A posteriori error estimates of fully discrete finite element method for Burgers equation in 2-D

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Abstract

Here, a posteriori error estimates of finite element method are derived for Burgers equation in two-dimension. By use of constructing an appropriate Burgers reconstruction, some posteriori estimates in $L^\infty(L^2)$, $L^\infty(H^1)$ and $L^2(L^2)$ norms with optimal order of convergence are established for the semidiscrete scheme. Furthermore, a fully discrete scheme with corresponding posteriori estimates are studied based on backward Euler scheme. Finally, some numerical results are presented to verify the performance of the established posteriori error estimators.

Keywords: A posteriori error estimates; Burgers equation; Burgers reconstruction; Backward Euler scheme; Adaptive algorithms


1. Introduction

We assume that $\Omega$ is a bounded polygonal domain in $\mathbb{R}^2$ with a sufficiently smooth boundary $\partial \Omega$ for our study in this paper. Consider the following Burgers equations

$$
\begin{align*}
\begin{cases}
u u_t - \Delta u + u \nabla \cdot u &= f \quad \text{in } \Omega \times (0, T], \\
u u &= 0 \quad \text{on } \partial \Omega \times (0, T], \\
u u &= u_0 \quad \text{on } \Omega \times \{0\},
\end{cases}
\end{align*}
$$

(1.1)

where $u$, $f$ and $u_0$ are the velocity, the prescribed body force, and the initial velocity respectively with the diffusion coefficient $\nu > 0$, and the finite time $T > 0$.

In the last decade years, there is a growing demand of designing and developing reliable and efficient space-time numerical algorithms for the parabolic partial
differential equations. Most of these methods are based on a posteriori error estimators due to the significant advantages of the adaptive method [2]. Although the theory of a posteriori error estimates of finite element method for elliptic problems is well-developed [3, 28], the theory for parabolic problems is less developed, and only a few results have been published until now, such as the spatial semidiscrete adaptivity [4, 5], the time semidiscrete adaptivity [18, 23] and space-time completely discrete scheme [12, 29, 30]. Combining the energy technique with the idea of elliptic reconstruction, Makridakis and Nochetto established an optimal a posteriori error estimates of semidiscrete finite element method for linear parabolic problem in [21]. Later, based on backward Euler method, Lakkis and Makridakis [20] extended these techniques to completely discrete cases. The role of the elliptic reconstruction in a posteriori error estimates is similar to the role played by elliptic projection introduced by Wheeler [33] for obtaining optimal a priori error estimates of finite element method for parabolic problems. Thanks to the elliptic reconstruction $\tilde{u}$, the error $u - u_h$ ($u_h$ is the numerical solution) can be split into two parts, one is $u - \tilde{u}$ and the other is $\tilde{u} - u_h$. The estimates of $\tilde{u} - u_h$ are based on a posteriori analysis of an elliptic problem, while the estimates of $u - \tilde{u}$ can be controlled by energy arguments in terms of the estimates of $\tilde{u} - u_h$. This analysis is further developed for linear parabolic problems by maximum norm estimates [11], discontinuous Galerkin method [15] or the mixed finite element method [22].

Although some results about evolution equations have been reported, for nonlinear case, there are still some open research issues. In this paper, we focus on the development of a posteriori error bounds for Burgers equation. Burgers equation is of interest primarily as a model for the unsteady incompressible Navier-Stokes equations. This equation has been investigated by many scientists [3, 6, 31]. There are examples of a posteriori error bounds for the steady Burgers equation [32] and incompressible Navier-Stokes equations [8, 9, 25, 26]. By energy method and an appropriate elliptic reconstruction, Makridakis et. al. have established a posteriori estimates with optimal norms for linear parabolic problem and unsteady Stokes equation in [11, 19, 20], respectively. Here, based on the techniques developed by Makridakis, we consider a posteriori error estimates of finite element method for two dimensional Burgers equation. By introducing an appropriate Burgers reconstruction, a posteriori error estimates with optimal order of convergence are derived in both semidiscrete and fully discrete formulations.

This article is organized as follows. In Section 2, we formulate the finite element method and recall some basic results. In Section 3, we present a posteriori error estimates of finite element method for spatial semidiscrete formulation. In Section 4, we present the fully discrete formulation and establish the corresponding a posteriori error estimates. Finally, we provide some numerical experiments to verify the performances of established error estimators.
2. Preliminaries

In this section, we present some classical results about Burgers equation and recall some important lemmas which are useful to derive our main results.

2.1. Basic notations and function setting for Burgers equation

For the mathematical setting of problem (1.1), we denote
\[ X = H^1_0(\Omega), \quad Y = L^2(\Omega), \quad D(A) = H^2(\Omega) \cap X, \]
where \( A \) is the Laplace operator \( Au = -\Delta u \). Standard notations are used for the Sobolev spaces with the norm and the seminorms in this work (see [1]). For all \( T > 0 \) and integer number \( n \geq 0 \), define
\[ H^n(0, T; W^{s,p}(\Omega)) = \{ v \in W^{s,p}(\Omega); \sum_{0 \leq i \leq n} \int_0^T (\frac{d^i}{dt^i} \| v \|_{s,p,\Omega})^2 dt < \infty \}, \]
with the norm given by
\[ \| v \|_{H^n(0, T; W^{s,p}(\Omega))} = \sum_{0 \leq i \leq n} \left[ \int_0^T \left( \frac{d^i}{dt^i} \| v \|_{s,p,\Omega} \right)^2 dt \right]^\frac{1}{2}. \]
Especially, as \( n = 0 \), we denote the norm as
\[ \| v \|_{0, T; L^2(W^{s,p}(\Omega))} = \left( \int_0^T \| v \|_{s,p,\Omega}^2 dt \right)^\frac{1}{2}. \]
Let
\[ L^\infty(0, T; W^{s,p}(\Omega)) = \{ v \in W^{s,p}(\Omega); \sup_{0 \leq t \leq T} \| v \|_{s,p,\Omega} < \infty \} \]
with the norm
\[ \| v \|_{L^\infty(0, T; W^{s,p}(\Omega))} = \sup_{0 \leq t \leq T} \| v \|_{s,p,\Omega}. \]

For the initial data \( u_0 \), we need the following assumption.
(A1). The initial velocity \( u_0 \in D(A) \) and \( f, f_t \in L^2(0, T; Y) \) are assumed to satisfy
\[ \| Au_0 \|_0 + \left( \int_0^T (\| f \|_0^2 + \| f_t \|_0^2) dt \right)^\frac{1}{2} \leq C. \]
Throughout of whole paper, the letter \( C > 0 \) denotes a generic constant, independent of mesh parameter and time step, and maybe different at different occurrences.

The continuous bilinear form \( a(\cdot, \cdot) \) on \( X \times X \) is defined by
\[ a(u, v) = \nu (\nabla u, \nabla v), \]
and the trilinear form by respectively

\[ b(u, v, w) = (u \nabla \cdot v, w) \quad \forall \, u, v, w \in X. \]

It is easy to verify that \( b(\cdot, \cdot, \cdot) \) satisfies the following important properties

\[ |b(u, v, w)| \leq N \| \nabla u \|_0 \| \nabla v \|_0 \| \nabla w \|_0, \]

for all \( u, v, w \in X \), where \( N = \sup_{0 \neq u, v, w \in X} \frac{|b(u, v, w)|}{\| \nabla u \|_0 \| \nabla v \|_0 \| \nabla w \|_0} \), and

\[ |b(u, v, w)| + |b(v, u, w)| + |b(w, u, v)| \leq C_0 \| \nabla u \|_0 \| Av \|_0 \| w \|_0, \]

for all \( u \in X, v \in D(A), w \in Y \).

For the subsequence convenience, we recall the Gronwall lemmas, which will be frequently used in the analysis of a posteriori error estimates (see [9, 10]).

