A large time stepping viscosity-splitting finite element method for the viscoelastic flows problem

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Abstract

In this article, a large time stepping viscosity-splitting scheme is considered for the viscoelastic flows problem. The temporal term is decomposed into a sequence of two steps (using decomposition of the viscosity). For the first step, a linear elliptic problem is solved with explicit scheme for the convection term (a linear system with a constant coefficient matrix is obtained and the computation becomes easy), At the second step, the problem has the structure of the Stokes problem. Both two problems satisfy the homogeneous Derichlet boundary conditions for the velocities. The main novelties of this work are the stability of numerical solutions under the condition $k_1\Delta t \leq 1$ with a positive constant k_1, and optimal error estimates for both velocity in $L^\infty(H^1)$ and $L^2(H^1)$ norms and pressure in $L^\infty(L^2)$ and $L^2(L^2)$ norms. In order to enlarge the time step, we introduce a diffusion term $\theta\Delta u$ in all steps of the schemes. Finally, some numerical results are provided to display the performance of our algorithm.

Keywords: viscoelastic flows problem; large time stepping; fractional-step method; stability; convergence; optimal error estimates.
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1. Introduction

Let Ω be an open bounded domain in $\mathbb{R}^2$ with smooth boundary $\partial\Omega$. Consider the following viscoelastic flows problem

$$u_t - \nu\Delta u + \nabla p + (u\cdot\nabla)u - \int_0^t \rho e^{-\delta(t-s)}\Delta u ds = f, \quad x\in\Omega,\ t>0, \rho\geq 0, \tag{1.1}$$

with incompressible condition

$$\text{div } u(t,x) = 0 \qquad \forall\ t\geq 0\ x\in\Omega, \tag{1.2}$$

and initial and boundary conditions

$$u(x,0) = u_0(x) \ \ x\in\Omega; \quad u|_{\partial\Omega} = 0 \ \text{ for } t\geq 0, \tag{1.3}$$

where $1/\delta$, $u = (u_1,u_2)^T$, p, f, and $u_0(x)$ represent the relaxation time, represents the velocity, the pressure, the prescribed external force, and the initial velocity respectively.

Equations (1.1)-(1.3) are used as a model in viscoelastic flows problem [24, 31] because equations (1.1)-(1.3) are the generalization of the initial boundary value problem of the Navier-Stokes equations. For the incompressible problem, the main difficulties are the coupling of the pressure and the incompressible conditions, and the nonlinearity term. The fractional step methods is one of the most widely used classical numerical time-discretization schemes [1, 30]. The method reduces the problem into two subproblems: one linear elliptic problem without incompressibility condition and one generalized Stokes problem.

The fractional step methods are different from the two-step projection methods which based on the projection of an intermediate velocity field onto the space of solehoidal vector fields, and originated by Chorin [3, 4] and Temam [32, 33] along with other methods, such as the pressure correction methods [14, 15], the matrix factorization methods [26, 27] et. al.. Generally speaking, the so-called Chorin-Temam projection scheme has time error estimates of $\mathcal{O}(\Delta t^{\frac{1}{2}})$ in $L^2(H^1)\cap L^\infty(L^2)$ for the velocity and $\mathcal{O}(\Delta t^{\frac{1}{2}})$ in $L^2(L^2)$ for the pressure. Later, with the help of a pressure correction term is added to the projection step, Shen improved these estimates to $\mathcal{O}(\Delta t)$ in $L^2(H^1)\cap L^\infty(L^2)$ for the intermediate velocity and $\mathcal{O}(\Delta t)$ in $L^2(L^2)$ for the pressure in [28, 29] based on some regularity hypotheses of the exact solution.

Another class of fractional step methods, called viscosity-splitting methods has also been researched. The convergence and stability of the fully discrete version of the so-called θ-method were given by Glowinski in [13] and Fernandez-Cara and Marin Beltran in [9]. The fractional step methods and operator-splitting scheme for numerical solution of Navier-Stokes problem were also considered by the well-known predictor-multicorrector algorithm in [5, 6, 8]. In this scheme, the time advancement is decomposed into a sequence of two steps: at the first step, a linear elliptic problem to be solved, while at the second step a Stokes problem to be considered. Two steps satisfy the full homogeneous Dirichlet boundary condition on the velocity. Optimal error estimates of $\mathcal{O}(\Delta t)$ in $L^2(H^1)\cap L^\infty(L^2)$ for the end-of-step velocity u^{n+1} and suboptimal bounds of $\mathcal{O}(\Delta t^{\frac{1}{2}})$ in $L^2(L^2)$ for the pressure p^{n+1} have been presented in [6]. Besides, numerical results of the viscosity-splitting scheme were performed in [2] for illustrating $\mathcal{O}(\Delta t)$ for both velocity and pressure. As a consequence, there exists a gap between the numerical analysis and numerical computations. In [16] the author has obtained the error estimates of $\mathcal{O}(\Delta t)$ in $L^\infty(H^1)$ for the velocity and in $L^2(L^2)$ for the pressure, where a weight at the initial time steps must be included to deduce the optimal error estimates for the pressure. To the best of the author's knowledge, this maybe the most perfect results related to the viscosity-splitting scheme for Navier-Stokes problem in semi-discrete form.

In this paper, a large time stepping viscosity-splitting fractional-step method is considered for the viscoelastic flows problem. We adopt the explicit/implicit formulation to handle the first step of splitting scheme, the advantage of using an implicit scheme for the linear terms and an explicit scheme for the nonlinear term is that a linear system with a constant coefficient matrix can be obtained which can save computational cost. Without introducing the weight for the initial steps, some new stabilities and optimal error estimates for both velocity and pressure are established by using Taylor expansion and other skills under some restriction about the time step, which improves the results of [2, 5, 6, 9, 8, 16]. Furthermore, we introduce a diffusion term $\theta\Delta u$ in all steps of our scheme. The purposes of the term $\theta\Delta u$ are to enlarge the time stepping and enhance numerical stability by choosing suitable parameter θ. The effectiveness of θ is analyzed in Section 6 and verified by numerous numerical experiments.

The rest of this paper is organized as follows. In Section 2, the notations and some basic results for equations (1.1)-(1.3) are recalled. In Section 3, the fractional-step schemes for the viscoelastic flows problem are established. In Section 4, the stability of numerical solutions are given. In Section 5, some error estimates for the intermediate velocity, the end-of-step velocity and the pressure are presented. In Section 6, many numerical results are provided to confirm the established theoretical findings. Finally, conclusions are made in the last section.

2. Preliminaries

In this section, we aim to describe some notations and results which will be frequently used in this paper. For the mathematical setting of the viscoelastic flows problem (1.1)-(1.3), we introduce the following Hilbert spaces:

$$X = H_0^1(\Omega)^2, \quad Y = L^2(\Omega)^2, \quad D(A) = H^2(\Omega)^2 \cap X,$$
$$M = L_0^2(\Omega) = \{q \in L^2(\Omega) : \int_\Omega q dx = 0\}.$$

The spaces $L^2(\Omega)^m$ $(m = 1, 2)$ are endowed with the standard L^2-scalar product $(\cdot, \cdot)$ and L^2-norm $\|\cdot\|_{0,\Omega}$. The spaces $H_0^1(\Omega)$ and X are equipped with the scalar product $(\nabla u, \nabla v)$ and norm $\|\nabla u\|_0^2$, for $\forall\, u, v \in\ H_0^1(\Omega)$ or X, respectively.

Next, set the closed subset V of X is given by

$$V = \{v \in X;\ \nabla \cdot v = 0\}$$

and denote the H be the closed subset of Y, i.e.,

$$H = \{v \in Y;\ \nabla \cdot v = 0,\ v \cdot n|_{\partial\Omega} = 0\}.$$

We denote $Au = -\Delta u$ a positive self-adjoint operator from $D(A)$ onto Y. Note that $A^\alpha(\alpha \in R)$ is well defined. In particular, there hold $D(A^{\frac{1}{2}}) = X,\ D(A^0) = Y$ and

$$(A^{1/2}u, A^{1/2}v) = (\nabla u, \nabla v), \qquad \forall\ u, v, \in X$$

It is well-known that there hold the following Gagliardo-Nirenberg inequalities

$$\|u\|_{L^4} \le c_0 \|u\|_0^{1/2} \|A^{1/2}u\|_0^{1/2}, \quad \|u\|_0 \le c_0 \|A^{1/2}u\|_0, \quad \forall\ u \in X, \tag{2.1}$$
$$\|u\|_{L^\infty} \le c_0 \|u\|_0^{\frac{1}{2}} \|Au\|_0^{\frac{1}{2}}, \quad \|\nabla v|_{L^4} \le c_0 \|A^{\frac{1}{2}}u\|_0^{\frac{1}{2}} \|Av\|_0^{\frac{1}{2}}, \ \ \forall\ u \in D(A), \tag{2.2}$$

where and in the following, c_0 and $c_i, (i = 1, \ldots)$ are positive constants only depending on Ω. We also use the letter c to denote a general positive constant which may stand for different values at its different occurrences.

We usually make the following assumption about the prescribed data (u_0, f) for problem (1.1)-(1.3) (see [19, 23]).
(A1). Assume that the initial velocity $u_0 \in D(A)$ with $\mathrm{div} u_0 = 0$ and the forcing functions $f, f_t \in L^\infty(0, T; Y)$ satisfy

$$\|u_0\|_2 + \sup_{t\in[0,T]} \{\|f\|_0 + \|f_t\|_0\} \le c.$$

Next, we make a regularity assumption on the Stokes problem [1, 2, 8, 16].
(A2) Assume that Ω is smooth such that there exists a unique solution $(v, q) \in X \times M$ of the following Stokes problem

$$-\Delta v + \nabla q = u, \quad \nabla \cdot v = 0, \quad \text{in } \Omega, \quad v|_{\partial\Omega} = 0 \tag{2.3}$$

for any prescribed $u \in H$. Furthermore, the solution $v = A^{-1}u$ satisfies

$$\|v\|_s = \|A^{-1}u\|_s \le c\|u\|_{s-2}, \quad s = 1, 2.$$

Form (2.3), it follows that $(A^{-1}u, u) = \|\nabla A^{-1}u\|_0$ and

$$\|v\|_{V'}^2 = (A^{-1}v, v), \quad \forall\, v \in H,$$

where V' is the dual space of V.
As for viscoelastic flows problem (1.1)-(1.3), we define the continuous bilinear forms $a(\cdot,\cdot)$ and $d(\cdot,\cdot)$ on $X \times X$ and $X \times M$, respectively, by

$$a(u, v) = \nu(\nabla u, \nabla v), \quad d(v, q) = (q, \mathrm{div} v), \quad \forall\, u, v \in X, \quad q \in M.$$

We also introduce the continuous trilinear form on $X \times X \times X$

$$b(u, v, w) = ((u \cdot \nabla)v, w), \quad \forall\, u, v, w \in X.$$

It is easy to verify that $b(\cdot,\cdot,\cdot)$ satisfies the following important properties (see [12, 18, 19, 23]):

$$b(u, v, w) = -b(u, w, v) \qquad \forall\, u, v, w \in X, \tag{2.4}$$
$$|b(u, v, w)| \le c_1\|Au\|_0\|A^{1/2}v\|_0\|w\|_0 \quad \forall\, u \in D(A), v \in X, w \in Y. \tag{2.5}$$

With above notations, the variational formulation of problem (1.1)-(1.3) can be formulated as follows: Find $(u, p) \in L^\infty(0, T; Y) \cap L^2(0, T; X) \times L^2(0, T; M)$ such that for all $t > 0$

$$(u_t, v) + B((u, p), (v, q)) + b(u, u, v) + J(t, u, v) = (f, v), \tag{2.6}$$

with the initial condition $u(0) = u_0$ and

$$B((u,p),(v,q)) = a(u,v) - d(p,v) + d(q,u),$$
$$J(t,u,v) = \rho e^{-\delta t}\int_0^t e^{\delta s}(Au(s),v)ds = \rho e^{-\delta t}\int_0^t e^{\delta s}(\nabla u(s),\nabla v)ds.$$

Assume that $f \in L^2(0,T;X')$ and $u_0 \in H$. Problem (1.1)-(1.3) has at least one solution (u,p) satisfying $u \in L^\infty(0,T;\Omega) \cap L^2(0,T;V)$. Uniqueness and regularity of the solution can also be proved by strengthening the assumptions on the data. In particular, we assume that u and p satisfy

(A3) $u \in L^\infty(0,T;H^2(\Omega)^2)$, $\nabla p \in L^\infty(0,T;Y)$,
(A4) $u_t \in L^2(0,T;X)$,
(A5) $\sqrt{t}u_{tt} \in L^2(0,T;Y)$,
(A6) $u_{tt} \in L^2(0,T;V')$.
(A7) $u_{ttt} \in L^2(0,T;V')$.

Note that all such assumptions are feasible. For example, (A3) and (A4) can be proved with assumptions $u_0 \in H^2(\Omega)^2 \cap V$, $f \in L^\infty(0,T;H)$ and $f_t \in L^1(0,T;H)$. When Ω is of class of C^2 or is a convex polygon, (A5) holds by [19] and [23]. Furthermore, (A6) holds by Shen in [28, 29] when they add some nonlocal compatibility conditions. A review of regularity results for Navier-Stokes equations and applications to error estimates for Euler-type scheme can be found in [18], where proof of (A7) was given.

We will frequently use the following discrete Gronwall lemmas [18, 19, 28].

Lemma 2.1. Let C_0 and a_k, b_k, c_k, d_k, for integers $k \geq 0$, be non-negative numbers such that

$$a_n + \triangle t\sum_{k=0}^{n} b_k \leq \Delta t\sum_{k=0}^{n-1} d_k a_k + \Delta t\sum_{k=0}^{n-1} c_k + C_0, \qquad \forall\ n \geq 1.$$

Then,

$$a_n + \triangle t\sum_{k=0}^{n} b_k \leq (\Delta t\sum_{k=0}^{n-1} c_k + C_0)\exp(\Delta t\sum_{k=0}^{n-1} d_k) \qquad \forall\ n \geq 1.$$

Finally, for error bounds of the numerical solutions, we recall the following regularity of solutions of problem (2.6).

Theorem 2.2. ([21, 25]) Assume that conditions of (A1)-(A2), and the uniqueness condition

$$\nu^{-2}N\|f_\infty\|_{-1} \leq 1, \text{ where } N = \sup_{u,v,w\in H_0^1(\Omega)^2}\frac{b(u,v,w)}{\|\nabla u\|_0\|\nabla v\|_0\|\nabla w\|_0} \qquad (2.7)$$

hold. Then, for all $s \geq 0$, the solution (u, p) of problem (2.6) satisfies

$$\|u(t)\|_0^2 + \|\nabla u\|_0^2 + \|Au\|_0^2 + \|u_t\|_0^2 + \|p\|_1^2 \leq \widehat{c}.$$

3. Viscosity splitting fractional-step method

One way of discretizing equations (1.1)-(1.3) in time is by viscosity splitting fractional-step method. In this scheme, the time advancement is generally decomposed into a sequence of two steps. For the time discretization of the integral term, we make analysis as in [25] and apply right rectangle rule to the integral term:

$$M^n(\phi) = \Delta t \rho \sum_{i=1}^{n} e^{-\delta(t_n - t_i)} \phi(t_i) \approx \rho \int_0^{t_n} e^{-\delta(t_n - t)} \phi(t) dt, \tag{3.1}$$

where Δt is the time stepsize. Due to Theorem 2.2, using the fact that $1 + \delta \triangle t \leq e^{\delta \triangle t} \leq c$ and $e^{-\delta t_n} \leq 1$, we have

$$\begin{aligned} \triangle t e^{-\delta t_n} \sum_{i=1}^{n} e^{\delta t_i} &\leq \triangle t e^{-\delta t_n} \cdot (e^{\delta t_1} + e^{\delta t_2} + \ldots + e^{\delta t_n}) \\ &= \triangle t \cdot (1 + e^{-\delta t_1} + e^{-\delta t_2} + \ldots + e^{-\delta t_{n-1}}) \leq C(1/\delta), \end{aligned} \tag{3.2}$$

$$|M^n(\nabla u)| \leq \rho \triangle t e^{-\delta t_n} \sum_{i=1}^{n} e^{\delta t_i} \|\nabla u(t_i)\|_0 \leq c(\rho) \triangle t e^{-\delta t_n} \sum_{i=1}^{n} e^{\delta t_i} \leq \widetilde{c}. \tag{3.3}$$

Next, we consider the following kind of two-step scheme for viscoelastic flows problem (1.1)-(1.3).

