A Modified Nonlinear Spectral Galerkin Method for the Equations of Motion Arising in the Kelvin-Voigt Fluids

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Abstract

In this paper, a variant of nonlinear Galerkin method is proposed and analysed for equations of motions arising in a Kelvin-Voigt model of viscoelastic fluids. Some new *a priori* bounds are obtained for the exact solution when the forcing function is independent of time or belongs to L^∞ in time. As a consequence, existence of a global attractor is shown. For the spectral Galerkin scheme, existence of a unique discrete solution to the semidiscrete scheme is proved and again existence of a discrete global attractor is established. Further, optimal error estimate in $L^\infty(\mathbf{L}^2)$ and $L^\infty(\mathbf{H}_0^1)$-norms are proved. Finally, a modified nonlinear Galerkin method is developed and optimal error bounds are derived. It is observed that optimum accuracy in $L^\infty(\mathbf{H}_0^1)$ is achieved when $N = O(n^3)$, for dimension $d = 2$, and $N = O(n^{5/2})$, for $d = 3$. Moreover, for optimal accuracy in $L^\infty(\mathbf{L}^2)$-norm, it is noted that $N = O(n^{3/2})$, when dimension $d = 2$, and $N = O(n^{5/4})$, in $d = 3$. It is further shown for both methods that error estimates are valid uniformly in time under uniqueness condition.

Keywords: *Viscoelastic fluids, Kelvin-Voigt model, nonlinear Galerkin method, spectral approximations, a priori error bound, uniform convergence in time .*
AMS 1991 Classification:

1. Introduction. The motion of an incompressible fluid in a bounded domain Ω in R^d ($d = 2, 3$) is described as in Chorin and Marsden [3] and Joseph [8] by the following system of partial differential equations:

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u} - \nabla\boldsymbol{\sigma} + \nabla p = \mathbf{F}(x,t), \;\; x \in \Omega, \; t > 0, \tag{1.1}$$

and

$$\nabla\cdot\mathbf{u} = 0, \;\; x \in \Omega, \, t > 0, \tag{1.2}$$

with appropriate initial and boundary conditions. Here $\boldsymbol{\sigma} = (\boldsymbol{\sigma}_{ik})$ denotes the stress tensor with $tr\boldsymbol{\sigma} = 0$, $\mathbf{u}$ represents the velocity vector, p is the pressure of the fluid and $\mathbf{F}$ is the external force. The defining relation between the stress tensor $\boldsymbol{\sigma}$ and the tensor of deformation velocities $\mathbf{D} = (\mathbf{D}_{ik}) = \frac{1}{2}(\mathbf{u}_{ix_k} + \mathbf{u}_{kx_i})$, called the equation of state or sometimes rheological equation, in fact establishes the type of fluid under consideration. In the fag end of the nineteenth century, models of viscoelastic fluids have been proposed which take into consideration the prehistory of the flow. One such model was proposed by Kelvin [11] and Voigt [19] and hence, this model is named after them. In this case, see Oskolkov [14], the rheological relation is expressed by

$$\boldsymbol{\sigma} = 2\nu(1+\kappa\nu^{-1}\frac{\partial}{\partial t})\mathbf{D}, \quad \nu,\kappa > 0, \tag{1.3}$$

where ν is the kinematic coefficient of viscosity and κ is the retardation time. On substitution of (1.3) in (1.1)–(1.2), we now obtain the following equations of motion for a Kelvin-Voigt fluid:

$$\mathbf{u}_t + (\mathbf{u}\cdot\nabla)\mathbf{u} - \kappa\Delta\mathbf{u}_t - \nu\Delta\mathbf{u} + \nabla p = \mathbf{F}(x,t), \;\; x\in\Omega, \;\; t>0, \tag{1.4}$$

and incompressibility condition

$$\nabla\cdot\mathbf{u} = 0, \;\; x\in\Omega, \, t>0, \tag{1.5}$$

with initial and homogeneous Dirichlet boundary conditions

$$\mathbf{u}(x,0) = \mathbf{u}_0 \;\text{ in } \Omega, \quad \mathbf{u} = 0 \text{ on } \partial\Omega, \;\; t\geq 0. \tag{1.6}$$

Here, Ω is a bounded domain in two or three dimensional Euclidean space R^d $(d=2,3)$ with smooth boundary $\partial\Omega$. For this particular setting, the Kelvin-Voigt fluid is characterized by the fact that after instantaneous removal of the stresses, the velocity of the flow does not vanish immediately, but fades out like $\exp(-\kappa^{-1}t)$ see Oskolkov [14].

In this paper, we first derive some *a priori* bounds for the solution of (1.4)–(1.6). Then we apply spectral Galerkin method to discretize in the direction of space, keeping time variable continuous and call it semi-discrete scheme. We establish optimal error estimates for the semi-discrete scheme. Finally, a variant of nonlinear spectral Galerkin method is analyzed and optimal error estimates in $L^\infty(L^2)$ as well as $L^\infty(\mathbf{H}^1)$-norms are derived.

It is to be noted that the system can be thought of as a perturbation of the Navier-Stokes equations in the sense that $-\kappa\Delta\mathbf{u}_t$ is added to the NS equations. Therefore, we would like to investigate '*how far the results on spectral Galerkin analysis for the Navier-Stokes equations (see, [17], [7]) can be carried over to the present case*'.

Based on the analysis of Ladyzenskaya [12], Oskolkov [14] has proved global existence of unique 'almost' classical solution in finite time interval for the initial and boundary value problem (1.4)–(1.6).

Earlier,Rautmann [17] has discussed spectral Galerkin method for Navier-Stokes equations and derived error estimates which may be thought of local in time in the sense that the constants involved in error analysis depend exponentially on time. Subsequently, Heywood [7] has analysed spectral Galerkin method for Navier-Stokes equations and established *a priori* error estimates in $L^\infty(\mathbf{H}^1)$-norm which is valid for uniform in time under the assumption that the exact solution is asymptotically stable. This analysis is further extended to the Kelvin-Vight system (1.4)–(1.6) by Oskolkov [15] and again error estimate is proved for $L^\infty(\mathbf{H}^1)$-norm.

While there are substantial literature available on nonlinear spectral Galerkin method applied to the Navier-Stokes system, see, He *et al.* [4]-[6] and in the context of Oldroyd model, see Cannon *et al.*[2], there is hardly any literature devoted to the system (1.4)–(1.6). Thus, the present investigation is a step towards achieving this. In this article, we not only discuss the optimal error estimates in $\mathbf{L}^2$ for the spectral Galerkin method, but also derive optimal *a priori* estimates for the nonlinear Galerkin method which is valid uniformly in time under uniqueness assumption. which is discussed in Section 3. We now summarise our major results in this article as follow:

- New regularity results for the system (1.4)–(1.6) are **derived by using a change of variable $\hat{\mathbf{u}} = e^{\alpha t}\mathbf{u}$** to take care of the forcing function which may be independent of time or may be L^∞ in time. As a consequence, existence of a global attractor is derived.
- For the spectral Galerkin method, existence of a unique discrete solution is proved using *a priori* estimates for the discrete problem and again as a remark existence of a global discrete attractor is shown. Further, optimal error estimates in $L^\infty(\mathbf{L}^2)$ and $L^\infty(\mathbf{H}_0^1)$-norms are established. It is also observed that super-convergence phenomenon in $L^\infty(\mathbf{H}_0^1)$-norm is derived, when spectral Galerkin approximation say $\mathbf{u}_{SG}$ is compared with the N^{th}-trucation say $\mathbf{u}_N$ of the spectral series. Under the assumption of uniqueness, error estimates are valid uniformly in time. **Analogous results were proved for Navier-Stokes Equation for Galerkin method by Archilla [1] and non linear Galerkin method by Novo [13]**.
- Finally, a modified nonlinear Galerkin method is applied to (1.4)–(1.6). It is observed that optimum accuracy in $L^\infty(\mathbf{H}_0^1)$ is achieved when $N = O(n^3)$, for dimension $d = 2$, and $N = O(n^{5/2})$, for $d = 3$. Moreover, for optimal accuracy in $L^\infty(\mathbf{L}^2)$-norm, it is noted that $N = O(n^{3/2})$, when dimension $d = 2$, and $N = O(n^{5/4})$, in $d = 3$. Like spectral Galerkin method, here also superconvergence phenomenon is observed.

Through out this article, C is considered as a generic positive constant which varies from time to time.

2. Preliminaries and *A priori* Bounds. In the first part of this Section, we present some preliminaries to be used in our subsequent analysis. Then we formulate the problem and discuss *a priori* bounds.

Let $H^m(\Omega)$ be standard Hilbert Sobolev spaces with norm $\|\cdot\|_m$. Let $L^p(\Omega)$, $1 \le p \le \infty$ be usual Lebesque measurable spaces with norm $\|\cdot\|_{L^p(\Omega)}$. When $p = 2$, we denote the norm on $L^2(\Omega)$ simply as $\|\cdot\|$ and inner-product as $(\cdot,\cdot)$. Further, let $H_0^1(\Omega) : \{\phi \in H^1(\Omega) : \phi = 0 \text{ on } \partial\Omega\}$. On $H_0^1(\Omega)$, the seminorm $\|\nabla\phi\|$ is in fact a norm which is equivalent to H^1-norm due to Poincaré inequality. We shall also use the following function spaces for the vector valued functions. Define

$$\mathbf{D}(\Omega) := \{\boldsymbol{\phi} \in (C_0^\infty(\Omega))^d : \ \nabla\cdot\boldsymbol{\phi} = 0 \text{ in } \Omega\},$$
$$\mathbf{H} := \text{the closure of } \mathbf{D}(\Omega) \text{ in } (L^2(\Omega))^d - \text{space}$$

and

$$\mathbf{V} := \text{the closure of } \mathbf{D}(\Omega) \text{ in } (H_0^1(\Omega))^d - \text{space}.$$

Note that since Ω is a bounded and some regularity assumptions on the boundary $\partial\Omega$, it is possible to characterize $\mathbf{V}$ as

$$\mathbf{V} := \{\boldsymbol{\phi} \in (H_0^1(\Omega))^d : \ \ \nabla\cdot\boldsymbol{\phi} = 0 \text{ in } \ \Omega\}.$$

The spaces of vector functions will be indicated with boldface letters, for instance $\mathbf{H}_0^1 = \mathbf{H}_0^1(\Omega) = (H_0^1(\Omega))^d$. The inner-product on $\mathbf{H}_0^1$ will be denoted by

$$(\nabla\boldsymbol{\phi}, \nabla\mathbf{w}) = \sum_{i=1}^{d} (\nabla\phi_i, \nabla w_i)$$

and norm

$$\|\nabla\boldsymbol{\phi}\| = (\sum_{i=1}^{d}\|\nabla\phi_i\|^2)^{\frac{1}{2}}.$$

Using Poincaré inequality, it can be shown that the norm on $\mathbf{H}_0^1$ is equivalent to $\mathbf{H}^1 = (H^1(\Omega))^d$- norm. Let $\mathbf{P}$ denote the orthogonal projection of $\mathbf{L}^2(\Omega)$ $(= (L^2(\Omega))^d)$ onto $\mathbf{V}$. Now the orthogonal complement $\mathbf{V}^\perp$ of $\mathbf{V}$ in $\mathbf{L}^2(\Omega)$ consists of functions $\boldsymbol{\phi}$ such that $\boldsymbol{\phi} = \nabla q$ for some $q \in H^1(\Omega)/R$. For more details, we refere to [18].