**Lemma 2.1.** Let \( g(t), h(t), y(t) \) be three locally integrable nonnegative functions on time interval \([0, \infty)\), such that for any fixed time \( t_0 \geq 0 \) and all \( t \geq t_0 \)

\[ y(t) + G(t) \leq C + \int_{t_0}^{t} h(s) ds + \int_{t_0}^{t} g(s) y(s) ds, \]

where \( G(t) \) is a nonnegative function on \([0, \infty)\), \( C \geq 0 \) is a constant. Then,

\[ y(t) + G(t) \leq \left( C + \int_{t_0}^{t} h(s) ds \right) \exp(\int_{t_0}^{t} g(s) ds). \]

With above notations, the variational formulation of problem (1.1) reads as: For all \( t \in (0, T] \), find \( u \in X \), such that for all \( v \in X \)

\[ (u_t, v) + a(u, v) + b(u, u, v) = (f, v). \]

**Remark 2.2.** For the bounded of the exact solution, it follows from \( v = u_t \) in (2.3), (2.2) and Lemma 2.1 that

\[ \nu \| \nabla u \|_0^2 + \int_0^t \| u_t(s) \|_0^2 ds \leq \exp\left( \frac{2C_0^2 \int_0^t \| Av \|_0^2 ds}{\nu} \right) \left( \nu \| \nabla u(0) \|_0^2 + 2 \int_0^t \| f \|_0^2 ds \right). \]

2.2. Finite element approximation

Let \( h > 0 \) be a real positive parameter. The finite element subspace \( X_h \) of \( X \) is characterized by \( T_h = T_h(\Omega) \), a partitioning of \( \Omega \) into triangles \( K \) assumed to be uniformly shape-regular as \( h \to 0 \), see [10] for details.
Definition 2.3 (Elliptic projection). For \( \forall \ v \in X \), we define a projection operator \( R_h \in X_h \) by
\[
a(v - R_h, v_h) = 0
\]
for all \( v_h \in X_h \). Furthermore, for any \( v \in D(A) \) operator \( R_h \) satisfies (see [10]):
\[
\|v - R_h\|_0 + h\|
\[
\|\|\nabla(v - R_h)\|_0\| \leq Ch^2.
\]
(2.5)

Now, we present the spatial semidiscrete finite element formulation for problem (1.1): Find \( u_h \in X_h \), for all \( t \in (0, T] \) such that
\[
(a(u_h, v_h) + b(u_h, u_h, v_h) = (f, v_h) \quad \forall \ v_h \in X_h.
\]
(2.6)

Since the bilinear form \( (\nabla u_h, \nabla v_h) \) is coercive on \( X_h \times X_h \), it generates an invertible operator \( A_h : X_h \to X_h \) through the definition (see [16, 17])
\[
(A_h u_h, v_h) = (A_{1/2}^h u_h, A_{1/2}^h v_h) = (\nabla u_h, \nabla v_h), \quad \forall \ u_h, v_h \in X_h.
\]

By using a similar argument to one used in [13, 14, 27] about the nonlinear convection-diffusion problem, we can obtain the following result.

Lemma 2.4. Assume that \( \Omega \) is a bounded polygonal domain with a sufficiently smooth boundary \( \partial \Omega \), under the assumption of (A1), there exists a constant \( C \) such that
\[
\|
\[
\|\nabla(u - u_h)\|_0^2 + \int_0^T \|\nabla(u - u_h)\|_0^2 dt \leq Ch^2.
\]

Furthermore, for the semidiscrete finite element scheme (2.6), the following smooth properties of \( u_h \) hold.

Lemma 2.5. Under the assumptions of Lemma 2.4, there exists a constant \( C \) such that
\[
\|
\[
\|A_h u_h\|_0^2 + \nu \|\nabla u_h\|_0^2 + \int_0^t \|A_h u_h(s)\|_0^2 ds + \int_0^t \|u_t(s)\|_0^2 ds \leq C.
\]

Proof. For the \( L^2 \)-norm of \( \|A_h u_h\|_0 \), there is
\[
\|
\[
\|A_h u_h\|_0 \leq \sup_{v_h \in L^2(\Omega)^2} \frac{\|(A_h u_h, v_h)\|_0}{\|v_h\|_0}.
\]

For \( \forall \ v_h \in L^2(\Omega)^2 \), we split \( v_h \) into two parts, i.e., there exist uniqueness \( v_h^1 \in X_h \) and \( v_h^2 \in X_h^\perp \), such that \( v = v_h^1 + v_h^2 \in X_h \oplus X_h^\perp = L^2(\Omega)^2 \). Then, according to the definition of \( A_h \), one finds
\[
(A_h u_h, v_h) = (A_h u_h, v_h^1) + (A_h u_h, v_h^2) = (A_h u_h, v_h^1).
\]
As a consequence, we have

\[ \|A_h u_h\|_0 = \sup_{0 \neq v_h \in L^2(\Omega)^2} \frac{\|(A_h u_h, v_h)\|_0}{\|v_h\|_0} = \sup_{0 \neq v_h \in X_h, 0 \neq v_h \in L^2(\Omega)^2} \frac{\|(A_h u_h, v^1_h)\|_0}{\|v_h\|_0} \]

\[ \leq \sup_{0 \neq v_h \in X_h, 0 \neq v_h \in L^2(\Omega)^2} \frac{\|(\nabla (u - u_h), \nabla v^1_h)\|_0 + \|(A u, v^1_h)\|_0}{\|v_h\|_0} \]

\[ \leq \sup_{0 \neq v_h \in X_h, 0 \neq v_h \in L^2(\Omega)^2} \frac{\|(\nabla (u - u_h))\|_0 \|\nabla v^1_h\|_0 + \|A u\|_0 \|v^1_h\|_0}{\|v_h\|_0} \]

\[ \leq C h^{-1} \|\nabla (u - u_h)\|_0 + \|A u\|_0. \]

It follows from Lemma 2.4 that

\[ \|A_h u_h\|_0 + \int_0^t \|A_h u_h(s)\|^2_0 ds \leq \tilde{C}. \quad (2.7) \]

Choosing \( v_h = u_{ht} \) in (2.6) and using (2.1), we obtain

\[ \|u_{ht}\|_0^2 + \frac{\nu}{2} \frac{d}{dt} \|\nabla u_h\|_0^2 = (f, u_{ht}) - b(u_h, u_h, u_{ht}) \]

\[ \leq \|f\|_0 \|u_{ht}\|_0 + C_0 \|A_h u_h\|_0 \|\nabla u_h\|_0 \|u_{ht}\|_0 \]

\[ \leq \|f\|_0^2 + C_0^2 \|A_h u_h\|_0^2 \|\nabla u_h\|_0^2 + \frac{1}{2} \|u_{ht}\|_0^2. \]

Integrating with respect to time from 0 to \( t \), using Lemma 2.1 and (2.7), we find that

\[ \nu \| \nabla u_h \|_0^2 + \int_0^t \| u_{ht}(s) \|_0^2 \, ds \leq \exp\left( \frac{2C_0^2 \tilde{C}}{\nu} \right) \left( \nu \| \nabla u_h(0) \|_0^2 + 2 \int_0^t \| f \|_0^2 \, ds \right). \quad (2.8) \]

We end this section by introducing some properties about \( L^2 \)-projection operators, which can be found in reference [10].

**Lemma 2.6.** Assume that there exist two \( L^2 \)-projection operators \( I_h : X \to X_h \) and \( I^h_r : X \to X^h_r \) (\( X^h_r \) will be defined in Section 4), which are defined by

\[ (\phi - I^h_r \phi, w_h) = 0 \quad \forall \ w_h \in X_h \text{ or } X^h_r, \quad \phi \in X \text{ (} I^h_r \text{ takes } I_h \text{ or } I_r^h). \]

Furthermore, if \( \phi \in D(A) \), the following properties hold

\[ h^i \| \phi - I^h_r \phi \|_{0,K} \leq C h^{i-2} \| \phi \|_{2,\omega_K} \quad (i = 0, 1), \quad \| \phi - I^h_r \phi \|_{0,E} \leq C h^{i/2} \| \phi \|_{1,\omega_K}, \]

where \( \omega_K = \cup_{K \cap K \neq \emptyset} K' \) and \( \omega_E = \cup_{E \cap K \neq \emptyset} K \) for \( \forall \ K, K' \in T_h \).
3. A posteriori error estimates for semidiscrete formulation

In this section, we present a posteriori error estimates of Burgers equation in spatial semidiscrete formulation (2.6), and establish the computable upper and lower bounds for numerical solution \( u_h \) in various of norms.

