ON THE CONVERGENCE RATE OF SEMI-GALERKIN APPROXIMATIONS FOR THE EQUATIONS OF VISCOUS FLUIDS IN THE PRESENCE OF DIFFUSION

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Abstract

We study the convergence rate of spectral semi-Galerkin approximations for the equations of motion of a nonhomogeneous viscous fluid in the presence of diffusion in a bounded domain. We find error estimates that are optimal in the H^1-norm as well as improved estimates in the L^2-norm.

Resumo

Estudamos a taxa de convergência das aproximações de semi-Galerkin espectrais para as equações do movimento de um fluido viscoso, não homogêneo na presença de difusão num domínio limitado. Encontramos estimativas de erro que são otimais na norma H^1 como também estimativas melhores na norma L^2.

1. Introduction

In this paper we will be working with the equations of motion of nonhomogeneous viscous incompressible fluids in the presence of diffusion. These equations are considered in a bounded domain $\omega \subset I\!\!R^3$, with boundary Γ, in a time interval $[0,T]$. To describe them let $u(x,t) \in I\!\!R^3, \rho(x,t) \in I\!\!R$ and $p(x,t) \in I\!\!R$

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denote, respectively, the unknown velocity, the density and the pressure of the fluid at a point $x \in \omega$, at time $t \in [0,T]$. Then, the governing equations are

$$
\begin{aligned}
&\rho\frac{\partial u}{\partial t} + \rho(u.\nabla)u - \mu\Delta u - \lambda[(u.\nabla)\nabla\rho + (\nabla\rho.\nabla)u] = -\nabla p + \rho f \\
&\text{div } u = 0 \\
&\frac{\partial \rho}{\partial t} + u.\nabla\rho - \lambda\Delta\rho = 0
\end{aligned}
\tag{1.1}
$$

together with the following boundary and initial conditions

$$
\begin{aligned}
u &= 0 \quad \text{on} \quad \Gamma \times (0,T), \\
u(x,0) &= u_0(x) \quad \text{in} \quad \omega, \\
\frac{\partial \rho}{\partial n} &= 0 \quad \text{on} \quad \Gamma \times (0,T) \\
\rho(x,0) &= \rho_0(x) \quad \text{in} \quad \omega.
\end{aligned}
\tag{1.2}
$$

Here $f(x,t)$ is the density by unit of mass of the external force acting on the fluid. The positive constants, μ and λ are the usual Newtonian viscosity and the diffusion coefficient, respectively. The symbols ∇, Δ and div denote the gradient, Laplacian and divergence operators; $n = n(x)$ is the unit outward normal to Γ. Also,

$$
(u.\nabla)u = \sum_{i=1}^{n} u_i \frac{\partial u}{\partial x_i}; \quad (u.\nabla)\nabla\rho = \sum_{i=1}^{n} u_i \frac{\partial}{\partial x_i}\nabla\rho; \tag{1.3}
$$

$$
(\nabla\rho.\nabla)u = \sum_{i=1}^{n} \frac{\partial \rho}{\partial x_i}\frac{\partial}{\partial x_i}u. \tag{1.4}
$$

For the derivation and physical discussion of equations (1.1) see Frank-Kamenestskii [9], Antoncev, Kazhikov and Monakhov [3], Prouse [16]. We observe that this model includes as a particular case the classical Navier-Stokes system, which has been much studied (see, for instance, the classical books by Ladyzhenkaya [14] and Temam [23] and the references there in). It also includes the reduced model of the nonhomogeneous Navier-Stokes equations, which has been less studied than the previous case (see for instance Simon [22], Kim [13], Ladyzhenkaya and Solonnikov [15], Salvi [20] Boldrini and Rojas-Medar [6]).

Concerning the generalized model of fluids considered in this paper Kazhikov and Smagulov [12] established the local existence of weak and strong solutions for (1.1) - (1.2) under certain assumptions by using the semi-Galerkin method, Salvi [19] also via that method proved the weak solution in a non-cylindrical domain.

Also, Beirão da Veiga [4], Secchi [21], established the local existence of strong solutions for a model that contain terms of order λ^2 by using linearization and fixed point argument. A more practical semi-Galerkin method was used by Damázio and Rojas-Medar [7].

In this work we are interested in establishing error estimates and convergence rates of the spectral semi-Galerkin approximations in several norms. By spectral semi-Galerkin approximations we mean that we make use of finite dimensional approximations for the velocity and infinite dimensional approximations for the density.

Before we describe our results, let us briefly comment related results.

Rautmann in [17] gave a systematic development of error estimates for spectral Galerkin approximations of the classical Navier-Stokes equations. Boldrini and Rojas-Medar gave analogous error estimates for a model of nonhomogeneous asymmetric fluids [5].

In this paper we consider the convergence rate of spectral semi-Galerkin approximations for the solutions of a more general fluid model (1.1)-(1.2). We show optimal rate of convergence in the H^1-norm (see Theorem 3.6). Differently to the case of the classical Navier-Stokes equations, for which optimal L^2-error estimates can be obtained (see Rojas-Medar and Boldrini [18]), in this case we are only able to obtain improved L^2-error estimates as compared to the trivial ones derived directly from the H^1-estimate (see Theorem 4.1).

The paper is organized as follows: in Section 2 we state some preliminary results that will be useful in the rest of the paper; we describe the approximation method and state the existence theorem that form the theoretical basis for the problem. In Section 3 we derive a H^1-error estimate for the velocity and a L^∞-error estimate for the density. In Section 4 we derive an improved L^2-error estimate for the velocity.

Finally, we would like to say that, as is usual in this context, to simplify the notation in the expressions we will denote by c a generic finite positive constant depending only on ω and on other fixed parameters of the problem (like the initial data) that may have different values in different expressions. To emphasize the fact that the constants are different we may use c_1, c_2, and so on.

