DISCRETE AND CONTINUOUS
DYNAMICAL SYSTEMS
Volume **27**, Number **4**, August **2010**

doi:10.3934/dcds.2010.27.1611

pp. **1611–1631**

TOPOLOGICAL PROPERTIES OF THE WEAK GLOBAL ATTRACTOR OF THE THREE-DIMENSIONAL NAVIER-STOKES EQUATIONS

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Abstract. The three-dimensional incompressible Navier-Stokes equations are considered along with its weak global attractor, which is the smallest weakly compact set which attracts all bounded sets in the weak topology of the phase space of the system (the space of square-integrable vector fields with divergence zero and appropriate periodic or no-slip boundary conditions). A number of topological properties are obtained for certain regular parts of the weak global attractor. Essentially two regular parts are considered, namely one made of points such that *all* weak solutions passing through it at a given initial time are strong solutions on a neighborhood of that initial time, and one made of points such that *at least one* weak solution passing through it at a given initial time is a strong solution on a neighborhood of that initial time. Similar topological results are obtained for the family of all trajectories in the weak global attractor.

1. **Introduction.** One of the current major open problems in mathematics is the well-posedness of the three-dimensional Navier-Stokes equations. The existence of weak solutions for all positive times as well as the existence and uniqueness of local strong solutions are well established but it is not known whether there exists a unique solution defined for all positive times starting from an arbitrary initial condition in a suitable function space.

For other equations where the well-posedness is well established, an important problem is the understanding of the dynamics and the asymptotic behavior of their solutions. This is the realm of the dynamical systems theory, be it finite or infinite dimensional, depending on the equation underlying the system. In the study of the asymptotic behavior, an important object is the so-called *global attractor*, which can

2000 *Mathematics Subject Classification.* 35Q30, 35B40.

Key words and phrases. Navier-Stokes equations, weak global attractor.

This work was partly supported by the National Science Foundation under the grants NSF-DMS-0604235 and NSF-DMS-0906440, by the Research Fund of Indiana University, and by CNPq, Brasília, Brazil, under the grant number 307077/2009-8.

usually be characterized as the smallest compact set attracting all bounded sets in the phase space of the system.

Despite the lack of a well-posedness result for the three-dimensional Navier-Stokes equations, it is still a natural question to ask what the dynamics and the asymptotic behaviors of their weak solutions are, despite the possibility that they are not unique with respect to the initial condition. In particular, it is natural to ask whether there exists some sort of global attractor in this case. Due to the lack of a well-defined semigroup associated with the solutions of the system, the classical theory of dynamical system does not apply directly. Nevertheless, it is still possible to adapt a number of results from the classical theory to this situation.

One of the first and main results in this direction was given in [12], in which an object called the *weak global attractor* was defined. It is the smallest compact set in the weak topology of the phase space which attracts all global weak solutions in the weak topology.

Later on, a number of results were developed with a different perspective, in which a well-defined semigroup exists in a trajectory space. A "point" in this trajectory space is a weak solution defined for all nonnegative times, and the semigroup is just the time-translation operator; see for instance [23, 2, 24]. The global attractor obtained in this way is sometimes called the *trajectory attractor*. The two approaches are connected, and the weak global attractor given in [12] is in fact the projection, at any given time, of the trajectory attractor. See Section 3.1 for more comments on this.

In [12] the weak global attractor was shown to possess a weakly open and dense subset made of points such that any weak solution passing through that point at a given initial time is a strong solution in a neighborhood of the initial time. It was called the "regular part" of the weak global attractor. The proof of this result was only sketched in [12] and we take the opportunity here to include the full details of the proof. We also introduce other such regular sets and study their topological properties, such as density and their Borel structure. Similar results are given in the trajectory space, for the family of all trajectories in the weak global attractor.

An important open question is whether the weak global attractor coincides with its regular part, i.e. whether at least within the global attractor all solutions are global strong (and unique) solutions. This would be a kind of asymptotic regularity question, and it is one of the major reasons we investigate here the nature of the regular parts of the weak global attractor.

A related asymptotic regularity problem is whether the stationary statistical solutions are carried by the regular part of the weak global attractor (see [17]). We recall here that stationary statistical solutions are suitable generalizations of the notion of invariant measures (see [5, 6, 14, 28, 29, 13]), and therefore are a natural object related to the asymptotic behavior of the system. A partial answer to this problem follows from one of the results given here and will be presented in the forthcoming work [10] dedicated to stationary statistical solutions (see [8] for a summary of the main results in [10], and also [9, 7] for time-dependent statistical solutions).

2. Preliminaries.

2.1. The Navier-Stokes equations and their mathematical setting. In this section we recall some fundamental results about individual solutions of the Navier-Stokes equations in space dimension three, for which the reader is referred to the works [3, 13, 15, 25, 26, 27].

The three-dimensional incompressible Navier-Stokes equations in Eulerian formulation are written in vectorial form as

$$\frac{\partial \mathbf{u}}{\partial t} - \nu\Delta\mathbf{u} + (\mathbf{u}\cdot\boldsymbol{\nabla})\mathbf{u} + \boldsymbol{\nabla}p = \mathbf{f}, \qquad \boldsymbol{\nabla}\cdot\mathbf{u} = 0. \tag{1}$$

The variable $\mathbf{u} = (u_1, u_2, u_3)$ denotes the velocity vector field; the term $\mathbf{f}$ represents the mass density of volume forces applied to the fluid and is assumed to be time-independent; the parameter $\nu > 0$ is the kinematic viscosity; and p is the kinematic pressure. We denote the space variable by $\mathbf{x} = (x_1, x_2, x_3)$ and the time variable by t.

We allow two kinds of boundary conditions: periodic and no-slip. In the periodic case we assume the flow is periodic with period L_i in each spatial direction $0x_i$, $i = 1, 2, 3$, and we set $\Omega = \Pi_{i=1}^3(0, L_i)$; we also assume that the averages of the flow and of the forcing term over Ω vanish, i.e.

$$\int_\Omega \mathbf{u}(\mathbf{x}, t)\ \mathrm{d}\mathbf{x} = 0, \qquad \int_\Omega \mathbf{f}(\mathbf{x})\ \mathrm{d}\mathbf{x} = 0.$$

In the no-slip case, we consider the flow on a bounded domain $\Omega \subset \mathbb{R}^3$, with a sufficiently smooth boundary $\partial\Omega$, and it is assumed that $\mathbf{u} = 0$ on $\partial\Omega$. Other boundary conditions such as those for periodic channel flows can be treated similarly.

In either the periodic or no-slip case one obtains, in appropriate function spaces, a functional equation formulation for the time-dependent velocity field $\mathbf{u} = \mathbf{u}(t)$ corresponding to the function $\mathbf{x} \in \Omega \mapsto \mathbf{u}(\mathbf{x}, t)$ at each time t.

For the precise definition of the function spaces, we first consider the space of test functions, which in the periodic case is given by

$$\mathcal{V}_{\mathrm{per}} = \left\{ \mathbf{u} = \mathbf{w}|_\Omega;\ \begin{array}{l} \mathbf{w} \in \mathcal{C}^\infty(\mathbb{R}^3)^3,\ \boldsymbol{\nabla}\cdot\mathbf{w} = 0,\ \int_\Omega \mathbf{w}(\mathbf{x})\ \mathrm{d}\mathbf{x} = 0,\ \mathbf{w}(\mathbf{x}) \text{ is periodic} \\ \text{with period } L_i \text{ in each direction } 0x_i \end{array} \right\},$$

while, in the no-slip case, it is given by

$$\mathcal{V}_0 = \left\{ \mathbf{u} \in \mathcal{C}_{\mathrm{c}}^\infty(\Omega)^3;\ \boldsymbol{\nabla}\cdot\mathbf{u} = 0 \right\},$$

where $\mathcal{C}_{\mathrm{c}}^\infty(\Omega)$ denotes the space of infinitely-differentiable real-valued functions with compact support in Ω. We let $\mathcal{V}$ stand for either $\mathcal{V}_{\mathrm{per}}$ or $\mathcal{V}_0$ depending on the case under consideration.

In each case the space H is defined as the completion of $\mathcal{V}$ under the $L^2(\Omega)^3$ norm. The space V is the completion of $\mathcal{V}$ under the $H^1(\Omega)^3$ norm. We identify H with its dual and consider the dual space V', so that $V \subseteq H \subseteq V'$, with the injections being continuous, and each space dense in the following one.

We denote the inner products in H and V respectively by

$$(\mathbf{u}, \mathbf{v})_{L^2} = \int_\Omega \mathbf{u}(\mathbf{x})\cdot\mathbf{v}(\mathbf{x})\ \mathrm{d}\mathbf{x}, \quad (\!(\mathbf{u}, \mathbf{v})\!)_{H^1} = \int_\Omega \sum_{i=1,2,3} \frac{\partial \mathbf{u}}{\partial x_i}\cdot\frac{\partial \mathbf{v}}{\partial x_i}\ \mathrm{d}\mathbf{x},$$

and the associated norms by $|\mathbf{u}|_{L^2} = (\mathbf{u}, \mathbf{u})_{L^2}^{1/2}$, $\|\mathbf{u}\|_{H^1} = (\!(\mathbf{u}, \mathbf{u})\!)_{H^1}^{1/2}$.

We denote by P_{LH} the (Leray-Helmhotz) orthogonal projector in $L^2(\Omega)^3$ onto the subspace H. The operator A below in (3) is the Stokes operator given by

$A\mathbf{u} = -P_{\mathrm{LH}}\Delta\mathbf{u}$, for $\mathbf{u} \in D(A)$, and with domain $D(A)$ defined as the closure of $\mathcal{V}$ in the space $H^2(\Omega)^3$ in the periodic case, and $D(A) = V \cap H^2(\Omega)^3$ in the Dirichlet case.

The Stokes operator is a positive self-adjoint operator on H; we denote its first eigenvalue by λ_1, and the following Poincaré inequality holds:

$$\lambda_1 |\mathbf{u}|_{L^2}^2 \leq \|\mathbf{u}\|_{H^1}^2, \tag{2}$$

for all $\mathbf{u} \in V$.

We define the bilinear term $B(\mathbf{u}, \mathbf{v}) = P_{\mathrm{LH}}((\mathbf{u} \cdot \boldsymbol{\nabla})\mathbf{v})$ associated with the inertial term. Taking the inner product in H of the bilinear term with a third vector field $\mathbf{w}$ yields the trilinear form

$$b(\mathbf{u}, \mathbf{v}, \mathbf{w}) = (B(\mathbf{u}, \mathbf{v}), \mathbf{w})_{L^2} = \int_\Omega [(\mathbf{u} \cdot \boldsymbol{\nabla})\mathbf{v}] \cdot \mathbf{w} \, \mathrm{d}\mathbf{x},$$

which is defined for $\mathbf{u}, \mathbf{v}, \mathbf{w}$ in V.

The functional equation reads then

$$\frac{\mathrm{d}\mathbf{u}}{\mathrm{d}t} + \nu A\mathbf{u} + B(\mathbf{u}, \mathbf{u}) = \mathbf{f}. \tag{3}$$

The forcing term $\mathbf{f}$ is assumed to belong to H, otherwise the term in the right hand side of (3) would be the Leray-Hopf projection of the term $\mathbf{f}$ in (1).