Let \( e_u = u_h - u \). It follows from (2.3) and (2.6) that \( e_u \) satisfies

\[
(e_{ut}, v) + a(e_u, v) + b(u_h, u_h, v) - b(u, u, v) = r(v),
\]

where the residual \( r(v) \) is given by

\[
r(v) = (u_{ht}, v) + \nu(\nabla u_h, \nabla v) + b(u_h, u_h, v) - (f, v).
\]

From (2.6), we know that \( r(v_h) = 0 \) for all \( v_h \in X_h \).

We split the error \( e_u \) into two parts

\[
e_u = u_h - u = (\bar{u} - u) - (\bar{u} - u_h) = \xi_u - \eta_u.
\]

Then, (3.1) can be rewritten as

\[
(\xi_{ut}, v) + a(\xi_u, v) + b(\xi_u, u_h, v) + b(u, \xi_u, v) = r(v) + (\eta_{ut}, v) + a(\eta_u, v) + b(\eta_u, u_h, v) + b(u, \eta_u, v).
\]

Now, we introduce the Burgers reconstruction \( \tilde{u} \in X \) of \( u_h(t) \) for all \( t \in (0, T] \).

**Definition 3.1 (Burgers reconstruction).** For given \( u_h \), we define the Burgers reconstruction \( \tilde{u}(t) \) satisfying

\[
a(\tilde{u} - u_h, v) + b(\eta_u, u_h, v) + b(u, \eta_u, v) = -r(v).
\]

**Remark 3.2.** For the given \( u_h, r \), from the continuity and coercivity of \( a(\cdot, \cdot) \), it is easy to know that (3.4) admits a unique solution \( \tilde{u} \in X \) for all \( t \in (0, T] \).

By the definition of (3.4), equation (3.3) can be transformed into

\[
(\xi_{ut}, v) + a(\xi_u, v) + b(\xi_u, u_h, v) + b(u, \xi_u, v) = (\eta_{ut}, v).
\]

3.1. Error estimates between exact solution and Burgers reconstruction

In this subsection, we derive some estimates about problem (3.5) by using the energy method and some standard techniques in the analysis of parabolic equations.

**Lemma 3.3.** Assume that \( \Omega \) is a bounded polygonal domain with a sufficiently smooth boundary \( \partial \Omega \). Let \( \xi_u \) be the solution of (3.5). Then, for all \( t \in (0, T] \), there exists a constant \( C \) such that

\[
\|\xi_u(t)\|_0^2 + \nu \int_0^t \|\nabla \xi_u(s)\|_0^2 ds \leq C \left( \|\xi_u(0)\|_0^2 + \int_0^t \|\eta_{ut}(s)\|_0^2 ds \right), \tag{3.6}
\]

\[
(\nu \|\nabla \xi_u(t)\|_0 + \int_0^t \|\xi_{ut}(s)\|_0 ds)^{\frac{1}{2}} \leq C \left( \nu^{1/2} \|\nabla \xi_u(0)\|_0 + \left( \int_0^t \|\eta_{ut}(s)\|_0^2 ds \right)^{\frac{1}{2}} \right). \tag{3.7}
\]

7
Proof. Choosing \( v = \xi_u \) in (3.5), by Cauchy inequality and (2.2), we obtain that

\[
\frac{1}{2} \frac{d}{dt} \| \xi_u \|_0^2 + \nu \| \nabla \xi_u \|_0^2 = (\eta_{ut}, \xi_u) - b(\xi_u, u_h, \xi_u) - b(u, \xi_u, \xi_u) \\
\leq \| \eta_{ut} \|_0 \| \xi_u \|_0 + C_0(\| A_h u_h \|_0 + \| A u \|_0) \| \nabla \xi_u \|_0 \| \xi_u \|_0 \\
\leq \left( \frac{C_0^2(\| A_h u_h \|_0^2 + \| A u \|_0^2)}{\nu} + \frac{1}{2} \right) \| \xi_u \|_0^2 + \frac{1}{2} \| \eta_{ut} \|_0^2 + \nu \| \nabla \xi_u \|_0^2.
\]

Kicking the last term, using Lemma 2.3 and integrating with respect to time from 0 to \( t \), we yield

\[
\| \xi_u(t) \|_0^2 + \nu \int_0^t \| \nabla \xi_u(s) \|_0^2 ds \leq \| \xi_u(0) \|_0^2 + \int_0^t \| \eta_{ut}(s) \|_0^2 ds + C \int_0^t \| \xi_u(s) \|_0^2 ds. \tag{3.8}
\]

The result in (3.6) comes from Lemma 2.1.

Secondly, taking \( v = A \xi_u \) in (3.5) and using Cauchy inequality, one finds

\[
\frac{1}{2} \frac{d}{dt} \| \nabla \xi_u \|_0^2 + \nu \| A \xi_u \|_0^2 = (\eta_{ut}, A \xi_u) - b(\xi_u, u_h, A \xi_u) - b(u, \xi_u, A \xi_u) \\
\leq \| \eta_{ut} \|_0 \| A \xi_u \|_0 + C_0(\| A_h u_h \|_0 + \| A u \|_0) \| \nabla \xi_u \|_0 \| A \xi_u \|_0 \\
\leq \frac{C_0^2}{\nu}(\| A_h u_h \|_0^2 + \| A u \|_0^2) \| \nabla \xi_u \|_0^2 + \frac{1}{\nu} \| \eta_{ut} \|_0^2 + \nu \| A \xi_u \|_0^2. \tag{3.9}
\]

Kicking the last term in (3.9), integrating it with respect to time from 0 to \( t \), using Lemmas 2.1 and 2.5, we obtain that

\[
\| \nabla \xi_u(t) \|_0^2 + \nu \int_0^t \| A \xi_u(s) \|_0^2 ds \leq C \left( \| \nabla \xi_u(0) \|_0^2 + \int_0^t \| \eta_{ut}(s) \|_0^2 ds \right). \tag{3.10}
\]

Thirdly, differentiating (3.5) with respect to time and choosing \( v = \xi_{ut} \), in terms of (2.1), (2.2) and Lemma 2.5, we obtain that

\[
\frac{1}{2} \frac{d}{dt} \| \xi_{ut} \|_0^2 + \nu \| \nabla \xi_{ut} \|_0^2 = (\eta_{utt}, \xi_{ut}) - b(\xi_{ut}, u_h, \xi_{ut}) - b(u, \xi_{ut}, \xi_{ut}) \\
\leq \| \eta_{utt} \|_0 \| \xi_{ut} \|_0 + C_0(\| A_h u_h \|_0 + \| A u \|_0) \| \nabla \xi_{ut} \|_0 \| \xi_{ut} \|_0 \\
\leq C_1 \left( \| \eta_{att} \|_0^2 + \| A \xi_{ut} \|_0^2 \| \xi_{ut} \|_0^2 + \nu \| \xi_{ut} \|_0^2 \right) + C_2 \left( \| A_h u_h \|_0^2 + \| A u \|_0^2 + 1 \right) \| \xi_{ut} \|_0^2 \leq \frac{\nu}{2} \| \nabla \xi_{ut} \|_0^2.
\]

Kicking the last term and integrating it with respect to time from 0 to \( t \), using Lemma 2.1, (3.6) and (3.10) we arrive at

\[
\| \xi_{ut} \|_0^2 + \nu \int_0^t \| \nabla \xi_{ut} \|_0^2 ds \leq C(\| \xi_u(0) \|_0^2 + \| \nabla \xi_u(0) \|_0^2 + \| \xi_{ut}(0) \|_0^2 + \| \eta_{utt}(0) \|_0^2 + \int_0^t \| \eta_{utt} \|_0^2 ds + \int_0^t \| \eta_{ut} \|_0^2 ds). \tag{3.11}
\]
Finally, taking $v = \xi_{ut}$ in (3.5) one finds that
\[
\|\xi_{ut}\|_{0}^{2} + \frac{\nu}{2} \frac{d}{dt} \|\nabla \xi_{u}\|_{0}^{2} = (\eta_{ut}, \xi_{ut}) - b(\xi_{u}, u_{h}, \xi_{ut}) - b(u, \xi_{u}, \xi_{ut}) \leq \|\eta_{ut}\|_{0} \|\xi_{ut}\|_{0} + C_{0}(\|A_{h}u_{h}\|_{0} + \|Au\|_{0})\|\nabla \xi_{u}\|_{0}\|\xi_{ut}\|_{0} \leq \|\eta_{ut}\|_{0}^{2} + C_{0}^{2}(\|A_{h}u_{h}\|_{2}^{2} + \|Au\|_{0}^{2})\|\nabla \xi_{u}\|_{0}^{2} + \frac{1}{2}\|\xi_{ut}\|_{0}^{2}.
\]

Kicking the last term, using Lemma 2.5 and integrating with respect to time from 0 to $t$, one finds
\[
\nu\|\nabla \xi_{u}(t)\|_{0}^{2} + \int_{0}^{t} \|\xi_{ut}(s)\|_{0}^{2} ds \leq \nu\|\nabla \xi_{u}(0)\|_{0}^{2} + 2\int_{0}^{t} \|\eta_{ut}(s)\|_{0}^{2} ds + C \int_{0}^{t} \|\nabla \xi_{u}(s)\|_{0}^{2} ds.
\]

By Lemma 2.1 and inequality $a^{2} + b^{2} \leq (a + b)^{2}$ ($a, b \geq 0$), we obtain the desired result (3.7). $\Box$

3.2. A posteriori error estimates for Burgers reconstruction

In this subsection, we derive a posteriori error estimates for Burgers equation in semidiscrete formulation (2.6). To achieve this aim, we need the following a posteriori estimates about $\eta_{u}$ and $\nabla \eta_{u}$ related to Burgers reconstruction (3.4).

