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ON A LINEARIZED BACKWARD EULER METHOD FOR THE EQUATIONS OF MOTION OF OLDROYD FLUIDS OF ORDER ONE*

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Abstract. In this paper, a linearized backward Euler method is discussed for the equations of motion arising in the Oldroyd model of viscoelastic fluids. Some new a priori bounds are obtained for the solution under realistically assumed conditions on the data. Further, the exponential decay properties for the exact as well as the discrete solutions are established. Finally, a priori error estimates in $\mathbf{H}^1$ and $\mathbf{L}^2$-norms are derived for the the discrete problem which are valid uniformly for all time $t > 0$.

Key words. viscoelastic fluids, Oldroyd model, a priori bounds, exponential decay, linearized backward Euler method, uniform convergence in time

AMS subject classifications. 35L70, 65M30, 76D05, 78A10

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1. Introduction.

The motion of an incompressible fluid in a bounded domain Ω in $\mathbb{R}^2$ is described by the system of partial differential equations

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u} - \nabla\cdot\boldsymbol{\sigma} + \nabla p = \mathbf{F}(x,t), \;\; x\in\Omega, \;\; t>0,$$

$$\nabla\cdot\mathbf{u} = 0, \;\; x\in\Omega, \, t>0,$$

with appropriate initial and boundary conditions. Here, $\boldsymbol{\sigma} = (\boldsymbol{\sigma}_{ik})$ denotes the stress tensor with $tr\boldsymbol{\sigma} = 0$, $\mathbf{u}$ represents the velocity vector, p is the pressure of the fluid, and $\mathbf{F}$ is the external force. The defining relation between the stress tensor $\boldsymbol{\sigma}$ and the rate of deformation tensor $\mathbf{D} = (\mathbf{D}_{ik}) = \frac{1}{2}(\mathbf{u}_{ix_k} + \mathbf{u}_{kx_i})$, called the equation of state or sometimes the rheological equation, in fact, establishes the type of fluids under consideration. When $\boldsymbol{\sigma} = 2\nu\mathbf{D}$ (using Newton's law) with ν the kinematic coefficient of viscosity, we obtain Newton's model of incompressible viscous fluid and the corresponding system is widely known as Navier–Stokes equations. This has been a basic model for describing the flow at moderate velocities of the majority of the incompressible viscous fluids encountered in practice. However, there are many fluids with complex microstructure, such as biological fluids, polymeric fluids, suspensions, and liquid crystals, which are used in the current industrial processes and show (nonlinear) viscoelastic behavior that cannot be described by the classical linear viscous Newtonian models. The departure from the Navier–Stokes behavior manifests itself in a variety of ways, such as non-Newtonian viscosity, stress relaxation, and nonlinear creeping. The model of rate type such as Oldroyd fluids (see [4], [23], [32]) can predict the stress relaxation as well as the retardation of deformation and, therefore,

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have become popular for describing polymeric suspension. In order to model the behavior of a dilute polymer solution in a Newtonian solvent, the extra stress tensor is often split into two components: a viscoelastic one and a purely viscous one. So the Oldroyd fluids of order one as it is known in the Russian literature (see [23], [2], [18]) are described by the defining relation

$$\left(1+\lambda\frac{\partial}{\partial t}\right)\boldsymbol{\sigma} = 2\nu\left(1+\kappa\nu^{-1}\frac{\partial}{\partial t}\right)\mathbf{D},$$

where λ, ν, κ are positive constants with $(\nu-\kappa\lambda^{-1}) > 0$. Here, ν denotes the kinematic viscocity, λ is the relaxation time, and κ represents the retardation time. In the form of an integral equation, we write the above defining relation as

$$\begin{aligned}\boldsymbol{\sigma}(x,t) = 2\kappa\lambda^{-1}\mathbf{D}(x,t) + 2\lambda^{-1}(\nu-\kappa\lambda^{-1})\int_0^t \exp(-\lambda^{-1}(t-\tau))\mathbf{D}(x,\tau)\,d\tau \\ +(\boldsymbol{\sigma}(x,0) - 2\kappa\lambda^{-1}\mathbf{D}(x,0))\exp(-\lambda^{-1}t).\end{aligned}$$

Now the equation of motion of the Oldroyd fluids of order one can be described most naturally by the system of integrodifferential equations

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u} - \mu\Delta\mathbf{u} - \int_0^t \beta(t-\tau)\Delta\mathbf{u}(x,\tau)\,d\tau + \nabla p = \mathbf{f}, \quad x\in\Omega,\ t>0, \tag{1.1}$$

and incompressibility condition

$$\nabla\cdot\mathbf{u} = 0, \quad x\in\Omega,\ t>0, \tag{1.2}$$

with initial and boundary conditions

$$\mathbf{u}(x,0) = \mathbf{u}_0, \quad x\in\Omega, \qquad \text{and} \qquad \mathbf{u}(x,t) = 0, \quad x\in\partial\Omega, \quad t\geq 0. \tag{1.3}$$

Here, Ω is a bounded domain in two-dimensional Euclidean space $\mathbb{R}^2$ with smooth boundary $\partial\Omega$, $\mu = \kappa\lambda^{-1} > 0$ and the kernel $\beta(t) = \gamma\exp(-\delta t)$, where $\gamma = \lambda^{-1}(\nu - \kappa\lambda^{-1})$ and $\delta = \lambda^{-1}$. For details of the physical background and its mathematical modeling, see [4], [17], [23], [24], and [32].

Throughout this paper, we shall assume that $\mu = 1$ and the nonhomogeneous term $\mathbf{f} = 0$. In fact, assuming conservative force, the function $\mathbf{f}$ can be absorbed in the pressure term.

As in Temam [28], we recast the above problem (1.1)–(1.3) as an abstract evolution equation in an appropriate function space setting. Let us denote by $H^m(\Omega)$ the standard Hilbert–Sobolev space and by $\|\cdot\|_m$ the norm defined on it. When $m=0$, we call $H^0(\Omega)$ as the space of square integrable functions $L^2(\Omega)$ with the usual norm $\|\cdot\|$ and inner product $(\cdot,\cdot)$. Further, let $H_0^1(\Omega)$ be the completion of $C_0^\infty(\Omega)$ with respect to $H^1(\Omega)$-norm. In fact, the seminorm $\|\nabla\phi\|$ on $H_0^1(\Omega)$ is a norm and is equivalent to H^1-norm. We also use the following function spaces for the vector valued functions.

Define

$$\mathbf{D}(\Omega) := \{\boldsymbol{\phi}\in(C_0^\infty(\Omega))^2 : \quad \nabla\cdot\boldsymbol{\phi} = 0 \ \text{ in } \Omega\},$$

$$\mathbf{H} := \text{the closure of } \mathbf{D}(\Omega) \text{ in } \ (L^2(\Omega))^2 - \text{space},$$

and

$$\mathbf{V} := \text{the closure of } \mathbf{D}(\Omega) \text{ in } \ (H_0^1(\Omega))^2 - \text{space}.$$

Note that under some smoothness assumptions on the boundary $\partial\Omega$, it is possible to characterize $\mathbf{V}$ as

$$\mathbf{V} := \{\boldsymbol{\phi} \in (H_0^1)^2 : \quad \nabla \cdot \boldsymbol{\phi} = 0 \;\text{ in }\; \Omega\}.$$

The spaces of vector functions are indicated by boldface letters, for instance, $\mathbf{H}_0^1 = (H_0^1(\Omega))^2$. The inner product on $\mathbf{H}_0^1$ is denoted by

$$(\nabla\boldsymbol{\phi}, \nabla\mathbf{w}) = \sum_{i=1}^{2} (\nabla\phi_i, \nabla w_i)$$

and the norm by

$$\|\nabla\boldsymbol{\phi}\| = \left(\sum_{i=1}^{2} \|\nabla\phi_i\|^2\right)^{\frac{1}{2}}.$$

Using the Poincaré inequality, it can be shown that the norm on $\mathbf{H}_0^1$ is equivalent to $\mathbf{H}^1 = (H^1(\Omega))^2$- norm. Let $\mathbf{P}$ denote the orthogonal projection of $\mathbf{L}^2(\Omega)$ $(= (L^2(\Omega))^2)$ onto $\mathbf{H}$. Now the orthogonal complement $\mathbf{V}^\perp$ of $\mathbf{V}$ in $\mathbf{L}^2(\Omega)$ consists of functions $\boldsymbol{\phi}$ such that $\boldsymbol{\phi} = \nabla p$ for some $p \in H^1(\Omega)/\mathbb{R}$. We define the Stokes operator $A\mathbf{v} = -\mathbf{P}\Delta\mathbf{v}, \quad \mathbf{v} \in D(A) = \mathbf{H}^2 \bigcap \mathbf{V}$. The Stokes operator is a closed linear self-adjoint and positive operator on $\mathbf{H}$ with densely defined domain $D(A)$ in $\mathbf{H}$. Note that its inverse is compact in $\mathbf{H}$; see [28]. Moreover, we set the sth power of A as A^s for every $s \in \mathbb{R}$. For $0 \le s \le 2$, $D(A^{s/2})$ is a Hilbert space with the inner product $(A^{s/2}\mathbf{v}, A^{s/2}\mathbf{w})$ and norm $\|A^{s/2}\mathbf{v}\| := (A^{s/2}\mathbf{v}, A^{s/2}\mathbf{v})^{1/2}$. For $\mathbf{v} \in D(A^{s/2}), \quad 0 \le s \le 2$, we note that $\|\mathbf{v}\|_s$ and $\|A^{s/2}\mathbf{v}\|$ are equivalent. We also define a bilinear operator $B(\mathbf{u}, \mathbf{v}) = \mathbf{P}((\mathbf{u} \cdot \nabla)\mathbf{v}),\ \mathbf{u}, \mathbf{v} \in \mathbf{V}$.

With the notations described above, we now rewrite the problem (1.1)–(1.3) in its abstract form as follows.

Find $\mathbf{u}(t) \in D(A)$ such that for $t \ge 0$

$$\frac{d\mathbf{u}}{dt}(t) + A\mathbf{u}(t) + B(\mathbf{u}(t), \mathbf{u}(t)) + \int_0^t \beta(t-s) A\mathbf{u}(s)\, ds = 0, \quad t > 0, \qquad \mathbf{u}(0) = \mathbf{u}_0. \tag{1.4}$$

In an Oldroyd fluid, the stresses after instantaneous cessation of the motion decay like $\exp(-\lambda^{-1}t)$, while the velocities of the flow after instantaneous removal of the stresses die out like $\exp(-\kappa^{-1}t)$. Therefore, it is of interest to discuss the exponential decay property of the solution of (1.4), and we derive these results in section 2. For some related studies in the decay of solution of the linear parabolic equations with memory, see [30] and [3].