The weak topology in H plays a crucial role due to the condition (iii) below in the Definition 2.1 of weak solutions. With that in mind, given a subset X of H, we denote by X_{w} this subset endowed with the weak topology of H. In particular, H_{w} denotes the space H endowed with its weak topology. The closed ball of radius R in H is denoted by $B_H(R)$. Since H is a separable Hilbert space, its weak topology is metrizable on bounded sets, and thus e.g. $B_H(R)_{\mathrm{w}}$ is metrizable for any $R < \infty$. Given a subset X of H, we denote by $\overline{X}^{\mathrm{w}}$ the closure of X in the weak topology of H.

We then have the following definition of weak solutions.

Definition 2.1. A (Leray-Hopf) weak solution on a time interval $I \subset \mathbb{R}$ is defined as a function $\mathbf{u} = \mathbf{u}(t)$ on I with values in H and satisfying the following properties:

i. $\mathbf{u} \in L^\infty_{\mathrm{loc}}(I; H) \bigcap L^2_{\mathrm{loc}}(I; V)$;
ii. $\partial\mathbf{u}/\partial t \in L^{4/3}_{\mathrm{loc}}(I; V')$;
iii. $\mathbf{u} \in \mathcal{C}(I; H_w)$, i.e. $\mathbf{u}$ is weakly continuous in H, which means that for every $\mathbf{v} \in H$, the function $t \mapsto (\mathbf{u}(t), \mathbf{v})_{L^2}$ is continuous from I into $\mathbb{R}$;
iv. $\mathbf{u}$ satisfies the functional equation (3) in the distribution sense on I, with values in V';
v. For almost all t' in I, $\mathbf{u}$ satisfies the following energy inequality:
$$\frac{1}{2}|\mathbf{u}(t)|_{L^2}^2 + \nu \int_{t'}^t \|\mathbf{u}(s)\|_{H^1}^2 \, \mathrm{d}s \leq \frac{1}{2}|\mathbf{u}(t')|_{L^2}^2 + \int_{t'}^t (\mathbf{f}, \mathbf{u}(s))_{L^2} \, \mathrm{d}s, \tag{4}$$
for all t in I with $t > t'$. The allowed times t' are characterized as the points of strong continuity from the right, in H, for $\mathbf{u}$, and their set is of total measure and denoted by $I'(\mathbf{u})$.
vi. If I is closed and bounded on the left, with its left end point denoted by t_0, then the solution is continuous in H at t_0 from the right, i.e. $\mathbf{u}(t) \to \mathbf{u}(t_0)$ in H as $t \to t_0^+$.

From now on, for notational simplicity, *a weak solution will always mean a Leray-Hopf weak solution.*

Remark 1. The following remarks are in order.

a. It is well known that given any initial time $t_0 \in \mathbb{R}$ and any initial condition $\mathbf{u}_0 \in H$, there exists at least one weak solution on $[t_0, \infty)$ satisfying $\mathbf{u}(t_0) = \mathbf{u}_0$.
b. Conditions (ii) and (iii) are actually consequences of (i) and (iv).
c. Assuming (i), condition (iv) is equivalent to

$$(\mathbf{u}(t), \mathbf{v})_{L^2} = (\mathbf{u}(s), \mathbf{v})_{L^2} + \int_s^t \{(\mathbf{f}, \mathbf{v})_{L^2} - \nu(\!(\mathbf{u}(\tau), \mathbf{v})\!)_{H^1} - b(\mathbf{u}(\tau), \mathbf{u}(\tau), \mathbf{v})\} \, \mathrm{d}\tau,$$

for every t, s in I and all $\mathbf{v}$ in V; see e.g. [25, Ch. 3, Section 1].
d. The allowed times $t' \in I'(\mathbf{u})$ in condition (v), which are the points of strong continuity for $\mathbf{u}$ in H from the right, can also be characterized as the Lebesgue points of the function $t \mapsto |\mathbf{u}(t)|_{L^2}^2$, in the sense that

$$\lim_{\tau \to 0^+} \frac{1}{\tau} \int_{t'}^{t'+\tau} |\mathbf{u}(t)|_{L^2}^2 \, \mathrm{d}t = |\mathbf{u}(t')|_{L^2}^2. \tag{5}$$

Since $t \mapsto |\mathbf{u}(t)|_{L^2}^2$ is locally integrable, these Lebesgue points $I'(\mathbf{u})$ form a set of full measure. In the case of a weak solution on an interval of the form $[t_0, t_1)$, since by condition (vi) the point t_0 is a point of strong continuity from the right, the estimate (9) is also valid for the initial time $t' = t_0$.
e. The energy inequality in integral form (4) in condition (v) can be interchanged with the assumption that $\mathbf{u}$ satisfies the following energy inequality in the distribution sense on I:

$$\frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} |\mathbf{u}(t)|_{L^2}^2 + \nu \|\mathbf{u}(t)\|_{H^1}^2 \leq (\mathbf{f}, \mathbf{u}(t))_{L^2}. \tag{6}$$

Assume e.g. that I is an interval of the form (t_0, t_1). It is elementary to see that (4) implies (6). Indeed, setting $t = t' + h$ in (4), we multiply (4) by $\varphi(t')$, where $\varphi \in \mathcal{D}(I)$, $\varphi \geq 0$, integrate in t' from t_0 to $t_1 - h$, and pass to the limit as $h \to 0$ using the Beppo-Levi theorem, with φ fixed; we find, since $|\mathbf{u}(\cdot)|_{L^2}^2$ and $\|\mathbf{u}(\cdot)\|_{H^1}^2$ are integrable functions and (4) holds for almost every t', that

$$\int_{t_0}^{t_1} \left\{ -\frac{1}{2} |\mathbf{u}(t)|_{L^2}^2 \varphi'(t) + \left[\nu \|\mathbf{u}(t)\|_{H^1}^2 - (\mathbf{f}, \mathbf{u}(t))_{L^2} \right] \varphi(t) \right\} \, \mathrm{d}t \leq 0,$$

for all $\varphi \in \mathcal{D}(I)$, $\varphi \geq 0$, which is exactly (6).

In order to show that (6) implies (4), let

$$\Phi(t) = \frac{1}{2} |\mathbf{u}(t)|_{L^2}^2 + \int_{t_0}^t \left[\nu \|\mathbf{u}(s)\|_{H^1}^2 - (\mathbf{f}, \mathbf{u}(s))_{L^2} \right] \, \mathrm{d}s.$$

Then (6) implies that $\mathrm{d}\Phi/\mathrm{d}t \leq 0$ in the distribution sense, i.e. $\mathrm{d}\Phi/\mathrm{d}t$ is a negative distribution. According to a result of Schwartz [21, 22], $\mathrm{d}\Phi/\mathrm{d}t = \mu$ is a negative measure. By a theorem of Riesz (see [18, Section 54, page 118]), if $\Psi(t)$ is the μ-measure of the interval (t_0, t), then Ψ is a decreasing function continuous on the right, and for every $\theta \in \mathcal{D}(I)$, $\langle \mu, \theta \rangle$ can be represented by a Stieltjes integral with respect to Ψ:

$$\langle \mu, \theta \rangle = \int_{t_0}^{t_1} \theta \, \mathrm{d}\Psi.$$

Integration by parts being legitimate [18], and $\theta(t_0) = \theta(t_1) = 0$, we have

$$-\int_{t_0}^{t_1} \Phi \theta' \, \mathrm{d}t = -\langle \Phi, \theta' \rangle = \langle \Phi', \theta \rangle = \langle \mu, \theta \rangle = \int_{t_0}^{t_1} \theta \, \mathrm{d}\Psi = -\int_{t_0}^{t_1} \Psi \theta' \, \mathrm{d}t.$$

The last equation being valid for every $\theta \in \mathcal{D}(I)$ shows that the difference $\Phi - \Psi$ is almost everywhere equal to a constant which we can take to be zero since Ψ is defined up to an additive constant. Denoting by $I_{\mathbf{u}}$ the set of total measure where $\Phi = \Psi$, then for every $t, t' \in I_{\mathbf{u}}$, $t' \leq t$, since Ψ is a decreasing function, we have

$$\Psi(t) = \Phi(t) \leq \Phi(t') = \Psi(t'), \tag{7}$$

which is (4) for such t and t'. Finally, if $t > t'$ is any number larger than t' in I and not necessarily in $I_{\mathbf{u}}$, we approximate t by a sequence of increasing numbers $t_j \in I_{\mathbf{u}}$, so that $\Phi(t_j) \leq \Phi(t')$. At the limit we obtain $\Phi(t) \leq \Phi(t')$, which is exactly (4), after observing that Φ is left lower-semicontinuous, like $|\mathbf{u}(\cdot)|_{L^2}^2$, as

$$|\mathbf{u}(t)|_{L^2}^2 \leq \liminf_{t_j \to t} |\mathbf{u}(t_j)|_{L^2}^2.$$

See also [13, Section II.7, Appendix II.A.2, and Appendix II.B.1] for discussions about this and a direct proof that (6) implies (4), without using the result of Schwartz.

f. Since $\mathbf{u}$ belongs to $L^2_{\text{loc}}(I; V)$, condition (v) implies, upon use of the Cauchy-Schwarz and Poincaré inequalities,

$$|\mathbf{u}(t)|_{L^2}^2 + \nu \int_{t'}^{t} \|\mathbf{u}(s)\|_{H^1}^2 \, \mathrm{d}s \leq |\mathbf{u}(t')|_{L^2}^2 + \frac{1}{\nu\lambda_1} |\mathbf{f}|_{L^2}^2 (t - t'), \tag{8}$$

for t' and t as in (v).

g. By using an appropriate sequence of test functions in the inequality (6) (see [13, Appendix II.B.1] for the details or [1, Proposition 7.3] for a different proof), one deduces that a weak solution on an arbitrary interval I also satisfies

$$|\mathbf{u}(t)|_{L^2}^2 \leq |\mathbf{u}(t')|_{L^2}^2 e^{-\nu\lambda_1(t-t')} + \frac{1}{\nu^2\lambda_1^2} |\mathbf{f}|_{L^2}^2 \left(1 - e^{-\nu\lambda_1(t-t')}\right), \tag{9}$$

for almost all t' in I and all t in I with $t' < t$. The allowed times t' are again the points $I'(\mathbf{u})$ in condition (v).

h. An important nondimensional parameter associated with the strength of the forcing term is the Grashof number

$$G = \frac{|\mathbf{f}|_{L^2}}{\nu^2 \lambda_1^{3/4}}. \tag{10}$$

Define

$$R_0 = \frac{|\mathbf{f}|_{L^2}}{\nu\lambda_1} = \frac{\nu G}{\lambda_1^{1/4}}. \tag{11}$$

Note that from (9) it follows that any weak solution defined on an interval unbounded on the right satisfies

$$\limsup_{t \to \infty} |\mathbf{u}(t)|_{L^2} \leq R_0. \tag{12}$$

i. A weak solution defined on $I = \mathbb{R}$ is called here a *global weak solution*. If $\mathbf{u}$ is a global weak solution which is uniformly bounded in H, then it follows from the energy estimate (9), by letting $t' \to -\infty$, that

$$|\mathbf{u}(t)|_{L^2} \leq R_0, \qquad \forall t \in \mathbb{R}. \tag{13}$$

j. Note that another concept of weak solutions (sometimes also called Leray-Hopf weak solutions) is one in which condition (vi) in the Definition 2.1 is not required. In this formulation, the initial condition still makes sense since the solution is still weakly continuous in H at the initial time. See [9] for details comparing these two particular formulations, and see Remark 4 for related discussions on this issue.

We turn now to the definition and a few properties of strong solutions.

Definition 2.2. A (Leray-Hopf) weak solution on an arbitrary interval I is called regular or a strong solution if it satisfies furthermore

vii. $\mathbf{u} \in \mathcal{C}(I; V)$.

