A posteriori error estimates of fully discrete finite element method for Burgers equation in 2-D☆

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Abstract

Here, a posteriori error estimates of finite element method are derived for Burgers equation in two-dimension. By use of constructing an appropriate Burgers reconstruction, some posteriori estimates in $L^\infty(L^2)$, $L^\infty(H^1)$ and $L^2(L^2)$ norms with optimal order of convergence are established for the semidiscrete scheme. Furthermore, a fully discrete scheme with corresponding posteriori estimates are studied based on backward Euler scheme. Finally, some numerical results are presented to verify the performance of the established posteriori error estimators.

Keywords: A posteriori error estimates; Burgers equation; Burgers reconstruction; Backward Euler scheme; Adaptive algorithms
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1. Introduction

We assume that Ω is a bounded polygonal domain in $\mathbb{R}^2$ with a sufficiently smooth boundary $\partial\Omega$ for our study in this paper. Consider the following Burgers equations

$$\begin{cases} u_t - \nu\Delta u + u\nabla\cdot u = f & \text{in } \Omega\times(0,T], \\ u = 0 & \text{on } \partial\Omega\times(0,T], \\ u = u_0 & \text{on } \Omega\times\{0\}, \end{cases} \tag{1.1}$$

where u, f and u_0 are the velocity, the prescribed body force, and the initial velocity respectively with the diffusion coefficient $\nu > 0$, and the finite time $T > 0$.

In the last decade years, there is a growing demand of designing and developing reliable and efficient space-time numerical algorithms for the parabolic partial

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differential equations. Most of these methods are based on a posteriori error estimators due to the significant advantages of the adaptive method [2]. Although the theory of a posteriori error estimates of finite element method for elliptic problems is well-developed [3, 28], the theory for parabolic problems is less developed, and only a few results have been published until now, such as the spatial semidiscrete adaptivity [4, 5], the time semidiscrete adaptivity [18, 23] and space-time completely discrete scheme [12, 29, 30]. Combining the energy technique with the idea of elliptic reconstruction, Makridakis and Nochetto established an optimal a posteriori error estimates of semidiscrete finite element method for linear parabolic problem in [21]. Later, based on backward Euler method, Lakkis and Makridakis [20] extended these techniques to completely discrete cases. The role of the elliptic reconstruction in a posteriori error estimates is similar to the role played by elliptic projection introduced by Wheeler [33] for obtaining optimal a priori error estimates of finite element method for parabolic problems. Thanks to the elliptic reconstruction $\widetilde{u}$, the error $u - u_h$ (u_h is the numerical solution) can be split into two parts, one is $u - \widetilde{u}$ and the other is $\widetilde{u} - u_h$. The estimates of $\widetilde{u} - u_h$ are based on a posteriori analysis of an elliptic problem, while the estimates of $u - \widetilde{u}$ can be controlled by energy arguments in terms of the estimates of $\widetilde{u} - u_h$. This analysis is further developed for linear parabolic problems by maximum norm estimates [11], discontinuous Galerkin method [15] or the mixed finite element method [22].

Although some results about evolution equations have been reported, for nonlinear case, there are still some open research issues. In this paper, we focus on the development of a posteriori error bounds for Burgers equation. Burgers equation is of interest primarily as a model for the unsteady incompressible Navier-Stokes equations. This equation has been investigated by many scientists [3, 6, 31]. There are examples of a posteriori error bounds for the steady Burgers equation [32] and incompressible Navier-Stokes equations [8, 9, 25, 26]. By energy method and an appropriate elliptic reconstruction, Makridakis et. al. have established a posteriori estimates with optimal norms for linear parabolic problem and unsteady Stokes equation in [11, 19, 20], respectively. Here, based on the techniques developed by Makridakis, we consider a posteriori error estimates of finite element method for two dimensional Burgers equation. By introducing an appropriate Burgers reconstruction, a posteriori error estimates with optimal order of convergence are derived in both semidiscrete and fully discrete formulations.

This article is organized as follows. In Section 2, we formulate the finite element method and recall some basic results. In Section 3, we present a posteriori error estimates of finite element method for spatial semidiscrete formulation. In Section 4, we present the fully discrete formulation and establish the corresponding a posteriori error estimates. Finally, we provide some numerical experiments to verify the performances of established error estimators.

2. Preliminaries

In this section, we present some classical results about Burgers equation and recall some important lemmas which are useful to derive our main results.

2.1. Basic notations and function setting for Burgers equation

For the mathematical setting of problem (1.1), we denote

$$X = H_0^1(\Omega), \quad Y = L^2(\Omega), \quad D(A) = H^2(\Omega) \cap X,$$

where A is the Laplace operator $Au = -\Delta u$. Standard notations are used for the Sobolev spaces with the norm and the seminorms in this work (see [1]). For all $T > 0$ and integer number $n \geq 0$, define

$$H^n(0,T; W^{s,p}(\Omega)) = \{v \in W^{s,p}(\Omega); \sum_{0\leq i\leq n} \int_0^T (\frac{d^i}{dt^i}\|v\|_{s,p,\Omega})^2 dt < \infty\},$$

with the norm given by

$$\|v\|_{H^n(0,T;W^{s,p}(\Omega))} = \sum_{0\leq i\leq n} [\int_0^T (\frac{d^i}{dt^i}\|v\|_{s,p,\Omega})^2]^{\frac{1}{2}}.$$

Especially, as $n = 0$, we denote the norm as

$$\|v\|_{0,T;L^2(W^{s,p}(\Omega))} = (\int_0^T \|v\|^2_{s,p,\Omega} dt)^{\frac{1}{2}}.$$

Let

$$L^\infty(0,T; W^{s,p}(\Omega)) = \{v \in W^{s,p}(\Omega); \text{ess} \sup_{0\leq t\leq T} \|v\|_{s,p,\Omega} < \infty\}$$

with the norm

$$\|v\|_{L^\infty(0,T;W^{s,p}(\Omega))} = ess \sup_{0\leq t\leq T} \|v\|_{s,p,\Omega}.$$

For the initial data u_0, we need the following assumption.

(**A1**). The initial velocity $u_0 \in D(A)$ and $f, f_t \in L^2(0,T;Y)$ are assumed to satisfy

$$\|Au_0\|_0 + (\int_0^T (\|f\|_0^2 + \|f_t\|_0^2) dt)^{\frac{1}{2}} \leq C.$$

Throughout of whole paper, the letter $C > 0$ denotes a generic constant, independent of mesh parameter and time step, and maybe different at different occurrences.

The continuous bilinear form $a(\cdot,\cdot)$ on $X \times X$ is defined by

$$a(u,v) = \nu(\nabla u, \nabla v),$$

and the trilinear form by respectively

$$b(u, v, w) = (u\nabla \cdot v, w) \quad \forall\ u, v, w \in X.$$

It is easy to verify that $b(\cdot, \cdot, \cdot)$ satisfies the following important properties

$$b(u, v, w)| \leq N\|\nabla u\|_0 \|\nabla v\|_0 \|\nabla w\|_0, \tag{2.1}$$

for all $u, v, w \in X$, where $N = \sup_{0\neq u,v,w\in X} \frac{|b(u,v,w)|}{\|\nabla u\|_0\|\nabla v\|_0\|\nabla w\|_0}$, and

$$|b(u, v, w)| + |b(v, u, w)| + |b(w, u, v)| \leq C_0\|\nabla u\|_0 \|Av\|_0 \|w\|_0, \tag{2.2}$$

for all $u \in X, v \in D(A),\ w \in Y$.

For the subsequence convenience, we recall the Gronwall lemmas, which will be frequently used in the analysis of a posteriori error estimates (see [9, 10]).

Lemma 2.1. *Let $g(t),\ h(t),\ y(t)$ be three locally integrable nonnegative functions on time interval $[0, \infty)$, such that for any fixed time $t_0 \geq 0$ and all $t \geq t_0$*

$$y(t) + G(t) \leq C + \int_{t_0}^{t} h(s)ds + \int_{t_0}^{t} g(s)y(s)ds,$$

where $G(t)$ is a nonnegative function on $[0, \infty),\ C \geq 0$ is a constant. Then,

$$y(t) + G(t) \leq \Big(C + \int_{t_0}^{t} h(s)ds\Big) \exp(\int_{t_0}^{t} g(s)ds).$$

With above notations, the variational formulation of problem (1.1) reads as: For all $t \in (0, T]$, find $u \in X$, such that for all $v \in X$

$$(u_t, v) + a(u, v) + b(u, u, v) = (f, v). \tag{2.3}$$

Remark 2.2. *For the bounded of the exact solution, it follows from $v = u_t$ in (2.3), (2.2) and Lemma 2.1 that*

$$\nu\|\nabla u\|_0^2 + \int_0^t \|u_t(s)\|_0^2 ds \leq \exp(\frac{2C_0^2 \int_0^t \|Au\|_0^2 ds}{\nu})\Big(\nu\|\nabla u(0)\|_0^2 + 2\int_0^t \|f\|_0^2 ds\Big).$$

2.2. Finite element approximation

Let $h > 0$ be a real positive parameter. The finite element subspace X_h of X is characterized by $\mathcal{T}_h = \mathcal{T}_h(\Omega)$, a partitioning of Ω into triangles K assumed to be uniformly shape-regular as $h \to 0$, see [10] for details.

