class of three dimensional Navier-Stokes equations with bounded delay in the convective term and in the external force. We discuss the existence and uniqueness of weak and strong solutions, the asymptotic behavior of strong solutions and the exponential stability of stationary solutions.

Navier-Stokes equations have received very much attention over the last decades due to their importance in the fluid dynamics and turbulence theory. The study of the Navier-Stokes equations with hereditary terms was initiated by Caraballo & Real in [1], where is studied the existence of weak solutions for the Navier-Stokes equations in which the forcing term contains some hereditary characteristics. Posteriorly, Caraballo & Real [2, 3], Taniguchi [11], Garrido & Marín-Rubio [4] and Marín-Rubio & Real [6, 7] investigated the asymptotic behavior of solutions to Navier-Stokes equations with different sorts of delay in the external forces.

In one dimension, Liu [5] considered the Burgers' equation with a time-delayed term of the form $u(t-\tau,x)u_x(t,x)$. This situation may appear when the trajectory of the fluid particles has a delay τ to follow the fluid. The motivation of introducing this kind of delay was well explored in the introduction of [5] and we will not repeat that discussion here, referring the reader to that article and to the references there contained. More recently, Tang & Wan [10] complemented the results of Liu [5].

2000 *Mathematics Subject Classification.* Primary: 35Q35, 76D05; Secondary: 35B40.

Key words and phrases. Navier-Stokes equations, bounded delay, asymptotic behavior.

S. Guzzo was partially supported by PICDT/Capes - Brazil. G. Planas has been partially supported by CNPq - Brazil, grant 303302/2009-7 and Fapesp - Brazil, grant 2007/51490-7.

The two-dimensional case, which corresponds to Navier-Stokes equations with time-delayed bilinear term, was analyzed by Planas & Hernández in [8].

The purpose of this paper is to extend the results in Planas & Hernández [8] to the three dimensional case. More precisely, we study a Navier-Stokes type system with delay of the form

$$\begin{aligned}\frac{\partial}{\partial t}u(t,x) - \nu\Delta u(t,x) + (u(t-\tau(t),x)\cdot\nabla)u(t,x) + \nabla p(t,x)\\ = f(t,x) + g(t,u_t), \qquad (t,x)\in(0,+\infty)\times\Omega, \quad (1)\end{aligned}$$

$$div\, u(t,x) = 0 \quad (t,x)\in(0,+\infty)\times\Omega, \quad (2)$$

$$u(t,x) = 0, \ \ (t,x)\in(0,+\infty)\times\partial\Omega, \quad (3)$$

$$u(0,x) = u^0(x), \ \ x\in\Omega, \quad (4)$$

$$u(t,x) = \phi(t,x), \ \ (t,x)\in(-h,0)\times\Omega, \quad (5)$$

where $\Omega\subset\mathbb{R}^3$ is an open bounded domain with smooth boundary $\partial\Omega$, $\nu>0$ is the viscosity, $u(\cdot)$ is the velocity field, $p(\cdot)$ is the hydrostatic pressure, $f(\cdot)+g(\cdot)$ is an external force which has hereditary characteristics, $u^0(\cdot)$ and $\phi:(-h,0)\times\Omega\to\mathbb{R}^3$ are the initial data where $h>0$ is a real number fixed; the history $u_t:[-h,0]\to\mathbb{R}^3$ is given by $u_t(s)=u(t+s)$ and $\tau:[0,+\infty)\to[0,h]$ is a suitable function.

We include now some definitions, properties and technicalities required to establish our results. Let us firstly introduce the spaces of divergence-free vector functions

$$\mathcal{V}=\{u\in(C_0^\infty(\Omega))^3; div\, u=0\}, \quad H=\bar{\mathcal{V}}^{(L^2(\Omega))^3} \text{ and } V=\bar{\mathcal{V}}^{(H_0^1(\Omega))^3}$$

where the notation $\overline{S}^W$ denotes the closure of a set S in a space W. The duality pairing between the spaces V and V' is denoted by $\langle\cdot,\cdot\rangle$ and $(\cdot,\cdot)$ represents the inner product in $(L^2(\Omega))^3$.

The Stokes operator $A:D(A)=(H^2(\Omega))^3\cap V\to V'$ is given by $Au=-P\Delta u$ where P denotes the orthogonal projection from $(L^2(\Omega))^3$ onto H. We also define the trilinear form $b:V\times V\times V\to\mathbb{R}$ by

$$b(u,v,w)=\sum_{i,j=1}^{3}\int_\Omega u_i\frac{\partial v_j}{\partial x_i}w_j dx.$$

From Robinson [9], we recall that $b(u,v,v)=0$ for all $u,\ v\in V$ and the following inequalities,

$$|b(u,v,w)|\le C_3\|u\|_{L^2}^{\frac14}\|u\|_V^{\frac34}\|v\|_V\|w\|_{L^2}^{\frac14}\|w\|_V^{\frac34}, \qquad u,v,w\in V, \quad (6)$$

$$|b(u,v,w)|\le C_3\|u\|_V\|v\|_V^{\frac12}\|Av\|_{L^2}^{\frac12}\|w\|_{L^2}, \qquad u\in V, v\in D(A), w\in H. \quad (7)$$

For additional details related to Navier-Stokes equations we refer the reader to [12].

To treat the system (1)-(5) we assume that g is a function defined from $[0,\infty)\times C([-h,0];V)$ into $(L^2(\Omega))^3$ and introduce the following conditions.

(H1) For all $\xi\in C([-h,0];V)$, the function $g(\cdot,\xi)$ is strongly measurable and $g(t,0)=0$ for all $t>0$.

(H2) There exists $L_g>0$ such that

$$\|g(t,\xi)-g(t,\eta)\|_{L^2}\le L_g\|\xi-\eta\|_{C([-h,0];V)},$$

for all $t>0$ and every $\xi,\eta\in C([-h,0];V)$.

(H3) There exists $C_g > 0$ such that

$$\int_0^t \|g(s,u_s) - g(s,v_s)\|_{L^2}^2 ds \le C_g \int_{-h}^t \|u(s) - v(s)\|_V^2 ds,$$

for all $t > 0$ and every $u, v \in C([-h,t];V)$.