Remark 2. The following remarks concerning strong solutions are in order.

a. It is well known that if $\mathbf{u}_0$ belongs to V, then there exists a local strong solution with $\mathbf{u}(t_0) = \mathbf{u}_0$, defined on some interval $[t_0, t_1)$, with $t_0 < t_1 \leq \infty$. Such a strong solution is unique among the class of all weak solutions $\mathbf{v}$ on $[t_0, t_1)$ with $\mathbf{v}(t_0) = \mathbf{u}_0$.
b. A strong solution on an interval I satisfies the energy equation

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|\mathbf{u}(t)|_{L^2}^2 + \nu\|\mathbf{u}(t)\|_{H^1}^2 = (\mathbf{f}, \mathbf{u}(t))_{L^2}, \tag{14}$$

as well as the enstrophy equation

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\mathbf{u}(t)\|_{H^1}^2 + \nu|A\mathbf{u}|_{L^2}^2 + b(\mathbf{u}, \mathbf{u}, A\mathbf{u}) = (\mathbf{f}, A\mathbf{u}(t))_{L^2}, \tag{15}$$

in the classical sense over the interval I. An estimate for the existence time $t_1 = t_1(\mathbf{u}_0)$ can be obtained by properly estimating the nonlinear term. In fact, using Hölder's inequality with L^6, L^3, and L^2, respectively, followed by Sobolev's, interpolation and Young's inequalities we find that

$$|b(\mathbf{u}, \mathbf{u}, A\mathbf{u})| \leq \frac{\nu}{4}|A\mathbf{u}|_{L^2}^2 + \frac{c_1}{\nu^3}\|\mathbf{u}\|_{H^1}^6, \tag{16}$$

for a suitable universal constant c_1. Using this estimate in the enstrophy equation, and applying the Cauchy-Scharwz to the forcing term we obtain the inequality

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\mathbf{u}(t)\|_{H^1}^2 + \frac{\nu}{2}|A\mathbf{u}|_{L^2}^2 \leq \frac{1}{\nu}|\mathbf{f}|_{L^2}^2 + \frac{c_1}{\nu^3}\|\mathbf{u}\|_{H^1}^6. \tag{17}$$

Let then

$$y = \nu^{2/3}|\mathbf{f}|_{L^2}^{2/3} + \|\mathbf{u}\|_{H^1}^2, \tag{18}$$

to find from the enstrophy equation that

$$y' \leq \frac{2c_0^2}{\nu^3}y^3, \tag{19}$$

where $c_0 = \max\{1, c_1^{3/2}\}$ is still a universal constant. As long as y is defined, we integrate the inequality above to find that

$$-\frac{1}{y(t)^2} + \frac{1}{y(s)^2} \leq \frac{4c_0^2}{\nu^3}(t - s). \tag{20}$$

Now, taking $s = t_0$, $y_0 = y(t_0)$, and

$$T(y_0) = \frac{3\nu^3}{16c_0^2 y_0^2}, \tag{21}$$

then we find that $y(t) \leq 2y_0$, for all $t_0 \leq t \leq t_0 + T(y_0)$, which yields that a strong solution starting in $\mathbf{u}_0 \in V$ at time t_0 exists at least up to the time $T_1(y_0) = t_0 + T(y_0)$. If the solution blows-up in V at a time t_1^*, then we let $t \to t_1^*$ and use that $y(t) \to \infty$ to deduce from (20) that $t_1^* \geq t_0 + \nu^3/(4c_0^2 y_0^2)$.

c. A strong solution on an open interval (t_1, t_2), $-\infty \leq t_1 < t_2 \leq \infty$ is analytic in time as a function from (t_1, t_2) into $D(A)$. In particular, if two strong solutions coincide at a given time, then they are equal on their interval of definition.

Concerning the regularity points of a weak solution we make the following definition.

Definition 2.3. Let $\mathbf{u}$ be a weak solution defined on an interval $I \subset \mathbb{R}$. Then a point $t \in I$ is called *singular* if $\mathbf{u}(t) \in H \setminus V$ and is called *regular* if $\mathbf{u}(t) \in V$. Morever, a regular point $t \in I$ is called a *point of interior regularity* if there exists a $\delta > 0$ such that $(t - \delta, t + \delta)$ is included in I and $\mathbf{u}$ restricted to $(t - \delta, t + \delta)$ is a strong solution.

2.2. **Properties of weak solutions.** Most of what follows in this section is due to and meant to overcome the difficulties of a possible lack of uniqueness of weak solutions. First, we will need to paste solutions together, according to the following result.

Lemma 2.4 (Pasting Lemma)**.** *Let $\mathbf{u}^{(1)}$ be a weak solution on an interval $(t_1, t_2]$ and $\mathbf{u}^{(2)}$ be a weak solution on an interval $[t_2, t_3)$, with $-\infty \leq t_1 < t_2 < t_3 \leq \infty$ and $\mathbf{u}^{(1)}(t_2) = \mathbf{u}^{(2)}(t_2)$. Then the function*

$$\tilde{\mathbf{u}}(t) = \begin{cases} \mathbf{u}^{(1)}(t), & t_1 < t < t_2, \\ \mathbf{u}^{(2)}(t), & t_2 \leq t < t_3, \end{cases} \tag{22}$$

is a weak solution on (t_1, t_3).

Proof. The assumption that $\mathbf{u}^{(2)}$ is strongly continuous from the right at t_2 guarantees that the energy inequality holds for $\mathbf{u}^{(2)}$ starting at t_2, so that the energy inequalities for $\mathbf{u}^{(1)}$ and $\mathbf{u}^{(2)}$ can also be concatenated, and condition (v) of the Definition 2.1 holds for $\mathbf{u}$. The other conditions are easy to check. □

Remark 3. In this result, it is important that $\mathbf{u}^{(2)}$ is strongly continuous from the right at t_2. If $\mathbf{u}^{(2)} \in \mathcal{C}([t_2, t_3); H_{\mathrm{w}})$ and is a (Leray-Hopf) weak solution only on (t_2, t_3), then the energy inequality (4) may not be valid for $\mathbf{u}^{(2)}$ at t_2, and the concatenated solution $\tilde{\mathbf{u}}$ may not be a Leray-Hopf weak solution on $[t_1, t_3)$ but only a weak solution in some broader sense (not specified here).

This simple remark leads us to the following *significant connection between the uniqueness of (Leray-Hopf) weak solutions and the existence of points t' at which the energy inequality (4) does not hold for $t > t'$.*

Proposition 1 (A sufficient condition for non-uniqueness of weak solutions)**.** *Assume that $\mathbf{u}$ is a (Leray-Hopf) weak solution of the Navier-Stokes equations on $[0, T)$ and that there exists $t' \in (0, T)$ at which $\mathbf{u}$ is not strongly continuous from the right*[1]*, in H. Then there exists another (Leray-Hopf) weak solution $\tilde{\mathbf{u}}$ on $[0, T)$ with $\tilde{\mathbf{u}}(0) = \mathbf{u}(0)$, and $\tilde{\mathbf{u}} \neq \mathbf{u}$.*

[1] Or, equivalently, (4) does not hold for this t'.

Proof. We define a new solution $\tilde{\mathbf{u}} \neq \mathbf{u}$ by using the concatenation procedure (22) above. We consider $t_1 = 0$, $t_2 = t'$, $t_3 = T$, and $\mathbf{u}^{(1)} = \mathbf{u}$ on $[0, t')$. We then take $\mathbf{u}^{(2)}$ to be a Leray-Hopf weak solution on $[t', T)$ with initial data $\mathbf{u}^{(2)}(t') = \mathbf{u}(t')$. By definition $\mathbf{u}^{(2)}$ is strongly continuous from the right at t', and hence so is $\tilde{\mathbf{u}}$. Therefore, $\mathbf{u} \neq \tilde{\mathbf{u}}$ and both are (Leray-Hopf) weak solutions on $[0, T)$ with the same initial data $\mathbf{u}(0)$. □

As a consequence of Proposition 1, we have the following remarkable result.

Proposition 2. *Let $\mathbf{u}_0 \in H$ and $T > 0$, and suppose $\mathbf{u}$ is the only (Leray-Hopf) weak solution on $[0, T)$ with $\mathbf{u}(0) = \mathbf{u}_0$. Then $\mathbf{u}$ is strongly continuous from the right in H everywhere on $[0, T)$, and, in particular, the energy inequality (4) holds for every t and t' in $[0, T)$, with $t \geq t'$.*

The following lemma will also be useful.

Lemma 2.5 (Compactness Lemma)**.** *Let $\{\mathbf{u}_j\}_{j\in\mathbb{N}}$ be a sequence of weak solutions on some interval $I = (t_1, t_2)$, $-\infty \leq t_1 < t_2 \leq \infty$, and suppose that this sequence is uniformly bounded in H. Then, there exists a subsequence $\{\mathbf{u}_{j'}\}_{j'}$ and a weak solution $\mathbf{u}(\cdot)$ on I such that $\mathbf{u}_{j'}$ converges to $\mathbf{u}$ in H_{w} uniformly on any compact interval in I. Moreover, if $\mathbf{u}$ is regular on $J = (t_3, t_4) \subset I$, then $\mathbf{u}_{j'}$ converges to $\mathbf{u}$ in V, uniformly on any compact interval in J.*

The proof of this lemma can be found in [4, Chapter 1].

Concerning the interior regularity points of a weak solution, as given in Definition 2.3, we have the following basic result.

Lemma 2.6. *Let $\mathbf{u}$ be a weak solution on an interval I and let t_0 belong to the interior of I. Then, the following statements are equivalent:*

i. t_0 *is a point of interior regularity for* $\mathbf{u}$*;*
ii. $\exists \lim_{t\to t_0^-} \|\mathbf{u}(t)\|_{H^1} < \infty$*;*
iii. $\limsup_{t\to t_0^-} \|\mathbf{u}(t)\|_{H^1} < \infty$*;*
iv. $\liminf_{t\to t_0^-} \|\mathbf{u}(t)\|_{H^1} < \infty$*.*

Proof. Implications (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) are trivial. We only need to prove that (iv) implies (i).

Suppose (iv) holds. Then, there exists $t_n \to t_0^-$ such that $\|\mathbf{u}(t_n)\|_{H^1}$ is bounded uniformly in n. Thus, the local existence and uniqueness result of strong solutions assures the existence of $\delta > 0$ such that $\mathbf{u}$ is regular on $[t_n, t_n + \delta)$, for all n (the estimate (21) for the interval of local existence depends on $\|\mathbf{u}(t_n)\|_{H^1}$, which is uniformly bounded, so that we can take a $\delta > 0$ independent of n). Then, for n sufficiently large, $t_n < t_0 < t_n + \delta$, which means that $\mathbf{u}$ is regular on the interval $(t_n, t_n + \delta)$ containing t_0. This implies that t_0 is a point of interior regularity for $\mathbf{u}$, which proves (i). □

2.3. **Regular and singular points.** Here we recall a few results concerning the structure of the regular and singular points of a weak solution, as given in Definition 2.3. For simplicity, we consider only global weak solutions $\mathbf{u} = \mathbf{u}(t)$, i.e. weak solutions defined for all $t \in \mathbb{R}$, but the results below can be extended to weak solutions defined on any interval $I \subset \mathbb{R}$ as explained at the end of this section.