We denote by $A = \mathbf{P}(-\Delta) = -\mathbf{P}\Delta$, the Stokes operator which is a self-adjoint, positive definite, and a closed linear operator on $\mathbf{V}$ with domain $\mathbf{H}^2\bigcap\mathbf{V}$. Note that A has a compact inverse. Let $\{\lambda_k\}$ be the sequence of eigenvalues with $0 < \lambda_1 \le \lambda_2 \le \ldots \le \lambda_k \le \ldots$, $\lambda_k \to \infty$ as $k \to \infty$, and let $\{\boldsymbol{\phi}^k\}$ be the corresponding eigenvectors of the Stokes operator A, i.e., $A\boldsymbol{\phi}^k = \lambda_k\boldsymbol{\phi}^k$. It is easy to check that $\{\boldsymbol{\phi}^k\}$ forms an orthogonal set in $\mathbf{H}$, $\mathbf{V}$ and $\mathbf{H}^2\bigcap\mathbf{V}$. Since A is positive definite, $A^{s/2}$ is defined on $\underline{\mathcal{D}(A^{s/2}) \supseteq \mathbf{H}^2\bigcap\mathbf{V},\ 0 < s \le 2.}$ Note that for all $\mathbf{v} \in \mathcal{D}(A^{s/2})$, $\|\mathbf{v}\|_s$ and $\|A^{s/2}\mathbf{v}\|$ are equivalent, again refer to [18]

Since λ_1 is the **first eigenvalue** of the Stokes operator, we have the following form of the Poincaré inequality: For $\boldsymbol{\phi} \in \mathbf{H}^2\bigcap\mathbf{V}$,

$$\|\boldsymbol{\phi}\| \le \lambda_1^{-\frac{1}{2}}\|A^{1/2}\boldsymbol{\phi}\| \le \lambda_1^{-1}\|A\boldsymbol{\phi}\|. \tag{2.1}$$

Following Temam [18], we rewrite the equations (1.1)–(1.3) in an *abstract form* as: Find $\mathbf{u}(t) \in \mathbf{V}$ such that for $t \in (0,\infty)$

$$\mathbf{u}_t + \kappa A\mathbf{u}_t + \nu A\mathbf{u} + B(\mathbf{u},\mathbf{u}) = \mathbf{f} \tag{2.2}$$

$$\mathbf{u}(0) = \mathbf{u}_0, \tag{2.3}$$

where $B(\mathbf{u},\mathbf{v}) = \mathbf{P}((\mathbf{u}\cdot\nabla)\mathbf{v})$ and $\mathbf{f} = \mathbf{PF}$.

In the sequel, we shall use the following estimate (see Temam [18]): For $\mathbf{u},\mathbf{v},\mathbf{w} \in \mathbf{V}$, we have $|(B(\mathbf{u},\mathbf{v}),\mathbf{w})| \le \|A^{1/2}\mathbf{u}\|\,\|A^{1/2}\mathbf{v}\|\,\|A^{1/2}\mathbf{w}\|$; so, there exist a positive constant M such that

$$M := \sup_{\mathbf{u},\mathbf{v},\mathbf{w}\in\mathbf{V}} \frac{|(B(\mathbf{u},\mathbf{v}),\mathbf{w})|}{\|A^{1/2}\mathbf{u}\|\,\|A^{1/2}\mathbf{v}\|\,\|A^{1/2}\mathbf{w}\|}. \tag{2.4}$$

2.1 *A Priori* Bounds.

In this subsection, we discuss some *a priori* bounds for the exact solution $\mathbf{u}$ of (2.2)-(2.3).

Lemma 2.1 *Assume that $\mathbf{f} \in \mathrm{L}^\infty(0,\infty,\mathbf{L}^2(\Omega))$, and $\mathbf{u}_0 \in \mathbf{V}$. Then, for $0 < \alpha \le \frac{\lambda_1\nu}{2(1+\kappa\lambda_1)}$ the solution $\mathbf{u}$ of (2.2)-(2.3) satisfies*

$$\begin{aligned}||\mathbf{u}(t)||^2 + \kappa||A^{1/2}\mathbf{u}(t)||^2 \;&+\; 2\beta\int_0^t e^{-2\alpha(t-s)}\|A^{1/2}\mathbf{u}(s)\|^2\,ds \le e^{-2\alpha t}\Big[||\mathbf{u}_0||^2 + \kappa||A^{1/2}\mathbf{u}_0||^2\Big]\\ &+\; \frac{1}{2\alpha\lambda_1\nu}||\mathbf{f}||^2_{\mathrm{L}^\infty(\mathbf{L}^2)}(1-e^{-2\alpha t}) = K_0(t) \le K_{0,\infty},\end{aligned}$$

where $\beta = \left(\frac{\nu}{2} - \alpha(\kappa + \frac{1}{\lambda_1}\right) \ge 0$ and $K_{0,\infty} = \sup_{t\in[0,\infty)} K_0(t)$.

Proof. Set $\hat{\mathbf{u}} = e^{\alpha t}\mathbf{u}$ in (2.2) to obtain

$$e^{\alpha t}\mathbf{u}_t + \kappa e^{\alpha t}A\mathbf{u}_t + \nu A\hat{\mathbf{u}} + e^{-\alpha t}B(\hat{\mathbf{u}},\hat{\mathbf{u}}) = \hat{\mathbf{f}}. \tag{2.5}$$

Taking the inner-product with (2.5) and $\hat{\mathbf{u}}$, we observe that $(B(\hat{\mathbf{u}},\hat{\mathbf{u}}),\hat{\mathbf{u}}) = 0$, and hence, we obtain

$$\frac{1}{2}\frac{d}{dt}\Big(||\hat{\mathbf{u}}||^2 + \kappa||A^{1/2}\hat{\mathbf{u}}||^2\Big) - \alpha\Big(||\hat{\mathbf{u}}||^2 + \kappa||A^{1/2}\hat{\mathbf{u}}||^2\Big) + \nu||A^{1/2}\hat{\mathbf{u}}||^2 = (\hat{\mathbf{f}},\hat{\mathbf{u}}). \tag{2.6}$$

Since $||\hat{\mathbf{u}}||^2 \leq \frac{1}{\lambda_1}||A^{1/2}\hat{\mathbf{u}}||^2$, it follows by using the Young's inequality that

$$(\hat{\mathbf{f}},\hat{\mathbf{u}}) \leq ||\hat{\mathbf{f}}||||\hat{\mathbf{u}}|| \leq \frac{1}{\sqrt{\lambda_1}}\|\hat{\mathbf{f}}\|\,\|A^{1/2}\hat{\mathbf{u}}\| \leq \frac{1}{2\lambda_1\nu}||\hat{\mathbf{f}}||^2 + \frac{\nu}{2}||A^{1/2}\hat{\mathbf{u}}||^2. \tag{2.7}$$

On substituting (2.7) in (2.6), we arrive at

$$\frac{d}{dt}\Big(||\hat{\mathbf{u}}||^2 + \kappa||A^{1/2}\hat{\mathbf{u}}||^2\Big) + 2(\frac{\nu}{2} - \alpha(\kappa + \frac{1}{\lambda_1}))||A^{1/2}\hat{\mathbf{u}}||^2 \leq \frac{1}{\lambda_1\nu}||\hat{\mathbf{f}}||^2. \tag{2.8}$$

Choose $\alpha > 0$ such that $(\frac{\nu}{2} - \alpha(\kappa + \frac{1}{\lambda_1})) = \beta \geq 0$, that is,

$$0 < \alpha \leq \frac{\lambda_1\nu}{2(1+\kappa\lambda_1)}.$$

Then integrate (2.8) with respect to time from 0 to t, to obtain

$$\begin{aligned} ||\mathbf{u}||^2 + \kappa||A^{1/2}\mathbf{u}||^2 \;&+\; \beta\int_0^t e^{-2\alpha(t-s)}\|A^{1/2}\mathbf{u}(s)\|^2\,ds \leq e^{-2\alpha t}\Big[||\mathbf{u}_0||^2 + \kappa||A^{1/2}\mathbf{u}_0||^2\Big] \\ &+\; \frac{1}{\lambda_1\nu}e^{-2\alpha t}\int_0^t e^{2\alpha s}||\mathbf{f}(s)||^2\,ds, \end{aligned} \tag{2.9}$$

and from which we conclude that

$$\begin{aligned} ||\mathbf{u}||^2 + \kappa||A^{1/2}\mathbf{u}||^2 \;&\leq\; e^{-2\alpha t}\Big[||\mathbf{u}_0||^2 + \kappa||A^{1/2}\mathbf{u}_0||^2\Big] \\ &+\; \frac{1}{2\alpha\lambda_1\nu}||\mathbf{f}||_{\mathrm{L}^\infty(\mathrm{L}^2)}(1 - e^{-2\alpha t}). \end{aligned}$$

This completes the rest of the proof. ■

Remark 2.1 *As a consequence of the above Lemma 2.1, we obtain from (2.8) with* $\alpha = 0$

$$\frac{d}{dt}\Big(||\mathbf{u}||^2 + \kappa||A^{1/2}\mathbf{u}||^2\Big) + 2\nu||A^{1/2}\mathbf{u}||^2 \leq \frac{1}{\lambda_1\nu}||\mathbf{f}||^2. \tag{2.10}$$

Integrating in time from t **to** *$t+T$, and using Lemma 2.1, we now arrive at for a fixed $T > 0$ and for all $t \geq 0$, the following estimate:*

$$\int_t^{t+T}\|A^{1/2}\mathbf{u}(s)\|^2\,ds \leq \mathbf{K_0(t)} + \frac{T}{\lambda_1\nu}||\mathbf{f}||^2_{\mathrm{L}^\infty(\mathbf{L}^2)}. \tag{2.11}$$

Remark 2.2 *When $\mathbf{f} \in L^2(0,\infty;\mathbf{L}^2(\Omega))$, then integrating (2.10) with respect to time from* 0 *to t, we find that*

$$\int_0^t\|A^{1/2}\mathbf{u}(s)\|^2\,ds \leq \mathbf{K_0(t)} \tag{2.12}$$

Lemma 2.2 *Under the weaker assumption on f, that is, $f \in L^\infty(0,\infty;V^*)$, the following a priori bound holds for* $0<\alpha\le \frac{\lambda_1\nu}{2(1+\kappa\lambda_1)}$

$$\begin{aligned} ||\mathbf{u}(t)||^2+\kappa||A^{1/2}\mathbf{u}(t)||^2 &+ 2\beta\int_0^t e^{-2\alpha(t-s)}\|A^{1/2}\mathbf{u}(s)\|^2\,ds \le e^{-2\alpha t}\Big[||\mathbf{u}_0||^2+\kappa||A^{1/2}\mathbf{u}_0||^2\Big] \\ &+ \frac{1}{2\alpha\nu}||\mathbf{f}||^2_{\mathrm{L}^\infty(\mathbf{V}^*)}(1-e^{-2\alpha t}) = \mathbf{K_0(t)}, \end{aligned}$$

where $||\mathbf{v}||_{\mathbf{V}^*} := ||A^{-1/2}\mathbf{v}||$, and $\beta=\Big(\frac{\nu}{2}-\alpha(\kappa+\frac{1}{\lambda_1}\Big)\ge 0$.