3.1. First Step

The first step of this scheme including viscous, convective effect and integral term, consists of finding an intermediate velocity $u^{n+\frac{1}{2}}$ such that

$$\begin{cases} \frac{u^{n+\frac{1}{2}} - u^n}{\Delta t} - (\theta + \nu) \Delta u^{n+\frac{1}{2}} + \theta \Delta u^n + (u^n \cdot \nabla) u^n \\ \qquad\qquad - \Delta t \rho \sum_{i=1}^{n} e^{-\delta(t_n - t_i)} \Delta u(t_i) = f(t_{n+1}), \\ u^{n+\frac{1}{2}}|_{\partial\Omega} = 0, \end{cases} \tag{3.4}$$

with $0 \leq n \leq N$ and $\theta > 0$ is a bounded parameter. The superscript n denotes the time level $t_n = n\Delta t$. For the nonlinear term, we only consider the explicit scheme. Of course, other approximation forms can also be taken such as the semi-implicit scheme $(u^n \cdot \nabla) u^{n+\frac{1}{2}}$ or the implicit scheme

$(u^{n+\frac{1}{2}}\cdot\nabla)u^{n+\frac{1}{2}}$. The advantage of using an explicit scheme for the nonlinear term is a linear system with a constant coefficient matrix which can save computational cost. As for the approximation of the body force term, the time average of f in $[t_n, t_{n+1}]$ can also be used.

The first step of this method can be considered as a linearized elliptic problem. The weak form of (3.4) can be written as

$$a_\theta(u^{n+\frac{1}{2}}, v) =< l_1, v > \qquad \forall\ v \in X,$$

where $a_\theta(u, v) = (u, v) + (\theta+\nu)\Delta t(\nabla u, \nabla v)$ is a bilinear continuous form on $X\times X$, and $l_1 = u^n+\Delta t\Big(f(t_{n+1})-\theta\Delta u^n-(u^n\cdot\nabla)u^n-\Delta t\rho\sum_{i=1}^{n} e^{-\delta(t_n-t_i)}\Delta u(t_i)\Big)$ is a known map. The trilinear form $b(u, v, w)$ is skew-symmetric in v and w if $u \in V$. The coerciveness of a_θ results from the skew-symmetric character of the approximation of the convective term and the presence of the Laplacian term, namely,

$$a_\theta(u, u) = (u, u) + (\theta + \nu)\Delta t(\nabla u, \nabla u) = \|u\|_0^2 + (\theta + \nu)\Delta t\|\nabla u\|_0^2 \geq c\|\nabla u\|_0^2.$$

The existence and uniqueness of $u^{n+\frac{1}{2}}$ is established by the Lax-Milgram theorem. Compared with the bilinear form in semi-implicit $a(u, v) = (u, v)+(\theta+\nu)\Delta t(\nabla u, \nabla v) + \Delta tb(u^n, u, v)$ or implicit scheme $a(u, v) = (u, v) + (\theta+\nu)\Delta t(\nabla u, \nabla v) + \Delta tb(u, u, v)$, the bilinear term $a_\theta(u, v)$ has better coercivity and stability. Furthermore, the stiffness matrix of system (3.4) does not change in every iteration. When ν is small, we can choose a bigger value of θ such that the time step can be enlarged in implementation. The same technique has been adopted to consider the Cahn-Hilliard equations in [22].

3.2. Second Step

Given $u^{n+\frac{1}{2}}$ from (3.4), the second step of the method consists of finding u^{n+1} and p^{n+1} such that

$$\begin{cases} \frac{u^{n+1}-u^{n+\frac{1}{2}}}{\Delta t} - (\theta+\nu)(\Delta u^{n+1} - \Delta u^{n+\frac{1}{2}}) + \nabla p^{n+1} = 0, \\ \mathrm{div} u^{n+1} = 0, \\ u^{n+1}|_{\partial\Omega} = 0. \end{cases} \tag{3.5}$$

The second step of this method can be considered as a mixed problem or the generalized Stokes problem with $f = 0$. Note that we can solve the system without imposing the boundary condition for the pressure. The weak form of problem (3.5) consists of finding $u^{n+1} \in X$ and $p^{n+1} \in M$ such that for all $(v, q) \in X \times M$

$$\begin{cases} a_\theta(u^{n+1}, v) + \Delta t d(v, p^{n+1}) =< l_2, v >, \\ d(u^{n+1}, q) = 0, \end{cases}$$

where $l_2 = u^{n+\frac{1}{2}} - \Delta t(\theta + \nu)\Delta u^{n+\frac{1}{2}} \in H^{-1}(\Omega)^2$ is a known map. If $d(\cdot, \cdot)$ satisfies the so-called *inf-sup* condition:

$$\beta \|q\|_0 \leq \inf_{0 \neq q \in M} \sup_{v \in X} \frac{d(v, q)}{\|\nabla v\|_0} \quad (\beta > 0 \text{ is a constant}),$$

the existence and uniqueness of u^{n+1} and p^{n+1} are guaranteed [12, 32].

By adding (3.4) and (3.5), we obtain that

$$\frac{u^{n+1} - u^n}{\Delta t} - \nu \Delta u^{n+1} - \theta(\Delta u^{n+1} - \Delta u^n) + (u^n \cdot \nabla) u^n + \nabla p^{n+1} \\ - \Delta t \rho \sum_{i=1}^{n} e^{-\delta(t_n - t_i)} \Delta u(t_i) = f(t_{n+1}). \quad (3.6)$$

From (3.6), note that the implicit treatment of the viscous term in u^{n+1} and u^n, and the intermediate velocity $u^{n+\frac{1}{2}}$ disappears. The term $\theta(\Delta u^{n+1} - \Delta u^n)$ is introduced as a stabilized term. It allows us to compute by the large time step and improve the numerical stability. Moreover, it is clear from (3.6) that at least for the linear problem, p^{n+1} keeps its meaning as an end-of-step pressure. One advantage of using the split scheme like (3.4) and (3.5) rather than a coupled (u, p) method, is that the decoupling of the convective effects from incompressibility, which allows the use of suitable approximations for each term.

Remark 3.1. Denote $d_t u^n = \frac{u^n - u^{n-1}}{\Delta t}$ and $d_t u^0$ is defined to satisfy

$$d_t u^0 - (\nu + \theta)\Delta u^0 + \nabla p^0 = f(t_0), \quad \text{div } u^0 = 0,$$

it is easy to verify that

$$\|u^0\|_0^2 + 2\Delta t(\nu + \theta)\|\nabla u^0\|_0^2 \leq \|f(t_0)\|_0^2 \Delta t^2, \quad (3.7)$$

$$\|d_t u^0\|_0^2 \leq (\nu + \theta)\|A u^0\|_0 + \|f(t_0)\|_0. \quad (3.8)$$

4. Convergence of the viscosity splitting fractional-step scheme

In order to obtain the error estimates for the numerical solutions, in this section, we firstly present some stability results of approximate solutions

$\{u^n\}_{n=1}^{N+1}$, $\{u^{n+\frac{1}{2}}\}_{n=1}^{N}$ and $\{p^n\}_{n=1}^{N+1}$ for schemes (3.4)-(3.5).

Lemma 4.1 The solutions u^n and $u^{n+\frac{1}{2}}$ are bounded in the sense that for $\forall\ 0 \le n \le N,\ N = -1, 0, 1, \ldots, [\frac{T}{\Delta t}] - 1$:

$$\|u^{N+1}\|_0^2 + \Delta t\theta\|\nabla u^{N+1}\|_0^2 + \Delta t\nu\sum_{n=0}^{N}\Big(\frac{1}{4}\|\nabla u^{n+\frac{1}{2}}\|_0^2 + \|\nabla u^{n+1}\|_0^2\Big) \le \gamma_0^2, \quad (4.1)$$

$$\Delta t\sum_{n=0}^{N}\Big(\frac{1}{2\nu}\|d_t u^{n+1}\|_0^2 + \nu\|Au^{n+1}\|_0^2\Big) + 2\|\nabla u^{N+1}\|_0^2 + 2\theta\Delta t\|Au^{N+1}\|_0^2 \le k_{01}, \quad (4.2)$$

$$\|d_t u^{N+1}\|_0^2 + \|Au^{n+1}\|_0^2 + \nu\Delta t\sum_{n=0}^{N}\|\nabla d_t u^{n+1}\|_0^2 + \theta\Delta t\|\nabla d_t u^{N+1}\|_0^2 \le k_{02}, \quad (4.3)$$

where

$$\begin{aligned}
\gamma_0^2 &= \|u^0\|_0^2 + \Delta t\theta\|\nabla u^0\|_0^2 + 4\nu^{-1}T(f_\infty^2 + \widehat{c}),\\
k_{01} &= \exp\Big(\nu^{-1}\Big(\frac{8}{\nu}\Big)^3 c_0^4\gamma_0^4\Big)\Big(2\|A^{1/2}u^0\|_0^2 + \frac{\nu}{4}\|Au^0\|_0^2\Delta t\\
&\quad + \frac{10}{\nu}T(f_\infty^2 + \widehat{c}^2) + 2\theta\Delta t\|Au^0\|_0^2\Big),\\
k_{02} &= (1 + 12(\theta+\nu)^{-2})\exp\Big(8\nu^{-1}c_1^2k_{01}\Big)\Big\{\|d_tu^0\|_0^2 + \theta\Delta t\|\nabla d_t u^0\|_0^2\\
&\quad + 4\nu^{-1}c_0^2T\sup_{0\le t\le T}\|f_t\|_0^2dt + 4\nu^{-1}(c_0^2+1)T\widehat{c}^2\Big\}\\
&\quad + 48(\theta+\nu)^{-4}c_0^4c_1^2k_{01}^2\gamma_0^2 + (\theta+\nu)^{-2}\Big(12f_\infty^2 + \theta^2 + C(1/\delta)\widehat{c}^2\Big),\\
k_1 &= 8\nu^{-1}c_1^2k_{02}, \qquad f_\infty = \sup_{1\le t\le T}\|f(t)\|_0.
\end{aligned}$$

Proof. We prove this lemma by using the induction. From (3.7)-(3.8), we know that (4.1)-(4.3) hold for $N = -1$. Assume that (4.1) holds for $N = 0, 1, \ldots, J$ with $1 \le J \le [\frac{T}{\Delta t}] - 2$. We need to prove (4.1) for $N = J+1$.

Taking inner product of (3.4) with $2\Delta t u^{n+\frac{1}{2}}$, we obtain that

$$\begin{aligned}
&(u^{n+\frac{1}{2}} - u^n, 2u^{n+\frac{1}{2}}) + 2\Delta t\nu\|\nabla u^{n+\frac{1}{2}}\|_0^2 + 2\Delta t^2\rho\sum_{i=1}^{n}e^{-\delta(t_n-t_i)}(\nabla u(t_i), \nabla u^{n+\frac{1}{2}})\\
&+\Delta t\theta(\nabla(u^{n+\frac{1}{2}} - u^n), 2\nabla u^{n+\frac{1}{2}}) + 2\Delta t b(u^n, u^n, u^{n+\frac{1}{2}}) = 2\Delta t(f(t_{n+1}), u^{n+\frac{1}{2}}). \quad (4.4)
\end{aligned}$$

By use of the identities

$$(a-b, 2a) = |a|^2 - |b|^2 + |a-b|^2 \quad \text{and} \quad 2(a,b) = |a|^2 + |b|^2 - |a-b|^2, (4.5)$$

equation (4.4) can be transformed into

$$\begin{aligned}&\|u^{n+\frac{1}{2}}\|_0^2-\|u^n\|_0^2+\|u^{n+\frac{1}{2}}-u^n\|_0^2+2\Delta t\nu\|\nabla u^{n+\frac{1}{2}}\|_0^2\\&+\Delta t\theta\Big(\|\nabla u^{n+\frac{1}{2}}\|_0^2-\|\nabla u^n\|_0^2+\|\nabla(u^{n+\frac{1}{2}}-u^n)\|_0^2\Big)+2\Delta tb(u^n,u^n,u^{n+\frac{1}{2}})\\&+2\Delta t^2\rho\sum_{i=1}^n e^{-\delta(t_n-t_i)}(\nabla u(t_i),\nabla u^{n+\frac{1}{2}})=2\Delta t(f(t_{n+1}),u^{n+\frac{1}{2}}).\end{aligned}\tag{4.6}$$

Multiplying (3.5) with $2\Delta tu^{n+1}$, noting the fact that $\nabla\cdot u^{n+1}=0$ and using (4.5) we have

$$\begin{aligned}&\|u^{n+1}\|_0^2-\|u^{n+\frac{1}{2}}\|_0^2+\|u^{n+1}-u^{n+\frac{1}{2}}\|_0^2\\&+\Delta t(\theta+\nu)\Big(\|\nabla u^{n+1}\|_0^2-\|\nabla u^{n+\frac{1}{2}}\|_0^2+\|\nabla(u^{n+1}-u^{n+\frac{1}{2}})\|_0^2\Big)=0.\end{aligned}\tag{4.7}$$

It follows from (4.6) with (4.7) that

$$\begin{aligned}&\|u^{n+1}\|_0^2-\|u^n\|_0^2+\|u^{n+1}-u^{n+\frac{1}{2}}\|_0^2+\|u^{n+\frac{1}{2}}-u^n\|_0^2+\Delta t\nu\|\nabla u^{n+\frac{1}{2}}\|_0^2\\&+\Delta t\theta\Big(\|\nabla u^{n+1}\|_0^2-\|\nabla u^n\|_0^2+\|\nabla(u^{n+1}-u^{n+\frac{1}{2}})\|_0^2+\|\nabla(u^{n+\frac{1}{2}}-u^n)\|_0^2\Big)\\&+\Delta t\nu\Big(\|\nabla u^{n+1}\|_0^2+\|\nabla(u^{n+1}-u^{n+\frac{1}{2}})\|_0^2\Big)+2\Delta tb(u^n,u^n,u^{n+\frac{1}{2}})\\&+2\Delta t^2\rho\sum_{i=1}^n e^{-\delta(t_n-t_i)}(\nabla u(t_i),\nabla u^{n+\frac{1}{2}})=2\Delta t(f(t_{n+1}),u^{n+\frac{1}{2}})\end{aligned}\tag{4.8}$$