2. Preliminaries

Let $\omega \subset I\!R^n, n = 2$ or 3, be a bounded domain with boundary Γ of class $C^{1,1}$.

We will consider the usual Sobolev spaces

$$W^{m,q}(D) = \{f \in L^q(D)/||\partial^\nu f||_{L^q(D)} < +\infty, |\nu| \leq m\},$$

$m = 0, 1, 2, \ldots, 1 \leq q \leq +\infty, D = \omega$ or $(0,T) \times \omega, 0 < T < +\infty$, with the usual norm. When $q = 2$, we denote $H^m(D) = W^{m,2}(D)$ and $H_0^m(D) =$ closure of $C_0^\infty(D)$ in $H^m(D)$. If B is a Banach space, we denote by $L^q(0,T;B)$ the Banach space of the B-valued functions defined on interval $(0,T)$ that are L^q-integrable in the sense of Bochner. We define

$$\begin{aligned} C_{0,\sigma}^\infty(\omega) &= \{v \in C_0^\infty(\omega)/\text{div } v = 0\}, \\ H = &\quad \text{closure of } C_{0,\sigma}^\infty(\omega) \text{ in } L^2(\omega), \\ V = &\quad \text{closure of } C_{0,\sigma}^\infty(\omega) \text{ in } H^1(\omega). \end{aligned}$$

It is possible to show that $V = \{v \in H_0^1(\omega)|\ \text{div } v = 0\}$. We recall the Helmholtz decomposition of vector fields: $L^2(\omega) = H \oplus G$, where $G = \{\phi|\phi = \nabla p, p \in H^1(\omega)\}$, with H and G orthogonal with respect to the L^2-inner product.

Throughout the paper P will denote the orthogonal projection from $L^2(\omega)$ onto H. Then, the operator $A : D(A) \hookrightarrow H$ given by $A = -P\Delta$ with $D(A) = H^2(\omega) \cap V$ is called the Stokes operator. It is well know that A is a positive definite self-adjoint operator characterized by

$$(Aw, v) = (\nabla w, \nabla v) \quad \forall w \in D(A), v \in V.$$

From now on, we denote the inner product in H (i.e., the L^2-inner product) by $(,)$. The general L^p-norm will be denote by $||\cdot||_{L^p}$; to simplify the notation, in the case $p = 2$ we denote the L^2-norm by $||\cdot||$.

We observe that for the regularity properties of the Stokes operator it is usually assumed that ω is of class C^3 to use Cattabriga's results [23]; instead of that, we use the stronger result of Amrouche and Girault [2], which implies, in particular, that when $Au \in L^2(\omega)$ then $u \in H^2(\omega)$ and $||u||_{H^2}$ and $||Au||$ are equivalent norms when ω is of class $C^{1,1}$. This will be enough for all the results in this paper.

We will denote respectively by φ_k and $\lambda_k (k \in I\!N)$ the eigenfunctions and eigenvalues of the Stokes operator defined on $V \cap H^2(\omega)$. It is well know that $\{\varphi_k\}_{k=1}$ form an orthogonal complete system in the spaces H, V and $V \cap H^2(\omega)$ with their usual inner products $(u, v), (\nabla u, \nabla v)$ and (Au, Av), respectively.

We denote by $V_k = \text{span } [\varphi_1, \ldots, \varphi_k]$ and by P_k the orthogonal projection from $L^2(\omega)$ onto V_k.

The following results can be found in Rautmann's paper [17].

Lemma 2.1. *If $v \in V$, then there holds*

$$||v - P_k v||^2 \leq \frac{1}{\lambda_{k+1}} ||\nabla v||^2.$$

Also, if $v \in V \cap H^2(\omega)$, we have

$$||\nabla v - \nabla P_k v||^2 \leq \frac{1}{\lambda_{k+1}} ||Av||^2 \qquad \text{and} \qquad ||v - P_k v||^2 \leq \frac{1}{\lambda_{k+1}^2} ||Av||^2.$$

With the above notation, we rewrite (1.1)-(1.2) as

$$\begin{aligned} &P(\rho u_t + \rho u.\nabla u - \lambda[(u.\nabla)\nabla\rho + (\nabla\rho.\nabla)u] - \rho f) + \mu A u = 0 \\ &\frac{\partial \rho}{\partial t} + u.\nabla\rho - \lambda\Delta\rho = 0 \\ &u(0) = u_0, \ \rho(0) = \rho_0, \ \ \frac{\partial\rho}{\partial n}\Big|_\Gamma = 0. \end{aligned} \tag{2.1}$$

(2.1) is equivalent to the weak form

$$\begin{aligned}
&(\rho u_t, v) + (\rho u.\nabla u, v) - \lambda((u.\nabla)\nabla\rho, v) - \lambda((\nabla\rho.\nabla)u, v)\\
&+\mu(Au, v) = (\rho f, v) \qquad \forall v \in V\\
&\frac{\partial\rho}{\partial t} + u.\nabla\rho - \lambda\Delta\rho = 0 \qquad \text{for } \ 0 < t < T\\
&u(0) = u_0, \ \rho(0) = \rho_0, \ \left.\frac{\partial\rho}{\partial n}\right|_\Gamma = 0.
\end{aligned} \tag{2.2}$$