Let $T > 0$ be given. Observe that, from assumptions (**H1**)-(**H3**), the mapping $G : C([-h,T];V) \to L^2(0,T;(L^2(\Omega))^3)$ defined by $G(u)(t) = g(t,u_t)$ is well defined and has a unique extension $\tilde{G} : L^2(-h,T;V) \to L^2(0,T;(L^2(\Omega))^3)$ which is uniformly continuous. From now on, $g(t,u_t) = \tilde{G}(u)(t)$ for $u \in L^2(-h,T;V)$ and the hypothesis (**H3**) becomes true for any $t \in [0,T]$ and $u, v \in L^2(-h,T;V)$.

We consider now an additional assumption.

(H4) If a sequence $(v_k)_{k\in\mathbb{N}}$ converges weakly to v in $L^2(-h,T;V)$ and strongly in $L^2(-h,T;H)$, then the sequence $(\tilde{G}(v_k))_{k\in\mathbb{N}}$ converges weakly to $\tilde{G}(v)$ in $L^2(0,T;V')$.

In this paper, $\tau \in C^1([0,+\infty);[0,h])$ is such that $\tau'(t) \le \tau^* < 1$ for all $t > 0$. We define $F(t,\psi) = \psi(-\tau(t))$ for $\psi \in L^2(-h,T;V)$.

By introducing the operator $B : V \times V \to V'$ given by

$$\langle B(u,v), w\rangle = b(u,v,w),$$

we can re-write the system (1)-(5) in the abstract form

$$\frac{du(t)}{dt} + \nu Au(t) + B(F(t,u_t),u(t)) = f(t) + g(t,u_t), \quad t > 0, \tag{8}$$

$$u(0) = u^0, \tag{9}$$

$$u(t) = \phi(t), \quad t \in (-h,0). \tag{10}$$

This paper is organized as follows. In the next section we study the existence of weak and strong solutions for (8)-(10). In the same section, we also discuss the exponential behavior of the strong solutions. In the last section, we prove under suitable assumptions that a time-dependent weak solution converges exponentially to the stationary solution.

2. **Existence of solutions.**

In this section we study the existence of weak and strong solutions for the system (8)-(10). We also discuss the exponential behavior of the strong solution.

We begin by considering the problem of the existence of weak solutions.

Definition 2.1. A function $u : [-h,T] \to H$ is said a weak solution of the system (8)-(10) on $[0,T]$ if $u \in L^2(-h,T;V) \cap L^\infty(0,T;H)$, $u(0) = u^0$, $u_0 = \phi$ and

$$\frac{d}{dt}(u(t),v) + \nu(\nabla u(t),\nabla v) + b(F(t,u_t),u(t),v) = \langle f(t),v\rangle + (g(t,u_t),v),$$

for all all $v \in V$ in the sense of distributions on $(0,T)$.

We can establish now our first existence result.

Theorem 2.2. *For any fixed $T > 0$, assume that $u^0 \in H$, $\phi \in L^2(-h,0;V)$, $f \in L^2(0,T;V')$ and that conditions* **(H1)-(H4)** *are verified. Then there exists a weak solution $u \in L^\infty(0,T;H) \cap L^2(-h,T;V)$ of the system* (8)-(10).

Proof. Let $\{w_j : j \in \mathbb{N}\}$ be the set of the eigenfunctions of the Stokes operator, $V_k = \operatorname{span}\{w_1, \dots, w_k\}$ and P_k be the projection from H into V_k.

For $k \in \mathbb{N}$, we introduce the approximate problems

$$\begin{aligned} \frac{d}{dt}(u^k(t), w_j) + &\nu \langle Au^k(t), w_j\rangle + b(F(t, u_t^k), u^k(t), w_j), \\ &= \langle f(t), w_j\rangle + (g(t, u_t^k), w_j), \quad 1 \le j \le k, \qquad (11) \\ u^k(0) &= P_k u^0, \qquad (12) \\ u^k(t) &= \phi^k(t), \quad t \in (-h, 0), \qquad (13) \end{aligned}$$

where $u^k(t) = \sum_{i=1}^{k} \gamma_{ik}(t) w_i$, $P_k u^0 = \sum_{i=1}^{k} \alpha_{ik} w_i$ and $\phi^k = \sum_{i=1}^{k} \beta_{ik}(t) w_i \to \phi$. The system (11)-(13) is a nonlinear ordinary functional differential equation with state in a finite dimensional space. Indeed, by defining $\Gamma_k(t) = (\gamma_{1k}(t), \dots, \gamma_{kk}(t))$, $\alpha_k = (\alpha_{1k}, \dots, \alpha_{kk})$ and $\beta_k(t) = (\beta_{1k}(t), \dots, \beta_{kk}(t))$, we can re-write the system (11)-(13) in the form

$$\begin{aligned} \frac{d}{dt}\Gamma_k(t) &= \Phi_1(\Gamma_k(t)) + \Phi_2(\Gamma_k(t - \tau(t)), \Gamma_k(t)) + \Phi_3(f(t)) + \Phi_4(g(t, u_t^k)), \qquad (14) \\ \Gamma_k(0) &= \alpha_k, \qquad (15) \\ (\Gamma_k)_0 &= \beta_k, \qquad (16) \end{aligned}$$

where the operators Φ_i, $i = 1, 2, 3, 4$, are defined in an obvious manner.

From the above definitions and our technical conditions, it is easy to see that there exist positive constants $\mathcal{C}_i$, $i = 1, 2, 3, 4$, such that

$$\begin{aligned} \|\Phi_1(\Gamma_k(t)) - \Phi_1(\bar\Gamma_k(t))\|_k &\le \mathcal{C}_1 \|\Gamma_k(t) - \bar\Gamma_k(t)\|_k, \\ \|\Phi_2(\Gamma_k(t - \tau(t)), \Gamma_k(t)) - \Phi_2(\bar\Gamma_k(t - \tau(t)), &\bar\Gamma_k(t))\|_k \\ &\le \mathcal{C}_2 \Big(\|\Gamma_k(t - \tau(t)) - \bar\Gamma_k(t - \tau(t))\|_k \|\Gamma_k(t)\|_k \\ &\quad + \|\bar\Gamma_k(t - \tau(t))\|_k \|\Gamma_k(t) - \bar\Gamma_k(t)\|_k \Big), \\ \|\Phi_3(f(t))\|_k &\le \mathcal{C}_3 \|f(t)\|_{V'}, \\ \|\Phi_4(g(t, u_t^k)) - \Phi_4(g(t, v_t^k))\|_k &\le \mathcal{C}_4 \|g(t, u_t^k) - g(t, v_t^k)\|_{L^2}, \end{aligned}$$

where $v^k(t) = \sum_{i=1}^{k} \bar\gamma_{ik}(t) w_i$, $\bar\Gamma_k(t) = (\bar\gamma_{1k}(t), \dots, \bar\gamma_{kk}(t))$ and $\|x\|_k = \sum_{i=1}^{k} |x_i|$. Moreover, it holds $\Phi_1(0) = \Phi_2(0, 0) = \Phi_4(0) = 0$.