Using (2.5) and (3.3) we deduce that

$$\begin{aligned}&2|b(u^n,u^{n+\frac{1}{2}},u^{n+\frac{1}{2}}-u^n)|\le 2c_1\|Au^n\|_0\|\nabla u^{n+\frac{1}{2}}\|_0\|u^{n+\frac{1}{2}}-u^n\|_0\\\le\ &\frac{\nu}{4}\|\nabla u^{n+\frac{1}{2}}\|_0^2+4\nu^{-1}c_1^2\|Au^n\|_0^2\|u^{n+\frac{1}{2}}-u^n\|_0^2,\\&2|\Delta t\rho\sum_{i=1}^n e^{\delta(t_n-t_i)}(\nabla u(t_i),\nabla u^{n+\frac{1}{2}})|\le 2\Delta t\rho\sum_{i=1}^n e^{\delta(t_n-t_i)}\|\nabla u(t_i)\|_0\|\nabla u^{n+\frac{1}{2}}\|_0\\\le\ &\frac{\nu}{4}\|\nabla u^{n+\frac{1}{2}}\|_0^2+4\nu^{-1}\Big(\Delta t\rho\sum_{i=1}^n e^{\delta(t_n-t_i)}\|\nabla u(t_i)\|_0\Big)^2,\\&2|(f(t_{n+1}),u^{n+\frac{1}{2}})|\Delta t\le 2\|f(t_{n+1})\|_0\|u^{n+\frac{1}{2}}\|_0\Delta t\\\le\ &\frac{\nu}{4}\|\nabla u^{n+\frac{1}{2}}\|_0^2+4\nu^{-1}\|f(t_{n+1})\|_0^2\Delta t.\end{aligned}$$

Combining above estimates with (4.8) yields

$$\begin{aligned} & \|u^{n+1}\|_0^2 - \|u^n\|_0^2 + \|u^{n+1} - u^{n+\frac{1}{2}}\|_0^2 + \frac{\Delta t\nu}{4}\|\nabla u^{n+\frac{1}{2}}\|_0^2 + \Delta t\nu\|\nabla u^{n+1}\|_0^2 \\ & +\Delta t\theta\Big(\|\nabla u^{n+1}\|_0^2 - \|\nabla u^n\|_0^2 + \|\nabla(u^{n+\frac{1}{2}} - u^n)\|_0^2\Big) + \Delta t(\theta+\nu)\|\nabla(u^{n+1} - u^{n+\frac{1}{2}})\|_0^2 \\ \leq\ & \Big(4\nu^{-1}c_1^2\|Au^n\|_0^2\Delta t - \frac{1}{2}\Big)\|u^{n+\frac{1}{2}} - u^n\|_0^2 + 4\nu^{-1}\|f(t_{n+1})\|_0^2\Delta t^2 \\ & +4\nu^{-1}\Delta t\Big(\Delta t\rho\sum_{i=1}^{n} e^{\delta(t_n-t_i)}\|\nabla u(t_i)\|_0\Big)^2, \end{aligned} \tag{4.9}$$

Due to the stability condition $k_1\Delta t \leq 1$ and the induction assumption on $N = -1, 0, 1, \ldots, [T/\Delta t] - 2$, we have

$$4\nu^{-1}c_1^2\|Au^n\|_0^2\Delta t - \frac{1}{2} \leq 4\nu^{-1}c_1^2k_{02}\Delta t - \frac{1}{2} \leq \frac{1}{2}\Delta tk_1 - \frac{1}{2} \leq 0, \quad \forall\ 0 \leq n \leq N. \tag{4.10}$$

Summing (4.9) for n from $n = 0$ to N, using (3.3) and (4.10) we obtain that

$$\begin{aligned} & \|u^{N+1}\|_0^2 + \Delta t\theta\|\nabla u^{N+1}\|_0^2 + \sum_{n=0}^{N}\|u^{n+1} - u^{n+\frac{1}{2}}\|_0^2 \\ & +\Delta t\nu\sum_{n=0}^{N}\Big(\frac{1}{4}\|\nabla u^{n+\frac{1}{2}}\|_0^2 + \|\nabla u^{n+1}\|_0^2 + \|\nabla(u^{n+1} - u^{n+\frac{1}{2}})\|_0^2\Big) \\ & +\Delta t\theta\sum_{n=0}^{N}\Big(\|\nabla(u^{n+\frac{1}{2}} - u^n)\|_0^2 + \|\nabla(u^{n+1} - u^{n+\frac{1}{2}})\|_0^2\Big) \\ \leq\ & \|u^0\|_0^2 + \Delta t\theta\|\nabla u^0\|_0^2 + 4\nu^{-1}\sum_{n=0}^{N}\|f(t_{n+1})\|_0^2\Delta t \\ & +4\nu^{-1}\sum_{n=0}^{N}\Delta t\Big(\Delta t\rho\sum_{i=1}^{n} e^{\delta(t_n-t_i)}\|\nabla u(t_i)\|_0\Big)^2 \\ \leq\ & \|u^0\|_0^2 + \Delta t\theta\|\nabla u^0\|_0^2 + 4\nu^{-1}T(f_\infty^2 + \widetilde{c}). \end{aligned} \tag{4.11}$$

Thanks to the induction assumption on $n = 0, 1, \ldots, N$, we finish the rest proof of the (4.1).

Next, for all $v \in V$ with $0 \leq n \leq N$, (3.6) can be rewritten as

$$\begin{aligned} & (d_tu^{n+1}, v) - \nu(\Delta u^{n+1}, v) - \theta\Big((\Delta u^{n+1} - \Delta u^n), v\Big) + b(u^n, u^n, v) \\ & \qquad -\Delta t\rho\sum_{i=1}^{n} e^{-\delta(t_n-t_i)}(\Delta u(t_i), v) = (f(t_{n+1}), v). \end{aligned} \tag{4.12}$$

Take $v = (\nu^{-1}d_t u^{n+1} + Au^{n+1})\Delta t$ in (4.12) to get

$$\begin{aligned}
&\nu^{-1}\Delta t\|d_t u^{n+1}\|_0^2 + \|\nabla u^{n+1}\|_0^2 - \|\nabla u^n\|_0^2 + \|\nabla(u^{n+1}-u^n)\|_0^2 + \nu\Delta t\|Au^{n+1}\|_0^2 \\
&+\theta\nu^{-1}\|\nabla(u^{n+1}-u^n)\|_0^2 + \frac{\theta\Delta t}{2}\Big(\|Au^{n+1}\|_0^2 - \|Au^n\|_0^2 + \|A(u^{n+1}-u^n)\|_0^2\Big) \\
&+b(u^n,u^n,\nu^{-1}d_t u^{n+1} + Au^{n+1})\Delta t = (f(t_{n+1}),\nu^{-1}d_t u^{n+1} + Au^{n+1})\Delta t \\
&+\Delta t^2\rho\sum_{i=1}^{n} e^{-\delta(t_n-t_i)}(\Delta u(t_i),\nu^{-1}d_t u^{n+1} + Au^{n+1}). \qquad (4.13)
\end{aligned}$$

By using (2.1)-(2.4) and (3.3) we obtain

$$\begin{aligned}
&|b(u^n,u^n,\nu^{-1}d_t u^{n+1} + Au^{n+1})| \\
\le\;& 2c_0\|A^{1/2}u^n\|_0\|u^n\|_0^{1/2}\|Au^n\|_0^{1/2}(\nu^{-1}\|d_t u^{n+1}\|_0 + \|Au^{n+1}\|_0) \\
\le\;& \frac{1}{4\nu}\|d_t u^{n+1}\|_0^2 + \frac{\nu}{4}\|Au^{n+1}\|_0^2 + \frac{8}{\nu}c_0^2\|A^{1/2}u^n\|_0^2\|u^n\|_0\|Au^n\|_0 \\
\le\;& \frac{1}{4\nu}\|d_t u^{n+1}\|_0^2 + \frac{\nu}{4}\|Au^{n+1}\|_0^2 + \frac{\nu}{8}\|Au^n\|_0^2 + \frac{1}{2}\Big(\frac{8}{\nu}\Big)^3 c_0^4\|A^{1/2}u^n\|_0^4\|u^n\|_0^2, \\
&\Big|\Delta t\rho\sum_{i=1}^{n} e^{-\delta(t_n-t_i)}\Big(Au(t_i),\nu^{-1}d_t u^{n+1} + Au^{n+1}\Big)\Big| \\
\le\;& \Delta t\rho\sum_{i=1}^{n} e^{-\delta(t_n-t_i)}\|Au(t_i)\|_0\Big(\nu^{-1}\|d_t u^{n+1}\|_0 + \|Au^{n+1}\|_0\Big) \\
\le\;& \frac{1}{4\nu}\|d_t u^{n+1}\|_0^2 + \frac{\nu}{16}\|Au^{n+1}\|_0^2 + \frac{5}{\nu}\Big(\Delta t\rho\sum_{i=1}^{n} e^{-\delta(t_n-t_i)}\|Au(t_i)\|_0\Big)^2, \\
&|(f(t_{n+1}),\nu^{-1}d_t u^{n+1} + Au^{n+1})| \\
\le\;& \frac{1}{4\nu}\|d_t u^{n+1}\|_0^2 + \frac{\nu}{16}\|Au^{n+1}\|_0^2 + \frac{5}{\nu}\|f(t_{n+1})\|_0^2.
\end{aligned}$$

Combining these estimates with (4.13) yields

$$\begin{aligned}
&(2\nu)^{-1}\Delta t\|d_t u^{n+1}\|_0^2 + 2\|\nabla u^{n+1}\|_0^2 - 2\|\nabla u^n\|_0^2 + 2\|\nabla(u^{n+1}-u^n)\|_0^2 \\
&+\nu\Delta t\Big(\frac{5}{4}\|Au^{n+1}\|_0^2 - \frac{1}{4}\|Au^n\|_0^2\Big) + 2\theta\nu^{-1}\|\nabla(u^{n+1}-u^n)\|_0^2 \\
&+\theta\Delta t\Big(\|Au^{n+1}\|_0^2 - \|Au^n\|_0^2 + \|A(u^{n+1}-u^n)\|_0^2\Big) \\
\le\;& \Big(\frac{8}{\nu}\Big)^3 c_0^4\|\nabla u^n\|_0^2\|u^n\|_0^2\|\nabla u^n\|_0^2\Delta t + \frac{10}{\nu}\|f(t_{n+1})\|_0^2\Delta t \\
&+\frac{10}{\nu}\Big(\Delta t\rho\sum_{i=1}^{n} e^{-\delta(t_n-t_i)}\|Au(t_i)\|_0\Big)^2\Delta t. \qquad (4.14)
\end{aligned}$$

Summing these above inequality for n from 0 to N, using the discrete Gronwall Lemma, Theorem 2.2 and (3.2), we arrive at

$$\begin{aligned}
&\Delta t\sum_{n=0}^{N}\Big((2\nu)^{-1}\|d_tu^{n+1}\|_0^2+\nu\|Au^{n+1}\|_0^2\Big)+2\|\nabla u^{N+1}\|_0^2+\theta\Delta t\|Au^{N+1}\|_0^2\\
&+2\sum_{n=0}^{N}\Big(\|\nabla(u^{n+1}-u^n)\|_0^2+\theta\nu^{-1}\|\nabla(u^{n+1}-u^n)\|_0^2+\frac{\theta\Delta t}{2}\|A(u^{n+1}-u^n)\|_0^2\Big)\\
&\le \exp\Big(\nu^{-1}\Big(\frac{8}{\nu}\Big)^3c_0^4\gamma_0^4\Big)\Big(2\|\nabla u^0\|_0^2+\frac{\nu}{4}\|Au^0\|_0^2\Delta t+\frac{10}{\nu}Tf_\infty^2\\
&+\frac{10}{\nu}T\Big(\Delta t\rho\sum_{i=1}^{n}e^{-\delta(t_n-t_i)}\|Au(t_i)\|_0\Big)^2+\theta\Delta t\|Au^0\|_0^2\Big).
\end{aligned}\tag{4.15}$$

Moreover, we deduce from (4.12) that

$$\begin{aligned}
&(d_{tt}u^{n+1},v)-\nu(Ad_tu^{n+1},v)-\theta\Big(A(d_tu^{n+1}-d_tu^n),v\Big)+b(d_tu^n,u^n,v)\\
&-\Delta t\rho\sum_{i=1}^{n}e^{-\delta(t_n-t_i)}(Au(t_i),v)+\Delta t\rho\sum_{i=1}^{n-1}e^{-\delta(t_{n-1}-t_i)}(Au(t_i),v)\\
&+b(u^{n-1},d_tu^n,v)=\frac{1}{\Delta t}\int_{t_n}^{t_{n+1}}(f_t,v)dt,\quad 1\le n\le N-1, && (4.16)\\
&(d_{tt}u^1,v)-(\nu+\theta)(Ad_tu^1,v)=0, && (4.17)
\end{aligned}$$

for all $v\in V$. From (4.17), it follows that

$$\|d_tu^1\|_0^2+\|d_{tt}u^1\|_0^2\Delta t^2+(\nu+\theta)\|\nabla d_tu^1\|_0^2=\|d_tu^0\|_0^2. \tag{4.18}$$

Taking $v=2d_tu^{n+1}\Delta t$ in (4.16) with $0\le n\le N$, we get

$$\begin{aligned}
&\|d_tu^{n+1}\|_0^2-\|d_tu^n\|_0^2+\|d_tu^{n+1}-d_tu^n\|_0^2+2\nu\Delta t\|\nabla d_tu^{n+1}\|_0^2\\
&+\theta\Delta t\Big(\|\nabla d_tu^{n+1}\|_0^2-\|\nabla d_tu^n\|_0^2+\|\nabla(d_tu^{n+1}-d_tu^n)\|_0^2\Big)\\
&+2\Delta t(1-e^{-\delta\Delta t})\Delta t\rho\sum_{i=1}^{n-1}e^{-\delta(t_{n-1}-t_i)}(Au(t_i),d_tu^{n+1})\\
&+2b(d_tu^n,u^n,d_tu^{n+1})\Delta t+2b(u^{n-1},d_tu^n,d_tu^{n+1})\Delta t\\
&-2\Delta t^2\rho(Au(t_n),d_tu^{n+1})=2\int_{t_n}^{t_{n+1}}(f_t,d_tu^{n+1})dt.
\end{aligned}\tag{4.19}$$

Using (2.4)-(2.5), we have

$$
\begin{aligned}
& 2|b(d_tu^n,u^n,d_tu^{n+1})|+|2b(u^{n-1},d_tu^n,d_tu^{n+1})| \\
\le\ & 2c_1\Big(\|Au^n\|_0+\|Au^{n-1}\|_0\Big)\|\nabla d_tu^{n+1}\|_0\|d_tu^n\|_0 \\
\le\ & \frac{\nu}{4}\|\nabla d_tu^{n+1}\|_0^2+8\nu^{-1}c_1^2\Big(\|Au^n\|_0^2+\|Au^{n-1}\|_0^2\Big)\|d_tu^n\|_0^2, \\
& 2|\int_{t_n}^{t_{n+1}}(f_t,d_tu^{n+1})dt|\le\frac{\nu\Delta t}{4}\|\nabla d_tu^{n+1}\|_0^2+4\nu^{-1}c_0^2\int_{t_n}^{t_{n+1}}\|f_t\|_0^2dt \\
& 2|\Delta t\rho\sum_{i=1}^{n-1}e^{-\delta(t_{n-1}-t_i)}(Au(t_i),d_tu^{n+1})| \\
\le\ & \frac{\nu}{4}\|\nabla d_tu^{n+1}\|_0^2+4\nu^{-1}\Big(\Delta t\rho\sum_{i=1}^{n-1}e^{-\delta(t_{n-1}-t_i)}\|Au(t_i)\|_0\Big)^2, \\
& 2|(Au(t_n),d_tu^{n+1})|\le\frac{\nu}{4}\|\nabla d_tu^{n+1}\|_0^2+4\nu^{-1}c_0^2\|Au(t_n)\|_0^2.
\end{aligned}
$$

