Semidiscrete Galerkin Method for Equations of Motion Arising in Kelvin-Voigt Model of Viscoelastic Fluid Flow

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Finite element Galerkin method is applied to equations of motion arising in the Kelvin–Voigt model of viscoelastic fluids for spatial discretization. Some new a priori bounds which reflect the exponential decay property are obtained for the exact solution. For optimal $L^\infty(\mathbf{L}^2)$ estimate in the velocity, a new auxiliary operator which is based on a modification of the Stokes operator is introduced and analyzed. Finally, optimal error bounds for the velocity in $L^\infty(\mathbf{L}^2)$ as well as in $L^\infty(\mathbf{H}_0^1)$-norms and the pressure in $L^\infty(L^2)$-norm are derived which again preserves the exponential decay property. Numer Methods Partial Differential Eq 29: 857–883, 2013

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I. INTRODUCTION

The motion of an incompressible fluid in a bounded domain Ω in $\mathbb{R}^2$ (or $\mathbb{R}^3$) is described by the following system of partial differential equations:

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} - \nabla \sigma + \nabla p = \mathbf{F}(x,t), \quad x \in \Omega, \ t > 0,$$

$$\nabla \cdot \mathbf{u} = 0, \quad x \in \Omega, \ t > 0,$$

with appropriate initial and boundary conditions. Here, $\sigma = (\sigma_{ik})_{1\le i,k\le 2}$ (or $\sigma = (\sigma_{ik})_{1\le i,k\le 3}$) denotes the stress tensor with $tr\ \sigma = 0$, $\mathbf{u} = (u_1, u_2)$ (or $\mathbf{u} = (u_1, u_2, u_3)$) represents the velocity

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vector, p is the pressure of the fluid and $\mathbf{F}$ is the external force. The defining relation between the stress tensor σ and the tensor of deformation velocities

$$\mathbf{D} = (\mathbf{D}_{ik}) = \frac{1}{2}(\mathbf{u}_{i,x_k} + \mathbf{u}_{k,x_i})$$

is called the equation of state or sometimes the rheological equation and it establishes the type of fluids under consideration. For example, when $\sigma = 2\nu\mathbf{D}$ (using Newton's law) where ν is the kinematic coefficient of viscosity, we derive Newton's model of incompressible viscous fluid and the corresponding system is popularly known as the Navier–Stokes system. This has been a basic model for describing the flow at moderate velocities of the majority of the incompressible viscous fluids that we have encountered in practice. However, in the mid-20th century, models of viscoelastic fluids which take into account the prehistory of the flow and are not subject to the Newtonian flow have been proposed. One such model is called Kelvin–Voigt model [1] and its rheological equation of state has the form:

$$\sigma = 2\nu\left(1 + \kappa\nu^{-1}\frac{\partial}{\partial t}\right)\mathbf{D}, \ \kappa, \nu > 0, \tag{1.1}$$

where ν is the kinematic coefficient of viscosity and κ is the retardation time, and is characterized by the fact that after instantaneous removal of the stress, the velocity of the fluid does not vanish instantaneously but dies out like $\exp(\kappa^{-1}t)$, see [1]. The coefficient κ is also called the time of relaxation of deformations. (1.1) differs from the Newtonian model in the sense that it has an additional term $\kappa\frac{\partial}{\partial t}\mathbf{D}$, that takes into account the relaxation property of the fluid. Apart from applications of this model in organic polymer and food industry, and so forth, the mechanisms of diffuse axonal injury that are unexplained by traumatic brain injury models proposed earlier are now based on Kelvin–Voigt model, see, [2–4] for more detailed description. Using the rheological relation (1.1), the equations of motion arising from the Kelvin–Voigt's model give rise to the following system of partial differential equations :

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u} - \kappa\Delta\mathbf{u}_t - \nu\Delta\mathbf{u} + \nabla p = \mathbf{f}(x,t), \ \ x \in \Omega, \ t > 0, \tag{1.2}$$

and incompressibility condition

$$\nabla\cdot\mathbf{u} = 0, \ \ x \in \Omega, \ t > 0, \tag{1.3}$$

with initial and boundary conditions

$$\mathbf{u}(x,0) = \mathbf{u}_0 \ \text{ in } \Omega, \qquad \mathbf{u} = 0, \ \text{ on } \partial\Omega, \ t \geq 0. \tag{1.4}$$

Here, Ω is a bounded domain in $\mathbb{R}^d, d = 2, 3$ with boundary $\partial\Omega$. Throughout this article, we assume that the right hand side function $\mathbf{f} = 0$. In fact, assuming conservative force, the function $\mathbf{f}$ can be absorbed in the pressure term.

Based on the analysis of Ladyzenskaya [5] for the solvability of the Navier Stokes equations, Oskolkov [6] and [1], proved the global existence of unique "almost" classical solution in finite time interval for the initial and boundary value problem (1.2)–(1.4). The investigations on solvability were further continued, see [7] and [8]. They have discussed the existence and uniqueness of a solution on the entire semiaxis $\mathbb{R}^+$.