The main focus of this paper is to discuss the linearized backward Euler method for time discretization of the system of equations (1.4). For the temporal discretization of the above abstract problem (1.4), let k denote the time step and $t_n = nk$. For smooth function ϕ defined on $[0, \infty)$, set $\phi^n = \phi(t_n)$ and $\bar{\partial}_t \phi^n = (\phi^n - \phi^{n-1})/k$. For the integral term, we apply the right rectangle rule as

$$q^n(\phi) = k \sum_{j=1}^{n} \beta_{n-j} \phi^j \approx \int_0^{t_n} \beta(t_n - s)\phi(s)\, ds, \tag{1.5}$$

where $\beta_{n-j} = \beta(t_n - t_j)$.

Now the linearized version of the backward Euler method applied to the problem (1.4) determines a sequence of functions $\{\mathbf{U}^n\}_{n\geq 0} \subset D(A)$ as solutions of

$$\bar{\partial}_t \mathbf{U}^n + A\mathbf{U}^n + B(\mathbf{U}^{n-1}, \mathbf{U}^n) + q^n(A\mathbf{U}) = 0, \;\; n > 0, \qquad \mathbf{U}^0 = \mathbf{u}_0. \tag{1.6}$$

The main objective of this paper is to derive the following result.

THEOREM 1. *Let $\mathbf{u}_0 \in D(A)$ and let $\mathbf{U}^n$ satisfy* (1.6). *Then there is a positive constant C independent of k but that may depend on $\|\mathbf{u}_0\|_2$ and Ω such that for some $k_0 > 0$ with $0 < k < k_0$ and for positive α with $0 < \alpha < \min(\delta, \lambda_1)$*

$$\|\mathbf{u}(t_n) - \mathbf{U}^n\|_1 \leq C(\|\mathbf{u}_0\|_2) e^{-\alpha t_n} k \left(t_n^{-1/2} + \log \frac{1}{k} \right),$$

where λ_1 is the least eigenvalue of the Stokes operator A.

Once Theorem 1 is proved, the proof of the following theorem becomes routine work. However, we shall indicate only the major steps without proving it in detail in the end of section 3.

THEOREM 2. *Under the assumptions of Theorem* 1, *there is a positive constant C independent of k but that may depend on $\|\mathbf{u}_0\|_2$ and Ω such that for some $k_0 > 0$ with $0 < k < k_0$ and $0 < \alpha < \min(\delta, \lambda_1)$*

$$\|\mathbf{u}(t_n) - \mathbf{U}^n\| \leq C(\|\mathbf{u}_0\|_2) e^{-\alpha t_n} k.$$

Based on the analysis of Ladyzenskaya [20] for the solvability of the Navier–Stokes equations, Oskolkov [24] proved the global existence of unique “almost” classical solutions in finite time interval for the initial and boundary value problem (1.1)–(1.3). The invesigations on solvability were further continued by the coworkers of Oskolkov [19] and Agranovich and Sobolevskii [1] under various sufficient conditions. In these articles, the regularity results are proved which are, in principle, based on some nonlocal compatibility conditions for the data at $t = 0$. Note that these compatibility conditions are either hard to verify or difficult to meet in practice. In case of Navier–Stokes equations, we refer to Heywood and Rannacher [14] for a similar kind of nonlocal conditions. In the present article, we have obtain some new a priori bounds for the solutions of (1.4) under realistically assumed conditions on the initial data. Recently, Sobolevskii [27] discussed the long-time behavior of solution under some stabilizing conditions on the nonhomogeneous forcing function using a combination of energy arguments and semigroup theoretic approach. When the forcing function is zero, we have derived, in sections 2 and 3, the exponential decay properties for the exact solution as well as for the discrete solution using only energy arguments.

For earlier works on the numerical approximations to the solutions of the problem (1.1)–(1.3), see [2] and [5]. While Akhmatov and Oskolkov [2] applied a finite difference scheme to the equation of motion arising in the Oldroyd model, Cannon et al. [5] analyzed a modified nonlinear Galerkin scheme for a periodic problem using spectral Galerkin procedure and discussed the rates of convergence for the semidiscrete approximations. Recently, Pani and Yuan [26] and He et al. [12] applied finite element methods to discretize the spatial variables and derived optimal error estimates for the problems (1.1)–(1.3) without using nonlocal compatibility conditions. In all these pappers [5], [26], [12], only semidiscrete approximations are discussed keeping the time variable continuous. In this article, we have proposed and analyzed a time discretization scheme based on linearized modification of the backward Euler method. Note that the results on higher order time discretization can easily be proved under the assumption that the exact solutions are sufficiently smooth when t is near 0. These regularity results as we have mentioned earlier entail nonlocal compatibility conditions for the initial data which cannot be verified in practice. Recently, in the context of Oldroyd B fluid, which is a generalization of Oldroyd fluid of order one, a second order Crank–Nicolson scheme [8] is used for the temporal discetization in conjuction with the finite element methods for spatial discretization under regularity requirements on the solutions which cannot be realistically assumed. Therefore, an attempt has been made in this paper to discuss the error estimates for the linearized modified backward Euler scheme (1.6) applied to (1.4) under realistically assumed conditions on the initial data. Finally, in section 4, we conclude with a summary and possible extensions.

The approach of the present article is influenced by the earlier results of Fujita and Mizutani [10], Thomée [29], and references therein on the approximation of semigroups for the parabolic problems; Okamoto [22] on the spatial discretization and Geveci [11] on the time discretization of the Navier–Stokes equations; and Thomée and Zhang [31] for the time discretization of the linear parabolic integrodifferential equations with nonsmooth initial data.

2. Some a priori estimates. For our future use, we make use of the positive definite property (see [21], for a definition) of the kernel β of the integral operator in (1.1). This can be seen as a consequence of the following lemma. For a proof, see Sobolevskii [27, p. 1601] and McLean and Thomeé [21].

LEMMA 3. *For arbitrary $\alpha > 0$, $t^* > 0$, and $\phi \in L^2(0, t^*)$, the following positive definite property holds:*

$$\int_0^{t^*} \left(\int_0^t \exp\left[-\alpha(t-s)\right]\phi(s)\, ds \right) \phi(t)\, dt \ge 0.$$

Since $\beta(t) = \gamma e^{-\delta t}$ with $\gamma > 0$, therefore, the above result is true for $\beta(t)$.

Below, we discuss some a priori bounds for the solution $\mathbf{u}$ of (1.4).

LEMMA 4. *Let $0 < \alpha < \min(\delta, \lambda_1)$ and $\mathbf{u}_0 \in \mathbf{L}^2(\Omega)$. Then, the following estimate holds:*

$$\|\mathbf{u}(t)\| \le e^{-\alpha t}\|\mathbf{u}_0\|, \ t > 0.$$

Moreover,

$$2\left(1 - \frac{\alpha}{\lambda_1}\right) \int_0^t e^{2\alpha\tau}\|A^{1/2}\mathbf{u}(\tau)\|^2 d\tau \le \|\mathbf{u}_0\|^2.$$

Proof. Setting $\hat{\mathbf{u}}(t) = e^{\alpha t}\mathbf{u}(t)$ for some $\alpha > 0$, we rewrite (1.4) as

$$\frac{d}{dt}\hat{\mathbf{u}} - \alpha\hat{\mathbf{u}} + e^{-\alpha t}B(\hat{\mathbf{u}}, \hat{\mathbf{u}}) + A\hat{\mathbf{u}} + \int_0^t \beta(t-\tau)e^{\alpha(t-\tau)}A\hat{\mathbf{u}}(\tau)\, d\tau = 0. \tag{2.1}$$

Form L^2-inner product between (2.1) and $\hat{\mathbf{u}}$. Note that $(B(\hat{\mathbf{u}}, \hat{\mathbf{u}}), \hat{\mathbf{u}}) = 0$, $(A\mathbf{u}, \mathbf{v}) = (A^{\frac{1}{2}}\mathbf{u}, A^{1/2}\mathbf{v})$, and $\|\hat{\mathbf{u}}\|^2 \le \lambda_1^{-1}\|A^{1/2}\hat{\mathbf{u}}\|^2$, where λ_1 is the least eigenvalue of the Stokes operator A. Then

$$\begin{aligned} \frac{d}{dt}\|\hat{\mathbf{u}}\|^2 &+ 2\left(1 - \frac{\alpha}{\lambda_1}\right)\|A^{1/2}\hat{\mathbf{u}}\|^2 \\ &+ 2\int_0^t \beta(t-\tau)e^{\alpha(t-\tau)}(A^{1/2}\hat{\mathbf{u}}(\tau), A^{1/2}\hat{\mathbf{u}}(\tau))d\tau \le 0. \end{aligned} \tag{2.2}$$

After integrating (2.2) with respect to time, the third term becomes nonnegative, since $\delta > \alpha$, and the second term on the left-hand side of (2.2) is also nonnegative if $\alpha < \lambda_1$. With $0 < \alpha < \min(\delta, \lambda_1)$, we find that

$$\|\hat{\mathbf{u}}\| \le \|\mathbf{u}_0\|.$$

Moreover,

$$2\left(1 - \frac{\alpha}{\lambda_1}\right)\int_0^t e^{2\alpha\tau}\|A^{1/2}\mathbf{u}(\tau)\|^2 d\tau \le \|\mathbf{u}_0\|^2.$$

This completes the rest of the proof. ∎

LEMMA 5. *Under the hypothesis of Lemma* 4, *the solution* $\mathbf{u}$ *of* (1.4) *satisfies*

$$\|A^{1/2}\mathbf{u}(t)\|^2 + e^{-2\alpha t}\int_0^t e^{2\alpha\tau}\|A\mathbf{u}(\tau)\|^2\, d\tau \le C(\|A^{1/2}u_0\|)e^{-2\alpha t}.$$