Definition 2.3 (Elliptic projection). *For $\forall\ v \in X$, we define a projection operator $R_h \in X_h$ by*

$$a(v - R_h, v_h) = 0 \tag{2.4}$$

for all $v_h \in X_h$. Furthermore, for any $v \in D(A)$ operator R_h satisfies (see [10]):

$$\|v - R_h\|_0 + h\|\nabla(v - R_h)\|_0 \le Ch^2. \tag{2.5}$$

Now, we present the spatial semidiscrete finite element formulation for problem (1.1): Find $u_h \in X_h$, for all $t \in (0, T]$ such that

$$(u_{ht}, v_h) + a(u_h, v_h) + b(u_h, u_h, v_h) = (f, v_h) \qquad \forall\ v_h \in X_h. \tag{2.6}$$

Since the bilinear form $(\nabla u_h, \nabla v_h)$ is coercive on $X_h \times X_h$, it generates an invertible operator $A_h : X_h \to X_h$ through the definition (see [16, 17])

$$(A_h u_h, v_h) = (A_h^{1/2} u_h, A_h^{1/2} v_h) = (\nabla u_h, \nabla v_h), \quad \forall\ u_h, v_h \in X_h.$$

By using a similar argument to one used in [13, 14, 27] about the nonlinear convection-diffusion problem, we can obtain the following result.

Lemma 2.4. *Assume that Ω is a bounded polygonal domain with a sufficiently smooth boundary $\partial\Omega$, under the assumption of* $(\mathbf{A1})$*, there exists a constant C such that*

$$\|\nabla(u - u_h)\|_0^2 + \int_0^T \|\nabla(u - u_h)\|_0^2 dt \le Ch^2.$$

Furthermore, for the semidiscrete finite element scheme (2.6), the following smooth properties of u_h hold.

Lemma 2.5. *Under the assumptions of Lemma 2.4, there exists a constant C such that*

$$\|A_h u_h\|_0^2 + \nu\|\nabla u_h\|_0^2 + \int_0^t \|A_h u_h(s)\|_0^2 ds + \int_0^t \|u_{ht}(s)\|_0^2 ds \le C.$$

Proof. For the L^2-norm of $\|A_h u_h\|_0$, there is

$$\|A_h u_h\|_0 \quad = \quad \sup_{v_h \in L^2(\Omega)^2} \frac{\|(A_h u_h, v_h)\|_0}{\|v_h\|_0}.$$

For $\forall\ v_h \in L^2(\Omega)^2$, we split v_h into two parts, i.e., there exist uniqueness $v_h^1 \in X_h$ and $v_h^2 \in X_h^\perp$, such that $v = v_h^1 + v_h^2 \in X_h \oplus X_h^\perp = L^2(\Omega)^2$. Then, according to the definition of A_h, one finds

$$(A_h u_h, v_h) = (A_h u_h, v_h^1) + (A_h u_h, v_h^2) = (A_h u_h, v_h^1).$$

As a consequence, we have

$$\begin{aligned}
\|A_h u_h\|_0 &= \sup_{0\neq v_h\in L^2(\Omega)^2} \frac{\|(A_h u_h, v_h)\|_0}{\|v_h\|_0} = \sup_{\substack{0\neq v_h^1\in X_h\\ 0\neq v_h\in L^2(\Omega)^2}} \frac{\|(A_h u_h, v_h^1)\|_0}{\|v_h\|_0}\\
&\leq \sup_{\substack{0\neq v_h^1\in X_h\\ 0\neq v_h\in L^2(\Omega)^2}} \frac{\|(\nabla(u-u_h), \nabla v_h^1)\|_0 + \|(Au, v_h^1)\|_0}{\|v_h\|_0}\\
&\leq \sup_{\substack{0\neq v_h^1\in X_h\\ 0\neq v_h\in L^2(\Omega)^2}} \frac{\|\nabla(u-u_h)\|_0\|\nabla v_h^1\|_0 + \|Au\|_0\|v_h^1\|_0}{\|v_h\|_0}\\
&\leq Ch^{-1}\|\nabla(u-u_h)\|_0 + \|Au\|_0.
\end{aligned}$$

It follows from Lemma 2.4 that

$$\|A_h u_h\|_0^2 + \int_0^t \|A_h u_h(s)\|_0^2 ds \leq \widetilde{C}. \tag{2.7}$$

Choosing $v_h = u_{ht}$ in (2.6) and using (2.1), we obtain

$$\begin{aligned}
\|u_{ht}\|_0^2 + \frac{\nu}{2}\frac{d}{dt}\|\nabla u_h\|_0^2 &= (f, u_{ht}) - b(u_h, u_h, u_{ht})\\
&\leq \|f\|_0\|u_{ht}\|_0 + C_0\|A_h u_h\|_0\|\nabla u_h\|_0\|u_{ht}\|_0\\
&\leq \|f\|_0^2 + C_0^2\|A_h u_h\|_0^2\|\nabla u_h\|_0^2 + \frac{1}{2}\|u_{ht}\|_0^2.
\end{aligned}$$

Integrating with respect to time from 0 to t, using Lemma 2.1 and (2.7), we find that

$$\nu\|\nabla u_h\|_0^2 + \int_0^t \|u_{ht}(s)\|_0^2 ds \leq \exp\left(\frac{2C_0^2\widetilde{C}}{\nu}\right)\Big(\nu\|\nabla u_h(0)\|_0^2 + 2\int_0^t \|f\|_0^2 ds\Big). \tag{2.8}$$

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We end this section by introducing some properties about L^2-projection operator, which can be found in reference [10].

Lemma 2.6. *Assume that there exist two L^2-projection operators $I_h : X \to X_h$ and $I_h^n : X \to X_h^n$ (X_h^n will be defined in Section 4), which are defined by*

$$(\phi - I_h^{\square}\phi, w_h) = 0 \ \ \forall\ w_h \in X_h \ \textit{or}\ X_h^n, \quad \phi \in X\ (I_h^{\square}\ \textit{takes}\ I_h\ \textit{or}\ I_h^n).$$

Furthermore, if $\phi \in D(A)$, the following properties hold

$$h^i\|\phi - I_h^{\square}\phi\|_{0,K} \leq Ch_K^{2-i}\|\phi\|_{2,\omega_K}\ (i = 0, 1), \quad \|\phi - I_h^{\square}\phi\|_{0,E} \leq Ch_E^{1/2}\|\phi\|_{1,\omega_K},$$

where $\omega_K = \cup_{K'\cap K\neq\emptyset} K'$ and $\omega_E = \cup_{E\cap K\neq\emptyset} K$ for $\forall\ K, K' \in \mathcal{T}_h$.

3. A posteriori error estimates for semidiscrete formulation

In this section, we present a posteriori error estimates of Burgers equation in spatial semidiscrete formulation (2.6), and establish the computable upper and lower bounds for numerical solution u_h in various of norms.

Let $e_u = u_h - u$. It follows from (2.3) and (2.6) that e_u satisfies

$$(e_{ut}, v) + a(e_u, v) + b(u_h, u_h, v) - b(u, u, v) = r(v), \tag{3.1}$$

where the residual $r(v)$ is given by

$$r(v) = (u_{ht}, v) + \nu(\nabla u_h, \nabla v) + b(u_h, u_h, v) - (f, v). \tag{3.2}$$

From (2.6), we know that $r(v_h) = 0$ for $\forall\ v_h \in X_h$.

We split the error e_u into two parts

$$e_u = u_h - u = (\widetilde{u} - u) - (\widetilde{u} - u_h) = \xi_u - \eta_u.$$

Then, (3.1) can be rewritten as

$$\begin{aligned}&(\xi_{ut}, v) + a(\xi_u, v) + b(\xi_u, u_h, v) + b(u, \xi_u, v)\\ &= r(v) + (\eta_{ut}, v) + a(\eta_u, v) + b(\eta_u, u_h, v) + b(u, \eta_u, v).\end{aligned} \tag{3.3}$$

Now, we introduce the Burgers reconstruction $\widetilde{u} \in X$ of $u_h(t)$ for all $t \in (0, T]$.

Definition 3.1 (Burgers reconstruction). *For given u_h, we define the Burgers reconstruction $\widetilde{u}(t)$ satisfying*

$$a(\widetilde{u} - u_h, v) + b(\eta_u, u_h, v) + b(u, \eta_u, v) = -r(v). \tag{3.4}$$

Remark 3.2. *For the given u_h, r, from the continuity and coercivity of $a(\cdot,\cdot)$, it is easy to know that* (3.4) *admits a unique solution $\widetilde{u} \in X$ for all $t \in (0, T]$.*

By the definition of (3.4), equation (3.3) can be transformed into

$$(\xi_{ut}, v) + a(\xi_u, v) + b(\xi_u, u_h, v) + b(u, \xi_u, v) = (\eta_{ut}, v). \tag{3.5}$$

3.1. Error estimates between exact solution and Burgers reconstruction

In this subsection, we derive some estimates about problem (3.5) by using the energy method and some standard techniques in the analysis of parabolic equations.