Proof. we now estimate the right had side of (2.6) as:

$$(\hat{\mathbf{f}},\hat{\mathbf{u}}) \le ||\hat{\mathbf{f}}||_{\mathbf{V}^*}\,||\hat{\mathbf{u}}||_{\mathbf{V}} \le \frac{1}{2\nu}||\hat{\mathbf{f}}||^2_{\mathbf{V}^*}+\frac{\nu}{2}||A^{1/2}\hat{\mathbf{u}}||^2. \tag{2.13}$$

Now substitute (2.13) in (2.6). Then use kickback **argument** and integrate with respect to time to complete the proof. ■

Using the above *a priori* bounds and applying Bubnov-Galerkin procedure, it is possible to prove the global existence of a unique solution $\mathbf{u}$ of (1.4)–(1.6) in $(0,\infty)$. We note that Oskolkov [14] has proved existence of a unique solution to (1.4)–(1.6) for finite time. However, the result in [14] can not be extended to prove global existence for $t\in(0,\infty)$ as the constant in the *a priori* bound tends to ∞ as $t\longrightarrow\infty$. Subsequently, Karazeeva *et al.* [10] have proved uniform bound, see (2.4) in page 97 under the condition that $\mathbf{f}\in L^1(0,\infty;\mathbf{L}^2(\Omega))$ not for the case $\mathbf{f}\in L^\infty(0,\infty;\mathbf{L}^2(\Omega))$ as claimed in Theorem 2.1.

Note that taking **limit superior** as time tends to infinity, we obtain:

$$\limsup_{t\longrightarrow\infty}\Big(||\mathbf{u}(t)||^2+\kappa||A^{1/2}\mathbf{u}(t)||^2\Big) \le \frac{1}{2\alpha\nu}||\mathbf{f}||^2_{\mathrm{L}^\infty(\mathbf{V}^*)}. \tag{2.14}$$

Moreover, we would like to obtain a slightly different estimate corresponding to (2.14), **which will be used** in Sections 3 and 4. Now again after substituting (2.13) in (2.6), we rewrite it as

$$\frac{d}{dt}\Big(||\hat{\mathbf{u}}||^2 + \kappa||A^{1/2}\hat{\mathbf{u}}||^2\Big)+\nu||A^{1/2}\hat{\mathbf{u}}||^2 \le \alpha\Big(||\hat{\mathbf{u}}||^2+\kappa||A^{1/2}\hat{\mathbf{u}}||^2\Big)+\frac{1}{\nu}||\hat{\mathbf{f}}||^2_{\mathbf{V}^*}. \tag{2.15}$$

Integrate (2.15) with respect to time from 0 to t, then **multiply by** $e^{-2\alpha t}$ to obtain

$$\begin{aligned} \Big(||\mathbf{u}||^2 &+ \kappa||A^{1/2}\mathbf{u}||^2\Big)+\nu e^{-2\alpha t}\int_0^t e^{2\alpha s}||A^{1/2}\mathbf{u}||^2\,ds \le e^{-2\alpha t}\Big(||\mathbf{u}_0||^2+\kappa||A^{1/2}\mathbf{u}_0||^2\Big) \\ &+ 2\alpha e^{-2\alpha t}\int_0^t e^{2\alpha s}\Big(||\mathbf{u}||^2+\kappa||A^{1/2}\mathbf{u}||^2\Big)+\frac{1}{\nu}e^{-2\alpha t}\int_0^t e^{2\alpha s}||\mathbf{f}||^2_{\mathbf{V}^*}\,ds. \end{aligned} \tag{2.16}$$

Now taking **limit superior** in (2.16) as $t\longrightarrow\infty$, we arrive at

$$\frac{\nu}{2\alpha}\limsup_{t\longrightarrow\infty}\|A^{1/2}\mathbf{u}(t)\|^2 \le \frac{1}{2\alpha\nu}||\mathbf{f}||^2_{\mathrm{L}^\infty(\mathbf{V}^*)},$$

and hence,

$$\limsup_{t\longrightarrow\infty}\|A^{1/2}\mathbf{u}(t)\| \le \frac{1}{\nu}||\mathbf{f}||_{\mathrm{L}^\infty(\mathbf{V}^*)}. \tag{2.17}$$

As a consequence of (2.14), we obtain below the following result.

Corollary 2.1 *There exists a bounded absorbing set*

$$\mathcal{B}_R : \{\mathbf{v} \in V : (\|\mathbf{v}\|^2 + \kappa \|A^{1/2}\mathbf{v}\|^2)^{1/2} \leq R\},$$

for the problem (2.2)-(2.3), that is, there exists $R > 0$ such that for any $\mathbf{u}_0 \in \mathbf{V}$, there is $t^* := t^*((\|\mathbf{u}_0\|^2 + \kappa\|A^{1/2}\mathbf{u}_0\|^2)^{1/2})$ such that for all $t \geq t^*$, the solution $\mathbf{u}(t)$ of the problem (2.2)-(2.3) satisfies $\mathbf{u}(t) \in \mathcal{B}_R$.

In fact R can be chosen as

$$R^2 := \frac{1}{\alpha\nu} ||\mathbf{f}||^2_{\mathrm{L}^\infty(\mathbf{V}^*)},$$

where α may be chosen as $\frac{\lambda_1 \nu}{2(1+\kappa\lambda_1)}$. Note that following the analysis of [10] or [9], we can show existence of a global attractor say $\mathcal{A} \subset \mathbf{V}$.

Below, we discuss *a priori* estimate for $\kappa\|A\mathbf{u}\|$ and $\|A^{1/2}\mathbf{u}\|$.

Lemma 2.3 *Assume, in addition to the hypotheses in Lemma 2.1, that $\mathbf{u}_0 \in \mathcal{D}(A)$. Then, there exists a positive constant C which may depend on the constant in Sobolev inequality and Sobolev imbedding such that the solution $\mathbf{u}$ satisfies the following* a priori *bound*

$$\begin{aligned} \|A^{1/2}\mathbf{u}(t)\|^2 &+ \kappa\|A\mathbf{u}(t)\|^2 + \beta \int_0^t e^{-2\alpha(t-s)} \|A\mathbf{u}(s)\|^2\, ds \leq e^{-2\alpha t}\Big[\|A^{1/2}\mathbf{u}_0\|^2 + \kappa\|A\mathbf{u}_0\|^2\Big] \\ &+ \Big(C\frac{K_{0,\infty}^3}{2\alpha\kappa^3} + \frac{1}{\alpha\nu}\|\mathbf{f}\|^2_{\mathrm{L}^\infty(\mathbf{L}^2)}\Big)(1 - e^{-2\alpha t}) =: K_1(t) \leq K_{1,\infty}, \end{aligned}$$

where $K_{0,\infty}$ and β are as in Lemma 2.1 and $K_{1,\infty} = \sup_{t\in[0,\infty)} K_1(t)$.

Proof. Form an inner-product between (2.5) and $A\hat{\mathbf{u}}$ to obtain

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\Big(\|A^{1/2}\hat{\mathbf{u}}\|^2 + \kappa\|A\hat{\mathbf{u}}\|^2\Big) &- \alpha\|A^{1/2}\hat{\mathbf{u}}\|^2 - \kappa\alpha\|A\hat{\mathbf{u}}\|^2 + \nu\|A\hat{\mathbf{u}}\|^2 \\ &= (\hat{\mathbf{f}}, A\hat{\mathbf{u}}) + (e^{-\alpha t}B(\hat{\mathbf{u}}, \hat{\mathbf{u}}), A\hat{\mathbf{u}}). \end{aligned} \tag{2.18}$$

For the nonlinear term on the right-hand side of (2.18), an application of Hölder's inequality with Ladyzhenskaya inequality

$$\|A^{1/2}\boldsymbol{\phi}\|_{\mathbf{L}^3} \leq C_S\|A^{1/2}\boldsymbol{\phi}\|^{1/2}\|A\boldsymbol{\phi}\|^{1/2}$$

yields

$$|(B(\hat{\mathbf{u}}, \hat{\mathbf{u}}), A\hat{\mathbf{u}})| \leq \|\hat{\mathbf{u}}\|_{\mathrm{L}^6}\|A^{1/2}\hat{\mathbf{u}}\|_{\mathrm{L}^3}\|A\hat{\mathbf{u}}\| \leq C(C_S, C_I)\|A^{1/2}\hat{\mathbf{u}}\|^{3/2}\|A\hat{\mathbf{u}}\|^{3/2},$$

where C_I is a positive constant in the Sobolev imbedding theorem. Thus, using Young's inequality, we obtain

$$\underline{|(e^{-\alpha t}(B\hat{\mathbf{u}}, \hat{\mathbf{u}}), A\hat{\mathbf{u}})|} \leq C(C_S, C_I)e^{-4\alpha t}\|A^{1/2}\hat{\mathbf{u}}\|^6 + \frac{\varepsilon}{2}\|A\hat{\mathbf{u}}\|^2.$$

Note that

$$|(\hat{\mathbf{f}}, A\hat{\mathbf{u}})| \leq \frac{1}{2\nu}\|\hat{\mathbf{f}}\|^2 + \frac{\nu}{2}\|A\hat{\mathbf{u}}\|^2.$$

Altogether, we now arrive at

$$\frac{1}{2}\frac{d}{dt}(\|A^{1/2}\hat{\mathbf{u}}\|^2 + \kappa\|A\hat{\mathbf{u}}\|^2) + \beta\|A\hat{\mathbf{u}}\|^2 \leq \frac{1}{2\nu}\|\hat{\mathbf{f}}\|^2 + Ce^{-4\alpha t}\|A^{1/2}\hat{\mathbf{u}}\|^6. \tag{2.19}$$

With $0 < \alpha < \frac{\lambda_1 \nu}{2(\lambda_1 \kappa + 1)}$, the coefficient β of the second term on the left-hand side of (2.19) becomes nonnegative and hence, we obtain

$$\frac{d}{dt}(\|A^{1/2}\hat{\mathbf{u}}\|^2 + \kappa\|A\hat{\mathbf{u}}\|^2) + \beta\|A\hat{\mathbf{u}}\|^2 \leq \frac{2}{\nu}\|\mathbf{f}\|^2 + Ce^{-4\alpha t}\|A^{1/2}\hat{\mathbf{u}}\|^6. \tag{2.20}$$