Arguing now as in the proof of [1, Theorem A1] we can prove that there exists a unique maximal solution $\Gamma_k \in C([0, t_k]; \mathbb{R}^k)$ of (14)-(16). In order to prove that $t_k = T$ for all $k \in \mathbb{N}$, we next obtain *a priori* estimates for the solutions Γ_k, $k \in \mathbb{N}$.

By multiplying (11) by $\gamma_{jk}(t)$ and adding in j, we get

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|u^k(t)\|_{L^2}^2 + \nu \|\nabla u^k(t)\|_{L^2}^2 + \int_\Omega F(t, u_t^k) \cdot \nabla u^k(t) u^k(t) dx \\ = \langle f(t), u^k(t)\rangle + (g(t, u_t^k), u^k(t)). \qquad (17) \end{aligned}$$

Using that $F(t, u_t^k)$ is divergence free and conditions **(H1)** and **(H2)**, by integrating (17) on $[0, t]$ we see that

$$\begin{aligned}
&\|u^k(t)\|_{L^2}^2 + 2\nu \int_0^t \|u^k(s)\|_V^2 ds \\
&\quad \le \|u^0\|_{L^2}^2 + \frac{1}{\nu}\int_0^t \|f(s)\|_{V'}^2 ds + \nu \int_0^t \|u^k(s)\|_V^2 ds \\
&\quad\quad + \frac{\nu}{2C_g}\int_0^t \|g(s, u_s^k)\|_{L^2}^2 ds + \frac{2C_g}{\nu}\int_0^t \|u^k(s)\|_{L^2}^2 ds \\
&\quad \le \|u^0\|_{L^2}^2 + \frac{1}{\nu}\int_0^t \|f(s)\|_{V'}^2 ds + \nu \int_0^t \|u^k(s)\|_V^2 ds \\
&\quad\quad + \frac{\nu}{2}\int_{-h}^0 \|\phi(s)\|_V^2 ds + \frac{\nu}{2}\int_0^t \|u^k(s)\|_V^2 ds + \frac{2C_g}{\nu}\int_0^t \|u^k(s)\|_{L^2}^2 ds,
\end{aligned}$$

from which we obtain

$$\begin{aligned}
&\|u^k(t)\|_{L^2}^2 + \frac{\nu}{2}\int_0^t \|u^k(s)\|_V^2 ds \\
&\quad \le \|u^0\|_{L^2}^2 + \frac{1}{\nu}\int_0^t \|f(s)\|_{V'}^2 ds + \frac{\nu}{2}\int_{-h}^0 \|\phi(s)\|_V^2 ds + \frac{2C_g}{\nu}\int_0^t \|u^k(s)\|_{L^2}^2 ds,
\end{aligned}$$

for all $t \in [0, t_k)$. Now, from the Gronwall inequality we infer that the sequence $(u^k)_{k\in\mathbb{N}}$ is bounded in the spaces $L^\infty(0, T; H)$ and $L^2(0, T; V)$, $t_k = T$ for all $k \in \mathbb{N}$ and that the sequence $(F(t, u_t^k))_{k\in\mathbb{N}}$ is bounded in $L^2(0, T; V)$.

Next, we obtain an *a priori* estimate for $\frac{d}{dt}u^k$ in the space $L^r(0, T; V')$ for some $r > 1$. From equation (11), for $v \in V$ with $\|v\|_V \le 1$ we see that

$$\begin{aligned}
\left\langle \frac{d}{dt}u^k(t), v \right\rangle &= \langle -\nu A u^k(t) - B(F(t, u_t^k), u^k(t)) + f(t) + g(t, u_t^k), P_k v\rangle \\
&\le \nu \|u^k(t)\|_V \|v\|_V + |b(F(t, u_t^k), u^k(t), v)| \\
&\quad + \|f(t)\|_{V'}\|v\|_V + C\|g(t, u_t^k)\|_{L^2}\|v\|_V.
\end{aligned}$$

Since the sequence $(u^k)_{k\in\mathbb{N}}$ is bounded in $L^2(0, T; V)$, from condition (**H3**) we have that the sequence $(g(t, u_t^k))_{k\in\mathbb{N}}$ is bounded in $L^2(0, T; (L^2(\Omega))^3)$. Since $(u^k)_{k\in\mathbb{N}}$ is bounded in $L^\infty(0, T; H)$ from (6) we find that

$$\begin{aligned}
\left| \int_\Omega F(t, u_t^k)\nabla u^k v dx \right| &= \left| -\int_\Omega F(t, u_t^k)\nabla v u^k dx \right| \\
&\le C\|F(t, u_t^k)\|_V \|u^k\|_V^{\frac{3}{4}} \|u^k\|_{L^2}^{\frac{1}{4}} \|v\|_V \\
&\le C\|F(t, u_t^k)\|_V \|u^k\|_V^{\frac{3}{4}}, \qquad (18)
\end{aligned}$$

where $C > 0$ is independent of $k \in \mathbb{N}$. By using the Young inequality with $p = \frac{7}{4}$ and $q = \frac{7}{3}$ and integrating (18) on $[0, t]$, we get

$$\int_0^t \|B(F(s,u_s^k),u^k)\|_{V'}^{\frac{8}{7}} ds \le C \int_0^t \|F(s,u_s^k)\|_V^{\frac{8}{7}} \|u^k(s)\|_V^{\frac{6}{7}} ds$$
$$\le \frac{4C^2}{7} \int_0^t \|F(s,u_s^k)\|_V^2 ds + \frac{3}{7}\int_0^t \|u^k(s)\|_V^2 ds,$$

which permits to conclude that the sequences $(B(F(t,u_t^k),u^k))_{k\in\mathbb{N}}$ and $(\frac{du^k}{dt})_{k\in\mathbb{N}}$ are bounded in $L^{\frac{8}{7}}(0,T;V')$.