An interval $J \subset \mathbb{R}$ is called an *interval of regularity* for $\mathbf{u}$ if $\mathbf{u}$ is regular on J, i.e. $\mathbf{u}|_J \in \mathcal{C}(J, V)$. An interval $J \subset \mathbb{R}$ is called a *maximal interval of regularity* for $\mathbf{u}$ if there is no interval of regularity strictly containing J. It follows from Lemma

2.6 that if J is a maximal interval of regularity, with end points α and β, $\alpha < \beta$, then it is necessarily open on the right, with

$$\liminf_{t\to\beta^-} \|\mathbf{u}(t)\|_{H^1} = \infty. \tag{23}$$

A maximal interval of regularity can be either open or closed on the left. In any case,

$$\liminf_{t\to\alpha^-} \|\mathbf{u}(t)\|_{H^1} = \infty. \tag{24}$$

But the lower limit of $\|\mathbf{u}(t)\|_{H^1}$ as $t \to \alpha^+$ may be either finite or infinite; and $\mathbf{u}(\alpha)$ may or may not belong to V. We have, in fact, the characterization

$$\mathbf{u}(\alpha) \notin V \textit{ if and only if } \liminf_{s\to\alpha^+} \|\mathbf{u}(s)\|_{H^1} = \infty. \tag{25}$$

Depending on whether $\mathbf{u}(\alpha)$ belongs to V or not, the maximal interval of regularity is either $[\alpha, \beta)$ or (α, β).

Using inequality (20) obtained from the enstrophy equation, we find that if J is a maximal interval of regularity with end points α and β, then $y(t)$ given in (18) is such that $y(t) \to \infty$ as $t \to \beta^-$ and

$$\frac{1}{y(s)^2} \le \frac{4c_0^2}{\nu^3}(\beta - s),$$

for $\alpha < s < \beta$. This can be written as

$$\frac{\nu^{3/2}}{(\beta - s)^{1/2}} \le 2c_0 \left(\nu^{2/3} |\mathbf{f}|_{L^2}^{2/3} + \|\mathbf{u}\|_{H^1}^2 \right), \tag{26}$$

for $\alpha < s < \beta$. Integrate (26) in s from α to β to find

$$\nu^{3/2}(\beta - \alpha)^{1/2} \le c_0 \left(\nu^{2/3} |\mathbf{f}|_{L^2}^{2/3} (\beta - \alpha) + \int_\alpha^\beta \|\mathbf{u}(t)\|_{H^1}^2 \,\mathrm{d}t \right). \tag{27}$$

The interval of definition of $\mathbf{u}$, which in this case is $\mathbb{R}$, can be classified according to the following:

$$\begin{aligned} &\mathcal{R} = \{t \in \mathbb{R};\ \mathbf{u}(t) \in V\}, \\ &\mathcal{R}^c = \{t \in \mathbb{R};\ \mathbf{u}(t) \notin V\}, \\ &\mathcal{O} = \{t \in \mathbb{R};\ \exists \varepsilon > 0,\ \mathbf{u} \in \mathcal{C}((t-\varepsilon, t+\varepsilon), V)\}. \end{aligned} \tag{28}$$

According to Definition 2.3, the points in $\mathcal{R}^c$ are the singular points, those in $\mathcal{R}$ are the regular points, and those in $\mathcal{O} \subset \mathcal{R}$ are the interior regularity points.

Since the solution belongs to V almost everywhere, the set $\mathcal{R}$ is of full Lebesgue measure (i.e. $\mathbb{R} \setminus \mathcal{R}$ is a null set) and dense in $\mathbb{R}$. The set $\mathcal{O}$ is open and can be written as a countable union of disjoint open intervals, say $\mathcal{O} = \bigcup_k (\alpha_k, \beta_k)$.

By the local existence of regular solutions it follows that $\mathcal{R} \subset \bigcup_k [\alpha_k, \beta_k)$, with $\mathcal{R} \setminus \mathcal{O} \subset \{\alpha_k\}_k$ at most countable[2]. In particular, it follows that $\mathcal{O}$ is also of full measure and dense in $\mathbb{R}$. Note that each α_k may or may not belong to $\mathcal{R}$, i.e. each $\mathbf{u}(\alpha_k)$ may or may not belong to V, according to (25). Depending on that, we have that either $[\alpha_k, \beta_k)$ or (α_k, β_k) is a maximal interval of regularity.

[2] In [13], it has been stated that $\mathcal{R}$ and $\mathcal{R} \setminus \mathcal{O}$ are equal to $\bigcup_k [\alpha_k, \beta_k)$ and $\{\alpha_k\}_k$, respectively, but the correct results are with the inclusions, as stated above.

If now $I \subset \mathbb{R}$ is a bounded interval, with length denoted $|I|$, and $N_I = \{k \in \mathbb{N};\ (\alpha_k, \beta_k) \subset I\}$, then the estimate (27) implies

$$\nu^{3/2} \sum_{k\in N_I} (\beta_k - \alpha_k)^{1/2} \leq c_0 \left(\nu^{2/3} |\mathbf{f}|_{L^2}^{2/3} |I| + \int_I \|\mathbf{u}(t)\|_{H^1}^2 \, \mathrm{d}t \right), \tag{29}$$

It can be showed that (27) and (29) imply that the 1/2-Hausdorf measure of the singular set $I \setminus \mathcal{O}$ is zero [16, 20, 11]. In fact, it can be showed that $I \setminus \mathcal{O}$ has fractal dimension less than or equal to 1/2; see [19].

The above estimates will be used here only for global weak solutions, but we remark that the facts presented in this section can be adapted to weak solutions defined on an arbitrary interval $I \subset \mathbb{R}$, taking care of the cases in which the endpoints of I coincide or not with the endpoints of a maximal interval of regularity.

2.4. **Trajectory spaces.** We define some basic "time-dependent" function spaces. First, we consider the spaces $\mathcal{C}(\mathbb{R}, H_{\mathrm{w}})$ and $\mathcal{C}(\mathbb{R}, B_H(R)_{\mathrm{w}})$, with $R > 0$, endowed with the topology of uniform weak convergence on compact intervals in $\mathbb{R}$. With this topology, the space $\mathcal{C}(\mathbb{R}, H_{\mathrm{w}})$ is a separable Hausdorff locally convex topological vector space, and $\mathcal{C}(\mathbb{R}, B_H(R)_{\mathrm{w}})$ is a Polish space (a separable and complete metrizable space).

For each $t_0 \in \mathbb{R}$, we define the projection operators

$$\begin{aligned} \Pi_{t_0} : \mathcal{C}(\mathbb{R}, H_{\mathrm{w}}) &\to H_{\mathrm{w}} \\ \mathbf{u} &\mapsto \Pi_{t_0}\mathbf{u} = \mathbf{u}(t_0), \end{aligned} \tag{30}$$

which are continuous, open, and surjective.

A space that plays a crucial role in the study of the asymptotic behavior of the solutions is that of the global weak solutions uniformly bounded in H:

$$\mathcal{W} = \left\{ \mathbf{u} \in \mathcal{C}(\mathbb{R}, H_{\mathrm{w}});\ \mathbf{u} \text{ is a weak solution on } \mathbb{R} \text{ with } \sup_{t\in\mathbb{R}} |\mathbf{u}(t)|_{L^2} < \infty \right\}, \tag{31}$$

endowed with the topology inherited from $\mathcal{C}(\mathbb{R}, H_{\mathrm{w}})$.

Proposition 3. *The space $\mathcal{W}$ is a compact metric space and is included in the space $\mathcal{C}(\mathbb{R}, B_H(R_0)_{\mathrm{w}})$, where R_0 is given in* (11).

Proof. From the uniform boundedness in H of an element $\mathbf{u}$ in $\mathcal{W}$ it follows from (13) that

$$|\mathbf{u}(t)|_{L^2} \leq R_0, \qquad \forall t \in \mathbb{R},\ \forall \mathbf{u} \in \mathcal{W}. \tag{32}$$

Thus $\mathcal{W}$ is a subset of $\mathcal{C}(\mathbb{R}, B_H(R_0)_{\mathrm{w}})$ and, hence, it is metrizable, and it suffices to prove sequential compactness. From the a priori estimate (8), we also have

$$\nu \int_{t_0}^{t_1} \|\mathbf{u}(t)\|_{H^1}^2 \, \mathrm{d}t \leq \frac{1}{\nu^2 \lambda_1^2} |\mathbf{f}|_{L^2}^2 + \frac{1}{\nu\lambda_1} |\mathbf{f}|_{L^2}^2 (t_1 - t_0), \ \ \forall t_0, t_1 \in \mathbb{R},\ t_0 < t_1,\ \forall \mathbf{u} \in \mathcal{W}. \tag{33}$$

From equation (3), and using Hölder's and Ladyzhenskaya's inequalities on the bilinear term, one finds, for a suitable universal constant c_2, that

$$\left\| \frac{\mathrm{d}\mathbf{u}}{\mathrm{d}t} \right\|_{V'} \leq \frac{1}{\lambda_1^{1/2}} |\mathbf{f}|_{L^2} + \nu \|\mathbf{u}\|_{H^1} + c_2^2 |\mathbf{u}|_{L^2}^{1/2} \|\mathbf{u}\|_{H^1}^{3/2}, \tag{34}$$

for every $\mathbf{u} \in \mathcal{W}$.

From (32), (33), and (34) it follows that $\mathcal{W}$ is equicontinuous in $\mathcal{C}(\mathbb{R}, V')$. Since V is separable and dense in H this implies that $\mathcal{W}$ is equicontinuous in the space $\mathcal{C}(\mathbb{R}, B_H(R)_{\mathrm{w}})$, and, moreover, $\{\mathbf{u}(t)\}_{\mathbf{u}\in\mathcal{W}} \subset B_H(R)_{\mathrm{w}}$ is relatively compact for each

$t \in \mathbb{R}$. Then, from the Arzela-Ascoli Theorem (and using a diagonalization process), it follows that $\mathcal{W}$ is relatively compact in $\mathcal{C}(\mathbb{R}, B_H(R)_{\rm w})$. From Lemma 2.5, $\mathcal{W}$ is also closed, thus $\mathcal{W}$ is a compact metric space. ☐

Now, taking into account that translations in time of global weak solutions are also global weak solutions, we have that

$$\Pi_t \mathcal{W} \text{ is independent of } t \in \mathbb{R}. \tag{35}$$

3. **The weak global attractor and its regular parts.**

3.1. **The weak global attractor.**

Let us consider the weak global attractor introduced in [12]. It is denoted by $\mathcal{A}_{\rm w}$ and is defined as the set of all points in H which belong to a global weak solution uniformly bounded in H on $\mathbb{R}$. By definition this set is directly related to the set $\mathcal{W}$ defined in the previous section and can be written as

$$\mathcal{A}_{\rm w} = \{\mathbf{u}_0 \in H;\ \exists \mathbf{u} \in \mathcal{W},\ \mathbf{u}(0) = \mathbf{u}_0\} = \Pi_0 \mathcal{W}.$$

Thanks to (35), we can also write

$$\mathcal{A}_{\rm w} = \Pi_{t_0} \mathcal{W}, \quad \forall t_0 \in \mathbb{R}. \tag{36}$$

Since $\mathcal{W}$ is compact in $\mathcal{C}(\mathbb{R}, H_{\rm w})$ and the projection operators are continuous it is clear that $\mathcal{A}_{\rm w}$ is compact in $H_{\rm w}$. It is in fact included and compact in $B_H(R_0)_{\rm w}$.