Proof. Forming L^2-inner product between (2.1) and $A\hat{\mathbf{u}}$, we obtain

$$(\hat{\mathbf{u}}_t, A\hat{\mathbf{u}}) + \|A\hat{\mathbf{u}}\|^2 + \int_0^t \beta(t-\tau)e^{\alpha(t-\tau)}(A\hat{\mathbf{u}}(\tau), A\hat{\mathbf{u}})\, d\tau = \alpha(\hat{\mathbf{u}}, A\hat{\mathbf{u}}) - e^{-\alpha t}(B(\hat{\mathbf{u}}, \hat{\mathbf{u}}), A\hat{\mathbf{u}}). \tag{2.3}$$

Note that

$$(\hat{\mathbf{u}}_t, A\hat{\mathbf{u}}) = \frac{1}{2}\frac{d}{dt}\|A^{1/2}\hat{\mathbf{u}}\|^2.$$

On integration of (2.3) with respect to time and using Lemma 3 along with the definition of β, it follows for $0 < \alpha \le \delta$ that

$$\begin{aligned}\|A^{1/2}\hat{\mathbf{u}}(t)\|^2 + 2\int_0^t \|A\hat{\mathbf{u}}(\tau)\|^2\, d\tau &\le \|A^{1/2}\mathbf{u}_0\|^2 + 2\alpha\int_0^t (\hat{\mathbf{u}}, A\hat{\mathbf{u}})\, d\tau \\ &\quad - 2\int_0^t e^{-\alpha\tau}(B(\hat{\mathbf{u}}, \hat{\mathbf{u}}), A\hat{\mathbf{u}})\, d\tau \\ &= \|A^{1/2}\mathbf{u}_0\|^2 + I_1 + I_2.\end{aligned} \tag{2.4}$$

To estimate $|I_1|$, we apply the Poincaré inequality and Cauchy–Schwarz inequality with $ab \le \frac{1}{2\epsilon}a^2 + \frac{\epsilon}{2}b^2$, $a, b \ge 0$, $\epsilon > 0$. Then the use of Lemma 4 yields

$$\begin{aligned}|I_1| &\le C(\alpha, \lambda_1, \epsilon)\int_0^t \|A^{1/2}\hat{\mathbf{u}}(\tau)\|^2\, d\tau + \epsilon\int_0^t \|A\hat{\mathbf{u}}(\tau)\|^2\, d\tau \\ &\le C(\alpha, \lambda_1, \epsilon)\|\mathbf{u}_0\|^2 + \epsilon\int_0^t \|A\hat{\mathbf{u}}(\tau)\|^2\, d\tau.\end{aligned} \tag{2.5}$$

For the estimation of I_2, we apply Hölder's inequality repeatedly with the form of the Sobolev inequality (see Temam [28])

$$\|\boldsymbol{\phi}\|_{L^4(\Omega)} \le C\|\boldsymbol{\phi}\|^{\frac{1}{2}}\, \|A^{1/2}\boldsymbol{\phi}\|^{\frac{1}{2}}, \quad \boldsymbol{\phi} \in \mathbf{H}^1(\Omega),$$

to obtain

$$\begin{aligned}|(B(\hat{\mathbf{u}}, \hat{\mathbf{u}}), A\hat{\mathbf{u}})| &\le \|B(\hat{\mathbf{u}}, \hat{\mathbf{u}})\|\|A\hat{\mathbf{u}}\| \\ &\le C\|\hat{\mathbf{u}}\|^{\frac{1}{2}}\|A^{1/2}\hat{\mathbf{u}}\|\|A\hat{\mathbf{u}}\|^{\frac{3}{2}}.\end{aligned}$$

Thus,

$$|I_2| \le C \int_0^t e^{-\alpha\tau} \|\hat{\mathbf{u}}\|^{\frac{1}{2}} \|A^{1/2}\hat{\mathbf{u}}\| \, \|A\hat{\mathbf{u}}\|^{\frac{3}{2}} \, d\tau.$$

An application of Young's inequality $ab \le \frac{a^p}{\epsilon^{p/q}} + \frac{\epsilon b^q}{q}$, $a, b \ge 0$, $\epsilon > 0$, and $\frac{1}{p} + \frac{1}{q} = 1$ with $p = 4$ and $q = \frac{4}{3}$ yields

$$|I_2| \le C(\epsilon) \int_0^t e^{-4\alpha\tau} \|\hat{\mathbf{u}}\|^2 \|A^{1/2}\hat{\mathbf{u}}\|^4 \, d\tau + \epsilon \int_0^t \|A\hat{\mathbf{u}}\|^2 \, d\tau. \tag{2.6}$$

Substituting (2.5)–(2.6) in (2.4), and using $\epsilon = \frac{1}{2}$, we find that

$$\|A^{1/2}\hat{\mathbf{u}}(t)\|^2 + \int_0^t \|A\hat{\mathbf{u}}(\tau)\|^2 \, d\tau \le C(\alpha, \lambda_1, \|A^{1/2}\mathbf{u}_0\|) + C \int_0^t e^{-4\alpha\tau} \|\hat{\mathbf{u}}\|^2 \|A^{1/2}\hat{\mathbf{u}}\|^4 \, d\tau.$$

An application of Gronwall's lemma yields

$$\|A^{1/2}\hat{\mathbf{u}}(t)\|^2 + \int_0^t \|A\hat{\mathbf{u}}(\tau)\|^2 \, d\tau \le C(\alpha, \lambda_1, \|A^{1/2}\mathbf{u}_0\|) \exp\left\{ C \int_0^t e^{-4\alpha\tau} \|\hat{\mathbf{u}}\|^2 \|A^{1/2}\hat{\mathbf{u}}\|^2 \, d\tau \right\}.$$

Using the a priori bounds in Lemma 4 for $0 < \alpha < \min(\delta, \lambda_1)$, we obtain the desired result. This completes the proof. $\square$

Remark 1. Based on the Faedo–Galerkin method and the a priori bounds derived in the above two lemmas, it is possible to prove the existence of global strong solutions to the problem (1.1)–(1.3). For a similar analysis in the case of Navier–Stokes equations, see Heywood [13], Temam [28], and Ladyzenskaya [20]. Since the analysis is quite standard, we state without proof the global existence theorem [25].

THEOREM 6. *Assume that* $\mathbf{u}_0 \in D(A)$. *Then for any given time* $T > 0$ *with* $0 < T \le \infty$, *there exists a unique strong solution* $\mathbf{u}$ *of* (1.4) *satisfying*

$$\mathbf{u} \in L^2(0, T; D(A)) \cap L^\infty(0, T; \mathbf{V}) \cap H^1(0, T; \mathbf{H}),$$

and the initial condition in the sense that

$$\|A^{1/2}(\mathbf{u}(t) - \mathbf{u}_0)\| \longrightarrow 0, \quad \text{as } t \longrightarrow 0.$$

Recently, Cannon et al. [5] proved existence of a global weak solution $\mathbf{u}$ satisfying

$$\mathbf{u} \in L^\infty(0, T; \mathbf{H}) \cap L^2(0, T; \mathbf{V}), \quad T > 0,$$

for a periodic problem, under the assumption that the forcing function $\mathbf{f} \in L^\infty(0, \infty; \mathbf{L}^2)$ and $\mathbf{u}_0 \in \mathbf{H}$. It is easy to extend our analysis to (1.1)–(1.3) with periodic boundary conditions and $\mathbf{f} = 0$.

Below, we derive some new regularity results without nonlocal assumptions on the data.

LEMMA 7. *Under the assumptions of Lemma* 4, *there is a positive constant* C *such that*

$$\|A\mathbf{u}(t)\| + \|\mathbf{u}_t\| \le C(\|A\mathbf{u}_0\|) e^{-\alpha t}, \quad t > 0, \tag{2.7}$$

and

$$\left(\int_0^t e^{2\alpha s} \|A^{1/2}\mathbf{u}_t(s)\|^2 \, ds \right)^{1/2} \le C(\|A\mathbf{u}_0\|). \tag{2.8}$$

Further, the following estimate holds:

$$\|A^{1/2}\mathbf{u}_t(t)\| + \left(\sigma(t)\int_0^t \sigma(s)\|A\mathbf{u}_t(s)\|^2\,ds\right)^{1/2} \le \frac{C(\|A\mathbf{u}_0\|)}{(\tau^*(t))^{1/2}}e^{-\alpha t}, \qquad t>0, \tag{2.9}$$

where $\sigma(t)=\tau^*(t)e^{2\alpha t}$ *and* $\tau^*(t)=\min(t,1)$.

Proof. From (2.1), we obtain

$$e^{\alpha t}\|\mathbf{u}_t\| \le \|A\hat{\mathbf{u}}\| + e^{-\alpha t}\|B(\hat{\mathbf{u}},\hat{\mathbf{u}})\| + \int_0^t \beta(t-s)e^{\alpha(t-s)}\|A\hat{\mathbf{u}}(s)\|\,ds. \tag{2.10}$$

Using the form of B and the Sobolev inequality, it follows that

$$\begin{aligned}\|B(\hat{\mathbf{u}},\hat{\mathbf{u}})\| &\le C\|\hat{\mathbf{u}}\|^{\frac12}\|A^{1/2}\hat{\mathbf{u}}\|\|A\hat{\mathbf{u}}\|^{\frac12}\\ &\le C\|\hat{\mathbf{u}}\|\|A^{1/2}\hat{\mathbf{u}}\|^2 + C\|A\hat{\mathbf{u}}\|.\end{aligned} \tag{2.11}$$

On squaring (2.10) and integrating with respect to time, we find from (2.11) that

$$\begin{aligned}\int_0^t e^{2\alpha s}\|\mathbf{u}_t\|^2\,ds \le C\Bigg[&\int_0^t \|A\hat{\mathbf{u}}\|^2\,ds + \int_0^t e^{-2\alpha s}\|\hat{\mathbf{u}}\|^2\|A^{1/2}\hat{\mathbf{u}}\|^4\,ds\\ &+\int_0^t \Big(\int_0^s \beta(s-\tau)e^{\alpha(s-\tau)}\|A\hat{\mathbf{u}}(\tau)\|\,d\tau\Big)^2\,ds\Bigg].\end{aligned} \tag{2.12}$$