From the above remarks and the Aubin-Lions compactness criterion, we infer there exist a subsequence of $(u^k)_{k\in\mathbb{N}}$ (which we still denote by $(u^k)_{k\in\mathbb{N}}$) and $u \in L^2(-h,T;V)\cap L^\infty(0,T;H)$ such that $u^k \to u$ in $L^2(-h,T;H)$ and $u^k \rightharpoonup u$ in $L^2(-h,T;V)$ as $k\to+\infty$. Moreover, from our assumptions, it is easy to see that

$$F(t,u_t^k) \to F(t,u_t) \quad \text{in} \quad L^2(0,T;H),$$
$$\tilde{G}(u^k) \rightharpoonup \tilde{G}(u) \quad \text{in} \quad L^2(0,T;V').$$

These convergences allow us to pass to the limit in (11)-(13) and conclude that $u(\cdot)$ is a weak solution. The proof is complete. □

Remark 1. Observe that if u is a weak solution of the system (8)-(10) on $[0,T]$ then u is a weakly continuous function from $[0,T]$ into H (see Lemma 1.4 in Chapter III of [12]).

We establish now an integral inequality satisfied by the weak solution constructed in Theorem 2.2 which will be useful in the study of the stability of the stationary solution. To this end, we introduce the following slight variant of condition (**H4**).

(H4*) If a sequence $(v_k)_{k\in\mathbb{N}}$ converges weakly to $v\in L^2(-h,T;V)$ and strongly in $L^2(-h,T;H)$, then the sequence $(\tilde{G}(v_k))_{k\in\mathbb{N}}$ converges weakly to $\tilde{G}(v)$ in $L^2(0,T;(L^2(\Omega))^3)$.

By multiplying (17) by $e^{\lambda t}$ with $\lambda\ge 0$ and integrating from 0 to t we see that

$$\frac{1}{2}e^{\lambda t}\|u^k(t)\|_{L^2}^2 - \frac{1}{2}\|u^k(0)\|_{L^2}^2 - \frac{1}{2}\int_0^t \lambda e^{\lambda s}\|u^k(s)\|_{L^2}^2 ds + \nu\int_0^t e^{\lambda s}\|u^k(s)\|_V^2 ds$$
$$= \int_0^t e^{\lambda s}\left(\langle f(s),u^k(s)\rangle + (g(s,u_s^k),u^k(s))\right) ds,$$

for all $t\in[0,T]$. Now, taking the liminf, and observing that u is weakly continuous from $[0,T]$ into H, it follows that

$$e^{\lambda t}\|u(t)\|_{L^2}^2 - \|u^0\|_{L^2}^2 - \int_0^t \lambda e^{\lambda s}\|u(s)\|_{L^2}^2 ds + 2\nu\int_0^t e^{\lambda s}\|u(s)\|_V^2 ds$$
$$\le 2\int_0^t e^{\lambda s}\left(\langle f(s),u(s)\rangle + (g(s,u_s),u(s))\right) ds, \quad (19)$$

for all $t\in[0,T]$, for any $\lambda\ge 0$.

2.1. **Existence and exponential behavior of strong solutions.** In the sequel, we discuss the existence and uniqueness of strong solutions for the system (8)-(10). To establish our results, we will assume that

$$g(t, u_t) = G(u(t - \rho(t)))$$

where $G : \mathbb{R}^3 \to \mathbb{R}^3$ is a Lipschitz continuous function with Lipschitz constant $L > 0$, $G(0) = 0$, $\rho \in C^1([0,\infty);[0,h])$, such that $\rho'(t) \leq \rho^* < 1$ for all $t \geq 0$. We note that under these conditions the function $g(\cdot)$ satisfies the conditions **(H1)-(H3)** and **(H4*)** (see [2, 8] for details) ensuring the existence of weak solutions to (8)-(10). By considering the development in Taniguchi [11], we also introduce the following condition for the function $f(\cdot)$.

Definition 2.3. Let $\theta \geq 0$ and $f \in L^2_{loc}([0,\infty);(L^2(\Omega))^3)$. We say that f is $M(\theta)$-integrable on $[0,\infty)$ if there exits $M(\theta) > 0$ such that

$$\int_0^t e^{\theta(s-t)} \|f(s)\|^2_{L^2} ds \leq M(\theta)$$

for every $t > 0$.

Next, we consider the following concept of strong solution.

Definition 2.4. A function $u : [-h, T] \to H$ is said to be a strong solution of the system (8)-(10) on $[0,T]$ if $u \in L^\infty(0,T;V) \cap L^2(0,T;D(A))$, $u(0) = u^0$, $u_0 = \phi$ and equation (8) is satisfied. We say that $u : [-h,\infty) \to H$ is a strong solution of the system (8)-(10), if $u(\cdot)$ is a strong solution of (8)-(10) on $[0,T]$ for all $T > 0$.

Remark 2. Observe that if $u^0 \in V$, $f \in L^2(0,T;(L^2(\Omega))^3)$, and u is a strong solution of the system (8)-(10) on $[0,T]$, then $PB(u,u)$ and du/dt belong to $L^2(0,T;H)$, therefore, $u \in C([0,T];V)$.

We are in position to state the following theorem. Let λ_1 be the first eigenvalue of the Stokes operator.

Theorem 2.5. *Let $f \in L^2_{loc}([0,\infty);(L^2(\Omega))^3)$. Assume that $(1-\rho^*)\nu^2\lambda_1^2 > 4L^2$, and there are $\theta > 0$ and $\kappa^* > 0$ such that $f(\cdot)$ is $M(\theta)$-integrable and*

$$\frac{4}{\nu^2} M^2(\theta) < \kappa^* < \frac{(1-\rho^*)\nu^4\lambda_1^2 - 4L^2\nu^2}{27(1-\rho^*)\lambda_1 C_3^4},$$
$$\frac{\theta}{\lambda_1} + \frac{2e^{h\theta}L^2}{(1-\rho^*)\nu\lambda_1^2} + \frac{27C_3^4\kappa^*}{2\nu^3\lambda_1} < \frac{\nu}{2}. \tag{20}$$

Then, for $u^0 \in V$ and $\phi \in L^\infty(-h,0;V)$ such that

$$\|\phi\|^2_{L^\infty(-h,0;V)} + \frac{2L^2}{\nu(1-\rho^*)}\|\phi\|^2_{L^2(-h,0;H)} + \|u^0\|^2_V + \frac{2}{\nu}M(\theta) < \sqrt{\kappa^*}, \tag{21}$$

there exists a unique strong solution $u(\cdot)$ of problem (8)-(10), that is, $u \in L^\infty_{loc}([0,\infty);V) \cap L^2_{loc}([0,\infty);D(A))$. Moreover,

$$\|u(t)\|^2_V \leq \Big(\|u^0\|^2_V + \frac{2L^2}{\nu(1-\rho^*)}\|\phi\|^2_{L^2(-h,0;H)}\Big)e^{-\theta t} + \frac{2}{\nu}M(\theta), \tag{22}$$

for all $t > 0$; $u \in C([-h,T];V)$ when $\phi \in C([-h,0];V)$ and $\phi(0) = u^0$. In particular, $u(\cdot)$ goes to zero exponentially if $f = 0$.