The spectral semi-Galerkin approximations for (u, ρ) are defined for each $k \in I\!N$ as the solution $(u^k, \rho^k) \in C^2([0,T], V_k) \times C^1(\overline{\omega} \times [0,T))$ of

$$\left.\begin{aligned}
&(\rho^k u^k_t, v) + (\rho^k u^k.\nabla u^k, v) + (Au^k, v)\\
-\ &\lambda((u^k.\nabla)\nabla\rho^k, v) - \lambda((\nabla\rho^k.\nabla)u^k, v) = (\rho^k f, v) \quad \forall v \in V_k\\
&\frac{\partial\rho^k}{\partial t} + u^k.\nabla\rho^k - \lambda\Delta\rho^k = 0 \qquad \forall (x,t) \in \omega \times (0,T)\\
&\frac{\partial\rho^k}{\partial n} = 0, \quad \forall x \in \Gamma\\
&u^k(x,0) = P_k u_0, \ \rho^k(0,x) = \rho_0(x), \quad \forall x \in \omega.
\end{aligned}\right\} \tag{2.3}$$

Before giving the Theorem of existence, we introduce the notation:

$$\left.\begin{aligned}
\widehat{\rho} \ &= \ \tfrac{1}{|\omega|}\textstyle\int_\omega \rho_0(x)dx\\
H^k_N \ &= \ \{\sigma \in H^k | \ \tfrac{\partial\sigma}{\partial n} = 0 \ \text{ on } \Gamma \text{ and } \ \textstyle\int_\omega \sigma(x)dx = 0\} \qquad k \geq 0
\end{aligned}\right\} \tag{2.4}$$

and consider the assumption $0 < \alpha \leq \rho_0(x) \leq \beta$ on the initial data.

By using these approximations, Kazhikov and Smagulov [12] proved existence of a weak solution for the above problem assuming that

$$\lambda > \frac{2\mu}{\beta - \alpha}. \tag{2.5}$$

In [4], H. Beirão da Veiga, proved existence and uniqueness of local and global solution of the problem (1.1)-(1.2) (with λ^2-term) without assumption (2.5) (Theorem A, pp. 345 in [4]). Damázio and Rojas-Medar also proved analogous results of [4] by using a more practical semi-Galerkin method [7].

By using the approximation (2.3), it can be proved that u^k, ρ^k converge in an appropriate sense to a solution (u, ρ) of (2.2) (or (1.1)-(1.2)) as $k \to \infty$. These conditions are given in the following Theorem, proved by Damázio and Rojas-Medar [7], on the existence and uniqueness of local strong solutions for problem (1.1)-(1.2).

Theorem 2.2. *Assume* $u_0 \in V \cap H^2(\omega), \rho_0 - \widehat{\rho} \in H^3_N,\ f \in L^2(0,T;H^1(\omega))$ *and* $f_t \in L^2(0,T;L^2(\omega))$. *Then there exists* $T_1 \in]0,T]$ *such that problem (1.1)-(1.2) is uniquely solvable in* $[0,T_1] \times \omega$.

Moreover the approximations u^k, ρ^k *satisfy the estimates*

$$\left.\begin{aligned}
&\alpha \leq \rho^k \leq \beta,\\
&||\nabla u^k(t)|| \leq F_1(t),\\
&||Au^k|| \leq F_2(t),\\
&||u^k_t||^2 + \int_0^t ||\nabla u^k_t||^2 ds \leq F_3(t),\\
&\int_0^t ||\nabla u^k(s)||^2_{L^\infty} ds \leq F_4(t),\\
&||\nabla \rho^k(t)||_{L^\infty} \leq F_5(t),\\
&||\Delta \rho^k||^2 + \int_0^t ||\rho^k(s)||^2_{H^3} ds \leq F_6(t),\\
&\int_0^t ||\nabla \rho^k_t||^2 ds \leq F_7(t).
\end{aligned}\right\} \tag{2.6}$$

Analogous estimates are verified by the solution (u, ρ).

The functions on the right-hand side depend on their argument t, *and in addition on* T_1 *and the fixed datum of problem. On the interval in question these functions are continuous in the variable* t.

We notice that H. Beirão da Veiga proved in [4] the regularity of the solution of problem (1.1)-(1.2) (with λ^2-terms) by linearization and fixed point argument, besides, he assumed that $u_0 \in V$ (naturally, the estimates given in [4] are weaker). Instead of that, we need estimates as in (2.6).

3. An optimal H^1-error estimates for the velocity

In this section we will prove some local in time optimal error estimates for the velocity in the H^1-norm. The optimality here means that the approximate solutions approach the solution of (2.2) at the best possible rate as measured by powers of the inverse of first discarded eigenvalue (λ_{k+1}) when one consideres approximations in the subspace V_k.

We start by working in a similar way as Heywood [10] for the case of the classical Navier-Stokes. For that, we define

Definition 3.1. Let (u, ρ) be the strong solution of problem (2.2) given by the Theorem 2.2 and (u^k, ρ^k) the approximate solution of problem (2.2). We define
i) $v^k = P_k u$
ii) $\theta^k = v^k - u^k$
iii) $E^k = u - v^k$
iv) $\pi^k = \rho - \rho^k$

In what follows we denote by $\overline{c} > 0$ a positive constant independent of k that may depend on the functions F_i given in the Theorem 2.2.