We integrate (2.20) with respect to time from 0 to t and use Lemma 2.1 to arrive at

$$\begin{aligned}\|A^{1/2}\mathbf{u}(t)\|^2 + \kappa\|A\mathbf{u}(t)\|^2 &+ \beta\int_0^t e^{-2\alpha(t-s)}\|A\mathbf{u}(s)\|^2\,ds \leq e^{-2\alpha t}\Big[\|A^{1/2}\mathbf{u}_0\|^2 + \kappa\|A\mathbf{u}_0\|^2\Big] \\ &+ Ce^{-2\alpha t}\int_0^t e^{2\alpha s}\|A^{1/2}\mathbf{u}\|^6 ds + \frac{2}{\nu}e^{-2\alpha t}\int_0^t e^{2\alpha s}\|\mathbf{f}\|^2 ds \\ &\leq e^{-2\alpha t}\Big[\|A^{1/2}\mathbf{u}_0\|^2 + \kappa\|A\mathbf{u}_0\|^2\Big] \\ &+ \Big(C\frac{K_{0,\infty}^3}{\alpha\kappa^3} + \frac{1}{\alpha\nu}\|\mathbf{f}\|_{L^\infty(\mathbf{L}^2)}\Big)(1 - e^{-2\alpha t}).\end{aligned}$$

This completes the rest of the proof. ∎

Remark 2.3 *If $\mathbf{f} = 0$ or if $\|e^{\alpha_0 t}\mathbf{f}(t)\|_{L^2(\Omega)} \leq C$, for some $\alpha_0 > 0$ and for all $t > 0$, we have exponential decay property for $\|\mathbf{u}\|, \|A^{1/2}\mathbf{u}\|$ and $\|A\mathbf{u}\|$, that is, for $t > 0$ and $0 < \alpha \leq \min(\frac{\lambda_1\nu}{2(1+\kappa\lambda_1)}, \alpha_0)$*

$$\|\mathbf{u}(t)\|,\ \ \|A^{1/2}\mathbf{u}(t)\|,\ \|A\mathbf{u}(t)\| = O(e^{-\alpha t}). \tag{2.21}$$

3. Semidiscrete Spectral Galerkin Method. In this section, we discuss a spectral Galerkin method and derive some *a priori* error estimates which are useful for our next section.

Let $\{\boldsymbol{\phi}_i\}_{i=1}^\infty$ be an orthogonal basis of $\mathbf{V}$ consisting of eigenvectors of A where $\{\lambda_i\}_{i=1}^\infty$ is the set of the corresponding eigenvalues. For $N \in \mathbf{N}$ consider $\mathbf{V}_N$ =Span$\{\boldsymbol{\phi}_1, \boldsymbol{\phi}_2, \ldots, \boldsymbol{\phi}_N\} \subset \mathbf{V}$ and let $\mathbf{P}_N$ be the orthogonal projection of $\mathbf{V}$ onto $\mathbf{V}_N$. Then the following estimates hold true for $\mathbf{P}_N$. For a proof, see Rautmann [17], [Lemmas 2.1, 2.3, 2.4].

Lemma 3.1 *For $\mathbf{v} \in \mathbf{V}$, we have*

$$\|\mathbf{v} - \mathbf{P}_N\mathbf{v}\|^2 \leq \frac{1}{\lambda_{N+1}}\|A^{1/2}\mathbf{v}\|^2.$$

Moreover, if $\mathbf{v} \in \mathbf{H}^2 \bigcap \mathbf{V}$, then

$$\|\mathbf{v} - \mathbf{P}_N\mathbf{v}\|^2 \leq \frac{1}{\lambda_{N+1}}\|A^{1/2}(\mathbf{v} - \mathbf{P}_N\mathbf{v})\|^2 \leq \frac{1}{\lambda_{N+1}^2}\|A\mathbf{v}\|^2.$$

Now, the standard Spectral Galerkin Method consists of determining an approximation $\mathbf{u}_{SG}(t) \in \mathbf{V}_N$ such that

$$\frac{d}{dt}\mathbf{u}_{SG} + \kappa A\frac{d}{dt}\mathbf{u}_{SG} + \nu A\mathbf{u}_{SG} + \mathbf{P}_N B(\mathbf{u}_{SG}, \mathbf{u}_{SG}) = \mathbf{P}_N\mathbf{f} \tag{3.1}$$

$$\mathbf{u}_{SG}(0) = \mathbf{P}_N\mathbf{u}_0. \tag{3.2}$$

Note that (3.1)-(3.2) can be written equivalently as

$$\begin{aligned}(\frac{d}{dt}\mathbf{u}_{SG}, \mathbf{v}) + \kappa(A^{1/2}\frac{d}{dt}\mathbf{u}_{SG}, A^{1/2}\mathbf{v}) &+ \nu(A^{1/2}\mathbf{u}_{SG}, A^{1/2}\mathbf{v}) + (B(\mathbf{u}_{SG}, \mathbf{u}_{SG}), \mathbf{v}) \\ &= (\mathbf{f}, \mathbf{v})\ \ \forall \mathbf{v} \in \mathbf{V}_N, \qquad (3.3) \\ (\mathbf{u}_{SG}(0), \mathbf{v}) &= (\mathbf{P}_N\mathbf{u}_0, \mathbf{v})\ \ \forall \mathbf{v} \in \mathbf{V}_N. \qquad (3.4)\end{aligned}$$

Lemma 3.2 *For a given $\mathbf{u}_{SG}(0) \in \mathbf{V}_N$, exists a unique solution $\mathbf{u}_{SG}(t) \in \mathbf{V}_N$, for all $t \geq 0$ satisfying (3.3)-(3.4).*

Proof. Since $\mathbf{V}_N$ is finite dimensional, (3.3)-(3.4) leads to a system of nonlinear ordinary differential equations with given initial condition. Using Picard's Theorem, there exists a unique solution $\mathbf{u}_{SG}(t)$ in $t \in (0, t_N)$ for some $0 < t_N$. In order to continue for all time, we need to establish an *a priori* bound for $\mathbf{u}_{SG}(t)$. Choose $\mathbf{v} = \mathbf{u}_{SG}$ in (3.3) and then proceed exactly as in the proof of Lemma 2.1 to obtain

$$\begin{aligned} \|\mathbf{u}_{SG}(t)\|^2 + \kappa\|A^{1/2}\mathbf{u}_{SG}(t)\|^2 &\leq e^{-2\alpha t}\Big[\|\mathbf{u}_{SG}(0)\|^2 + \kappa\|A^{1/2}\mathbf{u}_{SG}(0)\|^2\Big] \\ &+ \frac{1}{2\alpha\lambda_1\nu}\|\mathbf{f}\|^2_{\mathrm{L}^\infty(\mathrm{L}^2)}(1 - e^{-2\alpha t}). \end{aligned} \tag{3.5}$$

Since $\mathbf{u}_{SG}(0) = \mathbf{P}_N\mathbf{u}(0)$ then , $\|\mathbf{u}_{SG}(0)\|$ and $\|A^{1/2}\mathbf{u}_{SG}(0)\|$ are bounded by $\|A^{1/2}\mathbf{u}(0)\|$. Thus, $\mathbf{u}_{SG}(t)$ is bounded for all $t > 0$ and this completes the existence of a unique solution for all $t > 0$. This completes existence of a unique global solution to the problem (3.3)-(3.4). ∎

As a consequence of (3.5), there exists a bounded absorbing set

$$\mathcal{B}_{R_1} : \{\mathbf{u}_{SG} \in \mathbf{V}_N : (\|\mathbf{u}_{SG}\|^2 + \kappa\|A^{1/2}\mathbf{u}_{SG}\|^2)^{1/2} \leq R_1\},$$

for the problem (3.3)-(3.4), that is, there exists $R_1 > 0$ such that for any $\mathbf{u}_{SG}(0) \in \mathbf{V}_N$, there is $t^* := t^*((\|\mathbf{u}_{SG}(0)\|^2 + \kappa\|A^{1/2}\mathbf{u}_{SG}(0)\|^2)^{1/2})$ such that for all $t \geq t^{**}$, the solution $\mathbf{u}_{SG}(t)$ of the problem (3.3)-(3.4) satisfies $\mathbf{u}_{SG}(t) \in \mathcal{B}_{R_1}$. Now, set R_1 so that

$$R_1^2 := \frac{1}{\alpha\nu}||\mathbf{f}||^2_{\mathrm{L}^\infty(\mathbf{V}^*)},$$

where α may be chosen as $\frac{\lambda_1\nu}{2(1+\kappa\lambda_1)}$.

Therefore, we obtain easily the following result.

Lemma 3.3 *There exists a global attractor $\mathcal{A}_N \subset \mathbf{V}_N$ to the discrete problem (3.3)-(3.4), which attracts bounded sets in $\mathbf{V}_N$.*

We shall not pursue issues related to the dimension of global attractor (see, [9]), and convergence of discrete attractors etc. here as this is not the main purpose of this article.

For our future use, we discuss below the boundedness of $A\mathbf{u}_{SG}$. Note that following the proof technique of Lemma 2.2, we can easily obtain:

$$\frac{1}{2}\frac{d}{dt}(\|A^{1/2}\hat{\mathbf{u}}_{SG}\|^2 + \kappa\|A\hat{\mathbf{u}}_{SG}\|^2) + \beta\|A\hat{\mathbf{u}}_{SG}\|^2 \leq \frac{1}{2\nu}\|\mathbf{P}_N\mathbf{f}\|^2 + Ce^{-4\alpha t}\|A^{1/2}\hat{\mathbf{u}}_{SG}\|^6.$$

Then integrating with respect to t from 0 to t, and using (3.5), we obtain

$$\begin{aligned} \|A^{1/2}\mathbf{u}_{SG}(t)\|^2 &+ \kappa\|A\mathbf{u}_{SG}(t)\|^2 + \beta\int_0^t e^{-2\alpha(t-s)}\|A\mathbf{u}_{SG}(s)\|^2\,ds \\ &\leq e^{-2\alpha t}\Big[\|A^{1/2}\mathbf{P}_N\mathbf{u}_0\|^2 + \kappa\|A\mathbf{P}_N\mathbf{u}_0\|^2\Big] + \Big(C\frac{K_{0,\infty}^3}{\alpha\kappa^3} + \frac{1}{\alpha\nu}\|\mathbf{P}_N\mathbf{f}\|^2_{\mathrm{L}^\infty(\mathbf{L}^2)}\Big)(1 - e^{-2\alpha t}) \\ &\leq C\Big[\|A^{1/2}\mathbf{u}_0\|^2 + \kappa\|A\mathbf{u}_0\|^2\Big] + \Big(C\frac{K_{0,\infty}^3}{\alpha\kappa^3} + \frac{1}{\alpha\nu}\|\mathbf{f}\|^2_{\mathrm{L}^\infty(\mathbf{L}^2)}\Big). \end{aligned} \tag{3.6}$$

This completes the boundedness of $\|A\mathbf{u}_{SG}\|$.

3.1 Errors Estimates.

In this subsection, we discuss the error analysis of the semidiscrete spectral Galerkin approximation $\mathbf{u}_{SG}$.