Lemma 3.3. *Assume that Ω is a bounded polygonal domain with a sufficiently smooth boundary $\partial\Omega$. Let ξ_u be the solution of* (3.5). *Then, for all $t \in (0, T]$, there exists a constant C such that*

$$\|\xi_u(t)\|_0^2 + \nu\int_0^t \|\nabla\xi_u(s)\|_0^2 ds \le C\Big(\|\xi_u(0)\|_0^2 + \int_0^t \|\eta_{ut}(s)\|_0^2 ds\Big), \tag{3.6}$$

$$\Big(\nu\|\nabla\xi_u(t)\|_0 + \int_0^t \|\xi_{ut}(s)\|_0^2 ds\Big)^{\frac{1}{2}} \le C\Big(\nu^{1/2}\|\nabla\xi_u(0)\|_0 + \Big(\int_0^t \|\eta_{ut}(s)\|_0^2 ds\Big)^{\frac{1}{2}}\Big). \tag{3.7}$$

Proof. Choosing $v = \xi_u$ in (3.5), by Cauchy inequality and (2.2), we obtain that

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\xi_u\|_0^2 + \nu\|\nabla\xi_u\|_0^2 &= (\eta_{ut}, \xi_u) - b(\xi_u, u_h, \xi_u) - b(u, \xi_u, \xi_u) \\ &\leq \|\eta_{ut}\|_0\|\xi_u\|_0 + C_0(\|A_h u_h\|_0 + \|Au\|_0)\|\nabla\xi_u\|_0\|\xi_u\|_0 \\ &\leq \Big(\frac{C_0^2(\|A_h u_h\|_0^2 + \|Au\|_0^2\big)}{\nu} + \frac{1}{2}\Big)\|\xi_u\|_0^2 + \frac{1}{2}\|\eta_{ut}\|_0^2 + \frac{\nu}{2}\|\nabla\xi_u\|_0^2.\end{aligned}$$

Kicking the last term, using Lemma 2.3 and integrating with respect to time from 0 to t, we yield

$$\|\xi_u(t)\|_0^2 + \nu\int_0^t \|\nabla\xi_u(s)\|_0^2 ds \leq \|\xi_u(0)\|_0^2 + \int_0^t \|\eta_{ut}(s)\|_0^2 ds + C\int_0^t \|\xi_u(s)\|_0^2 ds. \quad (3.8)$$

The result in (3.6) comes from Lemma 2.1.

Secondly, taking $v = A\xi_u$ in (3.5) and using Cauchy inequality, one finds

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|\nabla\xi_u\|_0^2 + \nu\|A\xi_u\|_0^2 = (\eta_{ut}, A\xi_u) - b(\xi_u, u_h, A\xi_u) - b(u, \xi_u, A\xi_u) \\ \leq\ & \|\eta_{ut}\|_0\|A\xi_u\|_0 + C_0(\|A_h u_h\|_0 + \|Au\|_0)\|\nabla\xi_u\|_0\|A\xi_u\|_0 \\ \leq\ & \frac{C_0^2}{\nu}(\|A_h u_h\|_0^2 + \|Au\|_0^2)\|\nabla\xi_u\|_0^2 + \frac{1}{\nu}\|\eta_{ut}\|_0^2 + \frac{\nu}{2}\|A\xi_u\|_0^2. \quad (3.9)\end{aligned}$$

Kicking the last term in (3.9), integrating it with respect to time from 0 to t, using Lemmas 2.1 and 2.5. we obtain that

$$\|\nabla\xi_u(t)\|_0^2 + \nu\int_0^t \|A\xi_u(s)\|_0^2 ds \leq C\Big(\|\nabla\xi_u(0)\|_0^2 + \int_0^t \|\eta_{ut}(s)\|_0^2 ds\Big). \quad (3.10)$$

Thirdly, differentiating (3.5) with respect to time and choosing $v = \xi_{ut}$, in terms of (2.1), (2.2) and Lemma 2.5, we obtain that

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|\xi_{ut}\|_0^2 + \nu\|\nabla\xi_{ut}\|_0^2 \\ =\ & (\eta_{utt}, \xi_{ut}) - b(\xi_{ut}, u_h, \xi_{ut}) - b(\xi_u, u_{ht}, \xi_{ut}) - b(u_t, \xi_u, \xi_{ut}) - b(u, \xi_{ut}, \xi_{ut}) \\ \leq\ & \|\eta_{utt}\|_0\|\xi_{ut}\|_0 + C_0\Big((\|A_h u_h\|_0 + \|Au\|_0)\|\xi_{ut}\|_0 + \|A\xi_u\|_0\|u_{ht}\|_0\Big)\|\nabla\xi_{ut}\|_0 + C_0\|Au_t\|_0\|\nabla\xi_u\|_0\|\xi_{ut}\|_0 \\ \leq\ & C_1\Big(\|\eta_{utt}\|_0^2 + \|A\xi_u\|_0^2\|u_{ht}\|_0^2 + \|Au_t\|_0^2\|\nabla\xi_u\|_0^2\Big) + C_2\Big(\|A_h u_h\|_0^2 + \|Au\|_0^2 + 1\Big)\|\xi_{ut}\|_0^2 + \frac{\nu}{2}\|\nabla\xi_{ut}\|_0^2.\end{aligned}$$

Kicking the last term and integrating it with respect to time from 0 to t, using Lemma 2.1, (3.6) and (3.10) we arrive at

$$\begin{aligned}&\|\xi_{ut}\|_0^2 + \nu\int_0^t \|\nabla\xi_{ut}\|_0^2 ds \\ \leq\ & C\big(\|\xi_u(0)\|_0^2 + \|\nabla\xi_u(0)\|_0^2 + \|\xi_{ut}(0)\|_0^2 + \int_0^t \|\eta_{utt}\|_0^2 ds + \int_0^t \|\eta_{ut}\|_0^2 ds\Big). \quad (3.11)\end{aligned}$$

Finally, taking $v = \xi_{ut}$ in (3.5) one finds that

$$\begin{aligned}
\|\xi_{ut}\|_0^2 + \frac{\nu}{2}\frac{d}{dt}\|\nabla\xi_u\|_0^2 &= (\eta_{ut}, \xi_{ut}) - b(\xi_u, u_h, \xi_{ut}) - b(u, \xi_u, \xi_{ut}) \\
&\leq \|\eta_{ut}\|_0\|\xi_{ut}\|_0 + C_0(\|A_h u_h\|_0 + \|Au\|_0)\|\nabla\xi_u\|_0\|\xi_{ut}\|_0 \\
&\leq \|\eta_{ut}\|_0^2 + C_0^2(\|A_h u_h\|_0^2 + \|Au\|_0^2)\|\nabla\xi_u\|_0^2 + \frac{1}{2}\|\xi_{ut}\|_0.
\end{aligned}$$

Kicking the last term, using Lemma 2.5 and integrating with respect to time from 0 to t, one finds

$$\nu\|\nabla\xi_u(t)\|_0^2 + \int_0^t \|\xi_{ut}(s)\|_0^2 ds \leq \nu\|\nabla\xi_u(0)\|_0^2 + 2\int_0^t \|\eta_{ut}(s)\|_0^2 ds + C\int_0^t \|\nabla\xi_u(s)\|_0^2 ds.$$

By Lemma 2.1 and inequality $a^2 + b^2 \leq (a+b)^2$ $(a, b \geq 0)$, we obtain the desired result (3.7). ¶

3.2. A posteriori error estimates for Burgers reconstruction

In this subsection, we derive a posteriori error estimates for Burgers equation in semidiscrete formulation (2.6). To achieve this aim, we need the following a posteriori estimates about η_u and $\nabla\eta_u$ related to Burgers reconstruction (3.4).

Lemma 3.4. *Assume that Ω is a bounded polygonal domain with a sufficiently smooth boundary. Let $\widetilde{u}$ and u_h be the solutions of* (3.4) *and* (2.6), *respectively. Then, there exists a constant C such that*

$$\|\eta_u\|_0 \leq C\Big(\sum_{K\in\mathcal{T}_h} h_K^4\|u_{ht} - \nu\Delta u_h + u_h\nabla\cdot u_h - f\|_{0,K}^2 + \sum_{E\in\mathcal{E}_h} h_E^3\|[\nu\nabla u_h]\|_{0,E}^2\Big)^{1/2}, \quad (3.12)$$

$$\|\nabla\eta_u\|_0 \leq C\Big(\sum_{K\in\mathcal{T}_h} h_K^2\|u_{ht} - \nu\Delta u_h + u_h\nabla\cdot u_h - f\|_{0,K}^2 + \sum_{E\in\mathcal{E}_h} h_E\|[\nu\nabla u_h]\|_{0,E}^2\Big)^{1/2}. \quad (3.13)$$