Proof. Assume that $u^0 \in V$ and $\phi \in L^\infty(-h,0;V)$ verify the condition (21) and let $0<\kappa<\kappa^*$ be such that

$$\|\phi\|^2_{L^\infty(-h,0;V)} + \frac{2L^2}{\nu(1-\rho^*)}\|\phi\|^2_{L^2(-h,0;H)} + \|u^0\|^2_V + \frac{2}{\nu}M(\theta) < \sqrt{\kappa} < \sqrt{\kappa^*}.$$

Let $(u^k)_{k\in\mathbb{N}}$ be the sequence introduced in the proof of Theorem 2.2 and $W(\cdot,u^k): [0,\infty)\to\mathbb{R}^+$, $k\in\mathbb{N}$, be the function defined by

$$W(t,u^k) = e^{\theta t}\|u^k(t)\|^2_V + \frac{2}{\nu(1-\rho^*)}\int_{t-\rho(t)}^t e^{\theta s}e^{\theta h}\|G(u^k(s))\|^2_{L^2}ds.$$

Let $T>0$ be given. Next, we will show that $\|u^k\|_{L^\infty(0,T;V)} < \sqrt[4]{\kappa}$ for $k\in\mathbb{N}$. If we assume that this assertion is false, then there exist $k\in\mathbb{N}$ and $t_*\in[0,T]$ such that $\|u^k(t_*)\|_V = \sqrt[4]{\kappa}$ and $\|u^k(t)\|_V < \sqrt[4]{\kappa}$ for all $0\le t<t_*$. Then, for $t\in[0,t_*]$ we see that

$$\begin{aligned}
&\frac{d}{dt}W(t,u^k)\\
&\quad= \theta e^{\theta t}\|u^k(t)\|^2_V + 2e^{\theta t}\big(Au^k(t), \tfrac{d}{dt}u^k(t)\big)\\
&\quad+ \frac{2e^{\theta h}e^{\theta t}}{(1-\rho^*)\nu}\|G(u^k(t))\|^2_{L^2} - \frac{2e^{\theta t}}{\nu}e^{\theta(h-\rho(t))}\frac{(1-\rho'(t))}{1-\rho^*}\|G(u^k(t-\rho(t)))\|^2_{L^2}\\
&\quad\le \theta e^{\theta t}\|u^k(t)\|^2_V\\
&\quad+ 2e^{\theta t}\Big(Au^k(t), -\nu Au^k(t) - B(F(t,u^k_t),u^k(t)) + f(t) + G(u^k(t-\rho(t)))\Big)\\
&\quad+ \frac{2e^{\theta h}L^2e^{\theta t}}{(1-\rho^*)\nu}\|u^k(t)\|^2_{L^2} - \frac{2e^{\theta t}}{\nu}\|G(u^k(t-\rho(t)))\|^2_{L^2}.
\end{aligned}$$

By using (7), the Hölder and the Young inequalities, we can estimate the terms in the right hand side as follows

$$\begin{aligned}
2e^{\theta t}|\big(B(F(t,u^k_t),u^k(t)),Au^k(t)\big)|&\\
&\le 2e^{\theta t}C_3\|F(t,u^k_t)\|_V\|u^k(t)\|_V^{\frac12}\|Au^k(t)\|^{\frac32}_{L^2}\\
&\le \frac{27C_3^4e^{\theta t}}{2\nu^3}\|F(t,u^k_t)\|^4_V\|u^k(t)\|^2_V + \frac{\nu}{2}e^{\theta t}\|Au^k(t)\|^2_{L^2},\\
2e^{\theta t}|\big(Au^k(t),f(t)\big)| &\le \frac{\nu}{2}e^{\theta t}\|Au^k(t)\|^2_{L^2} + \frac{2}{\nu}e^{\theta t}\|f(t)\|^2_{L^2},
\end{aligned}$$

and

$$2e^{\theta t}|\big(Au^k(t),G(u(t-\rho(t)))\big)| \le \frac{\nu}{2}e^{\theta t}\|Au^k(t)\|^2_{L^2} + \frac{2}{\nu}e^{\theta t}\|G(u^k(t-\rho(t)))\|^2_{L^2}.$$

Therefore, we arrive at

$$\begin{aligned}
\frac{d}{dt}W(t,u^k) \le&\theta e^{\theta t}\|u^k(t)\|^2_V - \frac{\nu}{2}e^{\theta t}\|Au^k(t)\|^2_{L^2}\\
&+ \frac{27C_3^4e^{\theta t}}{2\nu^3}\|F(t,u^k_t)\|^4_V\|u^k(t)\|^2_V\\
&+ \frac{2e^{\theta t}}{\nu}\|f(t)\|^2_{L^2} + \frac{2e^{\theta h}L^2e^{\theta t}}{(1-\rho^*)\nu}\|u^k(t)\|^2_{L^2}.
\end{aligned}$$

Noting that

$$F(t,u_t^k) = u^k(t-\tau(t)) = \begin{cases} \phi^k(t-\tau(t)) & \text{if} \quad -h \le t-\tau(t) \le 0, \\ u^k(t-\tau(t)) & \text{if} \quad 0 < t-\tau(t) \le t_*, \end{cases}$$

we obtain that

$$\|F(t,u_t^k)\|_V^4 \le \max\{\|\phi\|^4_{L^\infty(-h,0;V)}, \|u^k\|^4_{L^\infty(0,t_*;V)}\} \le \kappa.$$

Thus,

$$\begin{aligned} &\frac{d}{dt}W(t,u^k) \\ &\le \left(\frac{\theta}{\lambda_1} - \frac{\nu}{2} + \frac{27C_3^4\kappa}{2\nu^3\lambda_1} + \frac{2e^{\theta h}L^2}{(1-\rho^*)\nu\lambda_1^2}\right) e^{\theta t}\|Au^k(t)\|^2_{L^2} + \frac{2}{\nu}e^{\theta t}\|f(t)\|^2_{L^2}. \end{aligned}$$