It follows from these estimates, (4.19), and (4.18) that

$$
\begin{aligned}
& \|d_tu^{n+1}\|_0^2-\|d_tu^n\|_0^2+\|d_tu^{n+1}-d_tu^n\|_0^2+\nu\Delta t\|\nabla d_tu^{n+1}\|_0^2 \\
& +\theta\Delta t\Big(\|\nabla d_tu^{n+1}\|_0^2-\|\nabla d_tu^n\|_0^2+\|\nabla(d_tu^{n+1}-d_tu^n)\|_0^2\Big) \\
\le\ & 8\nu^{-1}c_1^2\Big(\|Au^n\|_0^2+\|Au^{n-1}\|_0^2\Big)\|d_tu^n\|_0^2\Delta t+4\nu^{-1}c_0^2\int_{t_n}^{t_{n+1}}\|f_t\|_0^2dt \\
& +4\nu^{-1}\Delta t\Big(\Delta t\rho\sum_{i=1}^{n-1}e^{-\delta(t_{n-1}-t_i)}\|Au(t_i)\|_0\Big)^2+4\nu^{-1}c_0^2\Delta t\|Au(t_i)\|_0^2. \quad (4.20)
\end{aligned}
$$

Summing (4.20) from $n=0$ to N and applying the discrete Gronwall lemma yield that

$$
\begin{aligned}
& \|d_tu^{N+1}\|_0^2+\sum_{n=0}^{N}\|d_tu^{n+1}-d_tu^n\|_0^2+\nu\Delta t\sum_{n=0}^{N}\|\nabla d_tu^{n+1}\|_0^2 \\
& +\theta\Delta t\Big(\|\nabla d_tu^{N+1}\|_0^2+\sum_{n=0}^{N}\|\nabla(d_tu^{n+1}-d_tu^n)\|_0^2\Big) \qquad (4.21) \\
\le\ & \exp\Big(8\nu^{-1}c_1^2(\|Au^n\|_0^2+\|Au^{n-1}\|_0^2)\Big)\Big(\|d_tu^0\|_0^2+4\nu^{-1}c_0^2T\sup_{0\le t\le T}\|f_t\|_0^2dt \\
& +4\nu^{-1}T\Big(\Delta t\rho\sum_{i=1}^{n-1}e^{-\delta(t_n-t_i)}\|Au(t_i)\|_0\Big)^2+\theta\Delta t\|\nabla d_tu^0\|_0^2+4\nu^{-1}c_0^2T\|Au(t_i)\|_0^2\Big).
\end{aligned}
$$

Using again (2.1)-(2.5) and (4.12), we deduce that

$$
\begin{aligned}
(\nu+\theta)\|Au^{n+1}\|_0 \leq\ & \|d_t u^{n+1}\|_0 + \|f(t_{n+1})\|_0 + 2c_0 c_1^{1/2}\|\nabla u^n\|_0 \|u^n\|_0^{1/2}\|Au^n\|_0^{1/2} \\
& +\theta\|Au^n\|_0 + \Delta t\rho \sum_{i=1}^{n} e^{-\delta(t_n-t_i)}\|Au(t_i)\|_0
\end{aligned}
$$

If $\|Au^{n+1}\|_0 \leq \|Au^n\|_0$, thanks to Remark 3.1, we know that (4.3) holds. Otherwise, setting $k_* = \sup_{0\leq n\leq N+1}\|Au^n\|_0$ and using (4.1)-(4.2), then the above inequality gives

$$
\begin{aligned}
\|Au^{n+1}\|_0^2 \leq\ & k_*^2 \leq 12(\theta+\nu)^{-2}(\sup_{0\leq n\leq N}\|d_t u^{n+1}\|_0^2 + f_\infty^2) \\
& +48(\theta+\nu)^{-4}c_0^4 c_1^2 \sup_{0\leq n\leq N}\|\nabla u^n\|_0^4\|u^n\|_0^2 \\
& +(\theta+\nu)^{-2}\Big(\theta^2 + (\Delta t\rho\sum_{i=1}^{n-1} e^{-\delta(t_n-t_i)}\|Au(t_i)\|_0)^2\Big) \\
\leq\ & (\theta+\nu)^{-2}\Big(12 f_\infty^2 + \theta^2 + (\Delta t\rho\sum_{i=1}^{n-1} e^{-\delta(t_n-t_i)}\|Au(t_i)\|_0)^2\Big) \\
& +12(\theta+\nu)^{-2}\sup_{0\leq n\leq N}\|d_t u^{n+1}\|_0^2 + 48(\theta+\nu)^{-4}c_0^4c_1^2k_{01}^2\gamma_0^2. \quad (4.22)
\end{aligned}
$$

Combining (4.21) with (4.22) and using (4.2) yield (4.3) for $N+1$.

Lemma 4.2. In terms of estimates (4.11), (4.15) and (4.21) in Lemma 4.1, for every $0 \leq n \leq N,\ N = -1, 0, 1, \ldots, [\frac{T}{\Delta t}]-1$, there exist

$$
\begin{aligned}
& \sum_{n=0}^{N}\|u^{n+1} - u^{n+\frac{1}{2}}\|_0^2 + \Delta t\nu\sum_{n=0}^{N}\|\nabla(u^{n+1} - u^{n+\frac{1}{2}})\|_0^2 \\
& +\Delta t\theta\sum_{n=0}^{N}\Big(\|\nabla(u^{n+\frac{1}{2}} - u^n)\|_0^2 + \|\nabla(u^{n+1} - u^{n+\frac{1}{2}})\|_0^2\Big) \leq \gamma_0^2, \\
& \sum_{n=0}^{N}\Big(\|\nabla(u^{n+1}-u^n)\|_0^2 + \theta\nu^{-1}\|\nabla(u^{n+1}-u^n)\|_0^2 + \theta\Delta t\|A(u^{n+1}-u^n)\|_0^2\Big) \leq k_{01},
\end{aligned}
$$

and

$$
\sum_{n=0}^{N}\|d_t u^{n+1} - d_t u^n\|_0^2 + \theta\Delta t\sum_{n=0}^{N}\|\nabla(d_t u^{n+1} - d_t u^n)\|_0^2 \leq k_{02}.
$$

5. Error estimates

This section is devoted to present optimal error estimates for the velocity and pressure in the viscosity splitting fractional-step scheme introduced in Section 3. Here we only present the analysis of the explicit form for convection term. Of course, the similar error estimations can be obtained for other approximations.

5.1. Error estimates for the velocity

Let us define the semi-discrete velocity errors by

$$e^{n+1} = u(t_{n+1}) - u^{n+1}, \qquad e^{n+\frac{1}{2}} = u(t_{n+1}) - u^{n+\frac{1}{2}}$$

and the truncation error R^n by:

$$\frac{u(t_{n+1}) - u(t_n)}{\Delta t} - \nu \Delta u(t_{n+1}) + (u(t_{n+1}) \cdot \nabla) u(t_{n+1}) + \nabla p(t_{n+1}) \\ - \int_0^{t_{n+1}} \rho e^{-\delta(t_{n+1}-s)} \Delta u ds = f(t_{n+1}) + R^n \qquad (5.1)$$

respectively where

$$R^n = -\frac{1}{\Delta t} \int_{t_n}^{t_{n+1}} (t - t_n) u_{tt}(t) dt.$$

Firstly, we give the estimates of e^{n+1} and $e^{n+\frac{1}{2}}$ which show that both u^{n+1} and $u^{n+\frac{1}{2}}$ are order $\frac{1}{2}$ approximations to u in $L^\infty(Y)$ and in $L^2(X)$ respectively.

Lemma 5.1. Assume that conditions (A3)-(A6) hold. Then,

$$\|e^{N+1}\|_0^2 + \|e^{N+\frac{1}{2}}\|_0^2 + \sum_{n=0}^{N} \left(\|e^{n+1} - e^{n+\frac{1}{2}}\|_0^2 + \|e^{n+\frac{1}{2}} - e^n\|_0^2 \right) \\ + \Delta t(\theta + \nu) \sum_{n=0}^{N} \left\{ \|\nabla e^{n+1}\|_0^2 + \|\nabla e^{n+\frac{1}{2}}\|_0^2 + \|\nabla (e^{n+1} - e^{n+\frac{1}{2}})\|_0^2 \right\} \leq C \Delta t$$

for all $N = 0, 1, \ldots, [\frac{T}{\Delta t}] - 1$.

Proof. By subtracting (3.4) from (5.1), we have

$$\frac{e^{n+\frac{1}{2}} - e^n}{\Delta t} - (\theta + \nu) \Delta e^{n+\frac{1}{2}} + \theta(\Delta u(t_{n+1}) - \Delta u^n) \\ = (u^n \cdot \nabla) u^n - (u(t_{n+1}) \cdot \nabla) u(t_{n+1}) - \nabla p(t_{n+1}) + R^n \\ + \Delta t \rho \sum_{i=1}^{n} e^{-\delta(t_n - t_i)} \Delta u(t_i) - \int_0^{t_{n+1}} \rho e^{-\delta(t_{n+1}-s)} \Delta u(s) ds. \qquad (5.2)$$

For the sake of simplicity, we denote

$$z^n = u(t_{n+1}) - u(t_n) = \int_{t_n}^{t_{n+1}} u_t dt.$$

Then, $u(t_{n+1}) - u^n$ can be rewritten as

$$u(t_{n+1}) - u^n = u(t_{n+1}) - u(t_n) + u(t_n) - u^n = z^n + e^n.$$

Splitting the nonlinear terms on the right side of (5.2) as follows

$$\begin{aligned}&(u^n \cdot \nabla)u^n - (u(t_{n+1}) \cdot \nabla)u(t_{n+1}) \\ &= -(e^n \cdot \nabla)u^n - (z^n \cdot \nabla)u(t_n) - (u(t_n) \cdot \nabla)e^n - (u(t_{n+1}) \cdot \nabla)z^n,\end{aligned} \quad (5.3)$$

and the integral term can be transformed into

$$\begin{aligned}&\Delta t\rho \sum_{i=1}^{n} e^{-\delta(t_n-t_i)}\Delta u(t_i) - \int_0^{t_{n+1}} \rho e^{-\delta(t_{n+1}-s)}\Delta u(s)ds \\ &= \Delta t\rho \sum_{i=1}^{n} e^{-\delta(t_n-t_i)}\Delta u(t_i) - (\int_0^{t_n} + \int_{t_n}^{t_{n+1}})\rho e^{-\delta(t_{n+1}-s)}\Delta u(s)ds \\ &= \Delta t\rho \sum_{i=1}^{n} e^{-\delta(t_n-t_i)}\Delta u(t_i) - \int_0^{t_n} \rho e^{-\delta(t_{n+1}-s)}\Delta u(s)ds \\ &+ \int_0^{t_n} \rho e^{-\delta(t_{n+1}-s)}\Delta u(s)ds - \int_0^{t_{n+1}} \rho e^{-\delta(t_{n+1}-s)}\Delta u(s)ds.\end{aligned}$$

Taking the inner product of (5.2) with $2\Delta t e^{n+\frac{1}{2}}$, using (4.5), thanks to the above identities, we can change equation (5.2) to

$$\begin{aligned}&\|e^{n+\frac{1}{2}}\|_0^2 - \|e^n\|_0^2 + \|e^{n+\frac{1}{2}} - e^n\|_0^2 + 2\Delta t\nu\|\nabla e^{n+\frac{1}{2}}\|_0^2 \\ &+\theta\Delta t(\|\nabla e^{n+\frac{1}{2}}\|_0^2 - \|\nabla e^n\|_0^2 + \|\nabla(e^{n+\frac{1}{2}} - e^n)\|_0^2) \\ &+2\Delta t\Big(\int_0^{t_{n+1}} \rho e^{-\delta(t_{n+1}-s)}(\nabla u(s), \nabla e^{n+\frac{1}{2}})ds - \int_0^{t_n} \rho e^{-\delta(t_{n+1}-s)}(\nabla u(s), \nabla e^{n+\frac{1}{2}})ds\Big) \\ &= 2\Delta t(R^n, e^{n+\frac{1}{2}}) - 2\theta\Delta t(\Delta z^n, e^{n+\frac{1}{2}}) - 2\Delta t(\nabla p(t_{n+1}), e^{n+\frac{1}{2}}) - 2\Delta t b(e^n, u^n, e^{n+\frac{1}{2}}) \\ &-2\Delta t b(z^n, u(t_n), e^{n+\frac{1}{2}}) - 2\Delta t b(u(t_n), e^n, e^{n+\frac{1}{2}}) - 2\Delta t b(u(t_{n+1}), z^n, e^{n+\frac{1}{2}}) \\ &-2\Delta t\Big(\Delta t\rho \sum_{i=1}^{n} e^{-\delta(t_n-t_i)}(\nabla u(t_i), \nabla e^{n+\frac{1}{2}}) + \int_0^{t_n} \rho e^{-\delta(t_{n+1}-s)}(\nabla u(s), \nabla e^{n+\frac{1}{2}})ds\Big).\end{aligned} \quad (5.4)$$

It follows from (3.5) that

$$\frac{e^{n+1} - e^{n+\frac{1}{2}}}{\Delta t} - (\theta + \nu)(\Delta e^{n+1} - \Delta e^{n+\frac{1}{2}}) - \nabla p^{n+1} = 0. \quad (5.5)$$

Taking the inner product of (5.5) with $2\Delta t e^{n+1}$, and noticing the fact that $\nabla \cdot e^{n+1} = 0$, we have

$$\begin{aligned}
&\|e^{n+1}\|_0^2 - \|e^{n+\frac{1}{2}}\|_0^2 + \|e^{n+1} - e^{n+\frac{1}{2}}\|_0^2 \\
&+(\theta+\nu)\Delta t\Big(\|\nabla e^{n+1}\|_0^2 - \|\nabla e^{n+\frac{1}{2}}\|_0^2 + \|\nabla(e^{n+1} - e^{n+\frac{1}{2}})\|_0^2\Big) = 0. \quad (5.6)
\end{aligned}$$

Combining (5.4) with (5.6), one finds that

$$\begin{aligned}
&\|e^{n+1}\|_0^2 - \|e^n\|_0^2 + \|e^{n+1} - e^{n+\frac{1}{2}}\|_0^2 + \|e^{n+\frac{1}{2}} - e^n\|_0^2 + \Delta t\nu\Big(\|\nabla e^{n+\frac{1}{2}}\|_0^2 + \|\nabla e^{n+1}\|_0^2\Big) \\
&+\theta\Delta t\Big(\|\nabla e^{n+1}\|_0^2 - \|\nabla e^n\|_0^2 + \|\nabla(e^{n+\frac{1}{2}} - e^n)\|_0^2\Big) + (\theta+\nu)\Delta t\|\nabla(e^{n+1} - e^{n+\frac{1}{2}})\|_0^2 \\
&+2\Delta t\Big(\int_0^{t_{n+1}} \rho e^{-\delta(t_{n+1}-s)}(\nabla u(s), \nabla e^{n+\frac{1}{2}})ds - \int_0^{t_n} \rho e^{-\delta(t_{n+1}-s)}(\nabla u(s), \nabla e^{n+\frac{1}{2}})ds\Big) \\
&= 2\Delta t(R^n, e^{n+\frac{1}{2}}) - 2\theta\Delta t(\Delta z^n, e^{n+\frac{1}{2}}) - 2\Delta t(\nabla p(t_{n+1}), e^{n+\frac{1}{2}}) - 2\Delta t b(e^n, u^n, e^{n+\frac{1}{2}}) \\
&-2\Delta t\Big(\Delta t\rho\sum_{i=1}^{n} e^{-\delta(t_n-t_i)}(\nabla u(t_i), \nabla e^{n+\frac{1}{2}})ds + \int_0^{t_n} \rho e^{-\delta(t_{n+1}-s)}(\nabla u(s), \nabla e^{n+\frac{1}{2}})ds\Big) \\
&-2\Delta t b(z^n, u(t_n), e^{n+\frac{1}{2}}) - 2\Delta t b(u(t_n), e^n, e^{n+\frac{1}{2}}) - 2\Delta t b(u(t_{n+1}), z^n, e^{n+\frac{1}{2}}). \qquad (5.7)
\end{aligned}$$