Proof. Firstly, consider that $\Phi \in D(A)\cap X$ is the solution of the elliptic problem

$$a(\Phi, v) + b(u, v, \Phi) + b(v, u_h, \Phi) = (g, v), \quad (3.14)$$

where u and u_h are the solution of (1.1) and (2.6) respectively. According to the Lax-Milgram Theorem, we know that system (3.14) is well-posed and has a unique solution Φ satisfying (see [9, 10])

$$\|\Phi\|_2 \leq C\|g\|_0. \quad (3.15)$$

Choosing $v = \widetilde{u} - u_h$ in (3.14) and using (2.4), one arrives at

$$\begin{aligned}
(\eta_u, g) &= \nu(\nabla\Phi, \nabla\widetilde{u}) - \nu(\nabla\Phi, \nabla u_h) + b(u, \widetilde{u} - u_h, \Phi) + b(\widetilde{u} - u_h, u_h, \Phi) \\
&= \nu(\nabla\Phi, \nabla\widetilde{u}) - \nu(\nabla R_h, \nabla u_h) + b(u, \widetilde{u} - u_h, \Phi) + b(\widetilde{u} - u_h, u_h, \Phi) \\
&= \nu(\nabla(\Phi - R_h), \nabla\widetilde{u}) + b(u, \widetilde{u} - u_h, \Phi - R_h) + b(\widetilde{u} - u_h, u_h, \Phi - R_h) \\
&\quad + \nu(\nabla R_h, \nabla(\widetilde{u} - u_h)) + b(u, \widetilde{u} - u_h, R_h) + b(\widetilde{u} - u_h, u_h, R_h). \quad (3.16)
\end{aligned}$$

From the definition of (3.4), we know that

$$\nu(\nabla R_h, \nabla(\widetilde{u} - u_h)) + b(u, \widetilde{u} - u_h, R_h) + b(\widetilde{u} - u_h, u_h, R_h) = -r(R_h).$$

According to (3.2) with $v = R_h \in X_h$, equation (3.16) can be rewritten as

$$(\eta_u, g) = \nu(\nabla(\Phi - R_h), \nabla\widetilde{u}) + b(u, \widetilde{u} - u_h, \Phi - R_h) + b(\widetilde{u} - u_h, u_h, \Phi - R_h).$$

Using (2.4) again and noting (3.4), we deduce that

$$\begin{aligned}(\eta_u, g) &= \nu(\nabla(\Phi - R_h), \nabla\widetilde{u}) + b(u, \widetilde{u} - u_h, \Phi - R_h) + b(\widetilde{u} - u_h, u_h, \Phi - R_h) \\ &= \nu(\nabla(\Phi - R_h), \nabla(\widetilde{u} - u_h)) + b(u, \widetilde{u} - u_h, \Phi - R_h) + b(\widetilde{u} - u_h, u_h, \Phi - R_h) \\ &= -r(\Phi - R_h). \qquad (3.17)\end{aligned}$$

By (2.5), (3.2) and Green's formula, one finds

$$\begin{aligned}&(\eta_u, g) \\ = \ & (u_{ht} - \nu\Delta u_h + u_h\nabla\cdot u_h - f, \Phi - R_h) + \sum_{E\in\mathcal{E}_h}\int_E [\nu\nabla u_h]\cdot(\Phi - R_h)ds \qquad (3.18) \\ \le \ & \sum_{K\in\mathcal{T}_h} \|u_{ht} - \nu\Delta u_h + u_h\nabla\cdot u_h - f\|_{0,K}\|\Phi - R_h\|_{0,K} + \sum_{E\in\mathcal{E}_h} \|[\nu\nabla u_h]\|_{0,E}\|\Phi - R_h\|_{0,E} \\ \le \ & C\Big((\sum_{K\in\mathcal{T}_h} h_K^4 \|u_{ht} - \nu\Delta u_h + u_h\nabla\cdot u_h - f\|_{0,K}^2)^{\frac{1}{2}} + (\sum_{E\in\mathcal{E}_h} h_E^3 \|[\nu\nabla u_h]\|_{0,E}^2)^{\frac{1}{2}} \Big)\cdot\|\Phi\|_2.\end{aligned}$$

Using elliptic regularity (3.15) in (3.18), we deduce that

$$\frac{(\eta_u, g)}{\|g\|_0} \le C\Big((\sum_{K\in\mathcal{T}_h} h_K^4 \|u_{ht} - \nu\Delta u_h + u_h\nabla\cdot u_h - f\|_{0,K}^2)^{\frac{1}{2}} + (\sum_{E\in\mathcal{E}_h} h_E^3 \|[\nu\nabla u_h]\|_{0,E}^2)^{\frac{1}{2}} \Big). \quad (3.19)$$

Taking the supermum over g, we obtain the desired result (3.12).

By differentiating (3.18) with respect to time, using (3.2) and (3.4), following the proofs of (3.12), we can obtain that

$$\begin{aligned}\|\eta_{ut}\|_0 \ \le \ & C\Big((\sum_{K\in\mathcal{T}_h} h_K^4 \|u_{htt} - \nu\Delta u_{ht} + u_{ht}\nabla\cdot u_h + u_h\nabla\cdot u_{ht} - f_t\|_{0,K}^2)^{1/2} \\ & + (\sum_{E\in\mathcal{E}_h} h_E^3 \|[\nu\nabla u_{ht}]\|_0^2)^{1/2} \Big), \qquad (3.20)\end{aligned}$$

In order to obtain the estimate (3.13), we make the following assumption:

$$\frac{\nu}{2} \ge N\Big[\Big(\exp(\frac{2C_0^2\widetilde{C}}{\nu}) + \exp(\frac{2C_0^2\int_0^t \|Au\|_0^2 ds}{\nu})\Big)\Big(\|\nabla u_h(0)\|_0^2 + \frac{2}{\nu}\int_0^t \|f\|_0^2 ds\Big)\Big]^{\frac{1}{2}}. \ (3.21)$$

Taking $v = \eta_u$ in (3.4), and using the fact that $r_1(I_h\eta_u) = 0$, by Green formula and assumption (3.21), we obtain that

$$
\begin{aligned}
\nu\|\nabla\eta_u\|_0^2 &= -r(\eta_u) - b(\eta_u, u_h, \eta_u) - b(u, \eta_u, \eta_u) \\
&= -r(\eta_u - I_h\eta_u) - b(\eta_u, u_h, \eta_u) - b(u, \eta_u, \eta_u) \\
&\leq -r(\eta_u - I_h\eta_u) + N(\|\nabla u_h\|_0 + \|\nabla u\|_0)\|\nabla\eta_u\|_0^2 \\
&\leq (u_{ht} - \nu\Delta u_h + u_h\nabla\cdot u_h - f, \eta_u - I_h\eta_u) \\
&\quad + \sum_{E\in\mathcal{E}_h}\int_E [\nu\nabla u_h]\cdot(\eta_u - I_h\eta_u)ds + N(\|\nabla u_h\|_0 + \|\nabla u\|_0)\|\nabla\eta_u\|_0^2.
\end{aligned}
$$

By (2.8) and (3.21), we finish the rest of proof (3.12). ¶

As a consequence, combining Lemmas 3.3 and 3.4, we obtain the main theorem of a posteriori error estimates for Burgers equation in semidiscrete formulation.

Theorem 3.5. *Let Ω be a bounded polygonal domain with a sufficiently smooth boundary, u and u_h the solutions of* (2.3) *and* (2.6)*, respectively. Then, for all $t \in (0, T]$ there exists a constant C such that*

$$
\begin{aligned}
\|u_h - u\|_0^2 &\leq C\Big(\|u_h(0) - u_0\|_0^2 + \|\eta_u(0)\|_0^2 + \|\eta_u\|_0^2 + \int_0^t \|\eta_{ut}(s)\|_0^2 ds\Big), \\
\|\nabla(u_h - u)\|_0^2 &\leq C\Big(\|\nabla(u_h(0) - u_0)\|_0^2 + \|\nabla\eta_u(0)\|_0^2 + \|\nabla\eta_u\|_0^2 + \int_0^t \|\eta_{ut}(s)\|_0^2 ds\Big),
\end{aligned}
$$

where the estimates of $\|\eta_u\|_0, \|\nabla\eta_u\|_0$ and $\|\eta_{ut}\|_0$ are given by (3.12), (3.13) *and* (3.20)*, respectively.*

4. A posteriori estimates for fully discrete scheme

In this section, we consider a posteriori estimates of fully discrete approximation for Burgers equation (1.1) based on backward Euler scheme.

Set $0 = t_0 < t_1 < \cdots < t_N = T$, $I_n = (t_{n-1}, t_n]$ and denote $k_n = t_n - t_{n-1}$. For $\forall\, n \in [0, N]$, let $\mathcal{T}_n$ be a refinement of macrotriangulation which is a triangulation of the domain Ω that satisfies the same conformity and shape regularity assumptions made on its refinements (see [9] for details). Set

$$h_n(x) = \mathrm{diam}(K), \quad \text{where } K \in \mathcal{T}_n \text{ and } x \in K.$$

Given two compatible triangulations $\mathcal{T}_{n-1}$ and $\mathcal{T}_n$, namely, they are refinements of the same macrotriangulation, let $\hat{\mathcal{T}}_n$ be the finest common coarsening of $\mathcal{T}_n$ and $\mathcal{T}_{n-1}$, whose mesh size is given by $\hat{h}_n = \max(h_n, h_{n-1})$. For more information, we can refer to Appendix A of reference [20].