Theorem 3.1 *Let* $\mathbf{u}$ *be the solution of (2.2) and* $\mathbf{u}_{SG}$ *be the semi-discrete Galerkin approximation of* $\mathbf{u}$ *satisfying (3.1). Then, there is a positive constant* C *depending on* $\|A\mathbf{u}(0)\|$ *and* $\|\mathbf{f}\|_{\mathrm{L}^\infty(\mathrm{L}^2)}$ *such that for* $0 < \alpha < \frac{\lambda_1\nu}{2(\lambda_1\kappa+1)}$ *and for fixed* $T > 0$ *with* $t \in (0,T)$ *the following estimate holds*

$$\|\mathbf{u}(t) - \mathbf{u}_{SG}(t)\| \leq \frac{C}{\lambda_{N+1}} e^{Ct}. \tag{3.7}$$

Moreover, under the uniqueness assumption:

$$\frac{M}{\nu^2}\|\mathbf{f}\|_{\mathrm{L}^\infty(\mathbf{V}^*)} < 1, \tag{3.8}$$

where M *is given as in (2.4),the following uniform in time estimate holds for* $t > 0$ *:*

$$\|\mathbf{u}(t) - \mathbf{u}_{SG}(t)\| \leq \frac{C}{\lambda_{N+1}}. \tag{3.9}$$

Proof. Using the definition of $\mathbf{P}_N$, we write

$$\begin{aligned} \mathbf{u} - \mathbf{u}_{SG} &= (\mathbf{u} - \mathbf{P}_N\mathbf{u}) + (\mathbf{P}_N\mathbf{u} - \mathbf{u}_{SG}) \\ &= (\mathbf{u} - \mathbf{P}_N\mathbf{u}) + E_{SG}, \end{aligned}$$

where $\mathbf{E}_{SG} = \mathbf{P}_N\mathbf{u} - \mathbf{u}_{SG}$. Since the estimates of $\mathbf{u} - \mathbf{P}_N\mathbf{u}$ are known from Lemma 3.1, it is enough to estimate $\|\mathbf{E}_{SG}\|$. From (2.2) and (3.1), we obtain the following expression for the difference $\mathbf{E}_{SG}$

$$(\frac{d}{dt}\mathbf{E}_{SG}, \mathbf{v}) + \kappa(A^{1/2}\frac{d}{dt}\mathbf{E}_{SG}, A^{1/2}\mathbf{v}) + \nu(A^{1/2}\mathbf{E}_{SG}, A^{1/2}\mathbf{v}) = \Lambda(\mathbf{v}), \tag{3.10}$$

where $\Lambda(\mathbf{v}) = -\Big(B(\mathbf{u},\mathbf{u}) - B(\mathbf{u}_{SG},\mathbf{u}_{SG}), \mathbf{v}\Big)$.

Choose $\mathbf{v} = e^{2\alpha t}\mathbf{E}_{SG}(t)$ in the (3.10) and set $\hat{\mathbf{E}}_{SG}(t) = e^{\alpha t}\mathbf{E}_{SG}(t)$ to obtain

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\Big(\|\hat{\mathbf{E}}_{SG}\|^2 + \kappa\|A^{1/2}\hat{\mathbf{E}}_{SG}\|^2\Big) - \alpha\|\hat{\mathbf{E}}_{SG}\|^2 - \kappa\alpha\|A^{1/2}\hat{\mathbf{E}}_{SG}\|^2 &+ \nu\|A^{1/2}\hat{\mathbf{E}}_{SG}\|^2 \\ &= \Lambda(e^{\alpha t}\hat{\mathbf{E}}_{SG}). \end{aligned} \tag{3.11}$$

Note that

$$\begin{aligned} \Lambda(e^{\alpha t}\hat{\mathbf{E}}_{SG}) &= -e^{-\alpha t}\Big[(B((\hat{\mathbf{u}} - \mathbf{P}_N\hat{\mathbf{u}}), \hat{\mathbf{u}}), \hat{\mathbf{E}}_{SG}) + (B(\hat{\mathbf{E}}_{SG}, \hat{\mathbf{u}}), \hat{\mathbf{E}}_{SG}) \\ &+ (B(\hat{\mathbf{u}}_{SG}, \hat{\mathbf{u}} - \mathbf{P}_N\hat{\mathbf{u}}), \hat{\mathbf{E}}_{SG}) + (B(\hat{\mathbf{u}}_{SG}, \hat{\mathbf{E}}_{SG}), \hat{\mathbf{E}}_{SG})\Big]. \end{aligned} \tag{3.12}$$

Here, the last term on the right-hand side of (3.12) is zero. To estimate the first term on the right-hand side of (3.12), we use Hölders inequality and Sobolev inequality to obtain

$$\begin{aligned} |(B(\hat{\mathbf{u}} - \mathbf{P}_N\hat{\mathbf{u}}, \hat{\mathbf{u}}), \hat{E}_{SG})| &\leq C\|\hat{\mathbf{u}} - \mathbf{P}_N\hat{\mathbf{u}}\|\|A^{1/2}\hat{\mathbf{u}}\|_{\mathrm{L}^3}\|\hat{\mathbf{E}}_{SG}\|_{\mathrm{L}^6} \\ &\leq C\|\hat{\mathbf{u}} - \mathbf{P}_N\hat{\mathbf{u}}\|\|A\hat{\mathbf{u}}\|\ \|A^{1/2}\hat{\mathbf{E}}_{SG}\|, \end{aligned}$$

and hence, using the Young's inequality, we arrive at

$$e^{-\alpha t}|(B(\hat{\mathbf{u}}-\mathbf{P}_N\hat{\mathbf{u}},\hat{\mathbf{u}}),\hat{\mathbf{E}}_{SG})| \quad\leq\quad C(\varepsilon)e^{-2\alpha t}\|\hat{\mathbf{u}}-\mathbf{P}_N\hat{\mathbf{u}}\|^2\|A\hat{\mathbf{u}}\|^2+\frac{\varepsilon}{2}\|A^{1/2}\hat{\mathbf{E}}_{SG}\|^2.$$

Similarly for the second term on the right-hand side of (3.12), we find that

$$e^{-\alpha t}|(B(\hat{\mathbf{E}}_{SG},\hat{\mathbf{u}}),\hat{\mathbf{E}}_{SG})|\leq Ce^{-2\alpha t}\|\hat{\mathbf{E}}_{SG}\|^2\|A\hat{\mathbf{u}}\|^2+\frac{\varepsilon}{2}\|A^{1/2}\hat{\mathbf{E}}_{SG}\|^2.$$

For the third term on the right-hand side of (3.12), we note that

$$(B(\hat{\mathbf{u}}_{SG},\hat{\mathbf{u}}-\mathbf{P}_N\hat{\mathbf{u}}),\hat{\mathbf{E}}_{SG})=(B(\hat{\mathbf{u}}_{SG},\hat{\mathbf{E}}_{SG}),\hat{\mathbf{u}}-\mathbf{P}_N\hat{\mathbf{u}}),$$

and hence, using Sobolev inequality with Hölders inequality, we obtain

$$\begin{aligned}|e^{-\alpha t}(B(\hat{\mathbf{u}}_{SG},\hat{\mathbf{u}}-\mathbf{P}_N\hat{\mathbf{u}}),\hat{\mathbf{E}}_{SG})| &\leq Ce^{-\alpha t}\|A^{1/2}\hat{\mathbf{u}}_{SG}\|^{1/2}\|A\hat{\mathbf{u}}_{SG}\|^{1/2}\|A^{1/2}\hat{\mathbf{E}}_{SG}\|\|\hat{\mathbf{u}}-\mathbf{P}_N\hat{\mathbf{u}}\|\\ &\leq C(\varepsilon)e^{-2\alpha t}\|A^{1/2}\hat{\mathbf{u}}_{SG}\|\|A\hat{\mathbf{u}}_{SG}\|\|\hat{\mathbf{u}}-\mathbf{P}_N\hat{\mathbf{u}}\|^2+\frac{\varepsilon}{2}\|A^{1/2}\hat{\mathbf{E}}_{SG}\|^2.\end{aligned}$$

Thus,

$$\begin{aligned}|\Lambda(e^{\alpha t}\hat{\mathbf{E}}_{SG})| &\leq \frac{3\varepsilon}{2}\|A^{1/2}\hat{\mathbf{E}}_{SG}\|^2+Ce^{-2\alpha t}\|\hat{\mathbf{E}}_{SG}\|^2\|A^{1/2}\hat{\mathbf{u}}\|^2\\ &+ Ce^{-2\alpha t}(\|A\hat{\mathbf{u}}\|^2+\|A^{1/2}\hat{\mathbf{u}}_{SG}\|\,\|A\hat{\mathbf{u}}_{SG}\|)\|\hat{\mathbf{u}}-\mathbf{P}_N\hat{\mathbf{u}}\|^2.\end{aligned}$$

On substituting in (3.11), we arrive at

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}(\|\hat{\mathbf{E}}_{SG}\|^2+\kappa\|A^{1/2}\hat{\mathbf{E}}_{SG}\|^2) &+ (-\kappa\alpha+\nu-\frac{3\varepsilon}{2}-\frac{\alpha}{\lambda_1})\|A^{1/2}\hat{\mathbf{E}}_{SG}\|^2\\ &\leq C(\varepsilon)e^{-2\alpha t}\|A^{1/2}\hat{\mathbf{u}}\|^2\|\hat{\mathbf{E}}_{SG}\|^2\\ &+ C(\|A\mathbf{u}\|^2+\|A^{1/2}\mathbf{u}_{SG}\|\,\|A\mathbf{u}_{SG}\|)\|\hat{\mathbf{u}}-\mathbf{P}_N\hat{\mathbf{u}}\|^2. \quad (3.13)\end{aligned}$$

Choose $\varepsilon=\nu/3$ in (3.13) and then use *a priori* bounds from Lemmas 2.1-2.3 to obtain

$$\begin{aligned}\frac{d}{dt}\Big(\|\hat{\mathbf{E}}_{SG}\|^2 &+ \kappa\|A^{1/2}\hat{\mathbf{E}}_{SG}\|^2\Big)+(\nu-2\alpha(\frac{1}{\lambda_1}+\kappa))\,\|A^{1/2}\hat{\mathbf{E}}_{SG}\|^2\\ &\leq C\Big(\|A\mathbf{u}\|^2+\|A^{1/2}\mathbf{u}_{SG}\|\,\|A\mathbf{u}_{SG}\|\Big)\|\hat{\mathbf{u}}-\mathbf{P}_N\hat{\mathbf{u}}\|^2+C\|A^{1/2}\mathbf{u}\|^2\|\hat{\mathbf{E}}_{SG}\|^2\\ &\leq \frac{C(K_{0,\infty},K_{1,\infty},\kappa^{-1})}{\lambda_{N+1}^2}\|A\hat{\mathbf{u}}\|^2+C\|A^{1/2}\mathbf{u}\|^2\|\hat{\mathbf{E}}_{SG}\|^2. \quad (3.14)\end{aligned}$$

Integrating (3.14) with respect to time and choosing $0<\alpha\leq\frac{\lambda_1\nu}{2(1+\kappa\lambda_1)}$ we find that

$$\|\mathbf{E}_{SG}\|^2+\kappa\|A^{1/2}\mathbf{E}_{SG}\|^2 \quad\leq\quad C(K_{0,\infty},K_{1,\infty},\kappa^{-1})\frac{1}{\lambda_{N+1}^2}+C\int_0^t\|A^{1/2}\mathbf{u}\|^2\|\mathbf{E}_{SG}\|^2\,ds$$

again

$$\|\mathbf{E}_{SG}\|^2+\kappa\|A^{1/2}\mathbf{E}_{SG}\|^2 \quad\leq\quad C(K_{0,\infty},K_{1,\infty},\kappa^{-1})\frac{1}{\lambda_{N+1}^2}+C\int_0^t\|A^{1/2}\mathbf{u}\|^2(\|\mathbf{E}_{SG}\|^2+\kappa\|A^{1/2}\mathbf{E}_{SG}\|^2)\,ds$$

as $\mathbf{E}_{SG}(0)=0$.