**Lemma 3.4.** Assume that $\Omega$ is a bounded polygonal domain with a sufficiently smooth boundary. Let $\bar{u}$ and $u_{h}$ be the solutions of (3.4) and (2.6), respectively. Then, there exists a constant $C$ such that
\[
\|\eta_{u}\|_{0} \leq C\left( \sum_{K \in T_{h}} h_{K} \|u_{ht} - \nu \Delta u_{h} + u_{h} \nabla \cdot u_{h} - f\|_{0,K}^{2} + \sum_{E \in E_{h}} h_{E}^{3} \|\nu \nabla u_{h}\|_{0,E}^{2} \right)^{1/2},
\]
\[
\|\nabla \eta_{u}\|_{0} \leq C\left( \sum_{K \in T_{h}} h_{K}^{2} \|\nabla u_{ht} - \nu \Delta u_{h} + u_{h} \nabla \cdot u_{h} - f\|_{0,K}^{2} + \sum_{E \in E_{h}} h_{E} \|\nu \nabla u_{h}\|_{0,E}^{2} \right)^{1/2}.
\]

**Proof.** Firstly, consider that $\Phi \in D(A) \cap X$ is the solution of the elliptic problem
\[
a(\Phi, v) + b(u, v, \Phi) + b(v, u_{h}, \Phi) = (g, v), \tag{3.14}
\]
where $u$ and $u_{h}$ are the solution of (1.1) and (2.6) respectively. According to the Lax-Milgram Theorem, we know that system (3.14) is well-posed and has a unique solution $\Phi$ satisfying (see [9, 10])
\[
\|\Phi\|_{2} \leq C\|g\|_{0}. \tag{3.15}
\]
Choosing $v = \tilde{u} - u_{h}$ in (3.14) and using (2.4), one arrives at
\[
(\eta_{u}, g) = \nu(\nabla \Phi, \nabla \tilde{u}) - \nu(\nabla \Phi, \nabla u_{h}) + b(u, \tilde{u} - u_{h}, \Phi) + b(\tilde{u} - u_{h}, u_{h}, \Phi) = \nu(\nabla \Phi, \nabla \tilde{u}) - \nu(\nabla R_{h}, \nabla u_{h}) + b(u, \tilde{u} - u_{h}, \Phi) + b(\tilde{u} - u_{h}, u_{h}, \Phi) = \nu(\nabla (\Phi - R_{h}), \nabla \tilde{u}) + b(u, \tilde{u} - u_{h}, \Phi - R_{h}) + b(\tilde{u} - u_{h}, u_{h}, \Phi - R_{h}) + \nu(\nabla R_{h}, \nabla (\tilde{u} - u_{h})) + b(u, \tilde{u} - u_{h}, R_{h}) + b(\tilde{u} - u_{h}, u_{h}, R_{h}). \tag{3.16}
\]
From the definition of (3.4), we know that
\[ \nu(\nabla R_h, \nabla (\tilde{u} - u_h)) + b(u, \tilde{u} - u_h, R_h) + b(\tilde{u} - u_h, u_h, R_h) = -r(R_h). \]

According to (3.2) with \( v = R_h \in X_h \), equation (3.16) can be rewritten as
\[ (\eta_h, g) = \nu(\nabla (\Phi - R_h), \nabla \tilde{u}) + b(u, \tilde{u} - u_h, \Phi - R_h) + b(\tilde{u} - u_h, u_h, \Phi - R_h). \]

Using (2.4) again and noting (3.4), we deduce that
\[ (\eta_u, g) = \nu(\nabla (\Phi - R_h), \nabla \tilde{u}) + b(u, \tilde{u} - u_h, \Phi - R_h) + b(\tilde{u} - u_h, u_h, \Phi - R_h) \]
\[ = \nu(\nabla (\Phi - R_h), \nabla (\tilde{u} - u_h)) + b(u, \tilde{u} - u_h, \Phi - R_h) + b(\tilde{u} - u_h, u_h, \Phi - R_h) \]
\[ = -r(\Phi - R_h). \quad (3.17) \]

By (2.5), (3.2) and Green’s formula, one finds
\[ (\eta_u, g) = \nu(\nabla (\Phi - R_h), \nabla \tilde{u}) + b(u, \tilde{u} - u_h, \Phi - R_h) + b(\tilde{u} - u_h, u_h, \Phi - R_h) \]
\[ \leq \sum \left\Vert \nu(\nabla (\Phi - R_h), \nabla \tilde{u}) + b(u, \tilde{u} - u_h, \Phi - R_h) + b(\tilde{u} - u_h, u_h, \Phi - R_h) \right\Vert \Phi - R_h \right\Vert_{0,E} \]
\[ \leq C \left( \sum \left\Vert \nu(\nabla (\Phi - R_h), \nabla \tilde{u}) + b(u, \tilde{u} - u_h, \Phi - R_h) + b(\tilde{u} - u_h, u_h, \Phi - R_h) \right\Vert \right) \cdot \| \Phi \|_2. \quad (3.18) \]

Using elliptic regularity (3.15) in (3.18), we deduce that
\[ \frac{(\eta_u, g)}{\| g \|_0} \leq C \left( \sum \left\Vert h_K^4 \| u_{ht} - \nu \Delta u_h + u_h \nabla \cdot u_h - f \|_{0,K}^2 \right\Vert + \left( \sum \left\Vert h_K^3 [\nu \nabla u_h] \|_{0,E}^2 \right\Vert \right) \right). \quad (3.19) \]

Taking the supermum over \( g \), we obtain the desired result (3.12).

By differentiating (3.18) with respect to time, using (3.2) and (3.4), following the proofs of (3.12), we can obtain that
\[ \| \eta_{ht} \|_0 \leq C \left( \sum \left\Vert h_K^4 \| u_{ht} - \nu \Delta u_h + u_h \nabla \cdot u_h + u_h \nabla \cdot u_h - f \|_{0,K}^2 \right\Vert \right) + \left( \sum \left\Vert h_K^3 [\nu \nabla u_h] \|_{0,E}^2 \right\Vert \right)^{1/2}, \quad (3.20) \]

In order to obtain the estimate (3.13), we make the following assumption:
\[ \frac{\nu}{2} \geq N \left[ \left( \exp \left( \frac{2C_0^2 \nu}{\nu} \int_0^t \| A u \|^2 ds \right) \right) \left( \| \nabla u_h(0) \|^2_0 + \frac{2}{\nu} \int_0^t \| f \|^2 ds \right) \right]^{1/2}. \quad (3.21) \]

10
Let \( \eta \) be a refinement of macrotriangulation which is a triangulation of \( \Omega \) for Burgers equation (1.1) based on backward Euler scheme. Given two compatible triangulations \( T \) and \( \hat{T} \), respectively, we obtain that

\[

\begin{align*}
\nu \|\nabla \eta\|_0^2 &= -r(\eta) - b(\eta, u_h, \eta) - b(u, \eta, \eta) \\
&= -r(\eta - I_h \eta) - b(\eta, u_h, \eta) - b(u, \eta, \eta) \\
&\leq -r(\eta - I_h \eta) + N(\|\nabla u_h\|_0 + \|\nabla u\|_0) \|\nabla \eta\|_0^2 \\
&\leq (u_{ht} - \nu \Delta u_h + u_h \nabla \cdot u_h - f, \eta - I_h \eta) + \sum_{E \in \mathcal{E}_h} \int_E [\nu \nabla u_h] \cdot (\eta - I_h \eta) ds + N(\|\nabla u_h\|_0 + \|\nabla u\|_0) \|\nabla \eta\|_0^2.
\end{align*}
\]

By (2.8) and (3.21), we finish the rest of proof (3.12). \( \blacksquare \)

As a consequence, combining Lemmas 3.3 and 3.4, we obtain the main theorem of a posteriori error estimates for Burgers equation in semidiscrete formulation.