Denote

$$\partial_t\phi^n := \frac{1}{k_n}(\phi^n - \phi^{n-1}), \qquad f^n = f(t_n).$$

We set X_h^n as the finite element subspace of X defined over the triangulations $\mathcal{T}_n$.

Given $U^0 = I_h^n u_0$, find $\{U^n\}$ with $U^n \in X_h^n$ at $t = t_n$. For $\forall\ v \in X_h^n$ as $n = 0$

$$\nu(\nabla U^0, \nabla v) + b(U^0, U^0, v) = (f^0, v), \tag{4.1}$$

and for $n \in [1 : N]$

$$\frac{1}{k_n}(U^n - U^{n-1}, v) + \nu(\nabla U^n, \nabla v) + b(U^n, U^n, v) = (f^n, v). \tag{4.2}$$

For the existence, uniqueness and convergence of U^n of problem (4.2), following the guidelines provided in [13, 14], we have the following results.

Theorem 4.1. *Assume that Ω is a bounded polygonal domain with Lipschitz continuous boundary, under the assumption of* (**A1**)*. Then, there exists a constant C such that*

$$\nu\|\nabla(u - U^n)\|_0^2 + \int_0^T \nu\|\nabla(u - U^n)\|_0^2 dt \le Ch^2,$$

$$\|A_h U^n\|_0^2 + \nu\|\nabla U^n\|_0^2 + \int_0^t \|A_h U^n(s)\|_0^2 ds \le C.$$

Using a sequence of discrete values $\{U^n\}, n = 0, 1, 2, \ldots, N$, we define a continuous piecewise linear function $U(t)$ for $\forall\ t \in [0, T]$ by

$$U(t) = (1 - \frac{t - t_{n-1}}{k_n})U^{n-1} + \frac{t - t_{n-1}}{k_n}U^n, \quad t_{n-1} < t \le t_n,\ n = 1, 2, \ldots, N. \tag{4.3}$$

Note that the time derivative of U restricted to I_n is

$$U_t|_{I_n} = \partial_t U^n \quad \text{ for } \forall\ t \in I_n. \tag{4.4}$$

To motivate the use of Burgers reconstruction, we denote $e_u(t) = U(t) - u(t)$. For $\forall\ v \in X_h^n$, e_u satisfies

$$\begin{aligned} & (e_{ut}, v) + \nu(\nabla e_u, \nabla v) + b(U, U, v) - b(u, u, v) \\ = \ & \nu(\nabla(U - U^n), \nabla v) + (f^n - f, v) + \frac{1}{k_n}(I_h^n U^{n-1} - U^{n-1}, v) + \frac{1}{k_n}(U^n - I_h^n U^{n-1}, v) \\ & + \nu(\nabla U^n, \nabla v) - (f^n, v) + b(U, U, v) + b(U^n, U^n, v) - b(U^n, U^n, v). \end{aligned} \tag{4.5}$$

Define the residual r^n for $n = 1, 2, \ldots, N$ as

$$r^n(v) = \frac{1}{k_n}(U^n - I_h^n U^{n-1}, v) + \nu(\nabla U^n, \nabla v) + b(U^n, U^n, v) - (f^n, v). \tag{4.6}$$

Definition 4.2 (Burgers reconstruction). *For given U^n $(n = 0, 1, \ldots, N)$, find $\widetilde{u}^n \in X$ for $\forall\ v \in X$ such that*

$$\nu(\nabla(\widetilde{u}^n - U^n), \nabla v) + b(\widetilde{u}^n - U^n, U, v) + b(u, \widetilde{u}^n - U^n, v) = -r^n(v). \tag{4.7}$$

Combining (4.6) with (4.2) and Lemma 2.6, we know that

$$r^n(v_h) = 0 \qquad \text{for } \forall \;\; v_h \in X_h^n.$$

Note that U^n is Burgers reconstruction of $\widetilde{u}^n$ at $t = t_n$. Using a sequence of discrete values $\{\widetilde{u}^n\}$ $(n = 0, 1, \dots, N)$, for $\forall\ t \in [0, T]$, we define a continuous function of time as the continuous piecewise linear interpolation $\widetilde{u}(t)$:

$$\widetilde{u}(t) = (1 - \frac{t - t_{n-1}}{k_n})\widetilde{u}^{n-1} + \frac{t - t_{n-1}}{k_n}\widetilde{u}^n, \quad t_{n-1} < t \le t_n,\ n = 1, 2, \dots, N. \tag{4.8}$$

Furthermore, for $\forall\ v \in X$ and $\forall t \in [0, T]$, $\widetilde{u}$ satisfies

$$\nu(\nabla(\widetilde{u} - U), \nabla v) + b(\widetilde{u} - U, U, v) + b(u, \widetilde{u} - U, v) = -r(v), \tag{4.9}$$

where $r(v)$ is piecewise linear interpolation of $\{r^n\}_{n=1}^N$. In order to use (4.7), we split the error of e_u into two parts

$$e_u = (\widetilde{u} - u) - (\widetilde{u} - U) \triangleq \xi_u - \eta_u. \tag{4.10}$$

Thanks to (4.7), equations (4.5) can be rewritten as for $\forall\ v \in X$

$$\begin{aligned} & (\xi_{ut}, v) + \nu(\nabla\xi_u, \nabla v) + b(\xi_u, U, v) + b(u, \xi_u, v) \\ = \;& (\eta_{ut}, v) + \nu(\nabla(\widetilde{u} - \widetilde{u}^n), \nabla v) + (f^n - f, v) + \frac{1}{k_n}(I_h^n U^{n-1} - U^{n-1}, v) \\ & + b(U^n - u, U - U^n, v) + b(\widetilde{u} - \widetilde{u}^n, U, v) + b(u, \widetilde{u} - \widetilde{u}^n, v). \end{aligned} \tag{4.11}$$

Note that

$$\widetilde{u} - \widetilde{u}^n = -\frac{t_n - t}{k_n}(\widetilde{u}^n - \widetilde{u}^{n-1}) = -(t_n - t)\partial_t \widetilde{u}^n, \tag{4.12}$$

$$U - U^n = -\frac{t_n - t}{k_n}(U^n - U^{n-1}) = -(t_n - t)\partial_t U^n. \tag{4.13}$$

From equation (4.7), we deduce that

$$\begin{aligned} & \nu(\nabla\widetilde{u}^n, \nabla v) + b(\widetilde{u}^n, U, v) + b(u, \widetilde{u}^n, v) \\ = \;& \nu(\nabla U^n, \nabla v) - r^n(v) + b(U^n, U, v) + b(u, U^n, v). \end{aligned} \tag{4.14}$$

As a consequence, one finds

$$\begin{aligned} & \nu(\nabla(\widetilde{u}^n - \widetilde{u}^{n-1}), \nabla v) + b(\widetilde{u}^n - \widetilde{u}^{n-1}, U, v) + b(u, \widetilde{u}^n - \widetilde{u}^{n-1}, v) \\ = \;& \nu(\nabla(U^n - U^{n-1}), \nabla v) - (r^n(v) - r^{n-1}(v)) + b(U^n - U^{n-1}, U, v) + b(u, U^n - U^{n-1}, v). \end{aligned} \tag{4.15}$$

Substituting (4.13) and (4.15) into (4.11), we obtain that

$$\begin{aligned} & (\xi_{ut}, v) + \nu(\nabla\xi_u, \nabla v) + b(\xi_u, U, v) + b(u, \xi_u, v) \\ = \;& (\eta_{ut}, v) + \frac{1}{k_n}(I_h^n U^{n-1} - U^{n-1}, v) + (f^n - f, v) + \frac{t_n - t}{k_n}\Big((r^n(v) - r^{n-1}(v)) \\ & - \nu(\nabla(U^n - U^{n-1}), \nabla v) - b(U^n - U^{n-1}, U, v) - b(U^n, U^n - U^{n-1}, v)\Big). \end{aligned} \tag{4.16}$$

Next, we present some estimates about the errors between Burgers reconstruction (4.7) and the exact solution of (2.3) in various norms. In order to simplify the expressions, we introduce some notations as follows. Let

$$
\begin{aligned}
\mathscr{E}_1^m &= \sum_{n=1}^{m} \int_{t_{n-1}}^{t_n} \|f^n - f\|_0^2 ds, \quad \mathscr{E}_2^m = \sum_{n=1}^{m} \int_{t_{n-1}}^{t_n} \|\eta_{ut}(s)\|_0^2 ds, \\
\mathscr{E}_3^m &= \sum_{n=1}^{m} k_n \|k_n^{-1} h_n (I - I_h^n) U^{n-1}\|_0^2, \\
\mathscr{E}_4^m &= \sum_{n=1}^{m} k_n^3 \Big(\|\hat{h}_n \partial_t r_1^n\|_0^2 + \nu \|\nabla \partial_t U^n\|_0^2 + (\|A_h U\|_0^2 + \|A_h U^n\|_0^2) \|\partial_t U^n\|_0^2 \Big), \\
\mathscr{E}_5^m &= \|h_1 (I_h^1 - I)(\frac{1}{k_1} U^0)\|_0^2 + \sum_{n=2}^{m} k_n^2 \|\hat{h}_n \partial_t (I_h^n - I)(\frac{1}{k_n} U^{n-1})\|_0^2 + \|h_m (I_h^m - I)(\frac{1}{k_m} U^{m-1})\|_0^2.
\end{aligned}
$$