From condition (20) we have that

$$\Psi(\theta) = -\left(\frac{\theta}{\lambda_1} - \frac{\nu}{2} + \frac{27C_3^4\kappa}{2\nu^3\lambda_1} + \frac{2e^{\theta h}L^2}{(1-\rho^*)\nu\lambda_1^2}\right) > 0,$$

so that

$$\frac{d}{dt}W(t,u^k) + \Psi(\theta)e^{\theta t}\|Au^k(t)\|^2_{L^2} \le \frac{2}{\nu}e^{\theta t}\|f(t)\|^2_{L^2}.$$

Integrating this inequality on $[0,t]$ and using that f is $M(\theta)$-integrable, we see that

$$\begin{aligned} &e^{\theta t}\|u^k(t)\|_V^2 + \Psi(\theta)\int_0^t e^{\theta s}\|Au^k(s)\|^2_{L^2}ds \\ &\qquad \le \|u^k(0)\|_V^2 + \frac{2}{\nu(1-\rho^*)}\int_{-\rho(0)}^0 e^{\theta s}\|G(u^k(s))\|^2_{L^2}ds + \frac{2}{\nu}e^{\theta t}M(\theta) \\ &\qquad \le \|u^k(0)\|_V^2 + \frac{2L^2}{\nu(1-\rho^*)}\int_{-h}^0 \|\phi^k(s)\|^2_{L^2}ds + \frac{2}{\nu}e^{\theta t}M(\theta), \end{aligned} \tag{23}$$

and hence,

$$\|u^k(t)\|_V^2 \le \Big(\|u^0\|_V^2 + \frac{2L^2}{\nu(1-\rho^*)}\|\phi\|^2_{L^2(-h,0;H)}\Big)e^{-\theta t} + \frac{2}{\nu}M(\theta) < \sqrt{\kappa},$$

for all $0 \le t \le t_*$, which is a contradiction. Thus,

$$\sup_{k\in\mathbb{N}}\|u^k\|_{L^\infty(0,T;V)} \le \Big(\|u^0\|_V^2 + \frac{2L^2}{\nu(1-\rho^*)}\|\phi\|^2_{L^2(-h,0;H)}\Big)e^{-\theta t} + \frac{2}{\nu}M(\theta)$$

and $\|u^k\|_{L^\infty(0,T;V)} < \sqrt[4]{\kappa}$ for $k \in \mathbb{N}$.

From the above steps we infer that $u \in L^\infty(0,T;V)$ and that (22) is satisfied. Moreover, from (23) we find that

$$\Psi(\theta)\int_0^t \|Au^k(s)\|^2_{L^2}ds \le \|u^0\|_V^2 + \frac{2L^2}{\nu(1-\rho^*)}\|\phi\|^2_{L^2(-h,0;H)} + \frac{2}{\nu}e^{\theta t}M(\theta),$$

for all $k \in \mathbb{N}$, which implies that $u \in L^2(0,T;D(A))$ and completes the proof that $u(\cdot)$ is a strong solution.

The uniqueness is proved as usual. Let $u, v \in C([0,T];V) \cap L^2(0,T;D(A))$ be solutions of the system (8)-(10). For $w = u - v$ we get

$$\begin{aligned}\frac{d}{dt}w(t) + \nu Aw(t) &= -B(F(t,u_t) - F(t,v_t), u(t)) - B(F(t,v_t), w(t)) \\ &\quad + G(u(t-\rho(t))) - G(v(t-\rho(t))), \\ w(0) &= 0, \\ w(t) &= 0, \quad t \in (-h, 0).\end{aligned}$$

By taking the scalar product in L^2 with $w(t)$ we find

$$\begin{aligned}\frac{d}{dt}\|w(t)\|_{L^2}^2 + 2\nu\langle Aw(t), w(t)\rangle &= -2b(F(t,u_t) - F(t,v_t), u(t), w(t)) \\ &\quad + 2(G(u(t-\rho(t))) - G(v(t-\rho(t))), w(t)).\end{aligned}$$

By using the estimate (6) and the Young inequality we get

$$\begin{aligned}|b(F(t,u_t) - F(t,v_t), u(t), w(t))| &\leq \frac{\nu(1-\tau^*)}{2}\|F(t,u_t) - F(t,v_t)\|_V^2 + \frac{\nu}{2}\|w(t)\|_V^2 \\ &\quad + \frac{27C^4}{2\nu^7(1-\tau^*)^4}\|u\|_{L^\infty(0,T;V)}^8\|w(t)\|_{L^2}^2.\end{aligned}$$

Taking into account

$$\begin{aligned}&2|(G(u(t-\rho(t))) - G(v(t-\rho(t))), w(t))| \\ &\qquad \leq \frac{2}{\nu}\|G(u(t-\rho(t))) - G(v(t-\rho(t)))\|_{L^2}^2 + \frac{\nu}{2}\|w(t)\|_{L^2}^2,\end{aligned}$$

we obtain

$$\begin{aligned}\frac{d}{dt}\|w(t)\|_{L^2}^2 + 2\nu\|w(t)\|_V^2 &\leq \frac{\nu(1-\tau^*)}{2}\|F(t,u_t) - F(t,v_t)\|_V^2 + \frac{\nu}{2}\|w(t)\|_V^2 \\ &\quad + \frac{27C^4}{2\nu^7(1-\tau^*)^4}\|u\|_{L^\infty(0,T;V)}^8\|w(t)\|_{L^2}^2 \\ &\quad + \frac{2}{\nu}\|G(u(t-\rho(t))) - G(v(t-\rho(t)))\|_{L^2}^2 + \frac{\nu}{2}\|w(t)\|_{L^2}^2.\end{aligned}$$

Integrating this inequality from 0 to t and using the changes of variable $\eta = s - \tau(s)$ and $\eta = s - \rho(s)$ in the terms with delay give us

$$\begin{aligned}&\|w(t)\|_{L^2}^2 + 2\nu\int_0^t \|w(s)\|_V^2 ds \\ &\quad \leq \frac{\nu(1-\tau^*)}{2}\frac{1}{(1-\tau^*)}\int_0^t \|w(s)\|_V^2 ds + \nu\int_0^t \|w(s)\|_V^2 ds \\ &\quad + \frac{27C^4}{2\nu^7(1-\tau^*)^4}\|u\|_{L^\infty(0,T;V)}^8\int_0^t \|w(s)\|_{L^2}^2 ds + \frac{2L^2}{\nu(1-\rho^*)}\int_0^t \|w(s)\|_{L^2}^2 ds,\end{aligned}$$

and hence,

$$\begin{aligned}&\|w(t)\|_{L^2}^2+\nu\int_0^t\|w(s)\|_V^2ds\\&\qquad\leq\left(\frac{\nu}{2}+\frac{2L^2}{\nu(1-\rho^*)}+\frac{27C^4}{2\nu^7(1-\tau^*)^4}\|u\|_{L^\infty(0,T;V)}^8\right)\int_0^t\|w(s)\|_{L^2}^2ds.\end{aligned}$$