For these variables the following is true:

Lemma 3.2.

$$||\theta^k(t)||^2 + ||\pi^k(t)||^2 + C_0 \int_0^t (||\nabla\theta^k(s)||^2 + ||\nabla\pi^k(s)||^2) ds \leq \frac{\overline{c}}{\lambda_{k+1}^2} + \frac{\overline{c}}{\lambda_{k+1}}.$$

Proof. We observe that $v^k = P_k u$ satisfies

$$P_k(\rho u_t + \rho u.\nabla u - \rho f) - \lambda P_k((u.\nabla)\nabla\rho + (\nabla\rho.\nabla)u) + \mu A v^k = 0. \tag{3.1}$$

Subtracting (3.1) from (2.3) we obtain

$$\begin{aligned} &P_k[\pi^k(u_t + u.\nabla u - f) + \rho^k E_t^k + \rho^k\theta^k.\nabla u + \rho^k E^k.\nabla u \\ &+\rho^k u^k.\nabla\theta^k + \rho^k u^k.\nabla E^k] + P_k(\rho^k\theta_t^k) + \mu A\theta^k \\ &-\lambda P_k[(\theta^k.\nabla)\nabla\rho + (E^k.\nabla)\nabla\rho + (u^k.\nabla)\nabla\pi^k \\ &+(\nabla\pi^k.\nabla)u + (\nabla\rho^k.\nabla)\theta^k + (\nabla\rho^k.\nabla)E^k] = 0. \end{aligned} \tag{3.2}$$

By taking the inner product in $L^2(\omega)$ of the above identity with θ^k, we get

$$
\begin{aligned}
&(\rho^k\theta^k_t,\theta^k)+\mu||\nabla\theta^k||^2-(\pi^k(u_t+u.\nabla u-f),\theta^k)\\
&-(\rho^kE^k_t,\theta^k)-(\rho^k\theta^k.\nabla u,\theta^k)-(\rho^kE^k.\nabla u,\theta^k)-(\rho^ku^k.\nabla\theta^k,\theta^k) \qquad (3.3)\\
&-(\rho^ku^k.\nabla E^k,\theta^k)+\lambda((\theta^k.\nabla)\nabla\rho,\theta^k)+\lambda((E^k.\nabla)\nabla\rho,\theta^k)+\lambda((u^k.\nabla)\nabla\pi^k,\theta^k)\\
&+\lambda((\nabla\pi^k.\nabla)u,\theta^k)+\lambda((\nabla\rho^k.\nabla)\theta^k,\theta^k)+\lambda((\nabla\rho^k.\nabla)E^k,\theta^k)=0.
\end{aligned}
$$

We observe that

$$
\begin{aligned}
(\rho^k\theta^k_t,\theta^k) &= \frac{1}{2}\frac{d}{dt}||(\rho^k)^{1/2}\theta^k||^2-\frac{1}{2}(\rho^k_t\theta^k,\theta^k)\\
\left|\frac{1}{2}(\rho^k_t\theta^k,\theta^k)\right| &= \left|\frac{1}{2}\lambda(\Delta\rho^k\theta^k,\theta^k)-(u^k\cdot\nabla\rho^k\theta^k,\theta^k)\right|\\
&\le \lambda||\Delta\rho^k||||\theta^k||_{L^6}||\theta^k||_{L^3}+C||Au^k||||\nabla\rho^k||_{L^6}||\theta^k||_{L^3}||\theta^k||\\
&\le C||\theta^k||^2+2\epsilon||\nabla\theta^k||^2
\end{aligned}
$$

The other terms in (3.3) are estimated using Hölder and Young inequalities and Sobolev imbedding theorem. For instance,

$$
\begin{aligned}
&|\lambda((\nabla\rho^k.\nabla)E^k,\theta^k)|\\
&=\left|\lambda\sum_{i,j}\int_\omega\frac{\partial\rho^k}{\partial x_j}\frac{\partial E^k_i}{\partial x_j}\theta^k_i\right|\\
&=\left|-\lambda\sum_{i,j}\int_\omega\frac{\partial}{\partial x_j}\left(\frac{\partial\rho^k}{\partial x_j}\theta^k_i\right)E^k_i\right|\\
&\le\left|\lambda\sum_{i,j}\int_\omega\frac{\partial^2\rho^k}{\partial x_i^2}\theta^k_iE^k_i\right|+\left|\lambda\sum_{i,j}\int_\omega\frac{\partial\rho^k}{\partial x_j}\frac{\partial\theta^k_i}{\partial x_j}E^k_i\right|\\
&\le\lambda||\Delta\rho^k||_{L^4}||\theta^k||_{L^4}||E^k||+\lambda||\nabla\rho^k||_{L^\infty}||\nabla\theta^k||\ ||E^k||\\
&\le c_\varepsilon\lambda^2||\Delta\rho^k||^2_{L^4}||E^k||^2+c_\varepsilon\lambda||\nabla\rho^k||^2_{L^\infty}||E^k||^2+2\varepsilon||\nabla\theta^k||^2.
\end{aligned}
$$

The next step is considering the equation of π^k; we have

$$
\pi^k_t+\theta^k.\nabla\rho+E^k.\nabla\rho+u^k.\nabla\pi^k-\lambda\Delta\pi^k=0. \qquad (3.4)
$$

By taking the inner product in $L^2(\omega)$ of the above identity with π^k, we obtain

$$\begin{aligned}\frac{d}{dt}||\pi^k||^2 + \lambda||\nabla\pi^k||^2 &= -(\theta^k.\nabla\rho, \pi^k) - (E^k.\nabla\rho, \pi^k) \\ &\leq c_\delta||\Delta\rho||^2(||E^k||^2 + ||\theta||^2) + 2\delta||\nabla\pi^k||^2\end{aligned} \tag{3.5}$$

since

$$\int_\omega u^k.\nabla\pi^k\pi^k = \int_\omega u^k.\nabla\frac{|\pi^k|^2}{2} = -\int_\omega \text{div } u^k\frac{|\pi^k|^2}{2} = 0.$$