**Theorem 3.5.** Let \( \Omega \) be a bounded polygonal domain with a sufficiently smooth boundary, \( u \) and \( u_h \) the solutions of (2.3) and (2.6), respectively. Then, for all \( t \in (0, T] \) there exists a constant \( C \) such that

\[
\begin{align*}
\|u_h - u\|_0^2 &\leq C \left( \|u_h(0) - u_0\|_0^2 + \|\eta_u(0)\|_0^2 + \|\eta_u\|_0^2 + \int_0^t \|\eta_u(s)\|_0^2 ds \right), \\
\|\nabla (u_h - u)\|_0^2 &\leq C \left( \|\nabla (u_h(0) - u_0)\|_0^2 + \|\nabla \eta_u(0)\|_0^2 + \|\nabla \eta_u\|_0^2 + \int_0^t \|\eta_u(s)\|_0^2 ds \right),
\end{align*}
\]

where the estimates of \( \|\eta_u\|_0, \|\nabla \eta_u\|_0 \) and \( \|\eta_u\|_0 \) are given by (3.12), (3.13) and (3.20), respectively.

4. **A posteriori estimates for fully discrete scheme**

In this section, we consider a posteriori estimates of fully discrete approximation for Burgers equation (1.1) based on backward Euler scheme.

Set \( 0 = t_0 < t_1 < \cdots < t_N = T, I_n = (t_{n-1}, t_n) \) and denote \( k_n = t_n - t_{n-1} \). For \( \forall \ n \in [0, N] \), let \( T_n \) be a refinement of macrotriangulation which is a triangulation of the domain \( \Omega \) that satisfies the same conformity and shape regularity assumptions made on its refinements (see [9] for details). Set

\[
h_n(x) = \text{diam}(K), \quad \text{where } K \in T_n \text{ and } x \in K.
\]

Given two compatible triangulations \( T_{n-1} \) and \( T_n \), namely, they are refinements of the same macrotriangulation, let \( \hat{T}_n \) be the finest common coarsening of \( T_n \) and \( T_{n-1} \), whose mesh size is given by \( \hat{h}_n = \max(h_n, h_{n-1}) \). For more information, we can refer to Appendix A of reference [20].

Denote

\[
\partial_t \phi^n := \frac{1}{k_n} (\phi^n - \phi^{n-1}), \quad f^n = f(t_n).
\]
We set $X_h^n$ as the finite element subspace of $X$ defined over the triangulations $\mathcal{T}_n$.

Given $U^0 = I_h^n u_0$, find $\{U^n\}$ with $U^n \in X_h^n$ at $t = t_n$. For $\forall v \in X_h^n$ as $n = 0$

$$\nu(\nabla U^0, \nabla v) + b(U^0, U^0, v) = (f^0, v),$$  

(4.1)

and for $n \in [1 : N]$

$$\frac{1}{k_n}(U^n - U^{n-1}, v) + \nu(\nabla U^n, \nabla v) + b(U^n, U^n, v) = (f^n, v).$$  

(4.2)

For the existence, uniqueness and convergence of $U^n$ of problem (4.2), following the guidelines provided in [13, 14], we have the following results.

**Theorem 4.1.** Assume that $\Omega$ is a bounded polygonal domain with Lipschitz continuous boundary, under the assumption of (A1). Then, there exists a constant $C$ such that

$$\nu\|\nabla (u - U^n)\|_0^2 + \int_0^T \nu\|\nabla (u - U^n)\|_0^2 dt \leq Ch^2,$$

$$\|A_h U^n\|_0^2 + \nu\|\nabla U^n\|_0^2 + \int_0^t \|A_h U^n(s)\|_0^2 ds \leq C.$$

Using a sequence of discrete values $\{U^n\}, n = 0, 1, 2, \ldots, N$, we define a continuous piecewise linear function $U(t)$ for $\forall t \in [0, T]$ by

$$U(t) = (1 - \frac{t - t_{n-1}}{k_n})U^{n-1} + \frac{t - t_{n-1}}{k_n}U^n, \quad t_{n-1} < t \leq t_n, \quad n = 1, 2, \ldots, N. \quad (4.3)$$

Note that the time derivative of $U$ restricted to $I_n$ is

$$U_{t|I_n} = \partial_t U^n \quad \text{for} \quad \forall t \in I_n. \quad (4.4)$$

To motivate the use of Burgers reconstruction, we denote $e_u(t) = U(t) - u(t)$. For $\forall v \in X_h^n$, $e_u$ satisfies

$$(e_u, v) + \nu(\nabla e_u, \nabla v) + b(U, U, v) - b(u, u, v)$$

$$= \nu(\nabla (U - U^n), \nabla v) + (f^n - f, v) + \frac{1}{k_n}(I_h^n U^{n-1} - U^{n-1}, v) + \frac{1}{k_n}(U^n - I_h^n U^{n-1}, v)$$

$$+ \nu(\nabla U^n, \nabla v) - (f^n, v) + b(U, U, v) + b(U^n, U^n, v) - b(U^n, U^n, v). \quad (4.5)$$

Define the residual $r^n$ for $n = 1, 2, \ldots, N$ as

$$r^n(v) = \frac{1}{k_n}(U^n - I_h^n U^{n-1}, v) + \nu(\nabla U^n, \nabla v) + b(U^n, U^n, v) - (f^n, v). \quad (4.6)$$

**Definition 4.2 (Burgers reconstruction).** For given $U^n$ ($n = 0, 1, \ldots, N$), find $\tilde{u}^n \in X$ for $\forall v \in X$ such that

$$\nu(\nabla (\tilde{u}^n - U^n), \nabla v) + b(\tilde{u}^n - U^n, U, v) + b(u, \tilde{u}^n - U^n, v) = -r^n(v). \quad (4.7)$$
Combining (4.6) with (4.2) and Lemma 2.6, we know that

\[ r^n(v_h) = 0 \quad \text{for } \forall \ v_h \in X_h^n. \]

Note that \( U^n \) is Burgers reconstruction of \( \tilde{u}^n \) at \( t = t_n \). Using a sequence of discrete values \{\( \tilde{u}^n \)\} \( (n = 0, 1, \ldots, N) \) for \( \forall \ t \in [0, T] \), we define a continuous function of time as the continuous piecewise linear interpolation \( \tilde{u}(t) \):

\[ \tilde{u}(t) = (1 - \frac{t - t_{n-1}}{k_n})\tilde{u}^{n-1} + \frac{t - t_{n-1}}{k_n}\tilde{u}^{n}, \quad t_{n-1} < t \leq t_n, \ n = 1, 2, \ldots, N. \quad (4.8) \]

Furthermore, for \( \forall \ v \in X \) and \( \forall t \in [0, T] \), \( \tilde{u} \) satisfies

\[ \nu(\nabla(\tilde{u} - U), \nabla v) + b(\tilde{u} - U, U, v) + b(u, \tilde{u} - U, v) = -r(v), \quad (4.9) \]

where \( r(v) \) is piecewise linear interpolation of \{\( r^n \)\} \( n = 1 \). In order to use (4.7), we split the error of \( e_u \) into two parts

\[ e_u = (\tilde{u} - u) - (\tilde{u} - U) \triangleq \xi_u - \eta_u. \quad (4.10) \]

Thanks to (4.7), equations (4.5) can be rewritten as for \( \forall \ v \in X \)

\[ (\xi_{ut}, v) + \nu(\nabla \xi_u, \nabla v) + b(\xi_u, U, v) + b(u, \xi_u, v) = (\eta_{ut}, v) + \nu(\nabla (\tilde{u} - \tilde{u}^n), \nabla v) + (f^n - f, v) + \frac{1}{k_n} (I^n_h U^{n-1} - U^{n-1}, v) \]

\[ + b(U^n - u, U - U^n, v) + b(\tilde{u} - \tilde{u}^n, U, v) + b(u, \tilde{u} - \tilde{u}^n, v). \quad (4.11) \]