For the last term on the right-hand side of (2.12), use the form of β and Hölder's inequality to obtain

$$\begin{aligned}I &= \int_0^t \left(\int_0^s \beta(s-\tau)e^{\alpha(s-\tau)}\|A\hat{\mathbf{u}}(\tau)\|\,d\tau\right)^2 ds\\ &= \gamma^2\int_0^t \left(\int_0^s e^{-(\delta-\alpha)(s-\tau)}\|A\hat{\mathbf{u}}(\tau)\|\,d\tau\right)^2 ds\\ &\le \gamma^2\int_0^t \left(\int_0^s e^{-(\delta-\alpha)(s-\tau)}\,d\tau\right)\left(\int_0^s e^{-(\delta-\alpha)(s-\tau)}\|A\hat{\mathbf{u}}(\tau)\|^2\,d\tau\right) ds\\ &\le \frac{\gamma^2}{\delta-\alpha}\int_0^t \left(\int_0^s e^{-(\delta-\alpha)(s-\tau)}\|A\hat{\mathbf{u}}\|^2\,d\tau\right) ds.\end{aligned}$$

Using a change of variable, we find that

$$I \le \frac{\gamma^2}{\delta-\alpha}\int_0^t \left(\int_0^s e^{-(\delta-\alpha)\tau}\|A\hat{\mathbf{u}}(s-\tau)\|^2\,d\tau\right) ds.$$

Now a change in the order of integration yields

$$\begin{aligned}I &\le \frac{\gamma^2}{\delta-\alpha}\int_0^t e^{-(\delta-\alpha)\tau}\left(\int_\tau^t \|A\hat{\mathbf{u}}(s-\tau)\|^2\,ds\right) d\tau\\ &\le \frac{\gamma^2}{(\delta-\alpha)^2}\int_0^t e^{-(\delta-\alpha)(t-\tau)}\left(\int_0^t \|A\hat{\mathbf{u}}\|^2\,ds\right) d\tau,\end{aligned}$$

and hence,

$$I \le \left(\frac{\gamma}{\delta-\alpha}\right)^2\int_0^t \|A\hat{\mathbf{u}}(s)\|^2\,ds. \tag{2.13}$$

Using (2.13) in (2.12), we arrive at

$$\begin{aligned}(2.14)\qquad \int_0^t e^{2\alpha s}\|\mathbf{u}_t\|^2\, ds &\le C\left[\int_0^t \|A\hat{\mathbf{u}}\|^2\, ds + \int_0^t e^{-2\alpha s}\|\hat{\mathbf{u}}\|^2\|A^{1/2}\hat{\mathbf{u}}\|^4\, ds\right]\\ &\le C(\|A^{1/2}\mathbf{u}_0\|).\end{aligned}$$

Differentiate (1.4) with respect to time, and integrate by parts with respect to the temporal variable for the integral term to obtain

$$(2.15)\ \mathbf{u}_{tt} + A\mathbf{u}_t + \int_0^t \beta(t-s)A\mathbf{u}_s(s)\, ds = -(B(\mathbf{u}_t,\mathbf{u}) + B(\mathbf{u},\mathbf{u}_t)) - \beta(t)A\mathbf{u}_0.$$

Forming an inner product between (2.15) and $e^{2\alpha t}\mathbf{u}_t$, we arrive at

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|e^{\alpha t}\mathbf{u}_t\|^2 + e^{2\alpha t}\|A^{1/2}\mathbf{u}_t\|^2 + \int_0^t \beta(t-s)e^{\alpha(t-s)}(A^{1/2}e^{\alpha s}\mathbf{u}_s, A^{1/2}e^{\alpha t}\mathbf{u}_t)\, ds\\ (2.16)\qquad &= \alpha\|e^{\alpha t}\mathbf{u}_t\|^2 - e^{2\alpha t}\left((B(\mathbf{u}_t,\mathbf{u}) + B(\mathbf{u},\mathbf{u}_t),\mathbf{u}_t) - \beta(t)(A\mathbf{u}_0,\mathbf{u}_t)\right).\end{aligned}$$

Note that $(B(\hat{\mathbf{u}}, e^{\alpha t}\mathbf{u}_t), e^{\alpha t}\mathbf{u}_t) = 0$. Thus, it follows after integration of (2.16) with respect to time and using the positivity property of the kernel, i.e., Lemma 3 that

$$\begin{aligned}&e^{2\alpha t}\|\mathbf{u}_t\|^2 + 2\int_0^t e^{2\alpha s}\|A^{1/2}\mathbf{u}_t\|^2\, ds \le \|\mathbf{u}_t(0)\|^2 + 2\alpha\int_0^t e^{2\alpha s}\|\mathbf{u}_t\|^2\, ds\\ (2.17)\qquad &+ 2\int_0^t e^{-\alpha s}|B(e^{\alpha s}\hat{\mathbf{u}}_t, \hat{\mathbf{u}}), e^{\alpha s}\mathbf{u}_t)|\, ds + 2\gamma\|A\mathbf{u}_0\|\int_0^t e^{-(\delta-\alpha)s}\|e^{\alpha s}\mathbf{u}_t\|\, ds.\end{aligned}$$

The last term on the right-hand side of (2.17) is bounded by

$$(2.18)\qquad \le C(\alpha,\delta,\gamma)\left[\|A\mathbf{u}_0\|^2 + \int_0^t e^{2\alpha s}\|\mathbf{u}_t\|^2\, ds\right].$$

For the second term on the right-hand side of (2.17), we note with the help of Sobolev inequality that

$$\begin{aligned}&2\int_0^t e^{-\alpha s}|(B(e^{\alpha s}\hat{\mathbf{u}}_t, \hat{\mathbf{u}}), e^{\alpha s}\mathbf{u}_t)|\, ds \le C\sup_{0\le s\le t}\|A^{1/2}\hat{\mathbf{u}}(s)\|^4\int_0^t e^{-4\alpha s}(e^{2\alpha s}\|\mathbf{u}_t\|^2)\, ds\\ (2.19)\qquad &\qquad + \int_0^t e^{2\alpha s}\|A^{1/2}\mathbf{u}_t\|^2\, ds.\end{aligned}$$

On substitution of (2.18)–(2.19) in (2.17) and using Lemmas 4 and 5, we obtain

$$\begin{aligned}(2.20)\qquad &e^{2\alpha t}\|\mathbf{u}_t\|^2 + \int_0^t e^{2\alpha s}\|A^{1/2}\mathbf{u}_t\|^2\, ds\\ &\le C(\delta,\alpha)\left[\|\mathbf{u}_t(0)\|^2 + \|A\mathbf{u}_0\|^2 + \int_0^t e^{2\alpha s}\|\mathbf{u}_t\|^2\, ds\right].\end{aligned}$$

From the main equation (1.4), we have at $t = 0$, $\|\mathbf{u}_t(0)\| \le C(\|A\mathbf{u}_0\|)$, and hence, using (2.14) we find that

$$(2.21)\qquad \|\mathbf{u}_t\|^2 + e^{-2\alpha t}\int_0^t e^{2\alpha s}\|A^{1/2}\mathbf{u}_t(s)\|^2\, ds \le C(\|A\mathbf{u}_0\|)e^{-2\alpha t}.$$

To estimate $\|A\mathbf{u}(t)\|$, we now form an inner product between (2.1) and $A\hat{\mathbf{u}}(t)$ to obtain

$$\|A\hat{\mathbf{u}}\|^2 \le e^{\alpha t}\|\mathbf{u}_t\|\|A\hat{\mathbf{u}}\| + e^{-\alpha t}|(B(\hat{\mathbf{u}},\hat{\mathbf{u}}),A\hat{\mathbf{u}})| + \alpha\|\hat{\mathbf{u}}\|\|A\hat{\mathbf{u}}\| + \int_0^t \beta(t-s)e^{\alpha(t-s)}\|A\hat{\mathbf{u}}(s)\|\|A\hat{\mathbf{u}}(t)\|\,ds. \tag{2.22}$$

The first three terms on the right-hand side of (2.22) are bounded by

$$\le C(\epsilon)[\|\hat{\mathbf{u}}\|^2 + e^{2\alpha t}\|\mathbf{u}_t\|^2 + e^{-4\alpha t}\|\hat{\mathbf{u}}\|^2\|A^{1/2}\hat{\mathbf{u}}\|^4] + \epsilon\|A\hat{\mathbf{u}}\|^2.$$

For the last term on the right-hand side of (2.22), we have applied the Hölder's inequality with Sobolev inequality. Then the last term is bounded by

$$C(\gamma,\delta,\alpha,\epsilon)\int_0^t e^{2\alpha\tau}\|A\mathbf{u}(\tau)\|^2\,d\tau + \epsilon\|A\hat{\mathbf{u}}\|^2.$$

Note that we have used $e^{-2(\delta-\alpha)(t-s)} \le 1$. On substituting in (2.22), we choose $\epsilon = \frac{1}{4}$. An appeal to Lemmas 4 and 5 with the estimate (2.21) yields

$$\|A\hat{\mathbf{u}}\|^2 \le C(\|A\mathbf{u}_0\|),$$

and thus we complete the proof of (2.7)–(2.8).

In order to derive (2.9), we now differentiate (1.4) with respect to time and then form an inner product with $\sigma(t)A\mathbf{u}_t$, where $\sigma(t) = \tau^*(t)e^{2\alpha t}$, to obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}(\sigma(t)\|A^{1/2}\mathbf{u}_t\|^2) + \sigma(t)\|A\mathbf{u}_t\|^2 &= -\sigma(t)(A\mathbf{u},A\mathbf{u}_t) + \frac{1}{2}\sigma_t\|A^{1/2}\mathbf{u}_t\|^2 \\ &-\sigma(t)\int_0^t \beta_t(t-s)(A\mathbf{u}(s),A\mathbf{u}_t(t))\,ds - \tau^*(t)e^{-\alpha t}\big(B(e^{\alpha t}\mathbf{u}_t,\hat{\mathbf{u}}) \\ &+ B(\hat{\mathbf{u}},e^{\alpha t}\mathbf{u}_t), e^{\alpha t}A\mathbf{u}_t\big) = I_1 + I_2 + I_3 + I_4.\end{aligned} \tag{2.23}$$

For I_1, we use Young's inequality to arrive at

$$|I_1| \le \frac{\gamma^2}{2\epsilon}\tau^*(t)\|A\hat{\mathbf{u}}\|^2 + \frac{\epsilon}{2}\sigma(t)\|A\mathbf{u}_t\|^2. \tag{2.24}$$