Now, identity (3.3), inequality (3.5) and the above estimates imply the following differential inequality:

$$\frac{1}{2}\frac{d}{dt}(||\pi^k||^2 + ||(\rho^k)^{1/2}\theta^k||^2) + \mu||\nabla\theta^k||^2 + \lambda||\nabla\pi^k||^2$$

$$\begin{aligned}\leq\ & \frac{1}{2}||\rho^k||^2_{L^\infty}||E^k_t||^2 + \frac{1}{2}||\theta^k||^2(1 + ||\rho^k_t||_{L^\infty} + ||\nabla\rho^k||^2_{L^\infty}) \\ &+c_\varepsilon||\theta^k||^2[||\rho^k||^2_{L^\infty}||Au||^2 + ||\rho^k||^2_{L^\infty}||Au^k||^2 + \lambda^2||\nabla\rho||^2_{L^\infty} \\ &+\lambda^2||\nabla\rho^k||^2_{L^\infty}] + c_\delta||\theta^k||^2\lambda^2[||\nabla u^k||^2_{L^\infty} + ||\nabla u||^2_{L^\infty}] \\ &+c_\varepsilon||E^k||^2[||\nabla\rho^k||^2_{L^\infty}||\nabla u^k||^2 + ||\rho^k||^2_{L^\infty}||Au^k||^2 + ||\rho^k||^2_{L^\infty}||Au||^2 \\ &+\lambda^2||\Delta\rho^k||^2_{L^4} + \lambda^2||\nabla\rho^k||^2_{L^\infty} + \lambda^2||\nabla\rho||^2_{L^\infty}] \\ &+c_\varepsilon||\pi^k||^2\ ||u_t + u.\nabla u - f||^2_{L^4} + 10\varepsilon||\nabla\theta^k||^2 + 2\delta||\nabla\pi^k||^2 \\ &+\frac{1}{2}||\nabla\rho||^2_{L^\infty}||E^k||^2 + ||\pi^k||^2.\end{aligned}$$

By taking $\varepsilon = \frac{1}{20}\mu$, $\delta = \frac{1}{4}\lambda$, the above differential inequality yields the integral inequality

$$\begin{aligned}\frac{1}{2}||\pi^k(t)||^2 &+ \frac{1}{2}||(\rho^k(t))^{1/2}\theta^k(t)||^2 + \frac{\mu}{2}\int_0^t ||\nabla\theta^k(s)||^2ds \\ &+\frac{\lambda}{2}\int_0^t ||\nabla\pi^k(s)||^2ds \\ &\leq \frac{1}{2}||\pi^k(0)||^2 + \frac{1}{2}||(\rho^k(0))^{1/2}\theta^k(0)||^2 + c\int_0^t ||E^k_t(s)||^2ds \\ &+c\int_0^t ||\theta^k(s)||^2[1 + ||\nabla u^k(s)||^2_{L^\infty} + ||\nabla u(s)||^2_{L^\infty}]ds \\ &+c\int_0^t ||E^k(s)||^2ds + c\int_0^t ||\pi^k(s)||^2ds,\end{aligned}$$

thanks to the estimates given in the Theorem 2.2.

Now, using the Lemma 2.1 and the fact that $||\rho^k(t))^{1/2}\theta^k(t)||^2 \geq \alpha||\theta^k(t)||^2$, we obtain

$$\begin{aligned}||\pi^k(t)||^2 + \alpha||\theta^k(t)||^2 + \mu\int_0^t ||\nabla\theta^k(s)||^2 ds + \lambda\int_0^t ||\nabla\pi^k(s)||^2 ds \\ \leq \frac{ct}{\lambda_{k+1}^2} + \frac{c}{\lambda_{k+1}}\int_0^t ||\nabla u_t(s)||^2 ds + c\int_0^t ||\pi^k(s)||^2 ds \\ + \frac{c}{\alpha}\int_0^t \alpha||\theta^k(s)||^2[1 + ||\nabla u^k(s)||^2_{L^\infty} + ||\nabla u(s)||^2_{L^\infty}]ds.\end{aligned}$$

Applying the Gronwall's inequality, we obtain

$$\begin{aligned}||\pi^k(t)||^2 + \alpha||\theta^k(t)||^2 + \mu\int_0^t ||\nabla\theta^k(s)||^2 ds + \lambda\int_0^t ||\nabla\pi^k(s)||^2 ds \\ \leq \left[\tfrac{ct}{\lambda_{k+1}^2} + \tfrac{c}{\lambda_{k+1}}\int_0^t ||\nabla u_t(s)||^2 ds\right] \exp\left\{c\int_0^t [1 + ||\nabla u^k(s)||^2_{L^\infty} + ||\nabla u(s)||^2_{L^\infty}]ds\right\} \\ \leq \overline{c}\left[\tfrac{1}{\lambda_{k+1}^2} + \tfrac{1}{\lambda_{k+1}}\right]\end{aligned}$$

for all $t \in [0, T]$, thanks to the estimates given in the Theorem 2.2. This proves the stated result.

□

Remark 3.3. The term $\int_0^t ||E_t^k(s)||^2 ds$ could be estimated with optimal rate if $\int_0^T ||Au_t(s)||^2 ds \leq c$. In fact, in this case, we have

$$\int_0^t ||E_t^k(s)||^2 ds \leq \frac{c}{\lambda_{k+1}^2}\int_0^t ||Au_t(s)||^2 ds \leq \frac{\overline{c}}{\lambda_{k+1}^2},$$

thanks to the estimates given in the Lemma 2.1, and the rest of the analysis could be done as before, given as a final result an estimate of order $\dfrac{1}{\lambda_{k+1}^2}$ in the Lemma 3.2.

However, as it was pointed out by Heywood and Rannacher [11], even in the classical case of the Navier-Stokes equations (constant ρ) a condition like $\int_0^T ||Au_t(s)||^2 ds \leq c$ (which implies that $\int_0^T ||u_t(s)||^2_{H^2} ds \leq c$) would require that the initial condition satisfies a nonlocal compatibility condition (see Corollary 2.1 and condition (1.5) in [11]) which cannot be expected unless the initial condition is very special.

In the next section we improved the above estimate (see Theorem 4.1). Now, the above Lemma implies immediately the following L^2-error estimate for velocity and density.