Theorem 4.3. *Assume that Ω is a convex polygonal domain. Let u and U be the solutions of (2.3) and (4.2), respectively. Then, for $m \in [1, N]$ there exists a constant C such that*

$$
\|\xi_u(t_m)\|_0^2 + \int_0^{t_m} \nu \|\nabla \xi_u(s)\|_0^2 ds \le C \Big[\|e_u(0)\|_0^2 + \|\eta_u(0)\|_0^2 + \sum_{i=1}^{4} \mathscr{E}_i^m \Big], \tag{4.17}
$$

$$
\nu \|\nabla \xi_u\|_0^2 + \int_0^{t_j} \|\xi_{ut}\|_0^2 ds \le C \Big[\|e_u(0)\|_0^2 + \|\eta_u(0)\|_0^2 + \sum_{i=1}^{5} \mathscr{E}_i^m \Big]. \tag{4.18}
$$

Proof. Choosing $v = \xi_u$ in equation (4.16), and using Lemma 2.6, obtain that

$$
\begin{aligned}
&\frac{1}{2} \frac{d}{dt} \|\xi_u\|_0^2 + \nu \|\nabla \xi_u\|_0^2 \\
= \;& (\eta_{ut}, \xi_u) + (f^n - f, \xi_u) + \frac{1}{k_n} (I_h^n U^{n-1} - U^{n-1}, \xi_u) - b(\xi_u, U, \xi_u) - b(u, \xi_u, \xi_u) \\
& + (t_n - t) \Big(\partial_t r^n(\xi_u) - \nu (\nabla \partial_t U^n, \nabla \xi_u) - b(\partial_t U^n, U, \xi_u) - b(U^n, \partial_t U^n, \xi_u) \Big) \\
= \;& (\eta_{ut}, \xi_u) + (f^n - f, \xi_n) + \frac{1}{k_n} (I_h^n U^{n-1} - U^{n-1}, \xi_u - I_h^n \xi_u) - b(\xi_u, U, \xi_u) - b(u, \xi_u, \xi_u) \\
& + (t_n - t) \Big(\partial_t r^n(\xi_u) - \nu (\nabla \partial_t U^n, \nabla \xi_u) - b(\partial_t U^n, U, \xi_u) - b(U^n, \partial_t U^n, \xi_u) \Big) \\
\triangleq \;& T_1^n + T_2^n + T_3^n + T_4^n + T_5^n + T_6^n.
\end{aligned} \tag{4.19}
$$

Now, we estimate the right-hand side terms of (4.19) separately. For T_1^n and T_2^n, with Cauchy inequality, it is easy to see that

$$
\begin{aligned}
|(\eta_{ut}, \xi_u)| + |(f^n - f, \xi_n)| &\le \|\eta_{ut}\|_0 \|\xi_u\|_0 + \|f^n - f\|_0 \|\xi_u\|_0 \\
&\le \frac{1}{2} (\|\eta_{ut}\|_0^2 + \|f^n - f\|_0^2) + \frac{1}{2} \|\xi_u\|_0^2.
\end{aligned}
$$

For T_3^n, by Cauchy inequality and Lemma 2.6, we find that

$$\begin{aligned}|T_3^n| &\leq k_n^{-1}\|I_h^nU^{n-1}-U^{n-1}\|_0\|\xi_u-I_h^n\xi_u\|_0\\ &\leq \frac{3}{2\nu}\|k_n^{-1}h_n(I-I_h^n)U^{n-1}\|_0^2+\frac{\nu}{6}\|\nabla\xi_u\|_0^2.\end{aligned}$$

For T_4^n and T_5^n, by (2.2) and Theorem 4.1, we arrive at

$$\begin{aligned}|T_4^n|+|T_5^n| &\leq C_0(\|A_hU\|_0+\|Au\|_0)\|\nabla\xi_u\|_0\|\xi_u\|_0\\ &\leq \frac{3C_0^2}{2\nu}(\|A_hU\|_0^2+\|Au\|_0^2)\|\xi_u\|_0^2+\frac{\nu}{6}\|\nabla\xi_u\|_0^2.\end{aligned}$$

For T_6^n, using the fact that $r_1^n(v_h)=0$ for $\forall\ v_h\in X_h$, we have $(r_1^n-r_1^{n-1})(v_h)=0$ for all $v_h\in X^n\cap X^{n-1}$. Let $\hat{I}_h^n$ be the L^2-projection relative to the finest common coarsening $\hat{\mathcal{T}}_n$ of $\mathcal{T}_n$ and $\mathcal{T}_{n-1}$. For $\forall\ t\in(t_{n-1},t_n]$, we deduce that

$$\begin{aligned}|T_6^n| &\leq (t_n-t)\Big[\partial_tr^n(\xi_u-\hat{I}_h^n\xi_u)-\nu(\nabla\partial_tU^n,\nabla\xi_u)-b(\partial_tU^n,U,\xi_u)-b(U^n,\partial_tU^n,\xi_u)\Big]\\ &\leq \frac{3}{2\nu}k_n^2\Big(C_0^2(\|A_hU\|_0^2+\|A_hU^n\|_0^2)\|\partial_tU^n\|_0^2+\|\hat{h}_n\partial_tr^n\|_0^2+\nu\|\nabla\partial_tU^n\|_0^2\Big)+\frac{\nu}{6}\|\nabla\xi_u\|_0^2.\end{aligned}$$

Combining above inequalities with (4.19), integrating it with respect to time from 0 to t_m with $m\in[1:N]$, and using Lemma 2.1, we complete the proof of (4.17).

Next, we choose $v=\xi_{ut}$ in (4.16) and obtain

$$\begin{aligned}&\|\xi_{ut}\|_0^2+\frac{\nu}{2}\frac{d}{dt}\|\nabla\xi_u\|_0^2+b(\xi_u,U,\xi_{ut})+b(u,\xi_u,\xi_{ut})\\ =\;&(\eta_{ut},\xi_{ut})+(f^n-f,\xi_{ut})+\frac{1}{k_n}(I_h^nU^{n-1}-U^{n-1},\xi_{ut})+(t_n-t)\Big(\partial_tr^n(\xi_{ut})\\ &-\nu(\nabla\partial_tU^n,\nabla\xi_{ut})-b(\partial_tU^n,U,\xi_{ut})-b(U^n,\partial_tU^n,\xi_{ut})\Big)\\ =\;&(\eta_{ut},\xi_{ut})+(f^n-f,\xi_{ut})+\frac{1}{k_n}(I_h^nU^{n-1}-U^{n-1},\xi_{ut}-I_h^n\xi_{ut})\\ &+(t_n-t)\Big(\partial_tr^n(\xi_{ut})-(\nabla\partial_tU^n,\nabla\xi_{ut})-b(\partial_tU^n,U,\xi_{ut})-b(U^n,\partial_tU^n,\xi_{ut})\Big)\\ \triangleq\;&T_1^n+T_2^n+T_3^n+T_4^n.\end{aligned}\tag{4.20}$$

Integrating (4.20) with respect to time from 0 to t_m with $m\in[1:N]$, one gets

$$\begin{aligned}&\nu\|\nabla\xi_u(t_m)\|_0^2+2\int_0^{t_m}\|\xi_{ut}\|_0^2ds+2\int_0^{t_m}\Big(b(\xi_u,U,\xi_{ut})+b(u,\xi_u,\xi_{ut})\Big)ds\\ =\;&\nu\|\nabla\xi_u(0)\|_0^2+2\sum_{n=1}^m\int_{t_{n-1}}^{t_n}(T_1^n+T_2^n+T_3^n+T_4^n)ds.\end{aligned}\tag{4.21}$$

Set

$$\mathscr{F}^2(t_i)=\max_{0\leq t_i\leq t_m}\Big(\nu\|\nabla\xi_u(t_m)\|_0^2+\int_0^{t_i}\|\xi_{ut}\|_0^2ds\Big),\quad\text{for }\forall\ i\in[0:m].\tag{4.22}$$

By Cauchy inequality and integration by parts, it is straight forward to deduce that

$$\begin{aligned}
& |T_1^n|+|T_2^n|+|T_4^n| \\
\leq\ & \Big[\|\eta_{ut}\|_0+\|f^n-f\|_0+(t_n-t)\Big(\|\partial_t r^n\|_0+\|\Delta\partial_t U^n\|_0 \\
& +C_0(\|A_hU\|_0+\|A_hU^n\|_0)\|\nabla\partial_t U^n\|_0\Big)\Big]\|\xi_{ut}\|_0.
\end{aligned}$$