For the earlier results on the numerical approximations to the solutions of the problem (1.2)–(1.4), we refer to [9]. Under the condition that solution is asymptotically stable as $t \to \infty$, the authors of [9] have established the convergence of spectral Galerkin approximations for the semi axis $t \geq 0$. There is hardly any literature devoted to the analysis of the finite element Galerkin methods for the problem (1.2)–(1.4), and hence, the present investigation is a step toward achieving this objective. Therefore, in this article, we address the problem of finite element Galerkin approximations to (1.2)–(1.4). As system (1.2)–(1.3) differs from the Navier–Stokes system only by an additional term $\kappa \frac{\partial}{\partial t}\mathbf{D}$ which takes care of the relaxation property of the fluid, it is more pertinent to see "How far the results of the Navier–Stokes system [10] carry over to the present Kelvin–Voigt model ?" More precisely, our emphasis is to bring out the role played by this additional term. The main results of this article consist of

i. proving regularity results for the solution of (1.2)–(1.4), which are valid uniformly in time and even for (three-dimensional) domain.
ii. establishing the exponential decay property for the exact solution.
iii. obtaining optimal error estimates for the semidiscrete Galerkin approximations to the velocity in $L^\infty(\mathbf{L}^2)$-norm as well as in $L^\infty(\mathbf{H}_0^1)$-norm and to the pressure in $L^\infty(L^2)$-norm which also reflect the exponential decay property in time.

For the proof of (i) and (ii), we have made use of exponential weights for the derivation of the new regularity results. These weights also become crucial in establishing the results in item (iii). To derive optimal error estimates for the velocity in $L^\infty(\mathbf{L}^2)$-norm, we first split the error using a Galerkin approximation to a linearized Kelvin–Voigt model and then introduce a new auxiliary operator through a modification of the Stokes operator. Now making use of estimates derived for the auxiliary operator and the error estimates due to the linearized model, we recover the optimality of $L^\infty(\mathbf{L}^2)$ error estimates for the velocity. Finally, with the help of uniform inf-sup condition and error estimates for the velocity, we derive optimal error estimates for the pressure. Special care has been taken to preserve the exponential decay property even for the error estimates. For related articles in the context of Oldroyd viscoelastic model, we refer to [11–19].

The remaining part of this article is organized as follows. In Section II, we discuss the weak formulation and state the basic assumptions. Section III is devoted to development of a priori bounds for the exact solutions. In Section IV, we describe the semidiscrete Galerkin approximations and derive the main results required for error analysis. In Section V, we establish the optimal error estimates for the velocity. Section VI deals with the optimal error estimates for the pressure. In Section VII, results of numerical experiments, which validate the theoretical estimates, are presented.

II. PRELIMINARIES AND WEAK FORMULATION

In our subsequent analysis, we denote $\mathbb{R}^d$, $(d = 2, 3)$-valued function spaces using boldface letters. That is,

$$\mathbf{H}_0^1 = \left(H_0^1(\Omega)\right)^d, \quad \mathbf{L}^2 = (L^2(\Omega))^d, \quad \text{and} \quad \mathbf{H}^m = (H^m(\Omega))^d,$$

where $L^2(\Omega)$ is the space of square integrable functions defined in Ω with inner product $(\phi, \psi) = \int_\Omega \phi(x)\psi(x)\,dx$ and norm $\|\phi\| = (\int_\Omega |\phi(x)|^2\,dx)^{1/2}$. Further, $H^m(\Omega)$ is the standard

Hilbert Sobolev space of order $m \in \mathbb{N}^+$ with norm $\|\phi\|_m = (\sum_{|\alpha|\le m}\int_\Omega |D^\alpha \phi|^2\,dx)^{1/2}$. Note that $\mathbf{H}_0^1$ is equipped with a norm

$$\|\nabla \mathbf{v}\| = \left(\sum_{i,j=1}^{d}(\partial_j v_i, \partial_j v_i)\right)^{1/2} = \left(\sum_{i=1}^{d}(\nabla v_i, \nabla v_i)\right)^{1/2}.$$

We also introduce divergence free spaces, which are useful for our subsequent derivations:

$$\mathbf{J}_1 = \{\boldsymbol{\phi} \in \mathbf{H}_0^1 : \nabla \cdot \boldsymbol{\phi} = 0\}$$
$$\mathbf{J} = \{\boldsymbol{\phi} \in \mathbf{L}^2 : \nabla \cdot \boldsymbol{\phi} = 0 \ \text{ in } \ \Omega, \ \ \boldsymbol{\phi}\cdot \mathbf{n}|_{\partial\Omega} = 0 \ \text{ holds weakly}\},$$

where $\mathbf{n}$ is the outward normal to the boundary $\partial\Omega$ and $\boldsymbol{\phi}\cdot\mathbf{n}|_{\partial\Omega} = 0$ should be understood in the sense of trace in $\mathbf{H}^{-1/2}(\partial\Omega)$, see [20]. Let $H^m/\mathbb{R}$ be the quotient space consisting of equivalence classes of elements of H^m differing by constants, with norm $\|p\|_{H^m/\mathbb{R}} = \inf_{c\in\mathbb{R}} \|p + c\|_m$. Given any Banach Space X, let $L^p(0, T; X)$ denote the space of measurable X- valued functions $\boldsymbol{\phi}$ on $(0, T)$ such that

$$\int_0^T \|\boldsymbol{\phi}(t)\|_X^p dt < \infty \quad \text{if } \ 1 \le p < \infty,$$

and for $p = \infty$

$$ess \sup_{0<t<T} \|\boldsymbol{\phi}(t)\|_X < \infty \quad \text{if} p = \infty.$$

Further, let P be the orthogonal projection of $\mathbf{L}^2$ onto $\mathbf{J}$.