Using Gronwall's lemma, it now follows that

$$\|\mathbf{E}_{SG}(t)\|^2 \leq \frac{C(K_{0,\infty}, K_{1,\infty}, \kappa^{-1})}{\lambda_{N+1}^2} \exp(C\int_0^t \|A^{1/2}\mathbf{u}\|^2 ds)$$

A use of triangle inequality with Lemma 2.2 and Lemma 3.1 completes the proof of the estimate (3.7).

To derive (3.9), we note from the property of $\mathbf{P}_N$ and Lemma 2.2 that

$$\|\mathbf{u}(t) - \mathbf{P}_N\mathbf{u}(t)\| \leq \frac{C}{\lambda_{N+1}}\|A\mathbf{u}(t)\| \leq \frac{C}{\lambda_{N+1}}, \tag{3.15}$$

which is valid for all $t > 0$. Therefore, it is now enough to obtain an uniform in time estimate for $\mathbf{E}_{SG}$. We now estimate the nonlinear term (3.12) as folows:

$$\begin{aligned}
\Lambda(e^{\alpha t}\hat{\mathbf{E}}_{SG}) &= -e^{-\alpha t}\Big[(B((\hat{\mathbf{u}} - \mathbf{P}_N\hat{\mathbf{u}}), \hat{\mathbf{u}}), \hat{\mathbf{E}}_{SG}) + (B(\hat{\mathbf{E}}_{SG}, \hat{\mathbf{u}}), \hat{\mathbf{E}}_{SG}) + (B(\hat{\mathbf{u}}_{SG}, \hat{\mathbf{u}} - \mathbf{P}_N\hat{\mathbf{u}}), \hat{\mathbf{E}}_{SG})\Big] \\
&\leq Ce^{-\alpha t}\|\hat{\mathbf{u}} - \mathbf{P}_N\hat{\mathbf{u}}\|\Big(\|A\hat{\mathbf{u}}\| + \|A^{1/2}\hat{\mathbf{u}}_{SG}\|^{1/2}\|A\hat{\mathbf{u}}_{SG}\|^{1/2}\Big)\ \|A^{1/2}\hat{\mathbf{E}}_{SG}\| \\
&+ Me^{-\alpha t}\|A^{1/2}\hat{\mathbf{u}}\|\ \|A^{1/2}\hat{\mathbf{E}}_{SG}\|^2.
\end{aligned}$$

Note that for the third term on the right hand side of the nonlinear term (3.12), we have used the property (2.4). Now using the property of $\mathbf{P}_N$, Lemmas 2.1-2.2 ans Lemma 3.1, we arrive at

$$\Lambda(e^{\alpha t}\hat{\mathbf{E}}_{SG}) \leq C\frac{1}{\lambda_{N+1}}\|A\hat{\mathbf{u}}\|\ \|A^{1/2}\hat{\mathbf{E}}_{SG}\| + M\|A^{1/2}\mathbf{u}\|\ \|A^{1/2}\hat{\mathbf{E}}_{SG}\|^2. \tag{3.16}$$

On substituting (3.16)in (3.11), we then integrate from 0 to t to obtain

$$\begin{aligned}
\Big(\|\mathbf{E}_{SG}\|^2 &+ \kappa\|A^{1/2}\mathbf{E}_{SG}\|^2\Big) + 2e^{-2\alpha t}\int_0^t e^{2\alpha s}(\nu - M\|A^{1/2}\mathbf{u}\|)\,\|A^{1/2}\mathbf{E}_{SG}\|^2\, ds \\
&\leq 2\alpha e^{-2\alpha t}\int_0^t e^{2\alpha s}\Big(\|\mathbf{E}_{SG}\|^2 + \kappa\|A^{1/2}\mathbf{E}_{SG}\|^2\Big)\, ds \\
&+ C\frac{1}{\lambda_{N+1}}e^{-2\alpha t}\int_0^t e^{2\alpha s}\|A\mathbf{u}\|\ \|A^{1/2}\mathbf{E}_{SG}\|\, ds.
\end{aligned} \tag{3.17}$$

Taking **limit superior** as $t \longrightarrow \infty$ in (3.17) and using (2.17), we arrive at

$$\frac{1}{\nu}(1 - M\nu^{-2}\|\mathbf{f}\|_{\mathrm{L}^\infty(\mathbf{V}^*)})\ \limsup_{t\longrightarrow\infty}\|A^{1/2}\mathbf{E}_{SG}(t)\|^2 \leq C\frac{1}{\lambda_{N+1}}\limsup_{t\longrightarrow\infty}\|A^{1/2}\mathbf{E}_{SG}(t)\|,$$

and hence, from the uniqueness condition (3.8), we now obtain

$$\limsup_{t\longrightarrow\infty}\|A^{1/2}\mathbf{E}_{SG}(t)\| \leq C\frac{1}{\lambda_{N+1}}.$$

Thus,

$$\limsup_{t\longrightarrow\infty}\|\mathbf{E}_{SG}(t)\| \leq C\frac{1}{\lambda_{N+1}},$$

and

$$\limsup_{t\longrightarrow\infty}\|\mathbf{u}(t)-\mathbf{u}_{SG}(t)\|\leq C\frac{1}{\lambda_{N+1}}.$$

This completes the rest of the proof. ∎

As consequence of the Theorem 3.1, we obtain a super-convergence result for $\|A^{1/2}\mathbf{E}_{SG}(t)\|$ and we now derive easily the following estimate.

Corollary 3.1 *Under hypotheses of Theorem 3.1, there exists a positive constant C such that*

$$\|A^{1/2}(\mathbf{u}-\mathbf{u}_N)(t)\|\leq\frac{C}{\lambda_{N+1}^{1/2}}. \tag{3.18}$$

Remark 3.4 *We now note that the estimate (3.7) in Theorem 3.1 is valid for finite $T>0$, otherwise it blows up as $t\longrightarrow\infty$. When $\mathbf{f}=0$, then the uniqueness condition (3.8) is satisfied and the uniform-in-time estimate (3.9) holds. Moreover, it is easy to see from the proof of the Theorem 3.1 and the a priori bounds in Section 2 that*

$$\|(\mathbf{u}-\mathbf{u}_N)(t)\|\leq\frac{C(K_{0,\infty},K_{1,\infty},\kappa^{-1})}{\lambda_{N+1}}e^{-\alpha t},\;\; t>0.$$

Moreover, $\mathbf{f}\in\mathrm{L}^2(0,\infty;\mathrm{L}^2(\Omega))$, then the result (3.9) of the Theorem 3.1 is valid for all $t>0$ without the uniqueness assumption (3.8).

4. Nonlinear Galerkin Method. Let $\mathbf{V}_n=\operatorname{Span}\{\varphi_1,\cdots,\varphi_n\}$ and consider $\mathbf{P}_n:\mathbf{H}\longrightarrow\mathbf{V}_n$ be the orthogonal projection and $\mathbf{Q}_n=I-\mathbf{P}_n$ be the orthogonal complement. Consider the split $\mathbf{u}=\mathbf{p}+\mathbf{q}$ where $\mathbf{p}=\mathbf{P}_n u\in\mathbf{V}_n$ corresponds to the small eigenvalues, i.e., represents the large eddies of the flow and $\mathbf{q}=\mathbf{Q}_n u\in\mathbf{H}\backslash\mathbf{V}_n$ corresponds to large eigenvalues, represents the small eddies of the flow.

Apply $\mathbf{P}_n$ and $\mathbf{Q}_n$, respectively, to the abstract equations (2.2) to obtain

$$\frac{d\mathbf{p}}{dt}+\kappa A\frac{d\mathbf{p}}{dt}+\nu A\mathbf{p}+\mathbf{P}_nB(\mathbf{p}+\mathbf{q},\mathbf{p}+\mathbf{q})=\mathbf{P}_n\mathbf{f}, \tag{4.1}$$

and

$$\frac{d\mathbf{q}}{dt}+\kappa A\frac{d\mathbf{q}}{dt}+\nu A\mathbf{q}+\mathbf{Q}_nB(\mathbf{p}+\mathbf{q},\mathbf{p}+\mathbf{q})=\mathbf{Q}_n\mathbf{f}. \tag{4.2}$$

From the properties of the orthogonal projections and Lemma 3.1, we note that

$$||\mathbf{q}||=||(I-\mathbf{P}_n)\mathbf{u}||\leq\frac{1}{\lambda_{n+1}}||A\mathbf{u}||\leq\frac{C(K_{0,\infty},K_{1,\infty})}{\lambda_{n+1}},\;\; t\geq 0$$

and hence, $\mathbf{q}(t)$ carries a small part of the kinetic energy. Thus, it is reasonable to approximate the term $\mathbf{Q}_nB(\mathbf{p}+\mathbf{q},\mathbf{p}+\mathbf{q})$ as

$$\mathbf{Q}_nB(\mathbf{p}+\mathbf{q},\mathbf{p}+\mathbf{q})\approx\mathbf{Q}_n(B(\mathbf{p},\mathbf{p})+B(\mathbf{p},\mathbf{q})+B(\mathbf{q},\mathbf{p})). \tag{4.3}$$

The entire process leads to the following modified Nonlinear Galerkin Method: Find $(\mathbf{y},\mathbf{z})$ such that

$$\frac{d\mathbf{y}}{dt}+\kappa A\frac{d\mathbf{y}}{dt}+\nu A\mathbf{y}+\mathbf{P}_nB(\mathbf{y}+\mathbf{z},\mathbf{y}+\mathbf{z})=\mathbf{P}_n\mathbf{f}, \tag{4.4}$$

and

$$\frac{d\mathbf{z}}{dt} + \kappa A \frac{d\mathbf{z}}{dt} + \nu A\mathbf{z} + \mathbf{Q}_n[B(\mathbf{y}+\mathbf{z},\mathbf{y}) + B(\mathbf{y},\mathbf{z})] = \mathbf{Q}_n\mathbf{f}. \tag{4.5}$$

Since $\mathbf{z}(t) \in \mathbf{H} \setminus \mathbf{V}_n$, the equation (4.5) represents an infinite dimensional system. Thus, we have to approximate $\mathbf{z}$.