By using Cauchy inequality again, we get

$$\begin{aligned}
& 2\sum_{n=1}^{m}\int_{t_{n-1}}^{t^n}(T_1^n+T_2^n+T_4^n)ds \\
\leq\ & 2\Big\{\Big[\sum_{n=1}^{m}k_n^3\Big(C_0^2(\|A_hU\|_0^2+\|A_hU^n\|_0^2)\|\nabla\partial_t U^n\|_0^2+\|\partial_t r^n\|_0^2+\|\Delta\partial_t U^n\|_0^2\Big)\Big]^{1/2} \\
& +\Big(\sum_{n=1}^{m}\int_{t_{n-1}}^{t^n}\|\eta_{ut}\|_0^2ds\Big)^{1/2}+\Big(\sum_{n=1}^{m}\int_{t_{n-1}}^{t^n}\|f^n-f\|_0^2ds\Big)^{1/2}\Big\}\cdot\Big(\int_0^{t_m}\|\xi_{ut}\|_0^2ds\Big)^{1/2}.
\end{aligned}$$

For T_2^n, using integration for ξ_{ut} from t_{n-1} to t_n, and then applying summation by parts, we arrive at

$$\begin{aligned}
& \sum_{n=1}^{m}\int_{t_{n-1}}^{t_n}\frac{1}{k_n}(I_h^nU^{n-1}-U^{n-1},\xi_{ut})ds \\
=\ & \sum_{n=1}^{m}\frac{1}{k_n}(I_h^nU^{n-1}-U^{n-1},\xi_u^n-\xi_u^{n-1}) \\
=\ & \Big((I_h^m-I)(\frac{1}{k_m}U^{m-1}),\xi_u^m\Big)-\Big((I_h^1-I)(\frac{1}{k_1}U^0),\xi_u^0\Big) \\
& +\sum_{n=2}^{m}k_n\Big(\partial_t(I_h^n-I)(\frac{1}{k_n}U^{n-1}),\xi_u^{n-1}\Big).
\end{aligned}$$

From (2.2), we derive that

$$\begin{aligned}
& 2\int_0^{t_m}(b(\xi_u,U,\xi_{ut})+b(u,\xi_u,\xi_{ut}))ds \\
\leq\ & 2C_0\int_0^{t_m}(\|A_hU\|_0+\|Au\|_0)\|\nabla\xi_u\|_0\|\xi_{ut}\|_0ds \\
\leq\ & C\Big(\int_0^{t_m}(\|A_hU\|_0^2+\|Au\|_0^2)\|\nabla\xi_u\|_0^2ds\Big)^{\frac{1}{2}}\cdot(\int_0^{t_m}\|\xi_{ut}\|_0^2ds)^{1/2}.
\end{aligned}$$

By Lemma 2.6 and (4.22), one deduces that

$$\begin{aligned}
&\sum_{n=1}^{m}\int_{t_{n-1}}^{t_n}\frac{1}{k_n}(I_h^nU^{n-1}-U^{n-1},\xi_{ut})ds\\
=\;&\Big((I_h^m-I)(\frac{1}{k_m}U^{m-1}),\xi_u^m-I_h^m\xi_u^m\Big)-\Big((I_h^1-I)(\frac{1}{k_1}U^0),\xi_u^0-I_h^0\xi_u^0\Big)\\
&+\sum_{n=2}^{m}k_n\Big(\partial_t(I_h^n-I)(\frac{1}{k_n}U^{n-1}),\xi_u^{n-1}-\hat{I}_h^n\xi_u^{n-1}\Big)\\
\leq\;& C\Big(\|h_1(I_h^1-I)(\frac{1}{k_1}U^0)\|_0\|\nabla\xi_u^0\|_0+\|h_m(I_h^m-I)(\frac{1}{k_m}U^{m-1})\|_0\|\nabla\xi_u^m\|_0\\
&+\sum_{n=2}^{m}k_n\|\hat{h}_n\partial_t(I_h^n-I)(\frac{1}{k_n}U^{n-1})\|_0\|\nabla\xi_u^{n-1}\|_0\Big)\\
\leq\;& C\Big(\|h_1(I_h^1-I)(\frac{1}{k_1}U^0)\|_0+\sum_{n=2}^{m}k_n\|\hat{h}_n\partial_t(I_h^n-I)(\frac{1}{k_n}U^{n-1})\|_0\\
&+\|h_m(I_h^m-I)(\frac{1}{k_m}U^{m-1})\|_0\Big)\cdot\mathscr{F}(t_i).
\end{aligned}$$

Combining above inequalities with (4.21), Theorem 4.1 and replacing t_m by t_i, we complete the proof of (4.18).¶

Since (4.9) is quite similar in form to (3.4), we can prove the error estimates similar to Lemma 3.4 by following closely to the proof of Lemma 3.4. Here, we omit the proof.

Using required estimates of η_u and η_{ut} in Theorem 4.1, we obtain the final theorem of this section. Now, we firstly introduce some notations to be used in what follows. Set

$$\begin{aligned}
\mathscr{E}_6^m &= \sum_{E\in\mathcal{E}_h}h_E^3\|[\nabla U^0]\|_{0,E}^2+\sum_{K\in\mathcal{T}_h}h_K^4\|r_1^0\|_{0,K}^2,\\
\mathscr{E}_7^m &= \sum_{E\in\mathcal{E}_h}h_E^3\|[\nabla U^m]\|_{0,E}^2+\sum_{K\in\mathcal{T}_h}h_K^4\|r_1^m\|_{0,K}^2,\\
\mathscr{E}_8^m &= \sum_{n=1}^{m}k_n\Big(\sum_{E\in\mathcal{E}_h}h_E^3\|\partial_t[\nabla U^m]\|_{0,E}^2+\sum_{K\in\mathcal{T}_h}h_K^4\|\partial_t r_1^m\|_{0,K}^2\Big),\\
\mathscr{E}_9^m &= \sum_{E\in\mathcal{E}_h}h_E\|[\nabla U^m]\|_{0,E}^2+\sum_{K\in\mathcal{T}_h}h_K^2\|r_1^m\|_{0,K}^2.
\end{aligned}$$

Theorem 4.4. *Under the assumptions of Theorem 4.3 and* (**A1**)*, set u and U be the solution of* (2.3) *and* (4.2)*, respectively. Then, for $m\in[1,N]$, the following* *estimates hold*

$$\|U^m - u(t_m)\|_0^2 \le C\Big[\|e_u(0)\|_0^2 + \mathscr{E}_1^m + \mathscr{E}_3^m + \mathscr{E}_4^m\Big] + \sum_{i=6}^{8} \mathscr{E}_i^m,$$

$$\|\nabla(U-u)\|_0^2 + \int_0^{t_m} \|U_t - u_t\|_0^2 ds \le C\Big[\|e_u(0)\|_0^2 + \mathscr{E}_1^m + \sum_{i=3}^{6} \mathscr{E}_i^m + \mathscr{E}_8^m + \mathscr{E}_9^m\Big],$$

5. Numerical experiment

In this section, we provide some numerical results to verify the performance of the established posteriori error estimators. For the computed quantities like errors and indicators, we denote $\|e_u\|_1 = \|u - u_h\|_1$, the error indicator η and the number of triangulares in $\mathcal{T}_h$ (NT) which are output of the adaptive algorithm for a given tolerance ε. The experimental convergence rates are given by

$$\alpha_{e_u} = \frac{2 * log[\|e_u(\varepsilon_1)\|_0 / \|e_u(\varepsilon_2)\|_0]}{log[NT(\varepsilon_1)/NT(\varepsilon_1)]}, \qquad \alpha_\eta = \frac{2 * log[\eta(\varepsilon_1)/\eta(\varepsilon_2)]}{log[NT(\varepsilon_1)/NT(\varepsilon_1)]}.$$

The effectiveness index is defined as ration of a posteriori error bound and an approximate norm of the actual error, i.e., $\|e_u\|_1/\eta$. For a good estimator, this quantity should be a constant, independent of the mesh sizes and the time steps. Although our theoretical findings do not include a proof of efficiency, numerical experiments provide evidences of the efficiency of the estimators.

For the space-time algorithm, we follow the guideline provided in [22]. In order to keep the completeness of our work, we present the outline as follows.

Algorithm. Let $\widehat{\mathcal{T}}_0$ be a regular triangulation and ε the given tolerance.

(i). Compute on the shape-regular partition $\widehat{\mathcal{T}}_0$ with $t_0 = 0$.

(ii). Set an initial time step k_0, using U^0 to compute $\|u - U_h^0\|_1$ and η on $\widehat{\mathcal{T}}_0$.

(iii). Begin the time loop with obtained $\mathcal{T}_{n-1}, \widehat{\mathcal{T}}_n$ and U^{n-1}

(1) Set the time step $\widehat{t}_n = \min(t_{n-1} + k_n, T)$,

(2) Use $\widehat{k}_n$ to compute $\widehat{U}^n$ on $\widehat{\mathcal{T}}_n$, and then compute $\widehat{\eta}^n$,

(3) Set $t_n = \min(t_{n-1} + (\widehat{t}_n - t_{n-1})\varepsilon/\widehat{\eta}^n, T)$, and obtain that $k_n = t_n - t_{n-1}$,

(4) Use $\widehat{\mathcal{T}}_n, U^{n-1}$ and k_n to compute U^n and η^n,

(5) Adapt mesh $\widehat{\mathcal{T}}_n$ to obtain $\mathcal{T}_n$. Use U^{n-1} and k_n to compute U^n,

(6) For the next iteration, denote $\widehat{\mathcal{T}}_{n+1}$=$\mathcal{T}_n$ and $\widehat{k}_{n+1} = k_n$.