For the terms $\int_0^{t_{n+1}} \rho e^{-\delta(t_{n+1}-s)}(\nabla u(s), \nabla e^{n+\frac{1}{2}})ds - \int_0^{t_n} \rho e^{-\delta(t_{n+1}-s)}(\nabla u(s), \nabla e^{n+\frac{1}{2}})ds$ in left hand side of (5.7), according to the fact that $t_{n+1} = t_n + \Delta t$, we have $e^{-\delta t_{n+1}} = e^{-\delta(t_n+\Delta t)} = e^{-\delta t_n}e^{-\delta\Delta t} \le e^{-\delta t_n}$. Using (3.1) yields

$$\begin{aligned}
&\int_0^{t_{n+1}} \rho e^{-\delta(t_{n+1}-s)}(\nabla u(s), \nabla e^{n+\frac{1}{2}})ds - \int_0^{t_n} \rho e^{-\delta(t_{n+1}-s)}(\nabla u(s), \nabla e^{n+\frac{1}{2}})ds \\
&\ge \int_0^{t_{n+1}} \rho e^{-\delta(t_{n+1}-s)}(\nabla u(s), \nabla e^{n+\frac{1}{2}})ds - \int_0^{t_n} \rho e^{-\delta(t_n-s)}(\nabla u(s), \nabla e^{n+\frac{1}{2}})ds \\
&\approx \Delta t\rho\sum_{i=1}^{n+1} e^{-\delta(t_{n+1}-t_i)}(\nabla u(t_i), \nabla e^{n+\frac{1}{2}}) - \Delta t\rho\sum_{i=1}^{n} e^{-\delta(t_n-t_i)}(\nabla u(t_i), \nabla e^{n+\frac{1}{2}}) \\
&= \Delta t\rho\Big\{(e^{-\delta\Delta t} - 1)\sum_{i=1}^{n} e^{-\delta(t_n-t_i)}(\nabla u(t_i), \nabla e^{n+\frac{1}{2}}) + (\nabla u(t_{n+1}), \nabla e^{n+\frac{1}{2}})\Big\}. \quad (5.8)
\end{aligned}$$

In the same way, we have

$$
\begin{aligned}
& \int_0^{t_n} \rho e^{-\delta(t_{n+1}-s)}(\nabla u(s), \nabla e^{n+\frac{1}{2}})ds - \Delta t\rho \sum_{i=1}^{n} e^{-\delta(t_n-t_i)}(\nabla u(t_i), \nabla e^{n+\frac{1}{2}})ds \\
\leq\ & \int_0^{t_n} \rho e^{-\delta(t_n-s)}(\nabla u(s), \nabla e^{n+\frac{1}{2}})ds - \Delta t\rho \sum_{i=1}^{n} e^{-\delta(t_n-t_i)}(\nabla u(t_i), \nabla e^{n+\frac{1}{2}})ds \\
\leq\ & \rho \sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} e^{-\delta t_n}(t-t_{i-1}) \frac{\partial}{\partial t}(e^{\delta t}\nabla u, \nabla e^{n+\frac{1}{2}})dt \\
\leq\ & c\| \sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} e^{\delta(t-t_n)}(t-t_{i-1})(\delta \nabla u + \nabla u_t)dt\|_0 \|\nabla e^{n+\frac{1}{2}}\|_0 \\
\leq\ & c\Big(\sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} e^{2\alpha_0(t-t_n)}|t-t_{i-1}|^2 dt\Big)^{\frac{1}{2}} \\
& \qquad \times \Big(e^{-2\delta_0 t_n} \sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} e^{2\delta_0 t}(\|\nabla u\|_0^2 + \|\nabla u_t\|_0^2)dt\Big)^{\frac{1}{2}} \|\nabla e^{n+\frac{1}{2}}\|_0 \\
\leq\ & c\triangle t\Big(\int_0^{t_n} e^{2\alpha_0(t-t_n)}dt\Big)^{\frac{1}{2}} \cdot \Big(e^{-2\delta_0 t_n} \int_0^{t_n} e^{2\delta_0 t}(\|\nabla u\|_0^2 + \|\nabla u_t\|_0^2)dt\Big)^{\frac{1}{2}} \|\nabla e^{n+\frac{1}{2}}\|_0 \\
\leq\ & c\Delta t\|\nabla e^{n+\frac{1}{2}}\|_0. \ \Big(\text{where } \alpha_0 = \delta - \delta_0,\ 1 < \delta_0 < \frac{1}{2}\min\{\delta, \nu/c_0\}\Big).
\end{aligned} \tag{5.9}
$$

Thanks to (5.8) and (5.9), (5.7) can be transformed into

$$
\begin{aligned}
& \|e^{n+1}\|_0^2 - \|e^n\|_0^2 + \|e^{n+1} - e^{n+\frac{1}{2}}\|_0^2 + \|e^{n+\frac{1}{2}} - e^n\|_0^2 \\
& + \Delta t\nu\Big(\|\nabla e^{n+\frac{1}{2}}\|_0^2 + \|\nabla e^{n+1}\|_0^2\Big) + (\theta+\nu)\Delta t\|\nabla(e^{n+1} - e^{n+\frac{1}{2}})\|_0^2 \\
& + \theta\Delta t\Big(\|\nabla e^{n+1}\|_0^2 - \|\nabla e^n\|_0^2 + \|\nabla(e^{n+\frac{1}{2}} - e^n)\|_0^2\Big) \\
& \leq 2\Delta t(R^n, e^{n+\frac{1}{2}}) - 2\theta\Delta t(\Delta z^n, e^{n+\frac{1}{2}}) - 2\Delta t(\nabla p(t_{n+1}), e^{n+\frac{1}{2}}) \\
& + 2\Delta t^2\rho\Big\{(1 - e^{-\delta\Delta t})\sum_{i=1}^{n} e^{-\delta(t_n-t_i)}(\nabla u(t_i), \nabla e^{n+\frac{1}{2}}) + (\nabla u(t_{n+1}), \nabla e^{n+\frac{1}{2}})\Big\} \\
& - 2\Delta t b(e^n, u^n, e^{n+\frac{1}{2}}) - 2\Delta t b(z^n, u(t_n), e^{n+\frac{1}{2}}) - 2\Delta t b(u(t_n), e^n, e^{n+\frac{1}{2}}) \\
& - 2\Delta t b(u(t_{n+1}), z^n, e^{n+\frac{1}{2}}) - 2c\Delta t^2\|\nabla e^{n+\frac{1}{2}}\|_0.
\end{aligned} \tag{5.10}
$$

Now, we estimate the terms in the right hand side of (5.10) separately.

$$
\begin{aligned}
|2\Delta t(R^n, e^{n+\frac{1}{2}})| &\leq \frac{c}{\Delta t}\|\int_{t_n}^{t_{n+1}}(t-t_n)u_{tt}dt\|_0^2 + \frac{\nu\Delta t}{18}\|\nabla e^{n+\frac{1}{2}}\|_0^2 \\
&\leq c\Delta t\int_{t_n}^{t_{n+1}} t\|u_{tt}\|_0^2 dt + \frac{\nu\Delta t}{18}\|\nabla e^{n+\frac{1}{2}}\|_0^2,
\end{aligned}
$$

Using the fact that $\nabla\cdot e^n = 0$, we get

$$
\begin{aligned}
|2\Delta t(\nabla p(t_{n+1}), e^{n+\frac{1}{2}})| &= 2\Delta t|(\nabla p(t_{n+1}), e^{n+\frac{1}{2}} - e^n)| \\
&\leq \frac{1}{2}\|e^{n+\frac{1}{2}} - e^n\|_0^2 + 2\Delta t^2\|\nabla p(t_{n+1})\|_0^2
\end{aligned}
$$

With the help of (3.3), one finds

$$
\begin{aligned}
&\left|2\Delta t(1-e^{-\delta\Delta t})\Delta t\rho\sum_{i=1}^{n} e^{-\delta(t_n - t_i)}(\nabla u(t_i), \nabla e^{n+\frac{1}{2}})\right| \\
&\leq 2\widetilde{c}\Delta t(1-e^{-\delta\Delta t})\|\nabla e^{n+\frac{1}{2}}\|_0 \leq c\Delta t^3 + \frac{\nu\Delta t}{18}\|\nabla e^{n+\frac{1}{2}}\|_0^2
\end{aligned}
$$

$$
\begin{aligned}
\left|2\Delta t^2\rho(\nabla u(t_{n+1}), \nabla e^{n+\frac{1}{2}})\right| &\leq 2\Delta t^2\rho\|\nabla u(t_{n+1})\|_0\|\nabla e^{n+\frac{1}{2}}\|_0 \\
&\leq c\Delta t^3 + \frac{\nu\Delta t}{18}\|\nabla e^{n+\frac{1}{2}}\|_0^2, \\
|2\theta\Delta t(\Delta z^n, e^{n+\frac{1}{2}})| &\leq 2\theta\Delta t\|\nabla z^n\|_0\|\nabla e^{n+\frac{1}{2}}\|_0 \\
&\leq c\Delta t^2\int_{t_n}^{t_{n+1}}\|\nabla u_t\|_0^2 dt + \frac{\nu\Delta t}{18}\|\nabla e^{n+\frac{1}{2}}\|_0^2 \\
2c\Delta t^2\|\nabla e^{n+\frac{1}{2}}\|_0 &\leq c\Delta t^3 + \frac{\nu\Delta t}{18}\|\nabla e^{n+\frac{1}{2}}\|_0^2
\end{aligned}
$$

For the nonlinear term, with the help of Lemma 4.1, we have

$$
\begin{aligned}
|2\Delta t b(e^n, u^n, e^{n+\frac{1}{2}})| &\leq c\Delta t\|e^n\|_0\|Au^n\|_0\|\nabla e^{n+\frac{1}{2}}\|_0 \\
&\leq c\Delta t\|e^n\|_0^2 + \frac{\nu\Delta t}{18}\|\nabla e^{n+\frac{1}{2}}\|_0^2, \\
|2\Delta t b(z^n, u(t_n), e^{n+\frac{1}{2}})| &\leq c\Delta t\|z^n\|_0\|Au(t_n)\|_0\|\nabla e^{n+\frac{1}{2}}\|_0 \\
&\leq c\Delta t^2\int_{t_n}^{t_{n+1}}\|u_t\|_0^2 dt + \frac{\nu\Delta t}{18}\|\nabla e^{n+\frac{1}{2}}\|_0^2, \\
|2\Delta t b(u(t_n), e^n, e^{n+\frac{1}{2}})| &\leq c\Delta t\|e^n\|_0\|Au(t_n)\|_0\|\nabla e^{n+\frac{1}{2}}\|_0 \\
&\leq c\Delta t\|e^n\|_0^2 + \frac{\nu\Delta t}{18}\|\nabla e^{n+\frac{1}{2}}\|_0^2,
\end{aligned}
$$

$$
\begin{aligned}
|2\Delta t b(u(t_{n+1}), z^n, e^{n+\frac{1}{2}})| &\leq c\Delta t\|z^n\|_0\|Au(t_{n+1})\|_0\|\nabla e^{n+\frac{1}{2}}\|_0 \\
&\leq c\Delta t^2\int_{t_n}^{t_{n+1}}\|u_t\|_0^2 dt + \frac{\nu\Delta t}{18}\|\nabla e^{n+\frac{1}{2}}\|_0^2.
\end{aligned}
$$

From all these above inequalities we derive that

$$
\begin{aligned}
&\|e^{n+1}\|_0^2 - \|e^n\|_0^2 + \|e^{n+1} - e^{n+\frac{1}{2}}\|_0^2 + \frac{1}{2}\|e^{n+\frac{1}{2}} - e^n\|_0^2 \\
&+\theta\Delta t\Big(\|\nabla e^{n+1}\|_0^2 + \|\nabla(e^{n+\frac{1}{2}} - e^n)\|_0^2 - \|\nabla e^n\|_0^2\Big) \\
&+\frac{\Delta t\nu}{2}\|\nabla e^{n+\frac{1}{2}}\|_0^2 + \Delta t\nu\|\nabla e^{n+1}\|_0^2 + (\theta+\nu)\Delta t\|\nabla(e^{n+1} - e^{n+\frac{1}{2}})\|_0^2 \\
\leq\ & c\Delta t\int_{t_n}^{t_{n+1}} t\|u_{tt}\|_0^2 dt + 2\Delta t^2\|\nabla p(t_{n+1})\|_0^2 + c\Delta t\|e^n\|_0^2 + c\Delta t^3 \\
&+c\Delta t^2\int_{t_n}^{t_{n+1}}(\|u_t\|_0^2 + \|\nabla u_t\|_0^2)dt.
\end{aligned}
$$

Adding up the above inequality from $n = 0$ to N, we have

$$
\begin{aligned}
&\|e^{N+1}\|_0^2 + \sum_{n=0}^{N}\Big(\|e^{n+1} - e^{n+\frac{1}{2}}\|_0^2 + \frac{1}{2}\|e^{n+\frac{1}{2}} - e^n\|_0^2\Big) \\
&+\Delta t\sum_{n=0}^{N}\Big(\frac{\nu}{2}\|\nabla e^{n+\frac{1}{2}}\|_0^2 + \nu\|\nabla e^{n+1}\|_0^2 + (\theta+\nu)\|\nabla(e^{n+1} - e^{n+\frac{1}{2}})\|_0^2\Big) \\
&+\theta\Delta t\|\nabla e^{N+1}\|_0^2 + \theta\Delta t\sum_{n=0}^{N}\|\nabla(e^{n+\frac{1}{2}} - e^n)\|_0^2 \\
\leq\ & c\Delta t\int_0^T t\|u_{tt}\|_0^2 dt + 2\Delta t T\sup_{t\in[0,T]}\|\nabla p(t)\|_0^2 + c\Delta t\sum_{n=0}^{N}\|e^n\|_0^2 \\
&+c\Delta t^2\Big(\int_0^T(\|u_t\|_0^2 + \|\nabla u_t\|_0^2)dt + T\Big), \quad (\text{where } e^0 = 0). \qquad (5.11)
\end{aligned}
$$

Applying the discrete Gronwall lemma (Lemma 2.1) to (5.11) and using the regularity properties of the continuous solution, we obtain the desired result.