Now, from Gronwall Lemma we infer that $w=0$. The proof is complete. ☐

3. **Stability of stationary solutions.** In this section, we establish conditions under which the weak solution constructed in Section 2 converges exponentially to the solution of the stationary equation

$$\nu Au+B(u,u)=f+G(u). \tag{24}$$

In the rest of this section, we assume that the function $g(\cdot)$ satisfies the conditions in the Subsection 2.1. We note that under these conditions, there exists a weak solution of (8)-(10) which satisfies the energy inequality (19).

It was shown by Caraballo [2, Theorem 3.1] that there exists a stationary solution u_∞ to equation (24). Actually the proof is done in the two-dimensional case but holds as well in the three-dimensional case, this is clear from the argument used in the proof. We do not repeat the proof here, we only recall this result.

Theorem 3.1. *Suppose that $f\in V'$ and $\nu>\frac{L}{\lambda_1}$. Then, there exists a stationary solution $u_\infty\in V$ to the stationary problem (24). Moreover, if $C_1=C_1(\Omega)$ is the Sobolev embedding constant of V into $(L^4(\Omega))^3$ and $(\nu-\frac{L}{\lambda_1})^2>C_1^2\|f\|_{V'}$, then the solution is unique.*

The next result gives another condition ensuring the exponential convergence of weak solutions of (8)-(10) to the unique stationary solution.

Theorem 3.2. *Let $u^0\in H$, $\phi\in L^2(-h,0;V)$ and $f\in(L^2(\Omega))^3$. Suppose that $\nu>\frac{L}{\lambda_1}$ and*

$$2\nu-\frac{L}{\lambda_1}-\frac{C_1^2\|f\|_{V'}^2}{(\nu-\frac{L}{\lambda_1})^2}>\frac{L}{(1-\rho^*)\lambda_1}+\frac{C_1^2}{(1-\tau^*)}. \tag{25}$$

Then any weak solution u to problem (8)-(10) satisfying the energy inequality (19) converges exponentially, as $t\to\infty$, to the unique stationary solution u_∞ of (24). More precisely, there exist positive constants C and λ such that

$$\|u(t)-u_\infty\|_{L^2}^2\leq Ce^{-\lambda t}\left(\|u^0-u_\infty\|_{L^2}^2+\|\phi-u_\infty\|_{L^2(-h,0;V)}^2\right),\ \forall\, t\geq 0.$$

Proof. We establish an integral inequality that will lead to the decay estimate of the Theorem. Firstly, observe that u_∞ satisfies

$$\nu\|u_\infty\|_V^2=\langle f,u_\infty\rangle+(G(u_\infty),u_\infty).$$

We multiply by $2e^{\lambda s}$, with $\lambda\geq 0$ and $0\leq s\leq t$, and integrate from 0 to t

$$2\nu\int_0^t e^{\lambda s}\|u_\infty\|_V^2ds=2\int_0^t e^{\lambda s}\Big(\langle f,u_\infty\rangle+(G(u_\infty),u_\infty)\Big)ds.$$

We add the following relation

$$0=e^{\lambda t}\|u_\infty\|_{L^2}^2-\|u_\infty\|_{L^2}^2-\int_0^t\lambda e^{\lambda s}\|u_\infty\|_{L^2}^2ds$$

to obtain that u_∞ satisfies the energy equality

$$e^{\lambda t}\|u_\infty\|_{L^2}^2 - \|u_\infty\|_{L^2}^2 - \int_0^t \lambda e^{\lambda s}\|u_\infty\|_{L^2}^2 ds + 2\nu \int_0^t e^{\lambda s}\|u_\infty\|_V^2 ds = 2\int_0^t e^{\lambda s}\Big(\langle f, u_\infty\rangle + (G(u_\infty), u_\infty)\Big)ds. \quad (26)$$

Secondly, by arguing as in Lemma 3.6 of Chapter III of [12] and taking into account that $f \in (L^2(\Omega))^3$, we can obtain the following relation

$$\begin{aligned} e^{\lambda t}(u(t), u_\infty) - (u^0, u_\infty) - \int_0^t \lambda e^{\lambda s}(u_\infty, u(s))ds + 2\nu \int_0^t e^{\lambda s}(\nabla u(s), \nabla u_\infty)ds \\ = \int_0^t e^{\lambda s}\Big(\langle f, u_\infty + u(s)\rangle + (G(u_\infty), u(s)) + (G(u(s-\rho(s))), u_\infty)\Big)ds \\ - \int_0^t e^{\lambda s}(b(u_\infty, u_\infty, u(s)) + b(u(s-\tau(s)), u(s), u_\infty))ds. \quad (27) \end{aligned}$$

Finally, let us denote $w(t) = u(t) - u_\infty$. We now add the energy inequality (19) to (26) and subtract two times (27) from the corresponding inequality. Arranging terms we find the following integral inequality

$$\begin{aligned} e^{\lambda t}\|w(t)\|_{L^2}^2 - \|w(0)\|_{L^2}^2 - \int_0^t \lambda e^{\lambda s}\|w(s)\|_{L^2}^2 ds + 2\nu \int_0^t \|w(s)\|_V^2 ds \\ \leq 2\int_0^t e^{\lambda s}(G(u(s-\rho(s))) - G(u_\infty), w(s))ds \\ + 2\int_0^t e^{\lambda s} b(w(s-\tau(s)), w(s), u_\infty)ds, \quad (28) \end{aligned}$$

where we have used that

$$b(u(s-\tau(s)), u(s), u_\infty) + b(u_\infty, u_\infty, u(s)) = b(w(s-\tau(s)), w(s), u_\infty).$$

By using Hölder and Young inequalities, we estimate the first term on the r.h.s.