Theorem 3.4. *Suppose the assumptions of the Theorem 2.2 hold. Then, the approximations* u^k, ρ^k *satisfy*

$$||u(t) - u^k(t)||^2 + ||\rho(t) - \rho^k(t)||^2 \leq \overline{c}\Big[\frac{1}{\lambda_{k+1}^2} + \frac{1}{\lambda_{k+1}}\Big]$$

for any $t \in [0, T]$.

Proof. We have $u - u^k = E^k + \theta^k$. Then, by using the Lemmas 2.1 and 3.1, we get

$$||u(t) - u^k(t)||^2 \leq ||E^k(t)||^2 + ||\theta^k(t)||^2 \leq \overline{c}\Big[\frac{1}{\lambda_{k+1}^2} + \frac{1}{\lambda_{k+1}}\Big].$$

□

Now, we proceed to obtain higher order estimates.

Lemma 3.5.

$$\mu||\nabla\theta^k(t)||^2 + \lambda||\nabla\pi^k(t)||^2 + c\int_0^t [||\pi_t^k(s)||^2 + ||\theta_t^k(s)||^2]ds \leq \overline{c}\left[\frac{1}{\lambda_{k+1}^2} + \frac{1}{\lambda_{k+1}}\right].$$

Proof. Taking the inner product in $L^2(\omega)$ of the identity (3.2) with $v = \theta_t^k$, we obtain

$$\begin{aligned}
&\frac{\mu}{2}\frac{d}{dt}||\nabla\theta^k||^2 + ||(\rho^k)^{1/2}\theta_t^k||^2 = -(\pi^k(u_t + u.\nabla u - f).\theta_t^k) \\
&\quad -(\rho^k E_t^k, \theta_t^k) - (\rho^k\theta^k.\nabla u, \theta_t^k) - (\rho^k E^k.\nabla u, \theta_t^k) - (\rho^k u^k.\nabla\theta^k, \theta_t^k) \\
&\quad -(\rho^k u^k.\nabla E^k, \theta_t^k) + \lambda((\theta^k.\nabla)\nabla\rho, \theta_t^k) + \lambda((E^k.\nabla)\nabla\rho, \theta_t^k) \qquad (3.6) \\
&+\lambda((u^k.\nabla)\nabla\pi^k, \theta_t^k) \quad + \lambda((\nabla\pi^k.\nabla)u, \theta_t^k) + \lambda((\nabla\rho^k.\nabla)\theta^k, \theta_t^k) \\
&+\lambda((\nabla\rho^k.\nabla)E^k, \theta_t^k).
\end{aligned}$$

The terms on the right hand side are estimated by using Hölder and Young inequalities and Sobolev imbedding theorem.

On the other hand, multiplying the equality (3.4) by π_t^k and integrating over ω, we have for any $\delta > 0$,

$$\begin{aligned}||\pi_t^k||^2 + \frac{\lambda}{2}\frac{d}{dt}||\nabla\pi^k||^2 &= (\theta^k.\nabla\rho, \pi_t^k) + (E^k.\nabla\rho, \pi_t^k) + (u^k.\nabla\pi^k, \pi_t^k)\\ &\leq c_\delta||\nabla\rho||_{L^\infty}^2||\theta^k||^2 + c_\delta||\nabla\rho||_{L^\infty}^2||E^k||^2 + c_\delta||Au^k||^2||\nabla\pi^k||^2 + 3\delta||\nabla\pi_t^k||^2.\end{aligned}$$

The above inequality and equality (3.6) together with the obtained estimates (after choosing suitable $\varepsilon > 0$ and $\delta > 0$ and recalling that $||(\rho^k)^{1/2}\theta_t^k||^2 \geq \alpha||\theta_t^k||^2$), imply

$$\begin{aligned}&\alpha||\theta_t^k||^2 + ||\pi_t^k||^2 + \frac{\mu}{2}\frac{d}{dt}||\nabla\theta^k||^2 + \frac{\lambda}{2}\frac{d}{dt}||\nabla\pi^k||^2 \leq c||E_t^k||^2 + c||\theta^k||^2\\ &+c||E^k||^2 + c||\nabla\theta^k||^2 + \varphi_1(t) + c||\nabla E^k||^2\varphi_2(t) + c||\nabla\pi^k||^2\varphi_3(t),\end{aligned}$$

where

$$\begin{aligned}\varphi_1(t) &= ||\rho^k||_{L^\infty}^2||Au||^2 + ||\rho^k||_{L^\infty}^2||Au^k||^2|| + ||\nabla\rho^k||_{L^\infty}^2 + ||\Delta\rho||_{L^4}^2 + 1;\\ \varphi_2(t) &= ||\rho^k||_{L^\infty}^2||Au||^2 + ||\rho^k||_{L^\infty}^2||Au^k||^2 + ||\nabla\rho^k||_{L^\infty}^2 + ||\Delta\rho||_{L^4}^2;\\ \varphi_3(t) &= ||\nabla u||_{L^\infty}^2 + ||\nabla u^k||_{L^\infty}^2 + ||Au^k||^2 + ||\nabla u_t||^2 + ||Au||^4 + ||f||_{H^1}^2.\end{aligned}$$

Integrating the above inequality from 0 to t, we obtain

$$\begin{aligned}&2\alpha\int_0^t||\theta_t^k(s)||^2ds + 2\int_0^t||\pi_t^k(s)||^2ds + \mu||\nabla\theta^k(t)||^2 + \lambda||\nabla\pi^k(t)||^2\\ &\leq c\int_0^t||E_t^k(s)||^2ds + c\int_0^t||\theta^k(s)||^2ds + c\int_0^t||E^k(s)||^2ds\\ &+c\int_0^t||\nabla\theta^k(s)||^2\varphi_1(s)ds + c\int_0^t||\nabla E^k(s)||^2\varphi_2(s)ds + c\int_0^t||\nabla\pi^k(s)||^2\varphi_3(s)ds,\end{aligned}$$

since $||\nabla\theta^k(0)||^2 = ||\nabla\pi^k(0)||^2 = 0$.