Since $\sigma_t = \tau_t^* e^{2\alpha t} + 2\alpha\tau^* e^{2\alpha t}$ with $\tau^*,\ \tau_t^* \le 1$, we obtain

$$|I_2| \le C(\alpha)e^{2\alpha t}\|A^{1/2}\mathbf{u}_t\|^2. \tag{2.25}$$

To estimate I_4, a use of Sobolev inequality with Young's inequality yields

$$|I_4| \le C(\epsilon)e^{2\alpha t}\|A^{1/2}\mathbf{u}_t\|^2(\|A^{1/2}\hat{\mathbf{u}}\|\|A\hat{\mathbf{u}}\| + \|A^{1/2}\hat{\mathbf{u}}\|^2) + \epsilon\sigma(t)\|A\mathbf{u}_t\|^2. \tag{2.26}$$

Since $\beta_t(t-s) = -\frac{1}{\delta}\beta(t-s)$, we obtain a bound for I_3 as

$$|I_3| \le \frac{\gamma^2}{2\epsilon\delta^2}\tau^*\left(\int_0^t e^{-(\delta-\alpha)(t-s)}\|A\hat{\mathbf{u}}(s)\|\,ds\right)^2 + \frac{\epsilon}{2}\sigma(t)\|A\mathbf{u}_t\|^2, \tag{2.27}$$

and hence, integrating with respect to time and using the estimate (2.3) for I term, we find that

$$\begin{aligned}\int_0^t |I_3|\,ds &\le \frac{\gamma^2}{2\epsilon\delta^2}\tau^* I + \frac{\epsilon}{2}\int_0^t \sigma(s)\|A\mathbf{u}_t\|^2\,ds \\ &\le C(\gamma,\delta,\alpha,\epsilon)\tau^*(t)\int_0^t \|A\hat{\mathbf{u}}(s)\|^2\,ds + \frac{\epsilon}{2}\int_0^t \sigma(s)\|A\mathbf{u}_t(s)\|^2\,ds.\end{aligned} \tag{2.28}$$

Multiply (2.23) by 2 and integrate with respect to time. Substitute (2.24)–(2.28) in (2.23). With $\epsilon = \frac{1}{4}$, it now follows that

$$\begin{aligned}(2.29)\quad \sigma(t)\|A^{1/2}\mathbf{u}_t\|^2 + \int_0^t \sigma(s)\|A\mathbf{u}_t(s)\|^2\,ds \le C(\gamma,\delta,\alpha)\Bigg[\tau^*\int_0^t \|A\hat{\mathbf{u}}(s)\|^2\,ds \\ + \int_0^t e^{2\alpha s}\|A^{1/2}\mathbf{u}_t\|^2(\|A^{1/2}\hat{\mathbf{u}}\|\|A\hat{\mathbf{u}}\| + \|A^{1/2}\hat{\mathbf{u}}\|^4)\,ds\Bigg] \\ + \int_0^t e^{2\alpha s}\|A^{1/2}\mathbf{u}_t(s)\|^2\,ds.\end{aligned}$$

Using Lemmas 4 and 5 and the estimates (2.7) and (2.8) in (2.29), we obtain the required result (2.9), and this completes the rest of the proof. $\square$

Remark 2. The estimate for $\|A^{1/2}\mathbf{u}_t\|$ shows the singular behavior near $t = 0$ and also indicates the exponential decay property as $t \longrightarrow \infty$. In Lemma 7, the regularity results are derived without any nonlocal compatibility conditions.

3. Decay properties for the discrete solution and error estimates. In this section, we discuss the decay properties for the solution of the linearized backward Euler method. Finally, we derive a priori bounds for the error in H^1-norm and present briefly the error estimate in L^2-norm.

The right-hand rectangle rule q^n which is used to discretize the integral in (1.4) is positive in the sense that

$$k\sum_{n=1}^{J} q^n(\phi)\phi^n \ge 0 \quad \forall \phi = (\phi^1,\dots,\phi^J)^T.$$

For a proof, we refer to McLean and Thomée [21, pp. 40–42]. Moreover, the following Lemma is easy to prove using the line of proof of [21].

LEMMA 8. *For any $\alpha \ge 0$, $J > 0$, and sequence $\{\phi^n\}_{n=1}^{\infty}$, the following positivity property holds:*

$$k^2\sum_{n=1}^{J}\left(\sum_{j=1}^{n} e^{-\alpha(t_n-t_j)}\phi^j\right)\phi^n \ge 0.$$

LEMMA 9. *With $0 < \alpha < \min(\delta,\ \lambda_1)$, choose $k_0 > 0$ small so that for $0 < k \le k_0$*

$$(\lambda_1 k + 1) > e^{\alpha k}.$$

Then the discrete solution $\mathbf{U}^J$, $J \ge 1$ of (1.6) is exponentially stable in the following sense:

$$(3.1)\qquad \|\mathbf{U}^J\| + e^{-\alpha t_J}\left(k\sum_{n=1}^{J}\|A^{1/2}\hat{\mathbf{U}}^n\|^2\right)^{1/2} \le C(\lambda_1,\alpha)\,\|\mathbf{U}^0\|e^{-\alpha t_J},\quad J \ge 1,$$

and

$$(3.2)\qquad \|A^{1/2}\mathbf{U}^J\| \le C(\lambda_1,\alpha,\|A^{1/2}\mathbf{U}^0\|)e^{-\alpha t_J},\quad J \ge 1.$$

Proof. Setting $\hat{\mathbf{U}}^n = e^{\alpha t_n}\mathbf{U}^n$, we rewrite (1.6) as

$$e^{\alpha t_n}\bar{\partial}_t\mathbf{U}^n + A\hat{\mathbf{U}}^n + e^{-\alpha t_{n-1}}B(\hat{\mathbf{U}}^{n-1},\hat{\mathbf{U}}^n) + e^{\alpha t_n}q^n(A\mathbf{U}) = 0.$$

Note that

$$e^{\alpha t_n}\bar{\partial}_t \mathbf{U}^n = e^{\alpha k}\bar{\partial}_t \hat{\mathbf{U}}^n - \left(\frac{e^{\alpha k}-1}{k}\right)\hat{\mathbf{U}}^n.$$

On substitution and then multiplying the resulting equation by $e^{-\alpha k}$, we obtain

$$\begin{aligned}\bar{\partial}_t \hat{\mathbf{U}}^n - \left(\frac{1-e^{-\alpha k}}{k}\right)\hat{\mathbf{U}}^n + e^{-\alpha k}A\hat{\mathbf{U}}^n + e^{-\alpha t_n}B(\hat{\mathbf{U}}^{n-1},\hat{\mathbf{U}}^n)&\\ + \gamma e^{-\alpha k} k\sum_{j=1}^{n} e^{-(\delta-\alpha)(t_n-t_j)}A\hat{\mathbf{U}}^j = 0.&\end{aligned} \tag{3.3}$$

Forming an inner product between (3.3) and $\hat{\mathbf{U}}^n$, use

$$(B(\hat{\mathbf{U}}^{n-1},\hat{\mathbf{U}}^n),\hat{\mathbf{U}}^n) = 0, \quad \|\hat{\mathbf{U}}^n\|^2 \le \frac{1}{\lambda_1}\|A^{1/2}\hat{\mathbf{U}}^n\|^2, \quad \text{and} \quad (\bar{\partial}_t\hat{\mathbf{U}}^n,\hat{\mathbf{U}}^n) \ge \frac{1}{2}\bar{\partial}_t\|\hat{\mathbf{U}}^n\|^2$$

to obtain

$$\begin{aligned}\frac{1}{2}\bar{\partial}_t\|\hat{\mathbf{U}}^n\|^2 + \left(e^{-\alpha k} - \left(\frac{1-e^{-\alpha k}}{k}\right)\lambda_1^{-1}\right)\|A^{1/2}\hat{\mathbf{U}}^n\|^2&\\ + \gamma e^{-\alpha k}k\sum_{j=1}^{n} e^{-(\delta-\alpha)(t_n-t_j)}(A^{1/2}\hat{\mathbf{U}}^j, A^{1/2}\hat{\mathbf{U}}^n) \le 0.&\end{aligned} \tag{3.4}$$

With $0<\alpha<\min(\lambda_1,\delta)$, choose $0<k_0$ such that for $0<k<k_0$

$$(\lambda_1 k + 1) \ge e^{\alpha k}.$$

Then for $0<k\le k_0$, the coefficient of the second term on the left-hand side of (3.4), $\left(e^{-\alpha k} - (\frac{1-e^{-\alpha k}}{k})\lambda_1^{-1}\right)$, becomes positive. Multiplying (3.4) by $2k$ and summing from $n=1$ to J, the last term becomes nonnegative by Lemma 8 and thus we obtain the estimate (3.1).

For the estimate (3.2), we form an inner product between (3.3) and $A\hat{\mathbf{U}}^n$ and observe that

$$(\bar{\partial}_t\hat{\mathbf{U}}^n, A\hat{\mathbf{U}}^n) = (\bar{\partial}_t A^{1/2}\hat{\mathbf{U}}^n, A^{1/2}\hat{\mathbf{U}}^n) \ge \frac{1}{2}\bar{\partial}_t\|A^{1/2}\hat{\mathbf{U}}^n\|^2.$$

Altogether, we find that

$$\begin{aligned}\frac{1}{2}\bar{\partial}_t\|A^{1/2}\hat{\mathbf{U}}^n\|^2 + e^{-\alpha k}\|A\hat{\mathbf{U}}^n\|^2 + \gamma e^{-\alpha k}k\sum_{j=1}^{n} e^{-(\delta-\alpha)(t_n-t_j)}(A\hat{\mathbf{U}}^j, A\hat{\mathbf{U}}^n)&\\ \le \left(\frac{1-e^{-\alpha k}}{k}\right)(\hat{\mathbf{U}}^n, A\hat{\mathbf{U}}^n) - e^{-\alpha t_n}(B(\hat{\mathbf{U}}^{n-1},\hat{\mathbf{U}}^n), A\hat{\mathbf{U}}^n).&\end{aligned} \tag{3.5}$$

Multiplying (3.5) by $2k$ and summing from $n=1$ to J, the third term on the left-hand side becomes nonnegative by applying Lemma 8 as $0<\alpha<\delta$. Then, we obtain

$$\begin{aligned}\|A^{1/2}\hat{\mathbf{U}}^J\|^2 + 2ke^{-\alpha k}\sum_{n=1}^{J}\|A\hat{\mathbf{U}}^n\|^2 &\le \|A^{1/2}\mathbf{U}^0\|^2 + 2(1-e^{-\alpha k})k\sum_{n=1}^{J}|(\hat{\mathbf{U}}^n, A\hat{\mathbf{U}}^n)|\\ &\quad + 2e^{-\alpha k}k\sum_{n=1}^{J} e^{-\alpha t_{n-1}}|(B(\hat{\mathbf{U}}^{n-1},\hat{\mathbf{U}}^{n-1}), A\hat{\mathbf{U}}^n)|\\ &\le \|A^{1/2}\mathbf{U}^0\|^2 + I_1 + I_2.\end{aligned} \tag{3.6}$$