Note that

\[ \tilde{u} - \tilde{u}^n = -\frac{t_n - t}{k_n}(\tilde{u}^{n-1} - \tilde{u}^{n-1}) = -(t_n - t)\partial_t \tilde{u}^n, \quad (4.12) \]

\[ U - U^n = -\frac{t_n - t}{k_n}(U^{n-1} - U^{n-1}) = -(t_n - t)\partial_t U^n. \quad (4.13) \]

From equation (4.7), we deduce that

\[ \nu(\nabla \tilde{u}^n, \nabla v) + b(\tilde{u}^n, U, v) + b(u, \tilde{u}^n, v) = \nu(\nabla U^n, \nabla v) - r^n(v) + b(U^n, U, v) + b(u, U^n, v). \quad (4.14) \]

As a consequence, one finds

\[ \nu(\nabla (\tilde{u} - \tilde{u}^n), \nabla v) + b(\tilde{u} - \tilde{u}^n, U, v) + b(u, \tilde{u} - \tilde{u}^n, v) \]

\[ = \nu(\nabla (U^n - U^{n-1}), \nabla v) - (r^n(v) - r^{n-1}(v)) + b(U^n - U^{n-1}, U, v) + b(u, U^n - U^{n-1}, v). \quad (4.15) \]

Substituting (4.13) and (4.15) into (4.11), we obtain that

\[ (\xi_{ut}, v) + \nu(\nabla \xi_u, \nabla v) + b(\xi_u, U, v) + b(u, \xi_u, v) = (\eta_{ut}, v) + \frac{1}{k_n} (I^n_h U^{n-1} - U^{n-1}, v) + (f^n - f, v) + \frac{t_n - t}{k_n} \left((r^n(v) - r^{n-1}(v)) - \nu(\nabla (U^n - U^{n-1}), \nabla v) - b(U^n - U^{n-1}, U, v) - b(U^n, U^n - U^{n-1}, v)\right). \quad (4.16) \]
Next, we present some estimates about the errors between Burgers reconstruction (4.7) and the exact solution of (2.3) in various norms. In order to simplify the expressions, we introduce some notations as follows. Let

\[
\mathcal{E}_1^m = \sum_{n=1}^{m} \int_{t_{n-1}}^{t_n} \| f^n - f \|^2_0 ds, \quad \mathcal{E}_2^m = \sum_{n=1}^{m} \int_{t_{n-1}}^{t_n} \| \eta_{at}(s) \|^2_0 ds,
\]

\[
\mathcal{E}_3^m = \sum_{n=1}^{m} k_n \| \hat{h}_n(I - I^n_h) U^{n-1} \|^2_0,
\]

\[
\mathcal{E}_4^m = \sum_{n=1}^{m} k_n^2 \left( \| \hat{h}_n \partial_t r^n_u \|^2_0 + \| \hat{\nabla} \partial_t U^n \|^2_0 + \| \partial_t U^n \|^2_0 \right),
\]

\[
\mathcal{E}_5^m = \| h_1(I_h^n - I) \left( \frac{1}{k_1} U^0 \right) \|^2_0 + \sum_{n=1}^{m} k_n^2 \| \hat{h}_n \partial_t (I^n_h - I) \left( \frac{1}{k_m} U^{n-1} \right) \|^2_0 + \| h_m(I_h^m - I) \left( \frac{1}{k_m} U^{m-1} \right) \|^2_0.
\]

**Theorem 4.3.** Assume that \( \Omega \) is a convex polygonal domain. Let \( u \) and \( U \) be the solutions of (2.3) and (4.2), respectively. Then, for \( m \in [1, N] \) there exists a constant \( C \) such that

\[
\| \xi_u(t_m) \|^2_0 + \int_0^{t_m} \nu \| \nabla \xi_u(s) \|^2_0 ds \leq C \left[ \| e_u(0) \|^2_0 + \| \eta_u(0) \|^2_0 + \sum_{i=1}^{4} \mathcal{E}_i^m \right], \quad (4.17)
\]

\[
\nu \| \nabla \xi_u \|^2_0 + \int_0^{t_j} \| \xi_u(t_j) \|^2_0 ds \leq C \left[ \| e_u(0) \|^2_0 + \| \eta_u(0) \|^2_0 + \sum_{i=1}^{5} \mathcal{E}_i^m \right]. \quad (4.18)
\]

**Proof.** Choosing \( v = \xi_u \) in equation (4.16), and using Lemma 2.6, obtain that

\[
\begin{align*}
\frac{1}{2} \frac{d}{dt} \| \xi_u \|^2_0 + \nu \| \nabla \xi_u \|^2_0 &= (\eta_{at}, \xi_u) + (f^n - f, \xi_u) + \frac{1}{k_n} (I^n_h U^{n-1} - U^{n-1}, \xi_u) - b(U, \xi_u) - b(u, \xi_u) \\
&\quad + (t_n - t) \left( \partial_t r^n_u(\xi_u) - \nu (\hat{\nabla} \partial_t U^n, \nabla \xi_u) - b(\partial_t U^n, \xi_u) - b(U, \partial_t U^n, \xi_u) \right) \\
&= (\eta_{at}, \xi_u) + (f^n - f, \xi_u) + \frac{1}{k_n} (I^n_h U^{n-1} - U^{n-1}, \xi_u - I^n_h \xi_u) - b(U, \xi_u) - b(u, \xi_u) \\
&\quad + (t_n - t) \left( \partial_t r^n_u(\xi_u) - \nu (\hat{\nabla} \partial_t U^n, \nabla \xi_u) - b(\partial_t U^n, \xi_u) - b(U, \partial_t U^n, \xi_u) \right) \\
&\triangleq T_1^n + T_2^n + T_3^n + T_4^n + T_5^n.
\end{align*}
\]

(4.19)

Now, we estimate the right-hand side terms of (4.19) separately. For \( T_1^n \) and \( T_2^n \), with Cauchy inequality, it is easy to see that

\[
| (\eta_{at}, \xi_u) | + | (f^n - f, \xi_u) | \leq \| \eta_{at} \|_0 \| \xi_u \|_0 + \| f^n - f \|_0 \| \xi_u \|_0 \leq \frac{1}{2} (\| \eta_{at} \|^2_0 + \| f^n - f \|^2_0) + \frac{1}{2} \| \xi_u \|^2_0.
\]

14
For $T_3^n$, by Cauchy inequality and Lemma 2.6, we find that

$$\|T_3^n\| \leq k_n^{-1}\|I_h^n U^{n-1} - U^{n-1}\|_0 \|\xi_u - I_h^n \xi_u\|_0$$

$$\leq \frac{3}{2\nu}k_n^{-1}h_n(I - I_h^n)U^{n-1}\|_0^2 + \frac{\nu}{6}\|\nabla \xi_u\|_0^2.$$ 

For $T_4^n$ and $T_5^n$, by (2.2) and Theorem 4.1, we arrive at

$$|T_4^n| + |T_5^n| \leq C_0(\|A_hU\|_0 + \|Au\|_0)\|\nabla \xi_u\|_0 \|\xi_u\|_0$$

$$\leq \frac{3C_0^2}{2\nu}(\|A_hU\|_0^2 + \|A_{h\xi}U^n\|_0^2)\|\xi_u\|_0^2 + \frac{\nu}{6}\|\nabla \xi_u\|_0^2.$$

For $T_6^n$, using the fact that $r_h^n(v_h) = 0$ for $\forall v_h \in X_h$, we have $(r_1^n - r_1^{n-1})(v_h) = 0$ for all $v_h \in X^n \cap X^{n-1}$. Let $I_h^n$ be the $L^2$-projection relative to the finest common coarsening $\mathcal{T}_n$ of $\mathcal{T}_m$ and $\mathcal{T}_{n-1}$. For $\forall t \in (t_{n-1}, t_n]$, we deduce that

$$|T_6^n| \leq (t_n - t)\left(\partial_t v^n(\xi_u - I_h^n \xi_u) - \nu(\nabla \partial_t U^n, \nabla \xi_u) - b(\partial_t U^n, U, \xi_u) - b(U^n, \partial_t U^n, \xi_u)\right)$$

$$\leq \frac{3}{2\nu}k_n^2\left(C_0^2(\|A_hU\|_0^2 + \|A_{h\xi}U^n\|_0^2)\|\partial_t U^n\|_0^2 + \|\partial_t U^n\|_0^2 + \|\nabla \partial_t U^n\|_0^2) + \frac{\nu}{6}\|\nabla \xi_u\|_0^2.$$

Combining above inequalities with (4.19), integrating it with respect to time from 0 to $t_m$ with $m \in [1 : N]$, and using Lemma 2.1, we complete the proof of (4.17).