Consider $\mathbf{V}_N = \text{span}\{\varphi_1, \varphi_2, \dots, \varphi_n, \varphi_{n+1}, \dots, \varphi_N\}$. Let the finite dimensional approximation again called it as $\mathbf{z}(t) \in \mathbf{V}_N \setminus \mathbf{V}_n$ and $\mathbf{y}(t) \in \mathbf{V}_n$ satisfy

$$\frac{d\mathbf{y}}{dt} + \kappa A \frac{d\mathbf{y}}{dt} + \nu A\mathbf{y} + \mathbf{P}_n B(\mathbf{y}+\mathbf{z},\mathbf{y}+\mathbf{z}) = \mathbf{P}_n\mathbf{f}, \tag{4.6}$$

and

$$\frac{d\mathbf{z}}{dt} + \kappa A \frac{d\mathbf{z}}{dt} + \nu A\mathbf{z} + \mathbf{Q}_n^N\Big[B(\mathbf{y}+\mathbf{z},\mathbf{y}) + B(\mathbf{y},\mathbf{z})\Big] = \mathbf{Q}_n^N\mathbf{f} \tag{4.7}$$

with $\mathbf{y}(0) = \mathbf{P}_n\mathbf{u}_0$ and $\mathbf{z}(0) = \mathbf{Q}_n^N\mathbf{u}_0$, where $N > n$, and $\mathbf{Q}_n^N = \mathbf{P}_N - \mathbf{P}_n = \mathbf{Q}_n - \mathbf{Q}_N$. Note that for simplicity of notation, we still represent $\mathbf{y}(t)$ as the approximation of large eddies and $\mathbf{z}(t)$ as the approximate of small eddies. Thus, $\mathbf{u}_{MG}(t) = \mathbf{y}(t) + \mathbf{z}(t)$ is an approximation of $\mathbf{u}(t)$.

Following the arguments leading to the *a priori* bounds in (3.5) and (3.6), we easily obtain the following estimates for the discrete solution $\mathbf{u}_{MG}$ of (4.6)-(4.7): There exists a positive constant C such that

$$\begin{aligned} \|\mathbf{u}_{MG}(t)\|^2 + \kappa\|A^{1/2}\mathbf{u}_{MG}(t)\|^2 \;&\le\; C\Big[\|\mathbf{u}(0)\|^2 + \kappa\|A^{1/2}\mathbf{u}(0)\|^2\Big] \\ &+\; \frac{1}{2\alpha\lambda_1\nu}\|\mathbf{f}\|_{\mathrm{L}^\infty(\mathrm{L}^2)}, \end{aligned} \tag{4.8}$$

and

$$\begin{aligned} \|A^{1/2}\mathbf{u}_{MG}(t)\|^2 \;&+\; \kappa\|A\mathbf{u}_{MG}(t)\|^2 + \beta\int_0^t e^{-2\alpha(t-s)}\|A\mathbf{u}_{MG}(s)\|^2\,ds \\ &\le\; C\Big[\|A^{1/2}\mathbf{u}_0\|^2 + \kappa\|A\mathbf{u}_0\|^2\Big] + \Big(C\frac{K_{0,\infty}^3}{\alpha\kappa^3} + \frac{1}{\alpha\nu}\|\mathbf{f}\|_{\mathrm{L}^\infty(\mathbf{L}^2)}\Big). \end{aligned} \tag{4.9}$$

Remark 4.5 *Note that since $\frac{d\mathbf{q}}{dt}$ is also small, it is possible to drop it. However, the term $A\frac{d\mathbf{q}}{dt}$ may not be small, therefore, we prefer to keep both the time derivative terms in the modified nonlinear Galerkin method.*

4.1 *A Priori* Error Estimates .

In this subsection, we discuss *a priori* optimal error estimates for the modified nonlinear Galerkin scheme (4.6)-(4.7).

Using the projection $\mathbf{P}_N$, we now split

$$\begin{aligned} e_{MG}(t) &= \mathbf{u}(t) - \mathbf{u}_{MG}(t) \\ &= \Big(\mathbf{u}(t) - \mathbf{P}_N\mathbf{u}(t)\Big) + \Big(\mathbf{P}_N\mathbf{u}(t) - \mathbf{u}_{MG}(t)\Big) \\ &= \Big(\mathbf{u}(t) - \mathbf{P}_N\mathbf{u}(t)\Big) + \mathbf{E}_{MG}(t). \end{aligned}$$

Using the properties of $\mathbf{P}_n$ and $\mathbf{Q}_n^N$, we may rewrite whenever required in our analysis the term $\mathbf{E}_{MG}$ as $\mathbf{E}_{MG} = \mathbf{P}_n\mathbf{E}_{MG} + \mathbf{Q}_n^N\mathbf{E}_{MG}$. Note that $\mathbf{E}_{MG}(0) = 0$.

Theorem 4.1 *Let* $\mathbf{u}$ *be the solution of (2.2) and* $\mathbf{u}_{MG} = \mathbf{y} + \mathbf{z}$ *be the semi-discrete nonlinear Galerkin approximation of* $\mathbf{u}$ *satisfying (4.6)-(4.7). Then, there is a positive constant* C *depending on* $\|A\mathbf{u}_0\|$ *and* $\|\mathbf{f}\|_{\mathrm{L}^\infty(\mathrm{L}^2)}$ *such that for* $0 < \alpha < \frac{\lambda_1 \nu}{2(\lambda_1 \kappa + 1)}$ *and for fixed* $T > 0$ *with* $t \in (0, T)$ *the following estimate holds*

$$\|\mathbf{u}(t) - \mathbf{u}_{MG}(t)\| \le C\Big(\frac{1}{\lambda_{N+1}} + \frac{1}{\lambda_{n+1}^r}\Big) e^{CT}. \tag{4.10}$$

where $r = 3/2$, *if* $d = 2$ *and* $r = 5/4$, *when* $d = 3$. *Moreover, under the uniqueness assumption (3.8) the following uniform in time estimate holds for* $t > 0$ *:*

$$\|\mathbf{u}(t) - \mathbf{u}_{MG}(t)\| \le C\Big(\frac{1}{\lambda_{N+1}} + \frac{1}{\lambda_{n+1}^r}\Big). \tag{4.11}$$

Proof. Since the estimate of $\mathbf{u}(t) - \mathbf{P}_N \mathbf{u}(t)$ is known from Lemma 3.1, it is enough to derive the estimate of $\mathbf{E}_{MG}$. From (2.2) and (4.6)-(4.7), we easily obtain

$$\Big(\frac{d}{dt}\mathbf{E}_{MG}, \mathbf{v}\Big) + \kappa\Big(A^{1/2}\frac{d}{dt}\mathbf{E}_{MG}, A^{1/2}\mathbf{v}\Big) + \nu\Big(A^{1/2}\mathbf{E}_{MG}, A^{1/2}\mathbf{v}\Big) = \Lambda(\mathbf{v}), \tag{4.12}$$

where

$$\Lambda(\mathbf{v}) = -\Big(\mathbf{P}_N B(\mathbf{u},\mathbf{u}) - \mathbf{Q}_n^N B(y+z, y+z), \mathbf{v}\Big) - \Big(B(y+z, y) + B(y, z), \mathbf{v}\Big).$$

Choose $\mathbf{v} = e^{2\alpha t}\mathbf{E}_{MG} = e^{\alpha t}\hat{\mathbf{E}}_{MG}$ in (4.12) and rewrite the resulting equation as

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\Big(\|\hat{\mathbf{E}}_{MG}\|^2 + \kappa\|A^{1/2}\hat{\mathbf{E}}_{MG}\|^2\Big) - \alpha\Big(\|\hat{\mathbf{E}}_{MG}\|^2 &+ \kappa\|A^{1/2}\hat{\mathbf{E}}_{MG}\|^2\Big) + \nu\|A^{1/2}\hat{\mathbf{E}}_{MG}\|^2 \\ &= e^{\alpha t}\Lambda(\hat{\mathbf{E}}_{MG}), \end{aligned} \tag{4.13}$$

To estimate the term on the right-hand side of (4.13), we first rewrite it as

$$\begin{aligned} e^{\alpha t}\Lambda(\hat{\mathbf{E}}_{MG}) &= -e^{-\alpha t}\Big[\Big(B(\hat{\mathbf{E}}_{MG}, \hat{\mathbf{u}}), \hat{\mathbf{E}}_{MG}\Big) + \Big(B(\hat{\mathbf{u}}_{MG}, (I - \mathbf{P}_N)\hat{\mathbf{u}}), \hat{\mathbf{E}}_{MG}\Big) \\ &+ \Big(B((I-\mathbf{P}_N)\hat{\mathbf{u}}, \hat{\mathbf{u}}), \hat{\mathbf{E}}_{MG}\Big) + \Big(B(\hat{\mathbf{z}}, \hat{\mathbf{z}}), \mathbf{Q}_N^n \hat{\mathbf{E}}_{MG}\Big)\Big]. \end{aligned} \tag{4.14}$$

As in the proof of Theorem 3.1, we now estimate the terms on the right-hand side of (4.14) as follows. Note that using standard Hölders inequality, Sobolev imbedding theorem and Sobolev inequality, we obtain easily

$$e^{-\alpha t}|\Big(B(\hat{\mathbf{E}}_{MG}, \hat{\mathbf{u}}), \hat{\mathbf{E}}_{MG}\Big)| \le C(\varepsilon)e^{-2\alpha t}\|A\hat{\mathbf{u}}\|^2\|\hat{\mathbf{E}}_{MG}\|^2 + \frac{\varepsilon}{2}\|A^{1/2}\hat{\mathbf{E}}_{MG}\|^2. \tag{4.15}$$

Similarly, the second and third terms on the right-hand side of (4.14) can be estimated as

$$\begin{aligned} e^{-\alpha t} & \Big[|\Big(B(\hat{\mathbf{u}}_{MG}, (I-\mathbf{P}_N)\hat{\mathbf{u}}), \hat{\mathbf{E}}_{MG}\Big)| + |\Big(B((I-\mathbf{P}_N)\hat{\mathbf{u}}, \hat{\mathbf{u}}), \hat{\mathbf{E}}_{MG}\Big)|\Big] \\ \le & \; C(\varepsilon)e^{-2\alpha t}\Big(\|A\hat{\mathbf{u}}\|^2 + \|A\hat{\mathbf{u}}_{MG}\|^2\Big)\|(I-\mathbf{P}_N)\hat{\mathbf{u}}\|^2 + \frac{\varepsilon}{2}\|A^{1/2}\hat{\mathbf{E}}_{MG}\|^2 \\ \le & \; \frac{C(\varepsilon)}{\lambda_{N+1}^2} e^{-2\alpha t}\Big(\|A\hat{\mathbf{u}}\|^2 + \|A\hat{\mathbf{u}}_{MG}\|^2\Big) + \frac{\varepsilon}{2}\|A^{1/2}\hat{\mathbf{E}}_{MG}\|^2. \end{aligned} \tag{4.16}$$

For the fourth term on the right hand side of (4.14), we estimates it depending on d. When $d = 2$, we now estimate using generalized Holder's inequality and Ladyzhenskaya inequality

$$\|v\|_{L^4(\Omega)} \leq C\|v\|^{1/2}\,\|A^{1/2}v\|^{1/2}, \ \text{if}\, d = 2,$$

as

$$\begin{aligned} \Big|e^{-\alpha t}\Big(B(\hat{\mathbf{z}},\hat{\mathbf{z}}),\mathbf{Q}_N^n\hat{\mathbf{E}}_{MG}\Big)\Big| &\leq Ce^{-\alpha t}\|\hat{\mathbf{z}}\|_{L^4}^2\,\|A^{1/2}\hat{\mathbf{E}}_{MG}\| \\ &\leq Ce^{-\alpha t}\|\hat{\mathbf{z}}\|\,\|A^{1/2}\hat{bz}\|\,\|A^{1/2}\hat{\mathbf{E}}_{MG}\| \\ &\leq \frac{C(\varepsilon)}{\lambda_{n+1}^3}e^{-2\alpha t}||A\hat{\mathbf{u}}||^4 + \frac{\varepsilon}{2}||A^{1/2}\hat{\mathbf{E}}_{MG}||^2. \end{aligned} \tag{4.17}$$