To estimate I_1, we have by the mean value theorem $\frac{1-e^{-\alpha k}}{k} = \alpha e^{-\alpha k^*}$ for some $0 < k^* < k$, and hence, using (3.1), we find that

$$|I_1| \leq 2\alpha e^{-\alpha k^*} k \sum_{n=1}^{J} \|A^{1/2}\hat{\mathbf{U}}^n\|^2 \leq C(\lambda_1, \alpha)\|\mathbf{U}^0\|^2.$$

For I_2, a repeated use of Hölder's inequality with Sobolev inequality yields

$$\begin{aligned} e^{-\alpha t_{n-1}}|(B(\hat{\mathbf{U}}^{n-1}, \hat{\mathbf{U}}^n), A\hat{\mathbf{U}}^n)| \leq C e^{-\alpha t_{n-1}} \|\hat{\mathbf{U}}^{n-1}\|^{1/2} \|A^{1/2}\hat{\mathbf{U}}^{n-1}\|^{1/2} \\ \|A^{1/2}\hat{\mathbf{U}}^n\|^{1/2} \|A\hat{\mathbf{U}}^n\|^{3/2}. \end{aligned}$$

By an application of Young's inequality, it follows that

$$\begin{aligned} |I_2| &\leq Cke^{-\alpha k} \sum_{n=1}^{J} e^{-4\alpha t_{n-1}} (\|\hat{\mathbf{U}}^{n-1}\|^2 \|A^{1/2}\hat{\mathbf{U}}^{n-1}\|^2) \|A^{1/2}\hat{\mathbf{U}}^n\|^2 \\ &\quad + ke^{-\alpha k} \sum_{n=1}^{J} \|A\hat{\mathbf{U}}^n\|^2. \end{aligned}$$

Using the estimate $\|\hat{\mathbf{U}}^{n-1}\|$ and

$$k\|A^{1/2}\hat{\mathbf{U}}^{J-1}\|^2 \leq k \sum_{n=1}^{J} \|A^{1/2}\hat{\mathbf{U}}^n\|^2,$$

we easily find that from (3.1)

$$\begin{aligned} |I_2| &\leq C(\lambda, \alpha)\|\mathbf{U}^0\|^2 ke^{-\alpha k} \sum_{n=1}^{J-1} e^{-4\alpha t_{n-1}} \|A^{1/2}\hat{\mathbf{U}}^{n-1}\|^2 \|A^{1/2}\hat{\mathbf{U}}^n\|^2 \\ &\quad + C\|\mathbf{U}^0\|^4 e^{-\alpha k} e^{-4\alpha t_{J-1}} \, \|A^{1/2}\hat{\mathbf{U}}^J\|^2 + ke^{-\alpha k} \sum_{n=1}^{J} \|A\hat{\mathbf{U}}^n\|^2. \end{aligned}$$

Now substitute the estimates of I_1 and I_2 in (3.6). For small k, we note that $(1 - C\|\mathbf{U}^0\|^4 e^{-4\alpha k})$ can be made positive. Then apply discrete Gronwall's lemma with estimate (3.1) to complete the rest of the proof. ∎

3.1. Error analysis. Now we are ready to discuss the proof of our main result that is the proof of Theorem 1.

Let ε^n be the quadrature error associated with the quadrature rule (1.5) and for $\phi \in C^1[0, t_n]$, let it be given by

$$\varepsilon^n(\phi) := \int_0^{t_n} \beta(t_n - s)\phi(s)\, ds - q^n(\phi).$$

Note that the quadrature error ε^n satisfies

$$\begin{aligned} |\varepsilon^n(\phi)| &\leq Ck \int_0^{t_n} \left| \frac{\partial}{\partial s} \left(\beta(t_n - s)\phi(s)\right) \right| ds \\ &\leq Ck \int_0^{t_n} \left(|\beta_s(t_n - s)| \, |\phi(s)| + |\beta(t_n - s)| \, |\phi_s(s)| \right) | \, ds. \end{aligned} \tag{3.7}$$

For the proof of the main Theorem, we appeal to the semigroup theoretic approach; see Thomée [29], Fujita and Kato [9], and Okamoto [22]. It is well known that the Stoke's operator $-A$ generates an analytic semigroup, say, $E(t),\ t > 0$ on $\mathbf{H}$; see [28] or [9]. Moreover, the following estimates are also satisfied:

$$\|A^r E(t)\| \le Ct^{-r} e^{-\lambda_1 t}, \quad t > 0, \quad r > 0, \tag{3.8}$$

and for $r \in (0, 1]$, and $\mathbf{v} \in D(A^r)$, the domain of A^r,

$$\|(E(t) - I)\mathbf{v}\| \le C_r t^r \|A^r \mathbf{v}\|, \quad t > 0, \tag{3.9}$$

where C_r is a positive constant. For a proof, see [6, p. 383]. Further, we use the discrete semigroup E_k, which is given by

$$E_k = (I + kA)^{-1}.$$

Using spectral representation of A [29], the following estimate is easy to derive:

$$\|A^r E_k^n\| \le Ct_n^{-r} e^{-\lambda_1 t_n}, \quad t_n > 0, \quad 0 < r \le 1. \tag{3.10}$$

Now, using Duhamel's principle, (1.4) is written in an equivalent form as

$$\mathbf{u}(t) = E(t)\mathbf{u}_0 - \int_0^t E(t-s)\tilde{A}\mathbf{u}(s)\, ds - \int_0^t E(t-s)B(\mathbf{u}(s), \mathbf{u}(s))\, ds,$$

where for simplicity of symbol, we denote

$$\tilde{A}\mathbf{u}(t) = \int_0^t \beta(t-\tau)A\mathbf{u}(\tau)\, d\tau.$$

Similarly, using discrete semigroup $E_k = (I + kA)^{-1}$, we rewrite (1.6) as

$$\mathbf{U}^n = E_k^n \mathbf{u}_0 - \sum_{j=1}^n k E_k^{n-j+1} q^j(A\mathbf{U}) - \sum_{j=1}^n k E_k^{n-j+1} B(\mathbf{U}^{j-1}, \mathbf{U}^j).$$

Proof of Theorem 1. Note that the error $\mathbf{e}^n := \mathbf{u}(t_n) - \mathbf{U}^n$ is written in the form

$$\begin{aligned} \mathbf{e}^n &= \left(E(t_n) - E_k^n\right)\mathbf{u}_0 - \left(\int_0^{t_n} E(t_n - s)\tilde{A}\mathbf{u}(s)\, ds - \sum_{j=1}^n k E_k^{n-j+1} q^j(A\mathbf{U})\right) \\ &\quad - \left(\int_0^{t_n} E(t_n - s)B(\mathbf{u}(s), \mathbf{u}(s))\, ds - \sum_{j=1}^n k E_k^{n-j+1} B(\mathbf{U}^{j-1}, \mathbf{U}^j)\right) \\ &= I_1^n - I_2^n - I_3^n. \end{aligned} \tag{3.11}$$

Since $F_k^n := (E(t_n) - E_k^n)$ denotes the error operator for the purely parabolic problem, then following Thomée [29], we estimate $A^{1/2} I_1^n$ as

$$\|A^{1/2} I_1^n\| = \|A^{1/2} F_k^n \mathbf{u}_0\| \le C(\|A\mathbf{u}_0\|, \Omega)\frac{e^{-\alpha t_n}}{t_n^{1/2}} k. \tag{3.12}$$

In order to estimate $\|A^{1/2}I_2^n\|$, i.e., the memory term, we first rewrite I_2^n as

$$I_2^n = \left(\int_0^{t_n} E(t_n - s)\left(\tilde{A}\mathbf{u}(s) - \tilde{A}\mathbf{u}(t_n)\right) ds - \sum_{j=1}^{n} kE_k^{n-j+1}\left(q^j(A\mathbf{u}) - \tilde{A}\mathbf{u}(t_n)\right)\right)$$

$$+ \left(\int_0^{t_n} E(t_n - s)\, ds - \sum_{j=1}^{n} kE_k^{n-j+1}\right)\tilde{A}\mathbf{u}(t_n) \tag{3.13}$$

$$+\sum_{j=1}^{n} kE_k^{n-j+1} q^j(A\mathbf{e}) = I_{2,1}^n + I_{2,2}^n + I_{2,3}^n.$$

For $I_{2,2}^n$, we obtain using the semigroup property

$$\int_0^{t_n} E(t_n - s) - \sum_{j=1}^{n} kE_k^{n-j+1} = -F_k^n A^{-1},$$

and hence, using the definition of β, we arrive at

$$\begin{aligned}
\|A^{1/2}I_{2,2}^n\| &= \|A^{1/2}F_k^n A^{-1}\tilde{A}\mathbf{u}(t_n)\| \\
&\le Ck\frac{e^{-\lambda_1 t_n}}{t_n^{1/2}} e^{-\alpha t_n} \| \int_0^{t_n} e^{-(\delta-\alpha)(t_n-\tau)} A\hat{\mathbf{u}}(\tau)\, d\tau \| \\
&\le Ck\frac{e^{-\lambda_1 t_n}}{t_n^{1/2}} e^{-\alpha t_n}\left(\int_0^{t_n} \|A\hat{\mathbf{u}}(\tau)\|^2\, d\tau\right)^{1/2}.
\end{aligned}$$

An application of Lemma 5 yields for $0 < \alpha < \min(\lambda_1, \delta,)$

$$\|A^{1/2}I_{2,2}^n\| \le C(\|A^{1/2}\mathbf{u}_0\|)k\frac{e^{-\alpha t_n}}{t_n^{1/2}}.$$