Therefore, by using the estimates given in the Theorem 2.2 and Lemma 2.1, we obtain

$$\mu||\nabla\theta^k(t)||^2 + \lambda||\nabla\pi^k(t)||^2 + 2\alpha\int_0^t ||\theta_t^k(s)||^2 ds + 2\int_0^t ||\pi_t^k(s)||^2 ds$$
$$\leq \frac{c}{\lambda_{k+1}^2}\left[\int_0^t ||\nabla u_t(s)||^2 ds + \int_0^t \varphi_2(s)ds + F_2(t)T\right]$$
$$+\frac{c}{\lambda_{k+1}}[1 + F_2(t)T] + c\int_0^t (\varphi_1(s) + \varphi_3(s))[\mu||\nabla\theta^k(t)||^2 + \lambda||\nabla\pi^k(t)||^2]ds.$$

Now, we observe that the estimates given in Theorem 2.2, imply for all $t \in [0, T]$,

$$\int_0^t ||\nabla u_t(s)||^2 ds \leq c \quad , \quad \int_0^t \varphi_2(s)ds \leq c \quad , \quad \int_0^t [\varphi_1(s) + \varphi_3(s)]ds \leq c \ ,$$

consequently applying Gronwall's inequality we get the desired result.

□

Analogously as in the proof of Theorem 3.4 we obtain the following optimal H^1 estimate for the velocity.

Theorem 3.6. *Suppose the assumptions of the Theorem 2.2 hold. Then, the approximations u^k satisfy*

$$||\nabla u(t) - \nabla u^k(t)|| + \int_0^t ||u_t(s) - u_t^k(s)||^2 ds \leq \frac{\overline{c}}{\lambda_{k+1}}$$

for all $t \in [0, T]$.

□

Corollary 3.7.

$$\int_0^t ||\Delta\pi^k(s)||^2 ds \leq \frac{\overline{c}}{\lambda_{k+1}}$$

for all $t \in [0, T]$.

Proof. In fact, the equality (3.4) implies

$$\lambda\Delta\pi^k = \pi_t^k + \theta^k.\nabla\rho + E^k.\nabla\rho + u^k.\nabla\pi^k. \tag{3.7}$$

Consequently

$$\lambda||\Delta\pi^k||^2 \quad \leq \quad c(||\pi_t^k||^2 + ||\Delta\rho||^2(||\nabla\theta^k||^2 + ||\nabla E^k||^2) + ||Au^k||^2||\nabla\pi^k||^2).$$

Now, by integration in time from 0 to t, and using the lemmas 2.1 and 3.2, we obtained the desired result.

□

Corollary 3.8.

$$\int_0^t ||Au(s) - Au^k(s)||^2 ds \leq \frac{\overline{c}}{\lambda_{k+1}}$$

for all $t \in [0, T]$.

Proof. We start by considering the two following equations:

$$\begin{aligned} Au &= P[\rho u_t + \rho u \cdot \nabla u - \rho f - \lambda(u \cdot \nabla)\nabla\rho - \lambda(\nabla\rho \cdot \nabla)u] \\ Au^k &= P^k[\rho^k u_t^k + \rho^k u^k \cdot \nabla u^k - \rho^k f - \lambda(u^k \cdot \nabla)\nabla\rho^k - \lambda(\nabla\rho^k \cdot \nabla)u^k] \end{aligned}$$

and making their difference; then we take the norm of such terms. Let us see, for instance, how we can estimate one of those terms:

$$\begin{aligned} ||P\rho u\nabla u - P^k\rho^k u^k\nabla u^k||^2 &\leq c||(P - P^k)\rho u\nabla u||^2 + c||P^k\rho(u - u^k)\nabla u||^2 \\ &+c||P^k(\rho - \rho^k)u^k\nabla u||^2 + c||P^k\rho^k u^k\nabla(E^k + \theta^k)||^2 \\ &\leq \frac{\overline{c}}{\lambda_{k+1}}||\nabla(\rho u\nabla u)||^2 + c||(E^k + \theta^k)\nabla u||^2 \\ &+c||\pi^k u^k\nabla u^k||^2 + ||u^k\nabla(E^k + \theta^k)||^2 \\ &\leq \frac{\overline{c}}{\lambda_{k+1}}[||\Delta\rho||^2||Au||^4 + ||Au||^4] + c||\nabla E^k + \nabla\theta^k||Au||^2 \\ &+c||\nabla\pi^k||^2||Au||^4 + c||Au^k||^2||\nabla E^k + \nabla\theta^k||^2 \\ &\leq \frac{\overline{c}}{\lambda_{k+1}}, \end{aligned}$$

which implies that:

$$\int_0^t ||P\rho u\nabla u - P_k\rho^k u^k\nabla u^k||^2 ds \leq \frac{\overline{c}}{\lambda_{k+1}}.$$

After such estimates we conclude that

$$\int_0^t ||Au(s) - Au^k||^2 ds \leq \frac{\overline{c}}{\lambda_{k+1}}.$$

4. Improved L^2-error bounds

The L^2-estimates in Theorem 3.4 are not optimal; in fact it is expected to obtain a rate of convergence of order $1/\lambda_{k+1}^2$ instead of only $1/\lambda_{k+1}$. We were not able to do that, but in this section we will improved the L^2-estimates in Theorem 4.1 by using a bootstrap argument.