When $d = 3$, use again generalized Holder's inequality and Ladyzhenskaya inequality

$$\|v\|_{L^3(\Omega)} \leq C\|v\|^{1/2}\,\|A^{1/2}v\|^{1/2}, \ \text{if}\, d = 3,$$

as

$$\begin{aligned} \Big|e^{-\alpha t}\Big(B(\hat{\mathbf{z}},\hat{\mathbf{z}}),\mathbf{Q}_N^n\hat{\mathbf{E}}_{MG}\Big)\Big| &\leq Ce^{-\alpha t}\|\hat{\mathbf{z}}\|_{L^3}\,\|A^{1/2}\hat{\mathbf{z}}\|\,\|\hat{\mathbf{E}}_{MG}\|_{L^6} \\ &\leq Ce^{-\alpha t}\|\hat{\mathbf{z}}\|^{1/2}\,\|A^{1/2}\hat{bz}\|^{3/2}\,\|A^{1/2}\hat{\mathbf{E}}_{MG}\| \\ &\leq \frac{C(\varepsilon)}{\lambda_{n+1}^{5/2}}e^{-2\alpha t}||A\hat{\mathbf{u}}||^4 + \frac{\varepsilon}{2}||A^{1/2}\hat{\mathbf{E}}_{MG}||^2. \end{aligned} \tag{4.18}$$

Substituting (4.14)-(4.18) appropriately in (4.13) and then with $\varepsilon = \nu/3$, we obtain after integration with **respect** to time from 0 to t, and multiplying by $e^{-2\alpha t}$ with use of *a priori* bounds of Lemmas 2.1-2.2, Lemma 4.1 the following:

$$\begin{aligned} \|\mathbf{E}_{MG}(t)\|^2 + \kappa\|A^{1/2}\mathbf{E}_{MG}(t)\|^2 &\leq C(K_{0,\infty},K_{0,\infty})\Big(\frac{1}{\lambda_{N+1}^2} + \frac{1}{\lambda_{n+1}^{2r}}\Big) \\ &+ C\int_0^t \|A\mathbf{u}\|^2\,\|\mathbf{E}_{MG}\|^2\,ds, \end{aligned} \tag{4.19}$$

where $r = 3/2$, when $d = 2$ and $r = 5/4$, if $d = 3$. An application of Gronwall's Lemma with approximation property of $\mathbf{P}_N$ and triangle inequality yields the estimate (4.10).

In order to estimate (4.11), we again follow the proof technique of the second part of Theorem 3.1. Since the estimate of $\|\mathbf{u} - \mathbf{P}_N\mathbf{u}\|$ is known from Lemma 3.1 which is valid for all $t > 0$, it is, therefore, sufficient to obtain an uniform in time estimate for $\mathbf{E}_{MG}$. For nonlinear term (4.14), except for the first term on the right **hand side**, we estimate the remaining terms as in (4.16)-(4.17). For the first term, we estimate as follows:

$$e^{-\alpha t}|\Big(B(\hat{\mathbf{E}}_{MG},\hat{\mathbf{u}}),\hat{\mathbf{E}}_{MG}\Big)| \leq Me^{-2\alpha t}\|A^{1/2}\hat{\mathbf{u}}\|\,\|A^{1/2}\hat{\mathbf{E}}_{MG}\|^2. \tag{4.20}$$

We now estimate the nonlinear term (4.14) as folows:

$$\begin{aligned} \Lambda(e^{\alpha t}\hat{\mathbf{E}}_{MG}) &\leq Ce^{-\alpha t}\Big(\frac{1}{\lambda_{N+1}} + \frac{1}{\lambda_{n+1}^r}\Big)\,\|A\hat{\mathbf{u}}\|\,\|A^{1/2}\hat{\mathbf{E}}_{MG}\| \\ &+ Me^{-\alpha t}\,\|A^{1/2}\hat{\mathbf{u}}\|\,\|A^{1/2}\hat{\mathbf{E}}_{MG}\|^2. \end{aligned} \tag{4.21}$$

On substituting (4.21)in (4.13), we then integrate from 0 to t to obtain

$$\begin{aligned}\Big(\|\mathbf{E}_{MG}\|^2 &+ \kappa\|A^{1/2}\mathbf{E}_{MG}\|^2\Big) + 2e^{-2\alpha t}\int_0^t e^{2\alpha s}(\nu - M\|A^{1/2}\mathbf{u}\|)\,\|A^{1/2}\mathbf{E}_{MG}\|^2\,ds \\ &\leq 2\alpha e^{-2\alpha t}\int_0^t e^{2\alpha s}\Big(\|\mathbf{E}_{MG}\|^2 + \kappa\|A^{1/2}\mathbf{E}_{MG}\|^2\Big)\,ds \\ &+ C\Big(\frac{1}{\lambda_{N+1}} + \frac{1}{\lambda_{n+1}^r}\Big)e^{-2\alpha t}\int_0^t e^{2\alpha s}\|A\mathbf{u}\|\,\|A^{1/2}\mathbf{E}_{MG}\|\,ds. \qquad (4.22)\end{aligned}$$

Taking **limit superior** as $t \longrightarrow \infty$ in (4.22) and using (2.17), we arrive at

$$\frac{1}{\nu}(1 - M\nu^{-2}\|\mathbf{f}\|_{\mathrm{L}^\infty(\mathbf{V}^*)})\limsup_{t\longrightarrow\infty}\|A^{1/2}\mathbf{E}_{MG}(t)\|^2 \leq C\Big(\frac{1}{\lambda_{N+1}} + \frac{1}{\lambda_{n+1}^r}\Big)\limsup_{t\longrightarrow\infty}\|A^{1/2}\mathbf{E}_{MG}(t)\|,$$

and hence, from the uniqueness condition (3.8), we now obtain

$$\limsup_{t\longrightarrow\infty}\|A^{1/2}\mathbf{E}_{MG}(t)\| \leq C\Big(\frac{1}{\lambda_{N+1}} + \frac{1}{\lambda_{n+1}^r}\Big).$$

Thus,

$$\limsup_{t\longrightarrow\infty}\|\mathbf{E}_{MG}(t)\| \leq C\Big(\frac{1}{\lambda_{N+1}} + \frac{1}{\lambda_{n+1}^r}\Big),$$

and

$$\limsup_{t\longrightarrow\infty}\|\mathbf{u}(t) - \mathbf{u}_{MG}(t)\| \leq C\Big(\frac{1}{\lambda_{N+1}} + \frac{1}{\lambda_{n+1}^r}\Big).$$

This completes the rest of the proof. ∎

As consequence of Theorem 4.1, we obtain the following error estimate.

Corollary 4.1 *Under hypothese of Theorem 4.1, there exists a positive constant C such that*

$$\|A^{1/2}(\mathbf{u} - \mathbf{u}_{MG})(t)\| \leq C\Big(\frac{1}{\lambda_{N+1}^{1/2}} + \frac{1}{\lambda_{n+1}^r}\Big), \qquad (4.23)$$

where $r = 3/2$, if $d = 2$, and $r = 5/4$, when $d = 3$.

References

[1] Archilla, B. G., Novo, J. and Titi, E. S., An Approximation Inertial Manifolds Approach to Postprocessing the Galerkin Method for the Navier-Stokes equation, Math. Comput., 68 (1999) , pp. 893-911.

[2] Cannon, J. R., Ewing, R. E., He, Y. and Lin, Y., A modified nonlinear Galerkin method for the viscoelastic fluid motion equations, Intnl. J. Engg. Sc., 37 (1999), pp. 1643-1662.

[3] Chorin, A., Marsden, J., *A Mathematical Introduction to Fluid Mechanics*, 2nd. ed., Springer-Verlag, Berlin, 1990.

[4] He, Y., Hou, Y. and Li, K., Stability and convergence of optimum spectral nonlinear Galerkin methods, Math. Meth. Appl. Sc., 24 (2001), pp. 289-317.

[5] He, Y., Liu, K. M. and Sun, W., Multi-level spectral Galerkin method for the Navier-Stokes problem I : spatial discretization Numer. Math., 101 (2005), pp.501-522.

[6] He, Y., Liu, K. M., Multi-level spectral Galerkin method for the Navier-Stokes equations-II: time discretization, Advances in Computational Mathematics, 25 (2006), pp. 403-433.

[7] Heywood, J. G., An error estimate uniform in time for spectral Galerkin approximations of the Navier-Stokes problem, Pacific J. Math., vol 98 (1982), pp. 333-345.

[8] Joseph, D. D., *Fluid Dynamics of Viscoelastic Liquids*, Springer Verlag, New York, 1990.

[9] Kalantarov, V. K. and Titi, E. S., Global attractors and determining modes for the 3D Navier-Stokes-Voight equations, arXiv:0705.3972v1 [math.AP] 27 May 2007, pp 1-20.

[10] Karazeeva, N.A., Kotsiolis, A.A. and Oskolkov, A. P., On dynamical systems generated by initial boundary value problems for the equations of motion of linear viscoelastic fluids, Proceedings of the SteklovInstitute of Mathematics, vol.3 (1991), pp.73-108.

[11] Kelvin, W. T., On the theory viscoelastic fluids, Math. a. Phis. Pap., vol 3, (1875), 27-84.

[12] Ladyzenskaya, O. A., *The Mathematical Theory of Viscous Incompressible Flow*, Gordon and Breach, New York ,1969.

[13] Novo, J., Titi, E. and Wynne, S., Efficient methods using high acuracy approximate inertial manifolds, Numer. Math., 87, 2001, pp. 523 - 554.

[14] Oskolkov, A. P., Initial boundary value problems for the equations of motion of Kelvin-Voigt fluids and Oldroyd fluids, Proceedings of Steklov Institute of Mathematics, Vol.2 (1989), 137-182.

[15] Oskolkov, A. P., On an estimate uniform in the semi axis $t > 0$ for the rate of convergence of Galerkin approximations for the equations of motion of Kelvin-Voigt fluids, J. Soviet Math. 62 (1992), no. 3, 2802-2806.

[16] Oskolkov, A. P., Nonlocal problems for the equations of motion of the Kelvin-Voigt fluids, J. Math. Sci. 75 (1995), no.6, 2058-2078.

[17] Rautmann, R., On the convergence-rate of nonstationary Navier-Stokes approximations, Proc. IUTAM Symp, Paderborn (W. Germany) 1979, Springer Lecture Notes in Math., 771 (1980).

[18] Temam, R. , Navier-Stokes Equations, Theory and Numerical Analysis, North-Holland , Amsterdam, 1984.

[19] Voigt, W., Über die innere Reibung der festen Körper, insbesondere der Krystalle, Abh. Königl. Ges. Wiss. Göttingen 36, no.1, (1889).