Remark 5.1. In particular, Lemma 5.1 shows that the method provides uniformly stable velocity in X. Using the fact that $\|\nabla e^{n+\frac{1}{2}}\|_0 \leq c$, $\|\nabla e^{n+1}\|_0 \leq c$ and $u \in L^\infty(0, T; X)$, we know that there exists a positive constant c independent of the time step Δt such that for all $0 \leq n \leq N$

$$
\|\nabla u^{n+1}\|_0 \leq c, \quad \|\nabla u^{n+\frac{1}{2}}\|_0 \leq c.
$$

Lemma 5.2. For all $N = 0, 1, \ldots, [\frac{T}{\Delta t}] - 1$ and $\Delta t > 0$, under assumptions (A3)-(A6), we have

$$\|e^{N+1}\|_{V'}^2 + \theta\Delta t\|e^{N+1}\|_0^2 + \nu\Delta t\sum_{n=0}^{N}\Big(\|e^{n+1}\|_0^2 + \|e^{n+\frac{1}{2}}\|_0^2\Big) \le C\Delta t^2.$$

Proof. Setting the pressure error $r^{n+1} = p(t_{n+1}) - p^{n+1}$ and subtracting (3.6) from (5.1), we have

$$\begin{aligned}
&\frac{e^{n+1}-e^n}{\Delta t} - (\theta+\nu)\Delta e^{n+1} + \theta(\Delta z^n + \Delta e^n) + \nabla r^{n+1} \\
&= (u^n\cdot\nabla)u^n - (u(t_{n+1})\cdot\nabla)u(t_{n+1}) + R^n \\
&+\Delta t\rho\sum_{i=1}^{n} e^{-\delta(t_n-t_i)}\Delta u(t_i) - \int_0^{t_{n+1}}\rho e^{-\delta(t_{n+1}-s)}\Delta u(s)ds. \quad (5.12)
\end{aligned}$$

Taking the inner product of above equation with $2\Delta t A^{-1}e^{n+1}$, using the fact that $\nabla\cdot e^{n+1} = 0$ and the self-adjointness of A^{-1} we get

$$\begin{aligned}
&(e^{n+1}, A^{-1}e^{n+1}) - (e^n, A^{-1}e^n) + (e^{n+1}-e^n, A^{-1}(e^{n+1}-e^n)) \\
&-2(\theta+\nu)\Delta t(\Delta e^{n+1}, A^{-1}e^{n+1}) + 2\theta\Delta t(\Delta z^n + \Delta e^n, A^{-1}e^{n+1}) \\
&= 2\Delta t b(u^n, u^n, A^{-1}e^{n+1}) - 2\Delta t b(u(t_{n+1}), u(t_{n+1}), A^{-1}e^{n+1}) \\
&+2\Delta t(R^n, A^{-1}e^{n+1}) + 2\Delta t\Big(\Delta t\rho\sum_{i=1}^{n} e^{-\delta(t_n-t_i)}(\Delta u(t_i), A^{-1}e^{n+1}) \\
&-\int_0^{t_{n+1}}\rho e^{-\delta(t_{n+1}-s)}(\Delta u(s), A^{-1}e^{n+1})ds\Big). \quad (5.13)
\end{aligned}$$

Taking $u = e^{n+1}$ in (2.3), for the term $-2(\theta+\nu)\Delta t(Ae^{n+1}, A^{-1}e^{n+1}) + 2\theta\Delta t(\Delta e^n, A^{-1}e^{n+1})$, we can deal with them as follows

$$\begin{aligned}
&-2(\theta+\nu)\Delta t(\Delta e^{n+1}, A^{-1}e^{n+1}) + 2\theta\Delta t(\Delta e^n, A^{-1}e^{n+1}) \\
&= 2\nu\Delta t(e^{n+1}, -\Delta A^{-1}e^{n+1}) + 2\theta\Delta t\Big(e^{n+1}-e^n, -\Delta A^{-1}e^{n+1}\Big) \\
&= 2\nu\Delta t(e^{n+1}, e^{n+1} - \nabla q) + 2\theta\Delta t\Big(e^{n+1}-e^n, e^{n+1} - \nabla q\Big) \\
&= 2\nu\Delta t\|e^{n+1}\|_0^2 + 2\theta\Delta t(e^{n+1}-e^n, e^{n+1}) \\
&= 2\nu\Delta t\|e^{n+1}\|_0^2 + \theta\Delta t\Big(\|e^{n+1}\|_0^2 - \|e^n\|_0^2 + \|e^{n+1}-e^n\|_0^2\Big).
\end{aligned}$$

For the right hand side terms of equation (5.13), we can estimates them as follows

$$
\begin{aligned}
&|2\Delta t(R^n, A^{-1}e^{n+1})| \le 2\Delta t\|R^n\|_{V'}\|A^{-1}e^{n+1}\|_V = 2\Delta t\|R^n\|_{V'}\|e^{n+1}\|_{V'} \\
\le\ & \Delta t\|e^{n+1}\|_{V'}^2 + \Delta t\|R^n\|_{V'}^2 \le \Delta t\|e^{n+1}\|_{V'}^2 + \Delta t^2\int_{t_n}^{t_{n+1}} \|u_{tt}\|_{V'}^2 dt.
\end{aligned}
$$

For the nonlinear term, we can treat them as we have done in Lemma 5.1. Using splitting (5.3) we have

$$
\begin{aligned}
&|2\Delta t b(z^n, u(t_n), A^{-1}e^{n+1})| \\
\le\ & c\Delta t\|z^n\|_0\|\nabla u(t_n)\|_0\|A^{-1}e^{n+\frac{1}{2}}\|_2 \le c\Delta t^2\int_{t_n}^{t_{n+1}} \|u_t\|_0^2 dt + \frac{\nu\Delta t}{12}\|e^{n+1}\|_0^2, \\
&|2\Delta t b(u(t_n), e^n, A^{-1}e^{n+1})| \\
\le\ & c\Delta t\|e^n\|_0\|\nabla u(t_n)\|_0\|A^{-1}e^{n+1}\|_2 \le c\Delta t\|e^n\|_0^2 + \frac{\nu\Delta t}{12}\|e^{n+1}\|_0^2, \\
&|2\Delta t b(u(t_{n+1}), z^n, A^{-1}e^{n+1})| \\
\le\ & c\Delta t\|z^n\|_0\|\nabla u(t_{n+1})\|_0\|A^{-1}e^{n+1}\|_2 \le c\Delta t^2\int_{t_n}^{t_{n+1}} \|u_t\|_0^2 dt + \frac{\nu\Delta t}{12}\|e^{n+1}\|_0^2, \\
&|2\Delta t b(e^n, u^n, A^{-1}e^{n+1})| = |2\Delta t b(e^n, u(t_n), A^{-1}e^{n+1})| + |2\Delta t b(e^n, e^n, A^{-1}e^{n+1})| \\
&|2\Delta t b(e^n, u(t_n), A^{-1}e^{n+1})| \le c\Delta t\|e^n\|_0\|Au(t_n)\|_0\|A^{-1}e^{n+1}\|_1 \\
\le\ & c\Delta t(\|e^{n+1} - e^{n+\frac{1}{2}}\|_0 + \|e^{n+\frac{1}{2}} - e^n\|_0 + \|e^{n+1}\|_0)\|e^{n+1}\|_{V'} \\
\le\ & \frac{\nu\Delta t}{12}\|e^{n+1}\|_0^2 + c\Delta t\Big(\|e^{n+1} - e^{n+\frac{1}{2}}\|_0^2 + \|e^{n+\frac{1}{2}} - e^n\|_0^2 + \|e^{n+1}\|_{V'}^2\Big) \\
&|2\Delta t b(e^n, e^n, A^{-1}e^{n+1})| \le c\Delta t\|e^n\|_0\|\nabla e^n\|_0\|A^{-1}e^{n+1}\|_2 \\
\le\ & c\Delta t^{3/2}\|\nabla e^n\|_0\|e^{n+1}\|_0 \le c\Delta t^2\|\nabla e^n\|_0^2 + \frac{\nu\Delta t}{12}\|e^{n+1}\|_0^2.
\end{aligned}
$$

For the term $2\theta\Delta t(\Delta z^n, A^{-1}e^{n+1})$, we have

$$
|2\theta\Delta t(\Delta z^n, A^{-1}e^{n+1})| = 2\theta\Delta t|(z^n, e^{n+1})| \le \frac{\nu\Delta t}{12}\|e^{n+1}\|_0^2 + c\Delta t^2\int_{t_n}^{t_{n+1}} \|u_t\|_0^2 dt.
$$

For the last two terms in equation (5.13), we can treat them as we have done in Lemma 5.1. Finally, by using the above results, and adding (5.13) from $n = 0$ to N, one finds

$$
\begin{aligned}
& (e^{N+1}, A^{-1}e^{N+1}) + \sum_{n=0}^{N}(e^{n+1} - e^n, A^{-1}(e^{n+1} - e^n)) \\
& + \frac{\nu\Delta t}{2}\sum_{n=0}^{N}\|e^{n+1}\|_0^2 + \theta\Delta t\sum_{n=0}^{N}\|e^{n+1} - e^n\|_0^2 + \theta\Delta t\|e^{N+1}\|_0^2 \\
\leq \; & c\Delta t\sum_{n=0}^{N}\|e^{n+1}\|_{V'}^2 + c\Delta t^2\int_0^T \|u_{tt}\|_{V'}^2 dt + c\Delta t^2\int_0^T \|u_t\|_0^2 dt \\
& + 2c\Delta t^2 + c\Delta t\sum_{n=0}^{N}\left(\|e^{n+1} - e^{n+\frac{1}{2}}\|_0^2 + \|e^{n+\frac{1}{2}} - e^{n+1}\|_0^2\right).
\end{aligned}
$$

Using the regularity properties of the continuous solution and Lemma 5.1, we obtain that

$$
\begin{aligned}
& \|e^{N+1}\|_{V'}^2 + \sum_{n=0}^{N}\|e^{n+1} - e^n\|_{V'}^2 + \frac{\nu\Delta t}{2}\sum_{n=0}^{N}\|e^{n+1}\|_0^2 + \theta\Delta t\|e^{N+1}\|_0^2 \\
& + \theta\Delta t\sum_{n=0}^{N}\|e^{n+1} - e^n\|_0^2 \leq c\Delta t\sum_{n=0}^{N}\|e^{n+1}\|_{V'}^2 + c\Delta t^2.
\end{aligned}
$$

Applying the discrete Gronwall lemma (Lemma 2.1) to the above inequality yields

$$
\begin{aligned}
& \|e^{N+1}\|_{V'}^2 + \sum_{n=0}^{N}\|e^{n+1} - e^n\|_{V'}^2 + \frac{\nu\Delta t}{2}\sum_{n=0}^{N}\|e^{n+1}\|_0^2 \\
& + \theta\Delta t\sum_{n=0}^{N}\|e^{n+1} - e^n\|_0^2 + \theta\Delta t\|e^{N+1}\|_0^2 \leq c\Delta t^2. \qquad (5.14)
\end{aligned}
$$

For the intermediate velocity $u^{n+\frac{1}{2}}$, according to Lemma 5.1 and triangle inequality, we have

$$
\Delta t\nu\sum_{n=0}^{N}\|e^{n+\frac{1}{2}}\|_0^2 \leq \Delta t\nu\sum_{n=0}^{N}\left(\|e^{n+1}\|_0^2 + \|e^{n+1} - e^{n+\frac{1}{2}}\|_0^2\right) \leq c\Delta t^2 \quad (5.15)
$$

Combining (5.14) and (5.15) we complete the proof of Lemma 5.2.

Theorem 5.3. Assume that conditions (A3)-(A6) are valid. Then, for all $N = 0, 1, \ldots, [\frac{T}{\Delta t}] - 1$, there is

$$\|e^{N+1}\|_0^2 + \theta\Delta t\|\nabla e^{N+1}\|_0^2 + \Delta t\nu\sum_{n=0}^{N}\|\nabla e^{n+1}\|_0^2 \leq c\Delta t^2.$$

Proof. Taking the inner product of (5.12) with $2\Delta t e^{n+1}$ and using the fact that $\nabla\cdot e^{n+1} = 0$, we have

$$\begin{aligned}
&\|e^{n+1}\|_0^2 - \|e^n\|_0^2 + \|e^{n+1} - e^n\|_0^2 + 2\Delta t\nu\|\nabla e^{n+1}\|_0^2 \\
&+\theta\Delta t\Big(\|\nabla e^{n+1}\|_0^2 - \|\nabla e^n\|_0^2 + \|\nabla(e^{n+1} - e^n)\|_0^2\Big) \\
&= 2\Delta t b(u^n, u^n, e^{n+1}) - 2\Delta t b(u(t_{n+1}), u(t_{n+1}), e^{n+1}) - 2\theta\Delta t(\nabla z^n, \nabla e^{n+1}) \\
&+2\Delta t(R^n, e^{n+1}) + 2\Delta t\Big(\Delta t\rho\sum_{i=1}^{n} e^{-\delta(t_n - t_i)}(\Delta u(t_i), e^{n+1}) \\
&-\int_0^{t_{n+1}}\rho e^{-\delta(t_{n+1}-s)}(\Delta u(s), e^{n+1})ds\Big).
\end{aligned}\tag{5.16}$$

Now, we estimate the terms in the right hand side of (5.16) separately.

$$|2\Delta t(R^n, e^{n+1})| \leq 2\Delta t\|R^n\|_{V'}\|\nabla e^{n+1}\|_0 \leq \frac{\Delta t\nu}{12}\|\nabla e^{n+1}\|_0^2 + c\Delta t^2\int_{t_n}^{t_{n+1}}\|u_{tt}\|_{V'}^2 dt.$$

For the nonlinear terms, using splitting (5.3) yields

$$\begin{aligned}
&|2\Delta t b(z^n, u(t_n), e^{n+1})| \\
\leq\ & c\Delta t\|z^n\|_0\|Au(t_n)\|_0\|\nabla e^{n+1}\|_0 \leq c\Delta t^2\int_{t_n}^{t_{n+1}}\|u_t\|_0^2 dt + \frac{\nu\Delta t}{12}\|\nabla e^{n+1}\|_0^2, \\
&|2\Delta t b(u(t_n), e^n, e^{n+1})| \\
\leq\ & c\Delta t\|e^n\|_0\|Au(t_n)\|_0\|\nabla e^{n+1}\|_0 \leq c\Delta t\|e^n\|_0^2 + \frac{\nu\Delta t}{12}\|\nabla e^{n+1}\|_0^2, \\
&|2\Delta t b(u(t_{n+1}), z^n, e^{n+1})| \\
\leq\ & c\Delta t\|z^n\|_0\|Au(t_{n+1})\|_0\|\nabla e^{n+1}\|_0 \leq c\Delta t^2\int_{t_n}^{t_{n+1}}\|u_t\|_0^2 dt + \frac{\nu\Delta t}{12}\|\nabla e^{n+1}\|_0^2, \\
&|2\Delta t b(e^n, u^n, e^{n+1})| \\
\leq\ & c\Delta t\|e^n\|_0\|Au^n\|_0\|\nabla e^{n+1}\|_0 \leq c\Delta t\|e^n\|_0^2 + \frac{\nu\Delta t}{12}\|\nabla e^{n+1}\|_0^2.
\end{aligned}$$

For the last term

$$\begin{aligned}
|2\theta\Delta t(\nabla z^n, \nabla e^{n+1})| &\leq 2\theta\Delta t\|\nabla z^n\|_0\|\nabla e^{n+1}\|_0 \\
&\leq c\Delta t^2\int_{t_n}^{t_{n+1}}\|\nabla u_t\|_0^2 dt + \frac{\nu\Delta t}{12}\|\nabla e^{n+1}\|_0^2.
\end{aligned}$$

Thanks to the above inequalities, taking into account (5.8) and (5.9) and summing up (5.16) from $n = 0$ to N, we obtain

$$\begin{aligned} &\|e^{N+1}\|_0^2 + \sum_{n=0}^{N} \|e^{n+1} - e^n\|_0^2 + \Delta t\nu \sum_{n=0}^{N} \|\nabla e^{n+1}\|_0^2 \\ &+\theta\Delta t\Big(\|\nabla e^{N+1}\|_0^2 + \sum_{n=0}^{N} \|\nabla(e^{n+1} - e^n)\|_0^2\Big) \\ \leq\ & c\Delta t^2\Big(\int_0^T (\|u_{tt}\|_{V'}^2 + \|\nabla u_t\|_0^2 + \|u_t\|_0^2)dt + 1\Big) + c\Delta t \sum_{n=0}^{N} \|e^n\|_0^2. \quad (5.17) \end{aligned}$$

Finally, we obtain the desired results with application of Lemma 2.1 at (5.17).

5.2. Error estimates for the semidiscrete pressure

Now, we give the estimates for r^{n+1} which shows that p^{n+1} is order 1 approximations to p in $L^\infty(L^2)$ and $L^2(L^2)$ norms. In order to achieve this aim, we firstly provide some estimates about $d_t e^{n+1} = \frac{e^{n+1}-e^n}{\Delta t}$.