By definition, the weak global attractor $\mathcal{A}_{\rm w}$ is also invariant in the sense that if $\mathbf{u}_0 \in \mathcal{A}_{\rm w}$ and $\mathbf{u}$ is a global weak solution uniformly bounded in H with $\mathbf{u}(0) = \mathbf{u}_0$, then $\mathbf{u}(t) \in \mathcal{A}_{\rm w}$ for all $t \in \mathbb{R}$. Thanks to the Pasting Lemma 2.4, $\mathcal{A}_{\rm w}$ is also positively invariant in a stronger sense, namely, if $\mathbf{u}_0 \in \mathcal{A}_{\rm w}$ and $\mathbf{u}$ is any weak solution on an interval $[t_0, t_1]$ with $\mathbf{u}(t_0) = \mathbf{u}_0$, then $\mathbf{u}(t) \in \mathcal{A}_{\rm w}$ for all $t \in [t_0, t_1]$.

As proved in [12], $\mathcal{A}_{\rm w}$ has the property of attracting, in the weak topology, as $t \to \infty$, all solutions defined on an interval of the form $[t_0, \infty)$. In fact, it can be showed (see [13]) that this attraction is uniform with respect to initial conditions bounded in H. The proof of those results are based on Lemma 2.5.

It is not difficult to see that in fact $\mathcal{A}_{\rm w}$ is the smallest weakly compact set which attracts all bounded sets in the weak topology.

Remark 4. The set $\mathcal{W}$ itself can somehow be viewed as an attractor in trajectory space. More precisely, the restriction of the functions in $\mathcal{W}$ to a time interval of the form $[t_0, \infty)$, with $t_0 \in \mathbb{R}$, is the global attractor for the translation semigroup $\{\sigma_\tau\}_{\tau \geq 0}$, where for each $\tau \geq 0$, $(\sigma_\tau \mathbf{u})(t) = \mathbf{u}(t+\tau)$, for $t \geq t_0$, with the phase space taken e.g. as the subspace of $\mathcal{C}([t_0, \infty), H_{\rm w})$ made of weak solutions on the interval (t_0, ∞) (these are weak solutions on $[0, \infty)$ which do not satisfy condition (vi) in the Definition 2.1). For more details on this point of view, see the works [23, 24, 2], in which this problem is formulated in slightly different ways; in their work, Leray-Hopf weak solutions are in fact defined differently, avoiding in particular the condition (vi) in the Definition 2.1, and the topologies for the phase spaces of trajectories are also different.

3.2. **Regular parts of the weak global attractor.**

Two important "regular" subsets of $\mathcal{A}_{\rm w}$ to consider are $\mathcal{A}_{\rm reg}$ and $\mathcal{A}'_{\rm reg}$, defined by

$$\mathcal{A}_{\rm reg} = \left\{ \mathbf{u}_0 \in H; \begin{array}{l} \forall \mathbf{u} \in \mathcal{W} \text{ with } \mathbf{u}(0) = \mathbf{u}_0,\ \exists \delta_{\mathbf{u}} > 0 \text{ such that } \mathbf{u} \\ \text{is regular on } (-\delta_{\mathbf{u}}, \delta_{\mathbf{u}}) \end{array} \right\}. \tag{37}$$

and

$$\mathcal{A}'_{\rm reg} = \left\{ \mathbf{u}_0 \in H; \begin{array}{l} \exists \mathbf{u} \in \mathcal{W} \text{ with } \mathbf{u}(0) = \mathbf{u}_0 \text{ and } \exists \delta_{\mathbf{u}} > 0 \text{ such that} \\ \mathbf{u} \text{ is regular on } (-\delta_{\mathbf{u}}, \delta_{\mathbf{u}}) \end{array} \right\}. \tag{38}$$

Remark 5. The difference between $\mathcal{A}_{\rm reg}$ and $\mathcal{A}'_{\rm reg}$ is the following. If $\mathcal{A}'_{\rm reg} \setminus \mathcal{A}_{\rm reg}$ is not empty, then a point $\mathbf{u}_0$ in this set is a point such that there exists a global weak solution $\mathbf{u}$ on $\mathbb{R}$ with $\mathbf{u}(0) = \mathbf{u}_0$ which is a strong solution on some interval $(-\delta_{\mathbf{u}}, \delta_{\mathbf{u}})$, with $\delta_{\mathbf{u}} > 0$, and, furthermore, there exists another global weak solution $\mathbf{v}$ on $\mathbb{R}$ with $\mathbf{v}(0) = \mathbf{u}_0$ which is a strong solution on $[0, \delta)$, with $\delta = \delta_{\mathbf{u}}$ at least (and then $\mathbf{u} = \mathbf{v}$ on $[0, \delta)$), but $\mathbf{v}$ is not a strong solution on any interval $(\alpha, 0]$ with $\alpha < 0$ (and necessarily $\liminf_{t \to 0^-} \|\mathbf{v}(t)\|_{H^1} = \infty$).

The definition (37) of $\mathcal{A}_{\rm reg}$ is not the original definition given in [12]; the original definition is the one in (39) below. The advantage of the definition (37) is that it makes the relation between $\mathcal{A}_{\rm reg}$ and $\mathcal{A}'_{\rm reg}$ transparent. The following characterization result shows that both definitions are in fact equivalent.

Proposition 4. $\mathcal{A}_{\rm reg}$ *can be characterized by*

$$\mathcal{A}_{\rm reg} = \left\{ \mathbf{u}_0 \in H; \begin{array}{l} \exists \delta > 0 \textit{ and } \exists \mathbf{u} \in \mathcal{W} \textit{ with } \mathbf{u}(0) = \mathbf{u}_0, \textit{ such that } \mathbf{u} \textit{ is} \\ \textit{regular on } (-\delta, \delta) \textit{ and is unique on } (-\delta, \delta) \textit{ among all} \\ \textit{the global weak solutions in } \mathcal{A}_{\rm w} \textit{ with value } \mathbf{u}_0 \textit{ at time} \\ t = 0 \end{array} \right\}. \tag{39}$$

Moreover, each solution passing through $\mathbf{u}_0$ *at time* $t_0 = 0$ *has a maximal interval of regularity containing* $t_0 = 0$ *such that only the largest of such intervals is allowed to be (but not necessarily is) open on the left.*

Proof. Denote the set in the right hand side of (39) by $\tilde{\mathcal{A}}_{\rm reg}$. It follows from the definition that

$$\tilde{\mathcal{A}}_{\rm reg} \subset \mathcal{A}_{\rm reg}.$$

Hence, it suffices to show that $\mathcal{A}_{\rm reg} \subset \tilde{\mathcal{A}}_{\rm reg}$. Let then $\mathbf{u}_0 \in \mathcal{A}_{\rm reg}$. By definition, for any $\mathbf{u} \in \mathcal{W}$ with $\mathbf{u}(0) = \mathbf{u}_0$, there exists $\alpha_{\mathbf{u}} < 0 < \beta_{\mathbf{u}}$ such that $\mathbf{u}$ is regular on a maximal interval of regularity $I_{\mathbf{u}}$ with end points $\alpha_{\mathbf{u}}$ and $\beta_{\mathbf{u}}$. Due to the local existence and uniqueness of strong solutions forward in time, we must have $\beta_{\mathbf{u}} = \beta$ independent of $\mathbf{u} \in \mathcal{W} \cap \Pi_0^{-1}\{\mathbf{u}_0\}$, and $I_{\mathbf{u}}$ is open on the right end point. On the other end, let $\alpha = \inf\{\alpha_{\mathbf{u}},\ \mathbf{u} \in \mathcal{W} \cap \Pi_0^{-1}\{\mathbf{u}_0\}\}$. If $\mathbf{u} \in \mathcal{W} \cap \Pi_0^{-1}\{\mathbf{u}_0\}$ is such that $\alpha < \alpha_{\mathbf{u}}$, then we must have $I_{\mathbf{u}} = [\alpha_{\mathbf{u}}, \beta)$ closed on the left. In fact, if $\mathbf{v} \in \mathcal{W} \cap \Pi_0^{-1}\{\mathbf{u}_0\}$ is another solution with maximal interval of regularity $I_{\mathbf{v}}$ stricly larger than $I_{\mathbf{u}}$, which exists since $\alpha < \alpha_{\mathbf{u}}$, then $\mathbf{v}$ is regular on the interval $[\alpha_{\mathbf{u}}, \beta)$. By the time-analyticity of strong solutions, $\mathbf{u}$ and $\mathbf{v}$ must coincide on $(\alpha_{\mathbf{u}}, \beta)$. Since $\mathbf{u}$ and $\mathbf{v}$ are weakly continuous in H on $\mathbb{R}$, we must have $\mathbf{u}(\alpha_{\mathbf{u}}) = \mathbf{v}(\alpha_{\mathbf{v}})$ as well. Therefore, $\mathbf{u}$ must be strongly continuous on $[\alpha_{\mathbf{u}}, \beta)$. Thus, $I_{\mathbf{u}} = [\alpha_{\mathbf{u}}, \beta)$. Hence, only the "largest" maximal interval of regularity is allowed to be (but not necessarily is) open on the left.

Let us suppose that $\alpha' = \sup\{\alpha_{\mathbf{u}};\ \mathbf{u} \in \mathcal{W} \cap \Pi_0^{-1}\{\mathbf{u}_0\}\} = 0$. Hence, for every $n \in \mathbb{N}$, there exists $\mathbf{u}_n \in \mathcal{W} \cap \Pi_0^{-1}\{\mathbf{u}_0\}$ with $\alpha/n < \alpha_{\mathbf{u}_n}$. From the compactness of $\mathcal{W}$, there exists a convergent subsequence $\mathbf{u}_{n'} \to \mathbf{u}$ to some $\mathbf{u}$ in $\mathcal{W}$. Since $\mathbf{u}_n(0) = \mathbf{u}_0$ for all n, then $\mathbf{u}(0) = \mathbf{u}_0$. Since $\mathbf{u}_0 \in \tilde{\mathcal{A}}_{\rm reg}$, then there exist $\alpha'' < 0 < \beta''$ such that $\mathbf{u}$ is regular on (α'', β''). Due to Lemma 2.5, $\mathbf{u}_{n'}$ converges strongly to $\mathbf{u}$ on (α'', β''). This means in particular that, for sufficiently large n', $\mathbf{u}_{n'}$ is regular on $(\alpha''/2, \beta'/2')$. This implies that $\alpha_{\mathbf{u}_{n'}} \leq \alpha''/2$, which contradicts the fact that $\alpha_{\mathbf{u}_{n'}} \to 0$.

Hence, we must have $\alpha' < 0$. In that case, all solutions in $\mathcal{W} \cap \Pi_0^{-1}\{\mathbf{u}_0\}$ are regular on $(\alpha', \beta) \ni 0$, and, due to the analyticity of these solutions on (α', β), and the uniqueness of analytic solutions, they must all coincide on (α', β). This implies that $\mathbf{u}_0 \in \tilde{\mathcal{A}}_{\text{reg}}$, and the proof is complete. □

In view of (37) and (38) and the Lemma 2.6 we also have the following characterizations.

Lemma 3.1.

$$\mathcal{A}_{\text{reg}} = \left\{ \mathbf{u}_0 \in H;\ \forall \mathbf{u} \in \mathcal{W} \text{ such that } \mathbf{u}(0) = \mathbf{u}_0,\ \text{we have } \liminf_{t \to 0^-} \|\mathbf{u}(t)\|_{H^1} < \infty \right\};$$

$$\mathcal{A}'_{\text{reg}} = \left\{ \mathbf{u}_0 \in H;\ \exists \mathbf{u} \in \mathcal{W} \text{ such that } \mathbf{u}(0) = \mathbf{u}_0 \text{ and } \liminf_{t \to 0^-} \|\mathbf{u}(t)\|_{H^1} < \infty \right\}.$$

Lemma 3.1 implies that the complements of $\mathcal{A}'_{\text{reg}}$ and $\mathcal{A}_{\text{reg}}$ in $\mathcal{A}_{\text{w}}$ are related to solutions which blow-up in V. An estimate on how they blow up can be obtained with the inequality (26). Indeed, we have the following result.