The question about the possibility of obtaining this optimal rate of convergence in the L^2-norm was raised by Rautmann in [17], in the context of the classical (constant density) Navier-Stokes equations. This was answered in a positive way in that context by Rojas-Medar and Boldrini [18].

Theorem 4.1.

$$||u(t)-u^k(t)||^2+||\rho(t)-\rho^k(t)||^2+\int_0^t(||\nabla u(s)-\nabla u^k(s)||^2+||\nabla\rho(s)-\nabla\rho^k(s)||^2)ds$$

$$\leq \frac{\overline{c}}{\lambda_{k+1}^{3/2}},$$

for any $t\in[0,T]$.

Proof. We observe that the optimal rate is not obtained in the Theorem 3.4 because of the following term:

$$|\int_0^t(\rho^k E_t^k,\theta^k)ds|.$$

Now, by using the Lemma 3.5, we can estimate it as follows; by integration by parts with respect to t, and recalling that $\theta^k(0)=0$, we have

$$\begin{aligned}
\left|\int_0^t(\rho^k E_t^k,\theta^k)ds\right| &= \left|-\int_0^t(\rho_t^k E^k,\theta^k)ds-\int_0^t(\rho^k E^k,\theta_t^k)ds+(\rho(t)E^k(t),\theta^k(t)\right| \\
&\leq \int_0^t|(\rho_t^k E^k,\theta^k)|ds+\int_0^t|(\rho^k E^k,\theta_t^k)|ds+|(\rho(t)E^k(t),\theta^k(t)| \\
&\leq \int_0^t||\nabla\rho_t^k||_{L^4}||E^k||_{L^4}||\theta^k||ds+\int_0^t||\rho^k||_{L^\infty}||E^k||||\theta_t^k||ds+||\rho||_{L^\infty}||E^k||||\theta^k|| \\
&\leq \frac{c}{\lambda_{k+1}^{3/2}}+\frac{c}{\lambda_{k+1}}\left(\int_0^t||\theta_t^k||^2ds\right)^{1/2} \\
&\leq \frac{\overline{c}}{\lambda_{k+1}^{3/2}}.
\end{aligned}$$

□

By using the Theorem 4.1, we can improve the convergence rate for density. In fact, we get

Theorem 4.2.

$$||\nabla(\rho-\rho^k)(s)||^2+\int_0^t ||\rho_t(s)-\rho_t^k(s)||^2 ds \le \frac{\overline{c}}{\lambda_{k+1}^{3/2}},$$

for any $t\in[0,T]$.

Proof. From (3.7) we can see that:

$$\frac{d}{dt}||\nabla\pi^k||^2+c||\pi_t^k||^2 \quad \le \quad \frac{c}{\lambda_{k+1}^{3/2}}||\nabla\Delta\rho||^2+c||Au^k||^2||\nabla\pi^k||^2,$$

and integrating in time from 0 to t we get:

$$||\nabla\pi^k(t)||^2+c\int_0^t ||\pi_t^k||^2 ds \le \frac{c}{\lambda_{k+1}^{3/2}}+c\int_0^t ||Au^k||^2||\nabla\pi^k||^2 ds$$

since $||\nabla\pi^k(0)||=0$. Now, by using the Gronwall's lemma we obtain:

$$||\nabla\pi^k(t)||^2+c\int_0^t ||\pi_t^k||^2 ds \le \frac{c}{\lambda_{k+1}^{3/2}}\cdot\exp c\int_0^t ||Au^k||^2 ds$$

Corollary 4.3.

$$\int_0^t ||\Delta\pi^k||^2 ds \le \frac{\overline{c}}{\lambda_{k+1}^{3/2}},$$

for all $t\in[0,T]$.

Proof. It easily follows by using the above result on equation (3.7).

□

Proposition 4.4.

$$||\pi_t^k(t)||^2+\lambda\int_0^t ||\nabla\pi_s^k(s)||^2 ds \le \frac{c}{\lambda_{k+1}}$$

and

$$||\Delta\pi^k(t)||^2 \le \frac{c}{\lambda_{k+1}},$$

for all $t \in [0, T]$.

Proof. Differentiating (3.4) with respect to t and taking the inner product in $L^2(\omega)$ with π_t^k, we obtain:

$$\frac{1}{2}\frac{d}{dt}||\pi_t^k||^2 + \lambda||\nabla\pi_t^k||^2 = (\theta_t^k \cdot \nabla\rho, \pi_t^k) + (\theta^k \cdot \nabla\rho_t, \pi_t^k)$$
$$+(E_t^k \cdot \nabla\rho, \pi_t^k) + (E^k \cdot \nabla\rho_t, \pi_t^k) + (u_t^k \cdot \nabla\pi_t^k, \pi_t^k).$$

By using Hölder and Young inequalities, we estimate the right hand side of the above equality and then we get:

$$\frac{d}{dt}||\pi_t^k||^2 + \lambda||\nabla\pi_t^k||^2 \le c||\theta_t^k||^2 + \frac{c}{\lambda_{k+1}}||\nabla\rho_t||^2 + \frac{c}{\lambda_{k+1}}||\nabla u_t^k||^2 + \frac{c}{\lambda_{k+1}}.$$

Integrating from 0 to t and observing that

$$||\pi_t^k(0)|| \le \lambda||\Delta(\rho - \rho^k)(0)|| + ||(u - u^k)(0)||||\nabla\rho||_{L^6} + |||u^k||_{L^\infty}||\nabla(\rho - \rho^k)(0)||$$
$$\le c||\nabla(u - u^k)(0)|| \le c||\nabla\theta^k(0)|| \le \frac{c}{\lambda_{k+1}},$$

we obtain the result. The second estimate follows easily from the previous estimate and from (3.4).

Remark 4.3. The authors in [8] obtained analogous results to those presented in this work for a general model with λ^2-terms inclued.

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