Lemma 5.4. Assume that conditions (A3)-(A7) are valid. Then, for all $N = 0, 1, \ldots, [\frac{T}{\Delta t}] - 1$, we have

$$\begin{aligned} &\|d_t e^{N+1}\|_0^2 + \frac{1}{2}\sum_{n=0}^{N} \|d_t e^{n+1} - d_t e^n\|_0^2 + \nu\Delta t \sum_{n=0}^{N} \|\nabla d_t e^{n+1}\|_0^2 \\ &+\theta\Delta t\Big(\|\nabla d_t e^{N+1}\|_0^2 + \sum_{n=0}^{N} \|\nabla(d_t e^{n+1} - d_t e^n)\|_0^2\Big) \leq C\Delta t^2. \end{aligned}$$

Proof. From equation (5.12) we can obtain that for $\forall\ v \in V$

$$\begin{aligned} &(d_{tt} e^{n+1}, v) - (\theta + \nu)(\Delta d_t e^{n+1}, v) + \theta(\Delta d_t z^n + \Delta d_t e^n, v) \\ &= -b(d_t z^n, u(t_n), v) - b(z^{n-1}, d_t u(t_n), v) - b(d_t e^n, u^{n-1}, v) - b(e^n, d_t u^n, v) \\ &-b(d_t u^n, e^n, v) - b(u(t_{n-1}), d_t e^n, v) - b(d_t u(t_{n+1}), z^n, v) - b(u(t_n), d_t z^n, v) \\ &+(d_t R^n, v) + \Big[\Big(\Delta t\rho \sum_{i=1}^{n} e^{-\delta(t_n - t_i)} \Delta u(t_i), v\Big) - \Big(\Delta t\rho \sum_{i=1}^{n-1} e^{-\delta(t_{n-1} - t_i)} \Delta u(t_i), v\Big) \\ &+\Big(\int_0^{t_n} \rho e^{-\delta(t_n - s)} \Delta u(s) ds, v\Big) - \Big(\int_0^{t_{n+1}} \rho e^{-\delta(t_{n+1} - s)} \Delta u(s) ds, v\Big)\Big]\Big/\Delta t. (5.18) \end{aligned}$$

Choosing $v = 2\Delta t d_t e^{n+1}$ in (5.18) and using (3.1) and (4.5) we obtain

$$\begin{aligned}
&\|d_te^{n+1}\|_0^2 - \|d_te^n\|_0^2 + \|d_te^{n+1} - d_te^n\|_0^2 + \nu\Delta t\|\nabla d_te^{n+1}\|_0^2 \\
&+\theta\Delta t\Big(\|\nabla d_te^{n+1}\|_0^2 - \|\nabla d_te^n\|_0^2 + \|\nabla(d_te^{n+1} - d_te^n)\|_0^2\Big) \\
&\approx -2\Delta t\Big\{\theta(\nabla d_tz^n, \nabla d_te^{n+1}) - b(d_tz^n, u(t_n), d_te^{n+1}) - b(z^{n-1}, d_tu(t_n), d_te^{n+1}) \\
&-b(d_te^n, u^{n-1}, d_te^{n+1}) - b(e^n, d_tu^n, d_te^{n+1}) - b(d_tu^n, e^n, d_te^{n+1}) + (d_tR^n, d_te^{n+1}) \\
&-b(u(t_{n-1}), d_te^n, d_te^{n+1}) - b(d_tu(t_{n+1}), z^n, d_te^{n+1}) - b(u(t_n), d_tz^n, d_te^{n+1})\Big\} \\
&+2\Big[\Big(\Delta t\rho(e^{-\delta\Delta t} - 1)\sum_{i=1}^{n-1} e^{-\delta(t_{n-1}-t_i)}\Delta u(t_i) + \Delta t\rho\Delta u(t_n), d_te^{n+1}\Big) \\
&-\Big(\Delta t\rho(e^{-\delta\Delta t} - 1)\sum_{i=1}^{n} e^{-\delta(t_n-t_i)}\Delta u(t_i) + \Delta t\rho\Delta u(t_{n+1}), d_te^{n+1}\Big)\Big]. \qquad (5.19)
\end{aligned}$$

Now, we estimate the right-hand side terms separately. According to the definition of z^n, by using Taylor expansion we have

$$\begin{aligned}
d_tz^n = \frac{z^n - z^{n-1}}{\Delta t} &= \frac{u(t_{n+1}) - u(t_n) + u(t_{n-1}) - u(t_n)}{\Delta t} \\
&= \Big(u_{tt}(t_n) + \mathcal{O}(\Delta t^2)\Big)\Delta t. \qquad (5.20)
\end{aligned}$$

As a consequence, one finds that

$$\begin{aligned}
2\Delta t\theta|(\nabla d_tz^n, \nabla d_te^{n+1})| &\leq 2\Delta t^2\theta\|\nabla u_{tt}(t_n) + \mathcal{O}(\Delta t^2)\|_0\|\nabla d_te^{n+1}\|_0 \\
&\leq \frac{20}{\nu}\Delta t^3\theta^2\|\nabla u_{tt}(t_n) + \mathcal{O}(\Delta t^2)\|_0^2 + \frac{\nu}{20}\Delta t\|\nabla d_te^{n+1}\|_0.
\end{aligned}$$

For (d_tR^n, d_te^{n+1}), using the techniques that adopted by He in [18], we have

$$(d_tR^n, d_te^{n+1}) = -\frac{1}{\Delta t^2}\int_{t_n}^{t_{n+1}}(t - t_n)\int_{t-\Delta t}^{t}(u_{ttt}(s), d_te^{n+1})dsdt$$

for all $2 \leq n \leq N$. We deduce from above inequality that

$$\begin{aligned}
&|2\Delta t(d_tR^n, d_te^{n+1})| \leq 2\Delta t\|d_tR^n\|_0\|d_te^{n+1}\|_0 \leq c(\nu)\Delta t\|d_tR^n\|_0^2 + \frac{\nu}{20}\Delta t\|\nabla d_te^{n+1}\|_0^2 \\
\leq\ & c(\nu)\Delta t\Big[\Delta t^{-3/2}\Big(\int_{t_n}^{t_{n+1}}(t - t_n)^2\|\int_{t-\Delta t}^{t}u_{ttt}(s)ds\|_0^2dt\Big)^{1/2}\Big]^2 + \frac{\nu}{20}\Delta t\|\nabla d_te^{n+1}\|_0^2 \\
\leq\ & c\Delta t^2\int_{t_{n-1}}^{t_{n+1}}\|u_{ttt}\|_0^2dt + \frac{\nu}{20}\Delta t\|\nabla d_te^{n+1}\|_0^2.
\end{aligned}$$

For the nonlinear terms, with the help of (5.20) and the results of Theorem 5.3, we can estimate them as follows

$$
\begin{aligned}
& 2\Delta t|b(d_t z^n, u(t_n), d_t e^{n+1})| \leq 2\Delta t\|d_t z^n\|_0\|Au(t_n)\|_0\|\nabla d_t e^{n+1}\|_0 \\
\leq\ & 2\Delta t^2\|u_{tt}(t_n) + \mathcal{O}(\Delta t^2)\|_0\|Au(t_n)\|_0\|\nabla d_t e^{n+1}\|_0 \\
\leq\ & \frac{\nu}{20}\Delta t\|\nabla d_t e^{n+1}\|_0^2 + \frac{20}{\nu}\Delta t^3\|u_{tt}(t_n) + \mathcal{O}(\Delta t^2)\|_0^2\|Au(t_n)\|_0^2 \\
& 2\Delta t|b(z^{n-1}, d_t u(t_n), d_t e^{n+1})| \leq 2\Delta t\|z^{n-1}\|_0\|Ad_t u(t_n)\|_0\|\nabla d_t e^{n+1}\|_0 \\
\leq\ & 2\Delta t\|z^{n-1}\|_0\|Au_t(t_n) + \mathcal{O}(\Delta t)\|_0\|\nabla d_t e^{n+1}\|_0 \\
\leq\ & \frac{\nu}{20}\Delta t\|\nabla d_t e^{n+1}\|_0^2 + \frac{20}{\nu}\Delta t^2\int_{t_{n-1}}^{t_n}\|u_t\|_0^2 dt\|Au_t(t_n) + \mathcal{O}(\Delta t)\|_0^2 \\
& 2\Delta t|b(d_t e^n, u^{n-1}, d_t e^{n+1})| \leq 2\Delta t\|d_t e^n\|_0\|Au^{n-1}\|_0\|\nabla d_t e^{n+1}\|_0 \\
\leq\ & \frac{\nu}{20}\Delta t\|\nabla d_t e^{n+1}\|_0^2 + \frac{20}{\nu}\Delta t\|d_t e^n\|_0^2\|Au^{n-1}\|_0^2 \\
& 2\Delta t|b(e^n, d_t u^n, d_t e^{n+1})| \leq 2\Delta t\|e^n\|_0\|Ad_t u^n\|_0\|\nabla d_t e^{n+1}\|_0 \\
\leq\ & 2\Delta t\|e^n\|_0\|Au_t^n + \mathcal{O}(\Delta t)\|_0\|\nabla d_t e^{n+1}\|_0 \\
\leq\ & \frac{\nu}{20}\Delta t\|\nabla d_t e^{n+1}\|_0^2 + \frac{20}{\nu}\Delta t\|e^n\|_0^2\|Au_t^n + \mathcal{O}(\Delta t)\|_0^2, \\
& 2\Delta t|b(d_t u^n, e^n, d_t e^{n+1})| \leq \frac{\nu}{20}\Delta t\|\nabla d_t e^{n+1}\|_0^2 + \frac{20}{\nu}\Delta t\|e^n\|_0^2\|Au_t^n + \mathcal{O}(\Delta t)\|_0^2, \\
& 2\Delta t|b(u(t_{n-1}), d_t e^n, d_t e^{n+1})| \leq \frac{\nu}{20}\Delta t\|\nabla d_t e^{n+1}\|_0^2 + \frac{20}{\nu}\Delta t\|d_t e^n\|_0^2\|Au(t_{n-1})\|_0^2 \\
& 2\Delta t|b(d_t u(t_{n+1}), z^n, d_t e^{n+1})| \leq 2\Delta t\|Ad_t u(t_{n+1})\|_0\|z^n\|_0\|\nabla d_t e^{n+1}\|_0 \\
\leq\ & 2\Delta t\|Au_t(t_n) + \mathcal{O}(\Delta t)\|_0\|z^n\|_0\|\nabla d_t e^{n+1}\|_0 \\
\leq\ & \frac{\nu}{20}\Delta t\|\nabla d_t e^{n+1}\|_0^2 + \frac{20}{\nu}\Delta t^2\int_{t_{n-1}}^{t_n}\|u_t\|_0^2 dt\|Au_t(t_n) + \mathcal{O}(\Delta t)\|_0^2 \\
& 2\Delta t|b(u(t_n), d_t z^n, d_t e^{n+1})| \\
\leq\ & \frac{\nu}{20}\Delta t\|\nabla d_t e^{n+1}\|_0^2 + \frac{20}{\nu}\Delta t^3\|u_{tt}(t_n) + \mathcal{O}(\Delta t^2)\|_0^2\|Au(t_n)\|_0^2.
\end{aligned}
$$

Remark 5.1. In the estimation of fourth trilinear term, we have used the boundedness of $\|Au_t^n\|_0$ which can be proved by differentiating (1.1)-(1.2) with respect to time, with the split schemes (3.4)-(3.5) and the proof of Lemma 4.1. The bounded of $\|Au_t^n\|_0$ is similar to the results of (4.3). Here, we omit the proof.

For the last two terms in (5.19), we have

$$
\begin{aligned}
&2\Big[\Big(\Delta t\rho(e^{-\delta\Delta t}-1)\sum_{i=1}^{n-1}e^{-\delta(t_{n-1}-t_i)}\Delta u(t_i)+\Delta t\rho\Delta u(t_n),d_te^{n+1}\Big)\\
&-\Big(\Delta t\rho(e^{-\delta\Delta t}-1)\sum_{i=1}^{n}e^{-\delta(t_{n}-t_i)}\Delta u(t_i)+\Delta t\rho\Delta u(t_{n+1}),d_te^{n+1}\Big)\Big]\\
=\;&2\Delta t\rho(e^{-\delta\Delta t}-1)\Big((1-e^{-\delta\Delta t})\sum_{i=1}^{n-1}e^{-\delta(t_{n-1}-t_i)}\Delta u(t_i)-\Delta u(t_n),d_te^{n+1}\Big)\\
&+\Delta t\rho\Big(\int_{t_n}^{t_{n+1}}\Delta u_t dt,d_te^{n+1}\Big)\\
\le\;&C(\rho)\Delta t^2\|\nabla u_t\|_0\|\nabla d_te^{n+1}\|_0+C(\rho)\Delta t(e^{-\delta\Delta t}-1)\|\nabla u(t_n)\|_0\|\nabla d_te^{n+1}\|_0\\
&+(e^{-\delta\Delta t}-1)(1-e^{-\delta\Delta t})\|\Delta t\rho\sum_{i=1}^{n-1}e^{-\delta(t_{n-1}-t_i)}\nabla u(t_i)\|_0\|\nabla d_te^{n+1}\|_0\\
\le\;&\frac{\nu}{20}\Delta t\|\nabla d_te^{n+1}\|_0^2+C\Delta t^3\Big(\|\nabla u_t\|_0^2+\|\nabla u(t_n)\|_0^2+\|\Delta t\rho\sum_{i=1}^{n-1}e^{-\delta(t_{n-1}-t_i)}\nabla u(t_i)\|_0^2\Big).
\end{aligned}
$$

Combining above inequalities with (5.19) and summing from $n=0$ to N, we arrive at

$$
\begin{aligned}
&\|d_te^{N+1}\|_0^2+\frac{1}{2}\sum_{n=0}^{N}\|d_te^{n+1}-d_te^n\|_0^2+\nu\Delta t\sum_{n=0}^{N}\|\nabla d_te^{n+1}\|_0^2\\
&+\theta\Delta t\Big(\|\nabla d_te^{N+1}\|_0^2+\sum_{n=0}^{N}\|\nabla(d_te^{n+1}-d_te^n)\|_0^2\Big)\quad\Big(\text{Here } e^0=u(t_0)-u^0=0\Big)\\
\le\;&\frac{20T}{\nu}\Delta t^2\Big(\theta^2\|u_{tt}\|_1^2+\|u_{tt}\|_0^2+\|Au_t^n\|_0^2\Big)+\frac{20}{\nu}\Delta t^2\|Au_t\|_0^2\int_0^T\|u_t\|_0^2dt\\
&+c\Delta t^2\int_0^T\|u_{ttt}\|_0^2dt+\Delta t\sum_{n=0}^{N}\Big(1+\frac{40}{\nu}\|Au(t_{n-1})\|_0^2\Big)\|d_te^n\|_0^2.
\end{aligned}
$$

Thanks to Lemma 2.1, we obtain the desired results.

In Theorem 5.3, we have obtained the optimal error estimate for velocity in $L^2(H^1)$ norm. Next, thanks to the help of Theorem 5.3, we provide the optimal error estimate about velocity in $L^\infty(H^1)$ norm.