Lemma 3.2. *Let*

$$\Gamma(t) = \frac{\nu^{3/2}}{2c_0|t|^{1/2}} - \nu^{2/3}|\mathbf{f}|_{L^2}^{2/3}, \qquad t < 0, \tag{40}$$

where c_0 is as in (26). Then $\mathbf{u} \in \mathcal{W}$ is such that

$$\liminf_{t \to 0^-} \|\mathbf{u}(t)\|_{H^1} = \infty \tag{41}$$

if and only if

$$\|\mathbf{u}(t)\|_{H^1}^2 \geq \Gamma(t),\ \forall t < 0. \tag{42}$$

Proof. Clearly (42) implies (41) so we only need to show one implication. Let then $\mathbf{u} \in \mathcal{W}$ such that (41) holds. Then, it follows from Lemma 2.6 (see also (23)) that $t = 0$ is the right end point of a maximal interval of regularity for $\mathbf{u}$. Denoting by $\alpha_{\mathbf{u}} < 0$ the left end point of this interval of regularity, we obtain from (26) and (25) that

$$\|\mathbf{u}(t)\|_{H^1}^2 \geq \Gamma(t), \qquad \forall t \in [\alpha_{\mathbf{u}}, 0).$$

(If $\alpha_{\mathbf{u}}$ does not belong to the interval of regularity then the left-hand side above is infinite and the inequality also holds.)

Now consider $t < \alpha_{\mathbf{u}}$. If $\|\mathbf{u}(t)\|_{H^1}^2 = \infty$, then it is clearly larger than $\Gamma(t)$. On the other hand, if $\|\mathbf{u}(t)\|_{H^1}^2 < \infty$, then t belongs to some other maximal interval of regularity of the form $[\alpha', \beta')$ or (α', β'), hence the estimate applies with $\beta' < 0$ and, clearly,

$$\|\mathbf{u}(t)\|_{H^1}^2 \geq \Gamma(\beta' - t) \geq \Gamma(t).$$

Hence, (42) holds for all $t < 0$, which completes the proof. □

The following is a consequence of Lemmas 3.1 and 3.2.

Corollary 1. *For $\Gamma(t)$ as in (40), we have the following characterizations*

$$\mathcal{A}_{\text{w}} \setminus \mathcal{A}_{\text{reg}} = \left\{ \mathbf{u}_0 \in H;\ \exists \mathbf{u} \in \mathcal{W} \cap \Pi_0^{-1}\{\mathbf{u}_0\} \text{ such that } \|\mathbf{u}(t)\|_{H^1}^2 \geq \Gamma(t),\ \forall t < 0 \right\}, \tag{43}$$

$$\mathcal{A}_{\text{w}} \setminus \mathcal{A}'_{\text{reg}} = \left\{ \mathbf{u}_0 \in H;\ \forall \mathbf{u} \in \mathcal{W} \cap \Pi_0^{-1}\{\mathbf{u}_0\},\ \text{we have } \|\mathbf{u}(t)\|_{H^1}^2 \geq \Gamma(t),\ \forall t < 0 \right\}. \tag{44}$$

Concerning the complement of $\mathcal{A}_{\text{reg}}$ in $\mathcal{A}_{\text{w}}$, we also have the following remark.

Remark 6. Suppose $\mathbf{u}_0 \in \mathcal{A}_{\mathrm{w}} \setminus \mathcal{A}_{\mathrm{reg}}$ and let $-\infty \leq t_1 < t_2 < \infty$. Then, one can show that there exists a global weak solution $\mathbf{v}$ in $\mathcal{A}_{\mathrm{w}}$ with $\mathbf{v}(t_2) = \mathbf{u}_0$ and which is not regular on $(t_1, t_2]$. Moreover, if $\mathbf{u} \in \mathcal{W}$ is a weak solution with $\mathbf{u}(t_2) = \mathbf{u}_0$, then $\mathbf{v}$ can be chosen so that $\mathbf{v}(t) = \mathbf{u}(t)$, for all $t \geq t_2$. The proof follows easily by contradiction.

We now prove the following result concerning the regularity of $\mathcal{A}_{\mathrm{w}}$, a result which was misstated in [12].

Theorem 3.3. *The following statements are equivalent:*

i. All solutions in $\mathcal{A}_{\mathrm{w}}$ are global strong solutions;
ii. $\mathcal{A}_{\mathrm{w}} = \mathcal{A}_{\mathrm{reg}}$.
iii. $\mathcal{A}_{\mathrm{w}}$ is bounded in V;

Proof. From the definition (37) it is straightforward to see that (i) implies (ii). Let us prove that (ii) implies (iii). Assume then that $\mathcal{A}_{\mathrm{w}} = \mathcal{A}_{\mathrm{reg}}$ and suppose that $\mathcal{A}_{\mathrm{w}}$ is not bounded in V. Then there exists a sequence $\mathbf{u}_{0n} \in \mathcal{A}_{\mathrm{w}}$ such that $\|\mathbf{u}_{0n}\|_{H^1} \to \infty$. Let $\mathbf{u}_n \in \mathcal{W}$ be a sequence of global weak solutions with $\mathbf{u}_n(0) = \mathbf{u}_{0n}$. Since $\mathcal{W}$ is compact in $\mathcal{C}(\mathbb{R}, H_{\mathrm{w}})$, there exists a subsequence $\mathbf{u}_{n_k}$ which converges to some $\mathbf{u} \in \mathcal{W}$ in the topology of $\mathcal{W}$. Consequently, $\mathbf{u}_{0n_k} = \Pi_0 \mathbf{u}_{n_k} = \mathbf{u}_{n_k}(0) \to \mathbf{u}(0)$ weakly in H. Since $\mathbf{u}(0) \in \mathcal{A}_{\mathrm{w}} = \mathcal{A}_{\mathrm{reg}}$, then $\mathbf{u}$ is a strong solution on an interval $(-\delta, \delta)$, for some $\delta > 0$. Then, by Lemma 2.5, $\mathbf{u}_{n_k}(t)$ must converge to $\mathbf{u}(t)$ in V for t in $(-\delta, \delta)$. In particular, $\mathbf{u}_{0n_k} = \mathbf{u}_{n_k}(0)$ converges in V to $\mathbf{u}(0)$, which contradicts the fact that $\|\mathbf{u}_{0n_k}\|_{H^1} \to \infty$. Therefore, $\mathcal{A}_{\mathrm{w}}$ must be bounded in V.

In order to prove that (iii) implies (i), let us now suppose that $\mathcal{A}_{\mathrm{w}}$ is bounded in V. Let $\mathbf{u}$ be a global weak solution in $\mathcal{A}_{\mathrm{w}}$. If $\mathbf{u}$ were not a global strong solution, then there would exist a time $t_0 \in \mathbb{R}$ which is not a point of interior regularity for $\mathbf{u}$. Then, by Lemma 2.6, there exists a sequence $t_n \to t_0$, $t_n < t_0$ such that $\|\mathbf{u}(t_n)\|_{H^1} \to \infty$. But since each $\mathbf{u}(t_n) \in \mathcal{A}_{\mathrm{w}}$, this contradicts the assumption that $\mathcal{A}_{\mathrm{w}}$ is bounded in V. Therefore all solutions in $\mathcal{A}_{\mathrm{w}}$ must be global strong solutions. ☐

3.3. **Topological properties of the regular parts of the weak global attractor.** By definition, $\mathcal{A}_{\mathrm{reg}} \subset \mathcal{A}'_{\mathrm{reg}} \subset \mathcal{A}_{\mathrm{w}} \cap V$. It was stated in [12], with a sketch of the proof, that $\mathcal{A}_{\mathrm{reg}}$ is relatively open and dense in $\mathcal{A}_{\mathrm{w}}$ in the weak topology of H. Here, we give the complete proof of these results as well as supplementary properties.

The set $\mathcal{A}'_{\mathrm{reg}}$ is also weakly dense in $\mathcal{A}_{\mathrm{w}}$ since it contains $\mathcal{A}_{\mathrm{reg}}$, and $\mathcal{A}_{\mathrm{reg}}$ is dense, but the proof of this result can be given independently of this fact and is presented below. We also show that $\mathcal{A}'_{\mathrm{reg}}$ is a σ-compact set in H_{w}. We first address these two properties of $\mathcal{A}'_{\mathrm{reg}}$ and then address the structure of $\mathcal{A}_{\mathrm{reg}}$.

Theorem 3.4. *$\mathcal{A}'_{\mathrm{reg}}$ is dense in $\mathcal{A}_{\mathrm{w}}$ in the weak topology of H.*

Proof. Given any $\mathbf{u}_0 \in \mathcal{A}_{\mathrm{w}}$, there exists a solution $\mathbf{u} \in \mathcal{W}$ such that $\mathbf{u}(0) = \mathbf{u}_0$ and, moreover, the set of interior regularity points of $\mathbf{u}$ is dense in $\mathbb{R}$, so that there exists a sequence of interior regularity points $t_n \in \mathbb{R}$ of $\mathbf{u}$ such that $t_n \to 0$. By definition, each $\mathbf{u}(t_n)$ belongs to $\mathcal{A}'_{\mathrm{reg}}$ and, by the weak continuity of $\mathbf{u}$, we have that $\mathbf{u}(t_n)$ converges weakly to $\mathbf{u}_0$ in H. This shows that $\mathcal{A}'_{\mathrm{reg}}$ is weakly dense in $\mathcal{A}_{\mathrm{w}}$. ☐

Theorem 3.5. *$\mathcal{A}'_{\mathrm{reg}}$ is a σ-compact set in H_{w}.*

Proof. By definition, for any $\mathbf{u}_0 \in \mathcal{A}'_{\text{reg}}$, there exists $\mathbf{u} \in \mathcal{W}$ with $\mathbf{u}(0) = \mathbf{u}_0$ and which is strongly continuous in V on some neighborhood of $t = 0$. Hence, there exists $\delta, R > 0$ such that $\mathbf{u}$ restricted to $(-\delta, \delta)$ belongs to $\mathcal{C}((-\delta,\delta), B_V(R))$, where $B_V(R)$ is the closed ball in V of radius R and centered at the origin. Therefore, for any pair of sequences of positive numbers $\delta_n \to 0$, $R_m \to \infty$, we can write

$$\mathcal{A}'_{\text{reg}} = \bigcup_{n,m\in\mathbb{N}} \mathcal{A}'_{\text{reg}}(\delta_n, R_m), \tag{45}$$

where, for any $\delta, R > 0$,

$$\mathcal{A}'_{\text{reg}}(\delta, R) = \left\{\mathbf{u}_0 \in \mathcal{A}'_{\text{reg}};\ \exists \mathbf{u} \in \mathcal{W},\ \mathbf{u}(0) = \mathbf{u}_0, \mathbf{u}|_{(-\delta,\delta)} \in \mathcal{C}((-\delta,\delta); B_V(R))\right\}. \tag{46}$$

Since $\mathcal{W}$ is compact in $\mathcal{C}(\mathbb{R}, H_{\text{w}})$ and $\mathcal{C}((-\delta,\delta), B_V(R))$ is closed in $\mathcal{C}((-\delta,\delta), H_{\text{w}})$, it follows that each $\mathcal{A}'_{\text{reg}}(\delta, R)$ is compact in H_{w}. Hence $\mathcal{A}'_{\text{reg}}$ is a countable union of compact sets, that is $\mathcal{A}'_{\text{reg}}$ is a σ-compact set in H_{w}. □

Next, we show that $\mathcal{A}_{\text{reg}}$ is relatively weakly open in $\mathcal{A}_{\text{w}}$. In order to do that, we need the following lemma.