$$\begin{aligned} 2\int_0^t & e^{\lambda s}(G(u(s-\rho(s))) - G(u_\infty), w(s))ds \\ &\leq 2\int_0^t e^{\lambda s}\|G(u(s-\rho(s))) - G(u_\infty)\|_{L^2}\|w(s)\|_{L^2} ds \\ &\leq \int_0^t e^{\lambda s}\left(L\|w(s-\rho(s))\|_{L^2}^2 + \frac{L}{\lambda_1}\|w(s)\|_V^2\right)ds, \end{aligned}$$

and the last term on the r.h.s. by also using that $\|u_\infty\|_V \leq \frac{\|f\|_{V'}}{\left(\nu - \frac{L}{\lambda_1}\right)}$,

$$\begin{aligned} 2\int_0^t & e^{\lambda s} b(w(s-\tau(s)), w(s), u_\infty)ds \\ &\leq 2\int_0 e^{\lambda s} C_1\|w(s-\tau(s))\|_V\|w(s)\|_V C_1\|u_\infty\|_V ds \\ &\leq \int_0^t e^{\lambda s}\Big(\frac{C_1^2\|f\|_{V'}^2}{(\nu - \frac{L}{\lambda_1})^2}\|w(s)\|_V^2 + C_1^2\|w(s-\tau(s))\|_V^2\Big)ds. \end{aligned}$$

Plugging these two estimates in (28) and using the Poincaré inequality we deduce

$$e^{\lambda t}\|w(t)\|_{L^2}^2 \leq \|w(0)\|_{L^2}^2 + \Big(\frac{\lambda}{\lambda_1} - 2\nu + \frac{L}{\lambda_1} + \frac{C_1^2\|f\|_{V'}^2}{(\nu - \frac{L}{\lambda_1})^2}\Big)\int_0^t e^{\lambda s}\|w(s)\|_V^2 ds$$
$$+ \int_0^t e^{\lambda s}\Big(L\|w(s-\rho(s))\|_{L^2}^2 + C_1^2\|w(s-\tau(s))\|_V^2\Big)ds.$$

By using the changes of variable $\eta = s - \tau(s)$ and $\eta = s - \rho(s)$ in the two terms with delay, we find for any $t \geq 0$,

$$\begin{aligned} &e^{\lambda t}\|w(t)\|_{L^2}^2 \\ &\leq \|w(0)\|_{L^2}^2 + \left(\frac{Le^{\lambda h}}{(1-\rho^*)\lambda_1} + \frac{C_1^2 e^{\lambda h}}{(1-\tau^*)}\right)\int_{-h}^0 e^{\lambda s}\|w(s)\|_V^2 ds \\ &+ \left(\frac{\lambda}{\lambda_1} - 2\nu + \frac{L}{\lambda_1} + \frac{C_1^2\|f\|_{V'}^2}{(\nu - \frac{L}{\lambda_1})^2} + \frac{Le^{\lambda h}}{(1-\rho^*)\lambda_1} + \frac{C_1^2 e^{\lambda h}}{(1-\tau^*)}\right)\int_0^t e^{\lambda s}\|w(s)\|_V^2 ds. \end{aligned}$$

By assumption (25), there exists $\lambda > 0$ such that the constant in the last integral vanishes, and hence

$$e^{\lambda t}\|w(t)\|_{L^2}^2 \leq \|w(0)\|_{L^2}^2 + \left(\frac{Le^{\lambda h}}{(1-\rho^*)\lambda_1} + \frac{C_1^2 e^{\lambda h}}{(1-\tau^*)}\right)\|w(s)\|_{L^2(-h,0;V)}^2,$$

which implies

$$\|u(t) - u_\infty\|_{L^2}^2 \leq Ce^{-\lambda t}\left(\|u^0 - u_\infty\|_{L^2}^2 + \|\phi - u_\infty\|_{L^2(-h,0;V)}^2\right).$$

Moreover, it is clear from the above computations that the stationary solution is unique. The proof is complete. □

Acknowledgments. The authors would like to thank the anonymous referees for pointing out a mistake on an earlier version of the paper and for their helpful suggestions and comments which significantly contributed to improving the paper.

The authors thank to E. Hernández for interesting discussions about this subject.

REFERENCES

[1] T. Caraballo and J. Real, *Navier-Stokes equations with delays*, Proc. R. Soc. Lond. A, **457** (2001), 2441–2453.

[2] T. Caraballo and J. Real, *Asymptotic behavior of two-dimensional Navier-Stokes equations with delays*, Proc. R. Soc. Lond. A, **459** (2003), 3181–3194.

[3] T. Caraballo and J. Real, *Attractors for 2D-Navier-Stokes models with delays*, J. Differential Equations, **205** (2004), 271–297.

[4] M. J. Garrido-Atienza and P. Marín-Rubio, *Navier-Stokes equations with delays on unbounded domains*, Nonlinear Anal., **64** (2006), 1100–1118.

[5] W. Liu, *Asymptotic behavior of solutions of time-delayed Burgers' equation*, Discrete Contin. Dyn. Syst. Ser. B., **2** (2002), 47–56.

[6] P. Marín-Rubio and J. Real, *Attractors for 2D-Navier-Stokes equations with delays on some unbounded domains*, Nonlinear Anal., **67** (2007), 2784–2799.

[7] P. Marín-Rubio and J. Real, *Pullback attractors for 2D-Navier-Stokes equations with delays in continuous and sub-linear operators*, Discrete Contin. Dyn. Syst., **26** (2010), 989–1006.

[8] G. Planas and E. Hernández, *Asymptotic behaviour of two-dimensional time-delayed Navier-Stokes equations*, Discrete Contin. Dyn. Syst., **21** (2008), 1245–1258.

[9] J. C. Robinson, "Infinite-Dimensional Dynamical Systems: An Introduction to Dissipative Parabolic PDEs and the Theory of Global Attractors," Cambridge texts in applied mathematics, Cambridge University Press, 2001.

[10] Y. Tang and M. Wan, *A remark on exponential stability of time-delayed Burgers equation*, Discrete Contin. Dyn. Syst. Ser. B., **12** (2009), 219–225.

[11] T. Taniguchi, *The exponential behavior of Navier-Stokes equations with time delay external force*, Discrete Contin. Dyn. Syst., **12** (2005), 997–1018.

[12] R. Temam, "Navier-Stokes Equations: Theory and Numerical Analysis," Studies in Mathematics and its applications. Volume **2**, The Netherlands, 1984.

Received June 2010; revised September 2010.

E-mail address: smguzzo@gmail.com
E-mail address: gplanas@ime.unicamp.br