Lemma 5.5. Assume that conditions (A3)-(A7) are valid. Then, for all

$N = 0, 1, \ldots, [\frac{T}{\Delta t}] - 1$,

$$\|\nabla e^{N+1}\|_0^2 + \theta\Delta t\|Ae^{n+1}\|_0^2 + \sum_{n=0}^{N}\Big(\|\nabla(e^{n+1}-e^n)\|_0^2 + \nu\Delta t\|Ae^{n+1}\|_0^2\Big) \le c\Delta t^2.$$

Proof. Taking the inner product of (5.12) with $-2\Delta t A e^{n+1} \in V$, we have

$$\begin{aligned}
&\|\nabla e^{n+1}\|_0^2 - \|\nabla e^n\|_0^2 + \|\nabla(e^{n+1}-e^n)\|_0^2 + 2\Delta t\nu\|Ae^{n+1}\|_0^2 \\
&+\theta\Delta t\Big(\|Ae^{n+1}\|_0^2 - \|Ae^n\|_0^2 + \|A(e^{n+1}-e^n)\|_0^2\Big) \\
&= 2\Delta t b(u^n, u^n, Ae^{n+1}) - 2\Delta t b(u(t_{n+1}), u(t_{n+1}), Ae^{n+1}) - 2\theta\Delta t(Az^n, Ae^{n+1}) \\
&+2\Delta t(R^n, Ae^{n+1}) + 2\Delta t\Big[\Delta t\rho\sum_{i=1}^{n} e^{-\delta(t_n-t_i)}(\Delta u(t_i), Ae^{n+1}) \\
&-\int_0^{t_{n+1}} \rho e^{-\delta(t_{n+1}-s)}(\Delta u(s), Ae^{n+1})ds\Big]. \qquad (5.21)
\end{aligned}$$

Now, we estimate the terms in the right hand side of (5.21) separately.

$$|2\Delta t(R^n, Ae^{n+1})| \le 2\Delta t\|R^n\|_0\|Ae^{n+1}\|_0 \le \frac{\Delta t\nu}{12}\|Ae^{n+1}\|_0^2 + c\Delta t^2\int_{t_n}^{t_{n+1}}\|u_{tt}\|_0^2 dt.$$

For the nonlinear terms, using splitting (5.3) yields

$$\begin{aligned}
& |2\Delta t b(z^n, u(t_n), e^{n+1})| \\
\le\ & c\Delta t\|\nabla z^n\|_0\|Au(t_n)\|_0\|Ae^{n+1}\|_0 \le c\Delta t^2\int_{t_n}^{t_{n+1}}\|\nabla u_t\|_0^2 dt + \frac{\nu\Delta t}{12}\|Ae^{n+1}\|_0^2, \\
& |2\Delta t b(u(t_n), e^n, e^{n+1})| \\
\le\ & c\Delta t\|\nabla e^n\|_0\|Au(t_n)\|_0\|Ae^{n+1}\|_0 \le c\Delta t\|\nabla e^n\|_0^2 + \frac{\nu\Delta t}{12}\|Ae^{n+1}\|_0^2, \\
& |2\Delta t b(u(t_{n+1}), z^n, e^{n+1})| \\
\le\ & c\Delta t\|\nabla z^n\|_0\|Au(t_{n+1})\|_0\|Ae^{n+1}\|_0 \le c\Delta t^2\int_{t_n}^{t_{n+1}}\|\nabla u_t\|_0^2 dt + \frac{\nu\Delta t}{12}\|Ae^{n+1}\|_0^2, \\
& |2\Delta t b(e^n, u^n, e^{n+1})| \\
\le\ & c\Delta t\|\nabla e^n\|_0\|Au^n\|_0\|Ae^{n+1}\|_0 \le c\Delta t\|\nabla e^n\|_0^2 + \frac{\nu\Delta t}{12}\|Ae^{n+1}\|_0^2
\end{aligned}$$

For the last term

$$\begin{aligned}
|2\theta\Delta t(Az^n, Ae^{n+1})| &\le 2\theta\Delta t\|Az^n\|_0\|Ae^{n+1}\|_0 \\
&\le c\Delta t^2\int_{t_n}^{t_{n+1}}\|Au_t\|_0^2 dt + \frac{\nu\Delta t}{12}\|Ae^{n+1}\|_0^2,
\end{aligned}$$

Thanks to the above inequalities, taking into account (5.8) and (5.9) and summing up (5.16) from $n = 0$ to N, we obtain

$$
\begin{aligned}
&\|\nabla e^{N+1}\|_0^2 + \sum_{n=0}^{N} \|\nabla(e^{n+1} - e^n)\|_0^2 + \Delta t\nu \sum_{n=0}^{N} \|Ae^{n+1}\|_0^2 \\
&+\theta\Delta t\Big(\|Ae^{N+1}\|_0^2 + \sum_{n=0}^{N} \|A(e^{n+1} - e^n)\|_0^2\Big) \\
\leq\ & c\Delta t^2\Big(\int_0^T (\|u_{tt}\|_0^2 + \|Au_t\|_0^2)dt + 1\Big) + c\Delta t \sum_{n=0}^{N} \|\nabla e^n\|_0^2.
\end{aligned}
$$

Finally, we complete the proof by using the results of Theorem 5.3.

Now, we in the position of establishing the optimal error estimate for pressure in $L^\infty(L^2)$ and $L^2(L^2)$ norms based on the results presented in Lemmas 5.4 and 5.5.

Theorem 5.6. Assume that conditions (A3)-(A7) are valid. Then, for all $N = 0, 1, \ldots, [\frac{T}{\Delta t}] - 1$,

$$
\Delta t \sum_{n=0}^{N} \|p(t_{n+1}) - p^{n+1}\|_0^2 \leq c\Delta t^2.
$$

Furthermore, if $u_t \in L^\infty(0, T; X)$, then

$$
\|p(t_{n+1}) - p^{n+1}\|_0 \leq c\Delta t.
$$

Proof. We rewrite (5.12) as

$$
\begin{aligned}
-\nabla r^{n+1} \;=\;& d_t e^{n+1} - (\theta + \nu)\Delta e^{n+1} + \theta(\Delta z^n + \Delta e^n) - R^n \\
&+ \int_0^{t_{n+1}} \rho e^{-\delta(t_{n+1}-s)}\Delta u(s)ds - \Delta t\rho \sum_{i=1}^{n} e^{-\delta(t_n - t_i)}\Delta u(t_i) \\
&-(u^n \cdot \nabla)u^n + (u(t_{n+1}) \cdot \nabla)u(t_{n+1}). \qquad (5.22)
\end{aligned}
$$

Taking the inner product of (5.22) with an arbitrary $v \in X$ and using Poincare inequality, we have

$$
\begin{aligned}
&|\Big(d_t e^{n+1}, v\Big)| \leq \|d_t e^{n+1}\|_0 \|v\|_0 \leq c_0 \|d_t e^{n+1}\|_0 \|\nabla v\|_0, \\
&|\nu(\Delta e^{n+1}, v)| \leq \nu \|\nabla e^{n+1}\|_0 \|\nabla v\|_0, \\
&|\theta(\Delta(e^{n+1} - e^n), v)| \leq \theta \|\nabla(e^{n+1} - e^n)\|_0 \|\nabla v\|_0, \\
&|(R^n, v)| \leq \|R^n\|_0 \|v\|_0 \leq c\Delta t(\int_{t_n}^{t_{n+1}} t\|u_{tt}\|_0^2)^{1/2} \|\nabla v\|_0,
\end{aligned}
$$

$$|\theta(\Delta z^n, v)| \le c\Big(\Delta t \int_{t_n}^{t_{n+1}} \|\nabla u_t\|_0^2\Big)^{1/2} \|\nabla v\|_0.$$

For the nonlinear terms, we take the product with an arbitrary $v \in X$ and use the results provided in Lemma 4.1 to arrive at

$$\begin{aligned}
&|b(z^n, u(t_n), v)| \le c\|z^n\|_0 \|u(t_n)\|_2 \|\nabla v\|_0 \le c\Big(\Delta t \int_{t_n}^{t_{n+1}} \|u_t\|_0^2 dt\Big)^{1/2} \|\nabla v\|_0 \\
&|b(u(t_n), e^n, v)| \le c\|e^n\|_0 \|Au(t_n)\|_0 \|\nabla v\|_0 \le c\|e^n\|_0 \|\nabla v\|_0, \\
&|b(u(t_{n+1}), z^n, v)| \le c\|z^n\|_0 \|Au(t_{n+1})\|_0 \|\nabla v\|_0 \le c\Big(\Delta t \int_{t_n}^{t_{n+1}} \|u_t\|_0^2 dt\Big)^{1/2} \|\nabla v\|_0, \\
&|b(e^n, u^n, v)| \le c\|e^n\|_0 \|Au^n\|_0 \|\nabla v\|_0 \le c\|e^n\|_0 \|\nabla v\|_0.
\end{aligned}$$

Thus, thanks to (5.8) and (5.9), we obtain

$$\begin{aligned}
\|r^{n+1}\|_0 \quad \le \quad & \|d_t e^{n+1}\|_0 + c\Big\{ \|\nabla e^{n+1}\|_0 + \|\nabla e^n\|_0 + \|e^n\|_0 \\
& + \Big(\Delta t \int_{t_n}^{t_{n+1}} \|u_t\|_0^2 dt\Big)^{1/2} + \Delta t\Big(\int_{t_n}^{t_{n+1}} t\|u_{tt}\|_0^2 dt\Big)^{1/2}\Big\}. \quad (5.23)
\end{aligned}$$

Squaring (5.23) and summing it from $n = 0$ to N, with the results obtained in Lemmas 5.4 and 5.5, we obtain the desired result.

Furthermore, under the assumption of $u_t \in L^\infty(0, T; X)$ we know that

$$\begin{aligned}
&|\theta(\Delta z^n, v)| \le c\|\nabla u_t\|_0 \|\nabla v\|_0 \Delta t, \\
&|b(z^n, u(t_n), v)| \le c\|z^n\|_0 \|u(t_n)\|_2 \|\nabla v\|_0 \le c\|u_t\|_0 \|\nabla v\|_0 \Delta t, \\
&|b(u(t_{n+1}), z^n, v)| \le c\|z^n\|_0 \|Au(t_{n+1})\|_0 \|\nabla v\|_0 \le c\|u_t\|_0 \|\nabla v\|_0 \Delta t.
\end{aligned}$$

Thus, (5.23) can be changed into

$$\begin{aligned}
\|r^{n+1}\|_0 \quad \le \quad & \|d_t e^{n+1}\|_0 + c\Big\{ \|\nabla e^{n+1}\|_0 + \|\nabla e^n\|_0 + \|e^n\|_0 \\
& + \Delta t(\|u_t\|_0 + \|\nabla u_t\|_0) + \Delta t\Big(\int_{t_n}^{t_{n+1}} t\|u_{tt}\|_0^2 dt\Big)^{1/2}\Big\}.
\end{aligned}$$

Combining Lemmas 5.4 and 5.5, we obtain the optimal error estimate for pressure in $L^\infty(L^2)$ norm.

6. Numerical examples

In this section, we present some numerical results to show the effectiveness of fractional-step finite element method for the viscoelastic flows problem. We consider problem (1.1)-(1.3) on the unit square $\Omega = [0, 1]^2$ in all experiments.

We set $\nu = 0.002$, $\rho = \nu$, $1/\delta = 100\Delta t$. The exact solution for the velocity and pressure are

$$
\begin{aligned}
u_1 &= 10x^2(x-1)^2 y(y-1)(2y-1)e^{-2\nu\pi^2 t},\\
u_2 &= -10x(x-1)(2x-1)y^2(y-1)^2 e^{-2\nu\pi^2 t},\\
p &= 20(2x-1)(2y-1)e^{-4\nu\pi^2 t}.
\end{aligned}
$$

6.1. An analytical solution: Convergence validation

Firstly, we compare the numerical solution at $T = 0.01$ by using fractional-step method with different parameters θ and the standard Galerkin finite element method with Taylor-Hood element (P_2-P_1 element). From Tables 1 and 3, we know that, as time step $\Delta t = 0.0001$, the errors of velocity of fractional-step method and standard Galerkin method are almost the same, while the errors of pressure obtained by fractional-step method better than Galerkin method. Next, we set the time step $\Delta t = 0.001$, from Tables 2 and 3, we can see that the errors of both velocity and pressure obtained by two methods have the same accuracy. On the other hand, comparing Tables 1-3, the standard Galerkin method spends the least CPU-times than fractional-step scheme with different parameters in different time steps.

Secondly, we present the numerical results at $T = 0.1$ obtained by fractional-step method with parameters $\theta = 0, 5, 10, 100$ and standard Galerkin method at different time steps in Tables 4-7. From Tables 4-6, we can see that the more precise results of velocity can be obtained as the parameter θ increases, especially for the large time step, see Table 6 for details. Compared with Table 7, we know that the standard Galerkin method spends the least CPU-times. By choosing suitable parameter θ, we can obtain the good numerical results of velocity. On the other hand, the errors of pressure obtained by standard Galerkin method are undesired. From this view of point, the desired numerical results of both velocity and pressure can be got by fractional-step with suitable parameter θ in the large time steps.

6.2. The affection of parameter θ

In this subsection, we consider the affection of parameter θ for the time steps and stability to discrete system (3.6). Generally, the following linear algebra equations can be obtained from the discrete system of (3.6)

$$
\begin{pmatrix} A & -D \\ D^T & 0 \end{pmatrix} \begin{pmatrix} U \\ P \end{pmatrix} = \begin{pmatrix} F \\ 0 \end{pmatrix}, \tag{6.1}
$$

where the matrices A, D and G are deduced in the usual manner from the bilinear forms $a(\cdot,\cdot)$ and $d(\cdot,\cdot)$, F is the variation of the source term, trilinear and integrate terms. Here the matrix A can be split into two parts, namely

$$
\begin{aligned}
A &= \begin{pmatrix} 1-\nu\Delta t & a_{12} & \dots & a_{1n} \\ a_{21} & 1-\nu\Delta t & \dots & a_{2n} \\ \dots & & & \\ a_{n1} & a_{n2} & \dots & 1-\nu\Delta t \end{pmatrix} - \begin{pmatrix} \theta\Delta t & 0 & \dots & 0 \\ 0 & \theta\Delta t & \dots & 0 \\ \dots & & & \\ 0 & 0 & \dots & \theta\Delta t \end{pmatrix} \\
&\triangleq \overline{A} - B.
\end{aligned}
$$

Then, system (6.1) can be rewritten as

$$
\begin{pmatrix} \overline{A} & -D \\ D^T & 0 \end{pmatrix} \begin{pmatrix} U \\ P \end{pmatrix} - \begin{pmatrix} B & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} U \\ P \end{pmatrix} = \begin{pmatrix} F \\ 0 \end{pmatrix}. \tag{6.2}
$$

From (6.2), we obtain that $\overline{A}U - DP - \theta\Delta tU = F$. In order to get the optimal choice of θ, we denote

$$
R = F - \overline{A}U + DP + \theta\Delta tU.
$$

Taking $g = \frac{1}{2}R^TR$, the best choice of θ should satisfy $\frac{dg}{d\theta} = 0$, i.e.,

$$
\frac{1}{2}\Delta tU^T(F - \overline{A}U + DP + \theta\Delta tU) = 0. \tag{6.3}
$$

Solving equation (6.3), we obtain that

$$
\theta = \frac{U^T\overline{A}U - U^TDP - U^TF}{\Delta tU^TU} \tag{6.4}
$$

From the expression of (6.4), we know that the best θ not only depends on the exact solution u, p and ν ($\nu = \frac{1}{Re}$, Re the Reynolds number) but also on the relationship with time steps Δt. Figures 1-2 show the affection of different θ to the accuracy of velocity in H^1-norm with different time steps. From these Figures, we can see that the optimal θ is obtained from the view of numerical with fixed parameters ρ, δ and ν. Furthermore, From (6.4), when we fixed the exact solution and the corresponding parameters, the larger of Δt, the smaller of θ. Figures 1(b)-2(b) verify this fact that $\theta \approx 22$ when $\Delta t = 0.001$ while $\theta \approx 5$ when $\Delta t = 0.01$.

Next, we fix the time steps and the value of θ to consider the variation of the accuracy of velocity in H^1-norm with different Reynolds number. From Figures 3-4, we can see that the errors becomes smaller and smaller as the Reynolds numbers increase with different time steps. These Figures confirmed our theoretical findings and illustrated that suitable choices of θ can enhance the stability of the numerical scheme.

7. Conclusion

In this paper, we considered a fractional-step finite element method for the viscoelastic flows problem. Stability and convergence of the velocity and pressure are established under the stability condition about time step Δt. In order to enlarge the time step, we introduce a diffusion term $\theta \Delta u$ in all steps of the fractional-step schemes. Finally, some numerical results are provided to verify the efficiency of our algorithm.

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