Lemma 3.6. *Given* $\mathbf{u}_0$ *in* $\overline{\mathcal{A}_{\text{w}} \setminus \mathcal{A}_{\text{reg}}}^{\text{w}}$ *there exists a sequence* $\{\mathbf{u}_{0n}\}_{n\in\mathbb{N}}$ *such that*

$$\mathbf{u}_{0n} \in \mathcal{A}_{\text{w}} \cap V, \quad \mathbf{u}_{0n} \to \mathbf{u}_0 \text{ weakly in } H, \quad \|\mathbf{u}_{0n}\|_{H^1} \to \infty, \quad \text{as } n \to \infty. \tag{47}$$

Proof. For $\mathbf{u}_0 \in \mathcal{A}_{\text{w}} \setminus \mathcal{A}_{\text{reg}}$ the result follows directly from the characterization of this set in (43). Now, by a diagonalization process one can extend this property (of existence of the sequence satisfying (47)) to the closure of $\mathcal{A}_{\text{w}} \setminus \mathcal{A}_{\text{reg}}$. We use the fact that the weak topology is metrizable on $\mathcal{A}_{\text{w}}$, and denote by $\mathrm{d}_{\mathcal{A}_{\text{w}}}(\cdot,\cdot)$ a metric consistent with this topology. Then, given $\mathbf{u}_0$ in $\overline{\mathcal{A}_{\text{w}} \setminus \mathcal{A}_{\text{reg}}}^{\text{w}}$, we take a sequence $\{\mathbf{v}_{0n}\}_{n\in\mathbb{N}}$ in $\mathcal{A}_{\text{w}} \setminus \mathcal{A}_{\text{reg}}$ converging weakly to $\mathbf{u}_0$ in H. For each $n \in \mathbb{N}$, since $\mathbf{v}_{0n} \in \mathcal{A}_{\text{w}} \setminus \mathcal{A}_{\text{reg}}$, we use the previous step to find $\mathbf{u}_{0n}$ such that

$$\mathbf{u}_{0n} \in \mathcal{A}_{\text{w}} \cap V, \quad \mathrm{d}_{\mathcal{A}_{\text{w}}}(\mathbf{u}_{0n}, \mathbf{v}_{0n}) \le \frac{1}{n}, \quad \|\mathbf{u}_{0n}\|_{H^1} \ge n.$$

Thus, $\{\mathbf{u}_{0n}\}_n$ satisfies (47). □

We now prove that $\mathcal{A}_{\text{reg}}$ is relatively weakly open in $\mathcal{A}_{\text{w}}$.

Theorem 3.7. $\mathcal{A}_{\text{reg}}$ *is relatively open in* $\mathcal{A}_{\text{w}}$ *in the weak topology of* H.

Proof. Recall that $\mathcal{A}_{\text{w}}$ is weakly closed so that in order to show that $\mathcal{A}_{\text{reg}}$ is relatively weakly open in $\mathcal{A}_{\text{w}}$ it suffices to show that $\mathcal{A}_{\text{w}} \setminus \mathcal{A}_{\text{reg}}$ is weakly closed. Let then $\mathbf{u}_0 \in \overline{\mathcal{A}_{\text{w}} \setminus \mathcal{A}_{\text{reg}}}^{\text{w}}$. Since $\mathcal{A}_{\text{w}}$ is weakly closed it follows that $\mathbf{u}_0 \in \mathcal{A}_{\text{w}}$, and we only need to show that $\mathbf{u}_0$ does not belong to $\mathcal{A}_{\text{reg}}$.

Since $\mathbf{u}_0 \in \overline{\mathcal{A}_{\text{w}} \setminus \mathcal{A}_{\text{reg}}}^{\text{w}}$ it follows from Lemma 3.6 that there exists a sequence $\{\mathbf{u}_{0n}\}_{n\in\mathbb{N}}$ with

$$\mathbf{u}_{0n} \in \mathcal{A}_{\text{w}}, \quad \mathbf{u}_{0n} \rightharpoonup \mathbf{u}_0 \text{ weakly in } H, \quad \|\mathbf{u}_{0n}\|_{H^1} \to \infty, \qquad \text{as } n \to \infty.$$

Since $\mathbf{u}_{0n}$ and $\mathbf{u}_0$ belong to $\mathcal{A}_{\text{w}}$, there exist global weak solutions $\mathbf{u}^n = \mathbf{u}^n(t)$, $t \in \mathbb{R}$, and $\mathbf{u} = \mathbf{u}(t)$, $t \in \mathbb{R}$, such that $\mathbf{u}^n(0) = \mathbf{u}_{0n}$ and $\mathbf{u}(0) = \mathbf{u}_0$. Now, if $\mathbf{u}_0$ belonged to $\mathcal{A}_{\text{reg}}$, then $\mathbf{u}$ would be regular on some interval $(-\delta, \delta)$, $\delta > 0$, and hence, by Lemma 2.5, a subsequence of $\mathbf{u}^n$ would converge to $\mathbf{u}$ in V uniformly on any subinterval of $(-\delta, \delta)$. In particular, a subsequence of $\mathbf{u}^n(0) = \mathbf{u}_{0n}$ would converge to $\mathbf{u}(0) = \mathbf{u}_0$ in V. But this contradicts the fact that $\|\mathbf{u}_{0n}\|_{H^1} \to \infty$. Therefore, $\mathbf{u}_0$ cannot belong to $\mathcal{A}_{\text{reg}}$ and it must belong to $\mathcal{A}_{\text{w}} \setminus \mathcal{A}_{\text{reg}}$. This completes the proof that $\mathcal{A}_{\text{reg}}$ is open in $\mathcal{A}_{\text{w}}$. □

From the facts that $\mathcal{A}_{\mathrm{reg}}$ is relatively open in $\mathcal{A}_{\mathrm{w}}$ and that $\mathcal{A}_{\mathrm{w}}$ is relatively compact in H_{w} it is straigthforward to deduce the following result.

Corollary 2. *$\mathcal{A}_{\mathrm{reg}}$ is a Borel set in H.*

Let us now prove the density of $\mathcal{A}_{\mathrm{reg}}$ in $\mathcal{A}_{\mathrm{w}}$. First, we need the following lemma.

Lemma 3.8. *Let $\mathbf{u}_0 \in \mathcal{A}_{\mathrm{w}}$ and $\delta > 0$. Then, there exists $\mathbf{u} \in \mathcal{W}$ with $\mathbf{u}(0) = \mathbf{u}_0$ and $t \in [-\delta, 0]$ such that $\mathbf{u}(t) \in \mathcal{A}_{\mathrm{reg}}$.*

Proof. Let $\mathbf{u}_0 \in \mathcal{A}_{\mathrm{reg}}$ and $\delta > 0$ be given. Assume by contradiction that there is no solution $\mathbf{u}$ as claimed, i.e. any $\mathbf{u} \in \mathcal{W}$ with $\mathbf{u}(0) = \mathbf{u}_0$ satisfies $\mathbf{u}(t) \notin \mathcal{A}_{\mathrm{reg}}$ for all $t \in [-\delta, 0]$. In that case, given $n \in \mathbb{N}$, we will first construct a solution $\mathbf{u} \in \mathcal{W}$ with $\mathbf{u}(0) = \mathbf{u}_0$ and such that there exists $n+1$ times $s_0 < s_1 < \ldots < s_n$ which are not points of interior regularity for $\mathbf{u}$ and such that $s_n = 0$ and $s_j \in [-(n-j)\delta/n, -(n-j-1)\delta/n)$, for $j = 0, \ldots, n-1$. The construction is done in $n+1$ steps, starting from s_n and down to s_0.

First, let $\mathbf{u}^{(n)} \in \mathcal{W}$ be such that $\mathbf{u}^{(n)}(0) = \mathbf{u}_0$ and $t = 0$ is not a point of interior regularity for $\mathbf{u}^{(n)}$, which exists since $\mathbf{u}_0$ does not belong to $\mathcal{A}_{\mathrm{reg}}$. Now, take any $s_{n-1} \in [-\delta/n, 0)$ which is a point of strong continuity from the right in H for $\mathbf{u}^{(n)}$. Since by hypothesis $\mathbf{u}^{(n)}(s_{n-1})$ does not belong to $\mathcal{A}_{\mathrm{reg}}$, there exists a solution $\mathbf{v} \in \mathcal{W}$ with $\mathbf{v}(s_{n-1}) = \mathbf{u}^{(n)}(s_{n-1})$ such that s_{n-1} is not an interior regularity point for $\mathbf{v}$. This means that $\mathbf{v}(t)$ is not bounded in V as $t \to s_{n-1}^-$. Since s_{n-1} is a point of strong continuity from the right in H for $\mathbf{u}^{(n)}$, we are allowed to patch $\mathbf{v}$ before s_{n-1} with $\mathbf{u}^{(n)}$ after s_{n-1} to obtain a weak solution $\mathbf{u}^{(n-1)} \in \mathcal{W}$ with the properties that $\mathbf{u}^{(n-1)}(0) = \mathbf{u}_0$, and s_n and s_{n-1} are not points of interior regularity for $\mathbf{u}^{(n-1)}$. Proceeding by induction, we obtain for $j = 0, \ldots, n-1$, solutions $\mathbf{u}^{(j)} \in \mathcal{W}$ and times $s_j \in [-(n-j)\delta/n, -(n-j-1)\delta, n)$ with the properties that $\mathbf{u}^{(j)}(0) = \mathbf{u}_0$, and each s_k with $k = j, \ldots, n$ is not a point of interior regularity for $\mathbf{u}^{(j)}$.

At the last step, we obtain a weak solution $\mathbf{u} = \mathbf{u}^{(0)} \in \mathcal{W}$ with the property that $\mathbf{u}(0) = \mathbf{u}_0$, s_j is not a point of interior regularity for $\mathbf{u}$ for any $j = 0, \ldots, n$, and $s_j \in [-(n-j)\delta/n, -(n-j-1)\delta, n)$ for all $j = 0, \ldots, n-1$. This implies that any interval of regularity of $\mathbf{u}$ in the interval $[-\delta, 0]$ has to be of length less than $2\delta/n$. Let $\{(\alpha_k, \beta_k)\}_k$ be the connected components of the regular part of $\mathbf{u}$ on $[-\delta, 0]$. Thus we have $\beta_k - \alpha_k \leq 2\delta/n$. Then, using the estimates (29) and (33), we find that

$$\begin{aligned}\sum_k (\beta_k - \alpha_k) &\leq \sqrt{\frac{2\delta}{n}} \sum_k (\beta_k - \alpha_k)^{1/2} \\ &\leq \frac{c_0}{\nu^{3/2}} \sqrt{\frac{2\delta}{n}} \left(\nu^{2/3} |\mathbf{f}|_{L^2}^{2/3} \delta + \frac{1}{\nu^3 \lambda_1^2} |\mathbf{f}|_{L^2}^2 + \frac{1}{\nu^2 \lambda_1} |\mathbf{f}|^2 \delta \right).\end{aligned}$$

On the other hand, since the regular part of $\mathbf{u}$ is of full measure on $[-\delta, 0)$, we would have

$$\sum_k (\beta_k - \alpha_k) = \delta.$$

Therefore,

$$\sqrt{n} \leq \frac{c_0}{\nu^{3/2}} \sqrt{\frac{2}{\delta}} \left(\nu^{2/3} |\mathbf{f}|_{L^2}^{2/3} \delta + \frac{1}{\nu^3 \lambda_1^2} |\mathbf{f}|_{L^2}^2 + \frac{1}{\nu^2 \lambda_1} |\mathbf{f}|^2 \delta \right).$$

Since n is arbitrary this is a contradiction. This proves the result. ∎

As a consequence of Lemma 3.8 we have the following density result.