(iv) End the time loop and finish the computation.

Remark 5.1. *For simplifying the computation, we adopt Oseen iteration to treat the nonlinear term, and set all the constants C involved in indicator and ν equal to 1. For the time steps, as the strategy used in [24], we choose a maximum over the time step instead of summing up over different time steps.*

Consider the nonlinear parabolic problem

$$\begin{cases} u_t - \Delta u + u\nabla \cdot u = f & \text{in } \Omega \times (0,1] \\ u = g & \text{on } \partial\Omega \times (0,1] \\ u(0) = 0 & \text{on } \Omega \times \{0\}, \end{cases}$$

with $\Omega = [0,2]^2$ and $T = 1$. Choose f and g with the exact solution

$$u(x,y) = \frac{1}{10}\sin(\frac{\pi t}{2})\exp\Big\{-20\Big\|\Big(x-\frac{1}{2}, y-\frac{1}{2}\Big) - \frac{3}{4}\Big(\cos(\frac{\pi}{t}), \sin(\frac{\pi t}{2})\Big)\Big\|\Big\}$$

where $\|(x,y)\| = (x^2+y^2)^{1/2}$. We adopt the linear polynomial to seek the exact solution by backward Euler scheme in time. Table 1 presents the errors and convergence order with different tolerances ε. As expected, we can see that as ε becomes small, the errors go down while the numbers of triangulares increase quickly. The effectiveness index approaches 0.06, which is a constant independent of NT and time steps. Next, fixed the tolerance $\varepsilon = 0.04$, the meshes, the profiles of u_h and numerical solutions u_h are described in Figures 1-4 at different times. From these Figures, we can see that more meshes are concentrated on the critical region where the solution vary sharply. Figure 5 expresses the variation of time step as time increases with different tolerances ε. Generally speaking, the smaller of ε, the smaller of time step. Furthermore, as the time increases, the time step becomes smaller and smaller. Finally, we compare the profiles of solutions, including the exact solution and numerical solution with different ε in Figure 6. From these figures we can see that as the tolerance ε decreases, we can simulate the exact solution more precisely.

Table 1: Results obtained using space-time algorithm with linear polynomial.

ε	NT	$\|e_u\|_1$	η	$\|e_u\|_1/\eta$	α_{e_u}	α_η
0.08	1418	0.01707	0.3026	0.0564		
0.06	2175	0.01361	0.2385	0.0571	1.0590	1.1129
0.05	2634	0.01252	0.2156	0.0581	0.8719	1.0544
0.04	5164	0.009135	0.1501	0.0609	0.9365	1.0758

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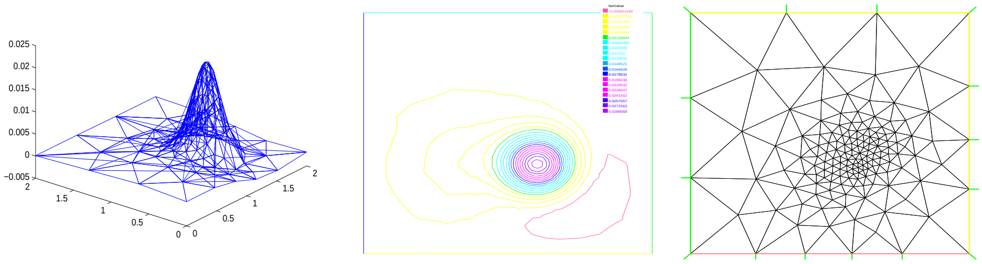
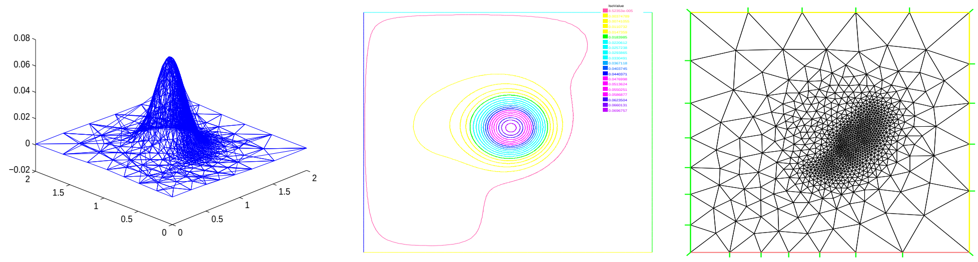


(a) (b) (c)

Figure 1: Meshes, the profiles of u_h and numerical solution u_h at time t=0.160751.



(a) (b) (c)

Figure 2: Meshes, the profiles of u_h and numerical solution u_h at time t=0.442934.

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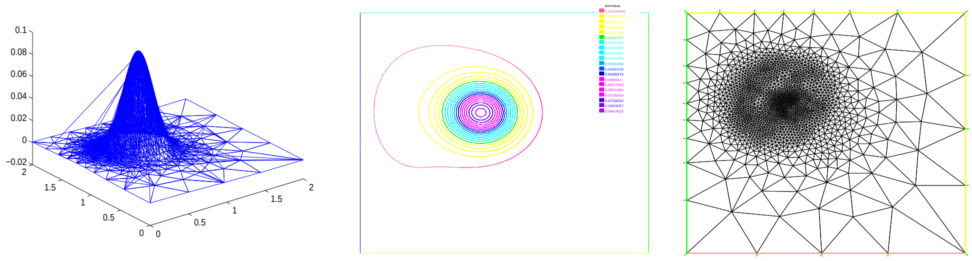
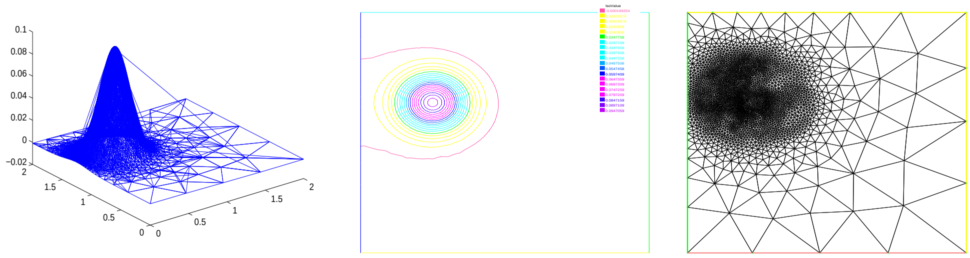


(a) (b) (c)

Figure 3: Meshes, the profiles of u_h and numerical solution u_h at time t=0.765945.



(a) (b) (c)

Figure 4: Meshes, the profiles of u_h and numerical solution u_h at time t=1.

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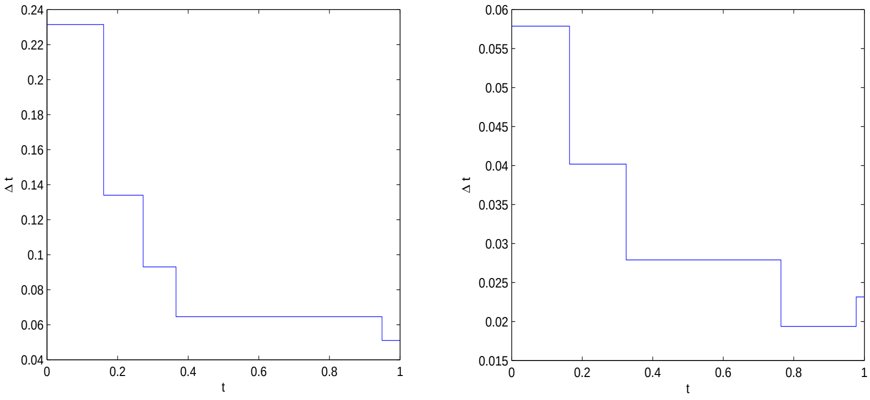
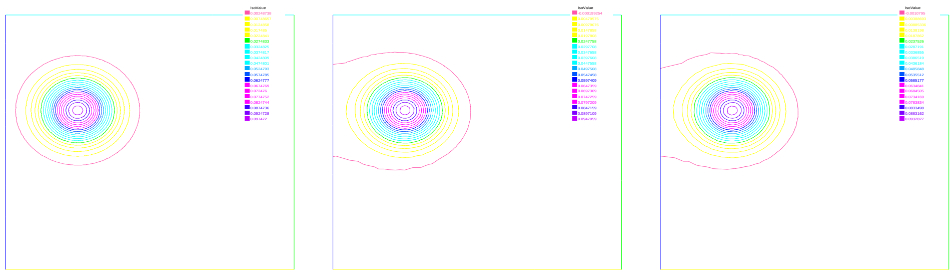


(a) (b)

Figure 5: The variable of time step with different ε. (a) $\varepsilon = 0.06$, (b) $\varepsilon = 0.04$.



(a) (b) (c)

Figure 6: The profiles of exact solution and numerical solution. (a) Exact solution, (b) $\varepsilon = 0.04$, (c) $\varepsilon = 0.08$.

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