For estimating $I_{2,3}^n$, we first use the change of variable and then the change of summation to obtain

$$\begin{aligned}
A^{1/2}I_{2,3}^n &= \sum_{j=0}^{n-1} kAE_k^{n-j}A^{-1/2}\sum_{i=1}^{j+1} k\beta_{j+1-i}A\mathbf{e}^i = \sum_{j=0}^{n-1} kAE_k^{n-j}\sum_{i=0}^{j} k\beta_{j-i}A^{1/2}\mathbf{e}^{i+1} \\
&= k\sum_{i=0}^{n-1}\left(\sum_{j=i}^{n-1} k\beta_{j-i}AE_k^{n-j}\right)A^{1/2}\mathbf{e}^{i+1} \\
&= k\sum_{i=0}^{n-1}\left(\sum_{j=i}^{n-1} k\beta_{n-i}AE_k^{n-j}\right)A^{1/2}\mathbf{e}^{i+1} \\
&\qquad -k\sum_{i=0}^{n-1}\left(\sum_{j=i}^{n-1} k(\beta_{n-i}-\beta_{j-i})AE_k^{n-j}\right)A^{1/2}\mathbf{e}^{i+1}.
\end{aligned}$$

For the first term on the right-hand side of $A^{1/2}I_{2,3}^n$, we have from the spectral property of the Stoke's operator and $r(\lambda)=(1+\lambda)^{-1}$:

$$\left\|k\sum_{j=i}^{n-1} AE_k^{n-j}\right\| = \sup_{\lambda\in Sp(A)}\left|\sum_{j=i}^{n-1} k\lambda r(k\lambda)^{n-j}\right| \le \sup_{\lambda>0}\sum_{j=i}^{n-1}\lambda r(\lambda)^{n-j}$$
$$\le \sup_{\lambda>0}\frac{\lambda r(\lambda)}{1-r(\lambda)} = 1,$$

where $Sp(A)$ is the spectrum of the Stokes operator A. For the second term on the right-hand side of $A^{1/2}I_{2,3}^n$, we use the smoothing property (3.8) of E_k^n, and therefore we obtain

$$\begin{aligned}
&\|A^{1/2}I_{2,3}^n\| \\
&\quad\le \gamma k\sum_{i=0}^{n-1} e^{-\delta(t_n-t_i)}\left\|k\sum_{j=i}^{n-1} AE_k^{n-j}\right\| \, \|A^{1/2}\mathbf{e}^{i+1}\| \\
&\quad+\gamma k\sum_{i=0}^{n-1}\left(\sum_{j=i}^{n-1} k|(e^{-\delta t_{n-i}}-e^{-\delta t_{j-i}})| \, \|AE_k^{n-j}\|\right)\|A^{1/2}\mathbf{e}^{i+1}\| \\
&\quad\le Cke^{-\alpha t_n}\sum_{i=0}^{n-1} e^{\alpha t_i}\|A^{1/2}\mathbf{e}^{i+1}\| \\
&\quad+Cke^{-\alpha t_n}\sum_{i=0}^{n-1} e^{\alpha t_i}\left(\sum_{j=i}^{n-1} ke^{-(\delta-\alpha)(t_j-t_i)}\frac{e^{-\delta(t_n-t_j)}-1}{(t_n-t_j)}e^{-(\lambda_1-\alpha)(t_n-t_j)}\right)\|A^{1/2}\mathbf{e}^{i+1}\|.
\end{aligned}$$

Using the meanvalue property of the exponential function, we find that

$$\left(\sum_{j=i}^{n-1} ke^{-(\delta-\alpha)(t_j-t_i)}\frac{e^{-\delta(t_n-t_j)}-1}{(t_n-t_j)}e^{-(\lambda_1-\alpha)(t_n-t_j)}\right)\le C,$$

and hence we arrive at

$$\|A^{1/2}I_{2,3}^n\|\le Ce^{-\alpha t_n}e^{-\alpha k}k\sum_{i=0}^{n} e^{\alpha t_i}\|A^{1/2}\mathbf{e}^i\|.$$

Now for the term $I_{2,1}^n$, we first rewrite it as

$$\begin{aligned}
I_{2,1}^n &= \sum_{j=1}^{n}\int_{t_{j-1}}^{t_j}(E(t_n-s)-E(t_{n-j+1}))\left(\tilde{A}\mathbf{u}(s)-\tilde{A}\mathbf{u}(t_n)\right)ds \\
&+\sum_{j=1}^{n}\int_{t_{j-1}}^{t_j}E(t_{n-j+1})\left(\tilde{A}\mathbf{u}(s)-\tilde{A}\mathbf{u}(t_j)\right)ds \\
&+\sum_{j=1}^{n} kF_k^{n-j+1}\left(\tilde{A}\mathbf{u}(t_j)-\tilde{A}\mathbf{u}(t_n)\right)+\sum_{j=1}^{n} kE_k^{n-j+1}\varepsilon^j(A\mathbf{u}) \\
&= M_1^n+M_2^n+M_3^n+M_4^n.
\end{aligned}$$

For M_1^n, we write it as

$$A^{1/2}M_1^n = \sum_{j=1}^{n} \int_{t_{j-1}}^{t_j} A^{3/2}E(t_n-s)A^{-1}\left(I-E(s-t_{j-1})\right)\left(\tilde{A}\mathbf{u}(s)-\tilde{A}\mathbf{u}(t_n)\right)\,ds.$$

Thus, using (3.8)–(3.9), we obtain

$$\begin{aligned}\|A^{1/2}M_1^n\| &\le \sum_{j=1}^{n} \int_{t_{j-1}}^{t_j} \|A^{3/2}E(t_n-s)\|\,\|A^{-1}\left(I-E(s-t_{j-1})\right)\left(\tilde{A}\mathbf{u}(s)-\tilde{A}\mathbf{u}(t_n)\right)\|\,ds \\ &\le Ck\int_0^{t_n} \frac{e^{-\lambda_1(t_n-s)}}{(t_n-s)^{3/2}}\|\tilde{A}\mathbf{u}(s)-\tilde{A}\mathbf{u}(t_n)\|\,ds.\end{aligned}$$

In order to estimate $\|\tilde{A}\mathbf{u}(s)-\tilde{A}\mathbf{u}(t_n)\|$, we note that

$$\tilde{A}\mathbf{u}(s)-\tilde{A}\mathbf{u}(t_n) = \int_0^s \left(\beta(s-\tau)-\beta(t_n-\tau)\right)A\mathbf{u}(\tau)\,d\tau - \int_s^{t_n}\beta(t_n-\tau)A\mathbf{u}(\tau)\,d\tau,$$

and hence, using the definition of β, the mean value theorem, $0<\alpha<\min(\lambda_1,\delta)$, and Lemma 7, we now obtain

$$\begin{aligned}\|\tilde{A}\mathbf{u}(s)-\tilde{A}\mathbf{u}(t_n)\| &\le \gamma e^{-\delta s}\left(1-e^{-\delta(t_n-s)}\right)\int_0^s e^{\delta\tau}\|A\mathbf{u}(\tau)\|\,d\tau \\ &\quad + \gamma\int_s^{t_n} e^{-\delta(t_n-\tau)}\|A\mathbf{u}(\tau)\|\,d\tau \\ &\le \delta\gamma(t_n-s)e^{-\alpha s}e^{-\delta s^*}\int_0^s e^{-(\delta-\alpha)(s-\tau)}\|e^{\alpha\tau}A\mathbf{u}(\tau)\|\,d\tau \\ &\quad + C(\|A\mathbf{u}_0\|,\gamma)\int_s^{t_n} e^{-\delta(t_n-\tau)}e^{-\alpha\tau}\,d\tau \\ &\le \delta\gamma(t_n-s)e^{-\alpha s}\left(\int_0^s e^{-2(\delta-\alpha)(s-\tau)}\,d\tau\right)^{1/2}\left(\int_0^s e^{2\alpha\tau}\|A\mathbf{u}(\tau)\|^2\,d\tau\right)^{1/2} \\ &\quad + C(\|A\mathbf{u}_0\|,\gamma)(t_n-s)e^{-\alpha s}.\end{aligned}$$

Using Lemma 5 and the boundedness of

$$\int_0^s e^{-2(\delta-\alpha)(s-\tau)}\,d\tau \le \frac{1}{2(\delta-\alpha)},$$

we arrive at

$$\|\tilde{A}\mathbf{u}(s)-\tilde{A}\mathbf{u}(t_n)\| \le C(\|A\mathbf{u}_0\|)(t_n-s)e^{-\alpha s}.$$

Therefore,

$$\begin{aligned}\|A^{1/2}M_1^n\| &\le C(\|A\mathbf{u}_0\|)ke^{-\alpha t_n}\int_0^{t_n}\frac{e^{-(\lambda_1-\alpha)(t_n-s)}}{(t_n-s)^{1/2}}\,ds \\ &\le C(\|A\mathbf{u}_0\|)ke^{-\alpha t_n}\int_0^{t_n}\frac{e^{-(\lambda_1-\alpha)\tau}}{\tau^{1/2}}\,d\tau \\ &\le C(\|A\mathbf{u}_0\|)ke^{-\alpha t_n}\int_0^{\infty}\frac{e^{-(\lambda_1-\alpha)\tau}}{\tau^{1/2}}\,d\tau \le C(\|A\mathbf{u}_0\|)ke^{-\alpha t_n}.\end{aligned}$$

To estimate M_2^n, we use the definition of $\tilde{A}$ and the property (3.8) to find that

$$\begin{aligned}\|A^{1/2}M_2^n\| &\le \sum_{j=1}^{n}\int_{t_{j-1}}^{t_j}\|A^{1/2}E(t_{n-j+1})\|\|\tilde{A}\mathbf{u}(s)-\tilde{A}\mathbf{u}(t_j)\|\,ds\\ &\le C\sum_{j=1}^{n}\int_{t_{j-1}}^{t_j}\frac{e^{-\lambda_1(t_n-t_{j-1})}}{(t_n-t_{j-1})^{1/2}}\|\tilde{A}\mathbf{u}(s)-\tilde{A}\mathbf{u}(t_j)\|\,ds.\end{aligned}$$

Since

$$\|\tilde{A}\mathbf{u}(s)-\tilde{A}\mathbf{u}(t_j)\|\le C(\|A\mathbf{u}_0\|)(t_j-s)e^{-\alpha s}\le C(\|A\mathbf{u}_0\|)ke^{-\alpha s},$$

we now obtain

$$\begin{aligned}\|A^{1/2}M_2^n\| &\le C(\|A\mathbf{u}_0\|)ke^{-\alpha t_n}\sum_{j=1}^{n}\frac{e^{-(\lambda_1-\alpha)(t_n-t_{j-1})}}{(t_n-t_{j-1})^{1/2}}\left(e^{\alpha t_{j-1}}\int_{t_{j-1}}^{t_j}e^{-\alpha s}\,ds\right)\\ &\le C(\|A\mathbf{u}_0\|)ke^{-\alpha t_n}\left(k\sum_{j=1}^{n}\frac{e^{-(\lambda_1-\alpha)(t_n-t_{j-1})}}{(t_n-t_{j-1})^{1/2}}\right)\\ &\le C(\|A\mathbf{u}_0\|)ke^{-\alpha t_n}.\end{aligned}$$

Note that we have used the boundedness of the summation term within the bracket.