Theorem 3.9. *$\mathcal{A}_{\rm reg}$ is dense in $\mathcal{A}_{\rm w}$ in the weak topology of H.*

Proof. Let $\mathbf{u}_0 \in \mathcal{A}_{\rm w}$. Since $B_H(R)_{\rm w}$ is metrizable, consider a metric $\mathrm{d}_{B_H(R)_{\rm w}}(\cdot,\cdot)$ which is consistent with the topology of $B_H(R)_{\rm w}$. According to Proposition 3 the space $\mathcal{W}$ is equicontinuous in $\mathcal{C}(\mathbb{R}, B_H(R)_{\rm w})$, hence given $\varepsilon > 0$, there exists $\delta > 0$ such that $\mathrm{d}_{B_H(R)_{\rm w}}(\mathbf{u}(t),\mathbf{u}_0) < \varepsilon$, for all $\mathbf{u} \in \mathcal{W} \cap \Pi_0^{-1}\{\mathbf{u}_0\}$ and any $|t| \leq \delta$.

Now, according to Lemma 3.8, there exists $\mathbf{u} \in \mathcal{W} \cap \Pi_0^{-1}\{\mathbf{u}_0\}$ and $t \in [-\delta, 0]$ such that $\mathbf{u}(t) \in \mathcal{A}_{\rm reg}$. Since $|t| \leq \delta$, we also have that $\mathrm{d}_{B_H(R)_{\rm w}}(\mathbf{u}(t),\mathbf{u}_0) < \varepsilon$. Since this holds for any $\varepsilon > 0$, we conclude that $\mathcal{A}_{\rm reg}$ is weakly dense in $\mathcal{A}_{\rm w}$. □

3.4. **On the structure of the set $\mathcal{W}$.** For each $t \in \mathbb{R}$, consider the sets

$$\mathcal{W}'_{\rm reg}(t) = \{\mathbf{u} \in \mathcal{W};\ \exists \delta = \delta_{\mathbf{u}} > 0, \text{ such that } \mathbf{u}|_{(t-\delta,t+\delta)} \in \mathcal{C}((t-\delta,t+\delta),V)\},$$

and

$$\mathcal{E}'(t) = \mathcal{W} \setminus \mathcal{W}'_{\rm reg}(t).$$

Notice that $\mathcal{W}'_{\rm reg}(t)$ is not necessarily equal to $\mathcal{W} \cap \Pi_t^{-1}\mathcal{A}'_{\rm reg}$. However, we have the following inclusions:

$$\mathcal{W} \cap \Pi_t^{-1}\mathcal{A}_{\rm reg} \subset \mathcal{W}'_{\rm reg}(t) \subset \mathcal{W} \cap \Pi_t^{-1}\mathcal{A}'_{\rm reg}. \tag{48}$$

Our aim in this section is to address the structure of the set $\mathcal{W}$. We show below that it can be written as a finite union of sets of the form $\mathcal{W}'_{\rm reg}(s_j)$, for appropriately chosen sets of times $s_0 < s_1 < \ldots < s_j \ldots < s_n$. The key condition is that we need elements in $\bigcap_{j=0}^n \mathcal{E}'(s_j)$ to be weak solutions such that s_j is not an interior regularity point for any j. This condition is obtained using the following lemma concerning weak solutions and which is essentially due to Leray [16].

Lemma 3.10. *Let $\mathbf{u}$ be a weak solution on $\mathbb{R}$ and consider the set $\mathcal{O}$ of interior regularity points of $\mathbf{u}$ defined in (28). Suppose $s_0 < s_1 < \ldots < s_n$ are such that $s_j \notin \mathcal{O}$ for $j = 1, \ldots, n-1$. Then,*

$$\nu^{3/2}\sum_{j=1}^n (s_j - s_{j-1})^{1/2} \leq c_0\left(\nu^{2/3}|\mathbf{f}|_{L^2}^{2/3}(s_n - s_0) + \int_{s_0}^{s_n} \|\mathbf{u}(t)\|_{H^1}^2\,\mathrm{d}t\right), \tag{49}$$

where c_0 is a nondimensional constant depending only on the shape of Ω (given according to (27)).

Proof. Let $\{(\alpha_k,\beta_k)\}_k$ be the connected components of the open set $\mathcal{O} \cap (s_0, s_n)$. By summing up the estimates (27) over k we find

$$\nu^{3/2}\sum_k (\beta_k - \alpha_k)^{1/2} \leq c_0\left(\nu^{2/3}|\mathbf{f}|_{L^2}^{2/3}(s_n - s_0) + \int_{s_0}^{s_n} \|\mathbf{u}(t)\|_{H^1}^2\,\mathrm{d}t\right). \tag{50}$$

Since each s_j, $j = 0, \ldots, n$, does not belong to any interval of regularity (α_k, β_k), if $(s_{j-1}, s_j)\bigcap(\alpha_k,\beta_k) \neq \emptyset$, then $(\alpha_k,\beta_k) \subset (s_{j-1}, s_j)$. Therefore, the summation $\sum_k (\beta_k - \alpha_k)^{1/2}$ can be written as

$$\sum_k (\beta_k - \alpha_k)^{1/2} = \sum_{j=1}^n \sum_{k \in K_j} (\beta_k - \alpha_k)^{1/2},$$

where

$$K_j = \{k;\ (\alpha_k,\beta_k) \subset (s_{j-1}, s_j)\}.$$

Note also that $(s_{j-1}, s_j) \setminus \left(\bigcup_{k\in K_j}(\alpha_k, \beta_k)\right)$ has Lebesgue measure zero. Therefore,

$$\sum_{k\in K_j}(\beta_k - \alpha_k) = s_j - s_{j-1}.$$

Thus,

$$\sum_{k\in K_j}(\beta_k - \alpha_k)^{1/2} \geq (s_j - s_{j-1})^{1/2}.$$

Then, we find

$$\sum_k (\beta_k - \alpha_k)^{1/2} = \sum_{j=1}^n \sum_{k\in K_j}(\beta_k - \alpha_k)^{1/2} \geq \sum_{j=1}^n (s_j - s_{j-1})^{1/2}.$$

Inserting this inequality into (50) proves (49). □

Based on Lemma 3.10, we have the following result.

Theorem 3.11. *Let* $-\infty < s_0 < s_1 < \ldots < s_n < \infty$ *be such that*

$$\nu^{3/2}\sum_{j=1}^n (s_j - s_{j-1})^{1/2} > c_0\left(\nu^{2/3}|\mathbf{f}|_{L^2}^{2/3}(s_n - s_0) + \frac{|\mathbf{f}|_{L^2}^2}{\nu^2\lambda_1}(s_n - s_0) + \frac{|\mathbf{f}|_{L^2}^2}{\nu^3\lambda_1^2}\right). \quad (51)$$

Then,

$$\bigcap_{j=0}^n \mathcal{E}'(s_j) = \emptyset \quad \textit{and} \quad \mathcal{W} = \bigcup_{j=0}^n \mathcal{W}'_{\mathrm{reg}}(s_j).$$

Proof. If $\bigcap_{j=0}^n \mathcal{E}'(s_j)$ were not empty, there would exist a global weak solution $\mathbf{u}$ for which s_j is not an interior regularity point for any j. Then, we apply Lemma 3.10 and the estimate

$$\int_{s_0}^{s_n} \|\mathbf{u}(t)\|_{H^1}^2 \, \mathrm{d}t \leq \frac{1}{\nu^3\lambda_1^2}|\mathbf{f}|_{L^2}^2 + \frac{1}{\nu^2\lambda_1}|\mathbf{f}|_{L^2}^2(s_n - s_0),$$

which follows from (33), to arrive at a contradiction. Moreover, $\mathcal{W} = \bigcup_{j=0}^n \mathcal{W}'_{\mathrm{reg}}(s_j)$ follows directly from $\bigcap_{j=0}^n \mathcal{E}'(s_j) = \emptyset$. □

Remark 7. By taking $\tau > 0$ arbitrary and $s_j - s_{j-1} = \tau/n$ it is not difficult to see that condition (51) holds for n sufficiently large, of the order of $G^2((\nu\lambda_1\tau)^{1/2} + (\nu\lambda_1\tau)^{-1/2})$. By taking in particular $\tau \sim (\nu\lambda_1)^{-1}$, then n is of order G^2.

Remark 8. Since the set of interior regularity points of a weak solution $\mathbf{u} \in \mathcal{W}$ is dense in $\mathbb{R}$, there exists $t_n \to t$ such that each t_n is a point of interior regularity for $\mathbf{u}$. Then, for each n, the translation $\mathbf{u}^{(n)}(s) = \mathbf{u}(s - t + t_n)$, $s \in \mathbb{R}$, belongs to $\mathcal{W}'_{\mathrm{reg}}(t)$, and the sequence $\mathbf{u}^{(n)}$ converges to $\mathbf{u}$ in $\mathcal{W}$, which proves that $\mathcal{W}'_{\mathrm{reg}}(t)$ is dense in $\mathcal{W}$, for any $t \in \mathbb{R}$.

Remark 9. Thanks to (36) we can regard Π_t restricted to $\mathcal{W}$ as a continuous map $\Pi_t|_{\mathcal{W}} : \mathcal{W} \to \mathcal{A}_{\mathrm{w}}$ and write $\mathcal{W} \cap \Pi_t^{-1}\mathcal{A}_{\mathrm{reg}} = (\Pi_t|_{\mathcal{W}})^{-1}\mathcal{A}_{\mathrm{reg}}$. Then, using that $\mathcal{A}_{\mathrm{reg}}$ is open in $\mathcal{A}_{\mathrm{w}}$ (Theorem 3.7), we find that $(\Pi_t|_{\mathcal{W}})^{-1}\mathcal{A}_{\mathrm{reg}}$ is open in $\mathcal{W}$.

Remark 10. We have not obtained a result for $\mathcal{W}$ similar to the one saying that $\mathcal{A}_{\mathrm{reg}}$ is open and dense in $\mathcal{A}_{\mathrm{w}}$. Remarks 8 and 9 say that $\mathcal{W}'_{\mathrm{reg}}(t)$ is dense in $\mathcal{W}$ and $\mathcal{W} \cap \Pi_t^{-1}\mathcal{A}_{\mathrm{reg}}$ is open, but it is not known whether $\mathcal{W} \cap \Pi_t^{-1}\mathcal{A}_{\mathrm{reg}}$ is also dense in $\mathcal{W}$. Another interesting regular set to consider is $\mathcal{Z} = \{\mathbf{u} \in \mathcal{W};\ I_{\mathrm{reg}}(\mathbf{u})$ is open and dense in $\mathbb{R}\}$, where $I_{\mathrm{reg}}(\mathbf{u}) = \{t \in \mathbb{R}; \mathbf{u}(t) \in \mathcal{A}_{\mathrm{reg}}\}$; the set $\mathcal{Z}$ is a $\mathcal{G}_\delta$ set in $\mathcal{W}$ and it would be interesting to find out whether $\mathcal{Z}$ is dense in $\mathcal{W}$. Ultimately, it would be important to know whether $\mathcal{W}$ is made of global strong solutions, which is exactly condition (i) in Theorem 3.3.

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Received for publication March 2010.

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