Next, we choose $v = \xi_u$ in (4.16) and obtain

$$\|\xi_u(t_m)\|_0^2 + \frac{\nu}{2}\int_0^{t_m}\|\nabla \xi_u(t)\|_0^2 + b(\xi_u, U, \xi_u) + b(u, \xi_u, \xi_u)$$

$$= (\eta U, \xi_u) + (f^n - f, \xi_u) + \frac{1}{k_n}(I_h^n U^{n-1} - U^{n-1}, \xi_u) + (t_n - t)\left(\partial_t v^n(\xi_u)$$

$$- \nu(\nabla \partial_t U^n, \nabla \xi_u) - b(\partial_t U^n, U, \xi_u) - b(U^n, \partial_t U^n, \xi_u)\right)$$

$$+ (t_n - t)\left(\partial_t v^n(\xi_u) - \nu(\nabla \partial_t U^n, \nabla \xi_u) - b(\partial_t U^n, U, \xi_u) - b(U^n, \partial_t U^n, \xi_u)\right)$$

$$\triangleq T_1^n + T_2^n + T_3^n + T_4^n. \quad (4.20)$$

Integrating (4.20) with respect to time from 0 to $t_m$ with $m \in [1 : N]$, one gets

$$\nu\|\nabla \xi_u(t_m)\|_0^2 + 2\int_0^{t_m}\|\xi_u(t)\|^2_0 ds + 2\int_0^{t_m}\left(b(\xi_u, U, \xi_u) + b(u, \xi_u, \xi_u)\right)ds$$

$$= \nu\|\nabla \xi_u(0)\|_0^2 + 2\sum_{n=1}^{m}\int_{t_{n-1}}^{t_n}(T_1^n + T_2^n + T_3^n + T_4^n)ds. \quad (4.21)$$

Set

$$\mathcal{F}^2(t_i) = \max_{0 \leq t_i \leq t_m} \left(\nu\|\nabla \xi_u(t_i)\|_0^2 + \int_0^{t_i}\|\xi_u(t)\|^2_0 dt\right), \quad \text{for } \forall i \in [0 : m]. \quad (4.22)$$
By Cauchy inequality and integration by parts, it is straight forward to deduce that

\[ |T_1^n| + |T_2^n| + |T_4^n| \leq \left[ \| \eta_{ut} \|_0 + \| f^n - f \|_0 + (t_n - t) \left( \| \partial_t r^n \|_0 + \| \Delta \partial_t U^n \|_0 \right) + C_0(\| A_h U \|_0 + \| A_h U^n \|_0) \| \nabla \partial_t U^n \|_0 \right] \| \xi_{ut} \|_0. \]

By using Cauchy inequality again, we get

\[ 2 \sum_{n=1}^{m} \int_{t_{n-1}}^{t_n} (T_1^n + T_2^n + T_4^n) ds \leq 2 \left\{ \sum_{n=1}^{m} \kappa_n^3 \left( C_0^2(\| A_h U \|_0^2 + \| A_h U^n \|_0^2) \| \nabla \partial_t U^n \|_0^2 + \| \partial_t r^n \|_0^2 + \| \Delta \partial_t U^n \|_0^2 \right) \right\}^{1/2} \]

\[ + \left( \sum_{n=1}^{m} \int_{t_{n-1}}^{t_n} \| \eta_{ut} \|_0^2 ds \right)^{1/2} + \left( \sum_{n=1}^{m} \int_{t_{n-1}}^{t_n} \| f^n - f \|_0^2 ds \right)^{1/2} \right\} \cdot \left( \int_0^{t_m} \| \xi_{ut} \|_0^2 ds \right)^{1/2}. \]

For \( T_2^n \), using integration for \( \xi_{ut} \) from \( t_{n-1} \) to \( t_n \), and then applying summation by parts, we arrive at

\[ \sum_{n=1}^{m} \int_{t_{n-1}}^{t_n} \frac{1}{\kappa_n} (I^n_h U^{n-1} - U^{n-1}, \xi_{ut}) ds \]

\[ = \sum_{n=1}^{m} \frac{1}{\kappa_n} (I^n_h U^{n-1} - U^{n-1}, \xi^0_{u} - \xi^0_{u}) \]

\[ = \left( (I^n_h - I)(\frac{1}{\kappa^n} U^{m-1}), \xi^m_{u} \right) - \left( (I^n_h - I)(\frac{1}{\kappa^0} U^0), \xi^0_{u} \right) \]

\[ + \sum_{n=2}^{m} \kappa_n \left( \partial_t (I^n_h - I)(\frac{1}{\kappa^n} U^{n-1}), \xi^{n-1}_{u} \right). \]

From (2.2), we derive that

\[ 2 \int_0^{t_m} (b(\xi_u, U, \xi_{ut}) + b(u, \xi_u, \xi_{ut})) ds \leq 2C_0 \int_0^{t_m} (\| A_h U \|_0 + \| A u \|_0) \| \nabla \xi_u \|_0 \| \xi_{ut} \|_0 ds \]

\[ \leq C \left( \int_0^{t_m} (\| A_h U \|_0^2 + \| A u \|_0^2) \| \nabla \xi_u \|_0^2 ds \right)^{1/2} \cdot \left( \int_0^{t_m} \| \xi_{ut} \|_0^2 ds \right)^{1/2}. \]

16
By Lemma 2.6 and (4.22), one deduces that

\[
\sum_{n=1}^{m} \int_{t_{n-1}}^{t_{n}} \frac{1}{k_{n}} (I_{h}^{m}U^{m-1} - U^{n-1}, \xi_{u}) ds
\]

\[
= \left( (I_{h}^{m} - I)(\frac{1}{k_{m}} U^{m-1}), \xi_{u}^{m} - I_{h}^{m} \xi_{u}^{m} \right) - \left( (I_{h}^{1} - I)(\frac{1}{k_{1}} U^{0}), \xi_{u}^{0} - I_{h}^{1} \xi_{u}^{0} \right)
\]

\[
+ \sum_{n=2}^{m} k_{n} \left( \partial_{h}(I_{h}^{n} - I)(\frac{1}{k_{n}} U^{n-1}), \xi_{u}^{n-1} - I_{h}^{n} \xi_{u}^{n-1} \right)
\]

\[
\leq C \left( \| h_{1}(I_{h}^{1} - I)(\frac{1}{k_{1}} U^{0}) \|_{0} \| \nabla \xi_{u}^{0} \|_{0} + \| h_{m}(I_{h}^{m} - I)(\frac{1}{k_{m}} U^{m-1}) \|_{0} \| \nabla \xi_{u}^{m} \|_{0}
\]

\[
+ \sum_{n=2}^{m} k_{n} \| \hat{h}_{n} \partial_{h}(I_{h}^{n} - I)(\frac{1}{k_{n}} U^{n-1}) \|_{0} \| \nabla \xi_{u}^{n-1} \|_{0} \right)
\]

\[
\leq C \left( \| h_{1}(I_{h}^{1} - I)(\frac{1}{k_{1}} U^{0}) \|_{0} + \sum_{n=2}^{m} k_{n} \| \hat{h}_{n} \partial_{h}(I_{h}^{n} - I)(\frac{1}{k_{n}} U^{n-1}) \|_{0}
\]

\[
+ \| h_{m}(I_{h}^{m} - I)(\frac{1}{k_{m}} U^{m-1}) \|_{0} \right) \cdot \mathcal{F}(t_{i}).
\]

Combining above inequalities with (4.21), Theorem 4.1 and replacing \( t_{m} \) by \( t_{i} \), we complete the proof of (4.18). \( \Box \)

Since (4.9) is quite similar in form to (3.4), we can prove the error estimates similar to Lemma 3.4 by following closely to the proof of Lemma 3.4. Here, we omit the proof.