In order to estimate M_3^n, we use the property of F_k^n and obtain

$$\|A^{1/2}M_3^n\|\le Ck^2\sum_{j=1}^{n}\frac{e^{-\lambda_1(t_n-t_{j-1})}}{(t_n-t_{j-1})^{3/2}}\|\tilde{A}\mathbf{u}(t_j)-\tilde{A}\mathbf{u}(t_n)\|.$$

As in the estimate of $\|A^{1/2}M_1^n\|$, we now find that

$$\begin{aligned}\|A^{1/2}M_3^n\| &\le C(\|A\mathbf{u}_0\|)ke^{-\alpha t_n}e^{-\alpha k}\left(k\sum_{j=1}^{n}\frac{e^{-(\lambda_1-\alpha)(t_n-t_{j-1})}}{(t_n-t_{j-1})^{1/2}}\right)\\ &\le C(\|A\mathbf{u}_0\|)ke^{-\alpha t_n}.\end{aligned}$$

Finally for M_4^n, we note that

$$\|A^{1/2}M_4^n\|\le\sum_{j=1}^{n}k\|AE_k^{n-j+1}\|\,\|\varepsilon^j(A^{1/2}\mathbf{u})\|.$$

Using (3.8), we obtain

$$\|A^{1/2}M_4^n\|\le\sum_{j=1}^{n}k\frac{e^{-\lambda_1(t_n-t_{j-1})}}{(t_n-t_{j-1})}\|\varepsilon^j(A\mathbf{u})\|.$$

To complete the estimate, we use (3.7) to compute the quadrature error $\|\varepsilon^j(A\mathbf{u})\|$ as

$$\|\varepsilon^j(A\mathbf{u})\|\le Ck\int_0^{t_j}\Big(|\beta_s(t_j-s)|\|A^{1/2}\mathbf{u}(s)\|+|\beta(t_j-s)|\|A^{1/2}\mathbf{u}_s(s)\|\Big)\,ds,$$

and hence we find from Lemma 6 that

$$\begin{aligned}\|\varepsilon^j(A\mathbf{u})\| &\le C(\|A\mathbf{u}_0\|)ke^{-\alpha t_j}\int_0^{t_j} e^{-(\delta-\alpha)(t_j-s)}\,ds\\ &\quad + Cke^{-\alpha t_j}\left(\int_0^{t_j} e^{-2(\delta-\alpha)(t_j-s)}\,ds\right)^{1/2}\left(\int_0^{t_j} e^{2\alpha s}\|A^{1/2}\mathbf{u}_s(s)\|^2\,ds\right)^{1/2}\\ &\le C(\|A\mathbf{u}_0\|)ke^{-\alpha t_j}.\end{aligned}$$

Thus, we arrive at

$$\begin{aligned}\|A^{1/2}M_4^n\| &\le C(\|A\mathbf{u}_0\|)ke^{-\alpha t_n}e^{-\alpha k}\left(k\sum_{j=1}^{n}\frac{e^{-(\lambda_1-\alpha)(t_n-t_{j-1})}}{(t_{n-j+1})}\right)\\ &\le C(\|A\mathbf{u}_0\|)ke^{-\alpha t_n}e^{-\alpha k}\left(k\sum_{j=1}^{n}\frac{1}{(t_{n-j+1})}\right)\\ &\le C(\|A\mathbf{u}_0\|)k\left(\log\frac{1}{k}\right)e^{-\alpha t_n}.\end{aligned}$$

All together, we therefore obtain

$$\begin{aligned}\|A^{1/2}I_2^n\| &\le C(\|A\mathbf{u}_0\|)e^{-\alpha t_n}k\left(1+\log\frac{1}{k}\right) + C(\|A^{1/2}\mathbf{u}_0\|)\frac{e^{-\alpha t_n}}{t_n^{1/2}}k\\ &\quad + Ce^{-\alpha t_n}k\sum_{i=0}^{n-1}e^{\alpha t_i}\|A^{1/2}\mathbf{e}^i\| + Cke^{-\alpha k}\|A^{1/2}\mathbf{e}^n\|.\end{aligned} \tag{3.14}$$

Finally, in order to estimate I_3^n involving the nonlinear term, we may split it as in Geveci [11] and apply Hölder's inequality, Sobolev imbedding theorem with Sobolev inequality. Lastly, with the help of Lemmas 4, 5, 7, and 9, we obtain

$$\begin{aligned}\|A^{1/2}I_3^n\| &\le C(\|A\mathbf{u}_0\|)\frac{e^{-\alpha t_n}}{t_n^{1/2}}k + C(\|A^{1/2}u_0\|)e^{-\alpha t_n}k^{1/4}\|A^{1/2}\mathbf{e}^n\|\\ &\quad + Ce^{-\alpha t_n}k\sum_{i=0}^{n-1}\frac{e^{\alpha t_i}}{(t_n-t_i)^{3/4}}\|A^{1/2}\mathbf{e}^i\|.\end{aligned} \tag{3.15}$$

On substituting (3.12), (3.14), and (3.15) in (3.9), we obtain, for sufficiently small k,

$$\begin{aligned}e^{\alpha t_n}\|A^{1/2}\mathbf{e}^n\| &\le C(\|A\mathbf{u}_0\|)\Bigg[k\left(t_n^{-1/2}+\log\frac{1}{k}\right)\\ &\quad + k\sum_{i=0}^{n-1}\left(\frac{1}{(t_n-t_i)^{3/4}}+1\right)e^{\alpha t_i}\|A^{1/2}\mathbf{e}^i\|\Bigg].\end{aligned} \tag{3.16}$$

Using the generalized discrete Gronwall's lemma (see Lemma 7.1 in [7]) and the arguments of Okamoto [22, p. 635], we complete the rest of the proof. ∎

The convergence in $\mathbf{L}^2$-norm now becomes a routine work. However, we indicate, below, only the major steps in the proof for achieving this result.

Proof of Theorem 2. From (3.9), the error e^n satisfies

$$\mathbf{e}^n = I_1^n - I_2^n - I_3^n.$$

Since a straightforward modification of $\mathbf{H}^1$-estimates of Geveci [11] yields the $\mathbf{L}^2$-estimates of I_1^n and I_3^n, it remains to estimate $\|I_2^n\|$. Note that the $\mathbf{L}^2$-estimates of $I_{2.2}^n$ and $I_{2,3}^n$ in (3.13) follow easily as

$$\begin{aligned}\|I_{2,2}^n\| &= \|F_k^n A^{-1}\tilde{A}\mathbf{u}(t_n)\| \\ &\le Cke^{-\lambda_1 t_n}\|\int_0^{t_n}\beta(t_n-s)A\mathbf{u}(s)\,ds\| \\ &\le Cke^{-\alpha t_n}\left(\int_0^{t_n}\|A\mathbf{u}(s)\|^2\,ds\right)^{1/2} \le C(\|A^{1/2}\mathbf{u}_0\|)ke^{-\alpha t_n}\end{aligned}$$

and

$$\|I_{2,3}^n\| = \|k\sum_{j=0}^{n-1}AE_k^{n-j}\sum_{i=0}^{j}k\beta_{j-i}\mathbf{e}^{i+1}\|.$$

We repeat the analysis for estimating $A^{1/2}I_{2,3}^n$ in Theorem 1, but now $\mathbf{e}^{i+1}$ is made free of $A^{1/2}$. Thus, we obtain

$$\|I_{2,3}^n\| \le Ce^{-\alpha t_n}k\sum_{i=0}^{n-1}e^{\alpha t_i}\|\mathbf{e}^i\| + Ck\|\mathbf{e}^n\|.$$

In order to estimate $I_{2,1}^n$, it is a routine matter to derive the estimates of $\|M_1^n\|$, $\|M_2^n\|$, and $\|M_3^n\|$. To complete the rest of the proof, we therefore need an estimate for $\|M_4^n\|$. Note that

$$\begin{aligned}\|M_4^n\| &\le \sum_{j=1}^{n}k\|A^{1/2}E_k^{n-j+1}\|\,\|\varepsilon^j(A^{1/2}\mathbf{u})\| \\ &\le Ck\sum_{j=1}^{n}\frac{e^{-\lambda_1(t_n-t_{j-1})}}{(t_n-t_{j-1})^{1/2}}\|\varepsilon^j(A^{1/2}\mathbf{u})\|.\end{aligned}$$

Using the estimate of $\|\varepsilon^j(A^{1/2}\mathbf{u})\|$ as in the proof of Theorem 1, we now obtain

$$\begin{aligned}\|M_4^n\| &\le C(\|A\mathbf{u}_0\|)ke^{-\alpha t_n}\left(k\sum_{j=1}^{n}\frac{e^{-(\lambda_1-\alpha)(t_n-t_{j-1})}}{(t_n-t_{j-1})^{1/2}}\right) \\ &\le C(\|A\mathbf{u}_0\|)ke^{-\alpha t_n}.\end{aligned}$$

Note that the summation in the bracket is bounded by a constant which is independent of k. This completes the rest of the proof. ∎

4. Conclusion. In this paper, we have proved new regularity results for the solutions which are valid for all time $t > 0$ without nonlocal compatibility conditions for the data and established the exponential decay property for the exact solution. Further, we have derived optimal error estimates in $\mathbf{H}^1$ and $\mathbf{L}^2$-norms for the linearized backward Euler scheme under realistically assumed conditions on the initial data. Here, the analysis is not complete as at each time level, we have still to solve an infinite dimensional problem. However, we can easily derive the error estimates for a completely discrete scheme by combining the present analysis with the semidiscrete results obtained in [26]. Since the problem (1.1)–(1.3) can be thought of as an integral perturbation of the Navier–Stokes equations, we would like to investigate *how far the results on finite element analysis combined with higher order time discretizations of the Navier–Stokes equations* [15], [16], [22] *can be carried over to the present case.* We shall pursue this in future. Finally, we note that we have discussd our results only for the two-dimensional problem and we can easily generalize the analysis of this paper to the problem in three-dimensional bounded domain under smallness conditions on the initial data.

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