Using required estimates of \( \eta_{u} \) and \( \eta_{ud} \) in Theorem 4.1, we obtain the final theorem of this section. Now, we firstly introduce some notations to be used in what follows. Set

\[
\mathcal{E}_{6}^{m} = \sum_{E \in \mathcal{E}_{h}} h_{E}^{3} \| \nabla U^{0} \|_{0,E}^{2} + \sum_{K \in \mathcal{T}_{h}} h_{K}^{4} \| r_{1}^{0} \|_{0,K}^{2},
\]

\[
\mathcal{E}_{7}^{m} = \sum_{E \in \mathcal{E}_{h}} h_{E}^{3} \| \nabla U^{m} \|_{0,E}^{2} + \sum_{K \in \mathcal{T}_{h}} h_{K}^{4} \| r_{1}^{m} \|_{0,K}^{2},
\]

\[
\mathcal{E}_{8}^{m} = \sum_{n=1}^{m} k_{n} \left( \sum_{E \in \mathcal{E}_{h}} h_{E}^{3} \| \partial_{h} \nabla U^{m} \|_{0,E}^{2} + \sum_{K \in \mathcal{T}_{h}} h_{K}^{4} \| \partial_{h} r_{1}^{m} \|_{0,K}^{2} \right),
\]

\[
\mathcal{E}_{9}^{m} = \sum_{E \in \mathcal{E}_{h}} h_{E}^{3} \| \nabla U^{m} \|_{0,E}^{2} + \sum_{K \in \mathcal{T}_{h}} h_{K}^{2} \| r_{1}^{m} \|_{0,K}^{2}.
\]

**Theorem 4.4.** Under the assumptions of Theorem 4.3 and (A1), set \( u \) and \( U \) be the solution of (2.3) and (4.2), respectively. Then, for \( m \in [1, N] \), the following
estimates hold
\[
\|U^m - u(t_m)\|_0^2 \leq C\left[\|e_u(0)\|_0^2 + \varepsilon_1^m + \varepsilon_3^m + \varepsilon_4^m\right] + \sum_{i=6}^{8} \varepsilon_i^m,
\]
\[
\|\nabla(U - u)\|_0^2 + \int_0^{t_m} \|U_t - u_t\|^2 ds \leq C\left[\|e_u(0)\|_0^2 + \varepsilon_1^m + \sum_{i=3}^{6} \varepsilon_i^m + \varepsilon_8^m + \varepsilon_9^m\right],
\]

5. Numerical experiment

In this section, we provide some numerical results to verify the performance of the established posteriori error estimators. For the computed quantities like errors and indicators, we denote \(\|e_u\|_1 = \|u - u_h\|_1\), the error indicator \(\eta\) and the number of triangulaires in \(\mathcal{T}_h\) (NT) which are output of the adaptive algorithm for a given tolerance \(\varepsilon\). The experimental convergence rates are given by
\[
\alpha_{e_u} = \frac{2 \ast \log[\|e_u(\varepsilon_1)\|_0/\|e_u(\varepsilon_2)\|_0]}{\log[NT(\varepsilon_1)/NT(\varepsilon_1)]}, \quad \alpha_\eta = \frac{2 \ast \log[\eta(\varepsilon_1)/\eta(\varepsilon_2)]}{\log[NT(\varepsilon_1)/NT(\varepsilon_1)]}.
\]

The effectiveness index is defined as ratio of a posteriori error bound and an approximate norm of the actual error, i.e., \(\|e_u\|_1/\eta\). For a good estimator, this quantity should be a constant, independent of the mesh sizes and the time steps. Although our theoretical findings do not include a proof of efficiency, numerical experiments provide evidences of the efficiency of the estimators.

For the space-time algorithm, we follow the guideline provided in [22]. In order to keep the completeness of our work, we present the outline as follows.

Algorithm. Let \(\mathcal{T}_0\) be a regular triangulation and \(\varepsilon\) the given tolerance.
(i). Compute on the shape-regular partition \(\mathcal{T}_0\) with \(t_0 = 0\).
(ii). Set an initial time step \(k_0\), using \(U^0\) to compute \(\|u - U^0\|_1\) and \(\eta\) on \(\mathcal{T}_0\).
(iii). Begin the time loop with obtained \(\mathcal{T}_{n-1}, \mathcal{T}_n\) and \(U^{n-1}\)
   (1) Set the time step \(t_n = \min(t_{n-1} + k_n, T)\),
   (2) Use \(k_n\) to compute \(\hat{U}^n\) on \(\mathcal{T}_n\), and then compute \(\hat{\eta}^n\),
   (3) Set \(t_n = \min(t_{n-1} + (\hat{t}_n - t_{n-1})\varepsilon/\hat{\eta}^n, T)\), and obtain that \(k_n = t_n - t_{n-1}\),
   (4) Use \(\mathcal{T}_n, U^{n-1}\) and \(k_n\) to compute \(U^n\) and \(\eta^n\),
   (5) Adapt mesh \(\mathcal{T}_n\) to obtain \(\mathcal{T}_{n+1}\). Use \(U^{n-1}\) and \(k_n\) to compute \(U^n\),
   (6) For the next iteration, denote \(\mathcal{T}_{n+1} = \mathcal{T}_n\) and \(\hat{k}_{n+1} = \hat{k}_{n}\).
(iv) End the time loop and finish the computation.

Remark 5.1. For simplifying the computation, we adopt Oseen iteration to treat the nonlinear term, and set all the constants \(C\) involved in indicator and \(\nu\) equal to 1. For the time steps, as the strategy used in [24], we choose a maximum over the time step instead of summing up over different time steps.
Consider the nonlinear parabolic problem

\[
\begin{cases}
  u_t - \Delta u + u \nabla \cdot u = f & \text{in } \Omega \times (0,1] \\
  u = g & \text{on } \partial \Omega \times (0,1] \\
  u(0) = 0 & \text{on } \Omega \times \{0\},
\end{cases}
\]

with \( \Omega = [0,2]^2 \) and \( T = 1 \). Choose \( f \) and \( g \) with the exact solution

\[
u(x, y) = \frac{1}{10} \sin\left(\frac{\pi t}{2}\right) \exp \left\{ -20\left\| (x - \frac{1}{2}, y - \frac{1}{2}) - \frac{3}{4} \left( \cos\left(\frac{\pi}{T}\right), \sin\left(\frac{\pi t}{2}\right) \right) \right\} \]

where \( \|(x, y)\| = (x^2 + y^2)^{1/2} \). We adopt the linear polynomial to seek the exact solution by backward Euler scheme in time. Table 1 presents the errors and convergence order with different tolerances \( \varepsilon \). As expected, we can see that as \( \varepsilon \) becomes small, the errors go down while the numbers of triangulares increase quickly. The effectiveness index approaches 0.06, which is a constant independent of \( NT \) and time steps. Next, fixed the tolerance \( \varepsilon = 0.04 \), the meshes, the profiles of \( u_h \) and numerical solutions \( u_h \) are described in Figures 1-4 at different times. From these Figures, we can see that more meshes are concentrated on the critical region where the solution vary sharply. Figure 5 expresses the variation of time step as time increases with different tolerances \( \varepsilon \). Generally speaking, the smaller of \( \varepsilon \), the smaller of time step. Furthermore, as the time increases, the time step becomes smaller and smaller. Finally, we compare the profiles of solutions, including the exact solution and numerical solution with different \( \varepsilon \) in Figure 6. From these figures we can see that as the tolerance \( \varepsilon \) decreases, we can simulate the exact solution more precisely.

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**References**


Figure 1: Meshes, the profiles of $u_h$ and numerical solution $u_h$ at time $t=0.160751$.

Figure 2: Meshes, the profiles of $u_h$ and numerical solution $u_h$ at time $t=0.442934$.


Figure 3: Meshes, the profiles of $u_h$ and numerical solution $u_h$ at time $t=0.765945$.

Figure 4: Meshes, the profiles of $u_h$ and numerical solution $u_h$ at time $t=1$.


Figure 5: The variable of time step with different $\varepsilon$. (a) $\varepsilon = 0.06$, (b) $\varepsilon = 0.04$.

Figure 6: The profiles of exact solution and numerical solution. (a) Exact solution, (b) $\varepsilon = 0.04$, (c) $\varepsilon = 0.08$.