Throughout this article, we make the following assumptions, which will be used in our subsequent analysis.

(**A1**). For $\mathbf{g} \in \mathbf{L}^2$, let $\{\mathbf{v} \in \mathbf{J}_1, q \in L^2/\mathbb{R}\}$ be the unique pair of solution to the steady state Stokes problem, see [20],

$$-\Delta \mathbf{v} + \nabla q = \mathbf{g},$$
$$\nabla \cdot \mathbf{v} = 0 \quad \text{in} \quad \Omega, \qquad \mathbf{v}|_{\partial\Omega} = 0$$

satisfying the following regularity result:

$$\|\mathbf{v}\|_2 + \|q\|_{H^1/\mathbb{R}} \le C\|\mathbf{g}\|. \tag{2.1}$$

Setting

$$-\tilde{\Delta} = -P\Delta : \mathbf{J}_1 \cap \mathbf{H}^2 \subset \mathbf{J} \to \mathbf{J}$$

as the Stokes operator, (**A1**) implies

$$\|\mathbf{v}\|_2 \le C\|\tilde{\Delta}\mathbf{v}\| \quad \forall \mathbf{v} \in \mathbf{J}_1 \cap \mathbf{H}^2. \tag{2.2}$$

It is easy to show that

$$\|\mathbf{v}\|^2 \le \lambda_1^{-1}\|\nabla \mathbf{v}\|^2 \ \ \forall \mathbf{v} \in \mathbf{H}_0^1(\Omega), \tag{2.3}$$

where λ_1^{-1} is a positive constant depending on the domain Ω. In fact this is known as Poincaré inequality with λ_1^{-1} as best possible positive constant. We again note that

$$\|\nabla \mathbf{v}\|^2 \le \lambda_1^{-1}\|\tilde{\Delta}\mathbf{v}\|^2 \quad \forall \mathbf{v} \in \mathbf{J}_1 \cap \mathbf{H}^2. \tag{2.4}$$

(**A2**). There exists a positive constant M, such that the initial velocity $\mathbf{u}_0$ satisfies

$$\mathbf{u}_0 \in \mathbf{H}^2 \cap \mathbf{J}_1 \quad \text{with} \quad \|\mathbf{u}_0\|_2 \le M.$$

Before stepping into the details, let us introduce the weak formulation of (1.2)–(1.4) with $\mathbf{f} = 0$.

Find a pair of functions $\{\mathbf{u}(t), p(t)\} \in \mathbf{H}_0^1 \times L^2/\mathbb{R},\ \ t > 0$, such that

$$\begin{aligned}(\mathbf{u}_t, \boldsymbol{\phi}) + \kappa(\nabla \mathbf{u}_t, \nabla \boldsymbol{\phi}) + \nu(\nabla \mathbf{u}, \nabla \boldsymbol{\phi}) + (\mathbf{u}\cdot\nabla\mathbf{u}, \boldsymbol{\phi}) &= (p, \nabla\cdot\boldsymbol{\phi}) \quad \forall \boldsymbol{\phi} \in \mathbf{H}_0^1,\\ (\nabla\cdot\mathbf{u}, \chi) &= 0 \quad \forall \chi \in L^2,\end{aligned} \tag{2.5}$$

with $\mathbf{u}(0) = \mathbf{u}_0$.

Equivalently, find $\mathbf{u}(t) \in \mathbf{J}_1$ such that

$$(\mathbf{u}_t, \boldsymbol{\phi}) + \kappa(\nabla \mathbf{u}_t, \nabla \boldsymbol{\phi}) + \nu(\nabla \mathbf{u}, \nabla \boldsymbol{\phi}) + (\mathbf{u}\cdot\nabla\mathbf{u}, \boldsymbol{\phi}) = 0 \quad \forall \boldsymbol{\phi} \in \mathbf{J}_1, \ \ t > 0 \tag{2.6}$$

with $\mathbf{u}(0) = \mathbf{u}_0$.

From time to time, we make use of the following result to deal with the nonlinear term in our problem.

Lemma 2.1. *Let $\nabla\cdot\mathbf{v} = 0$ in Ω and $\boldsymbol{\phi}, \mathbf{w} \in \mathbf{H}^1(\Omega)$. Then,*

$$(\mathbf{v}\cdot\nabla\mathbf{w}, \boldsymbol{\phi}) + (\mathbf{v}\cdot\nabla\boldsymbol{\phi}, \mathbf{w}) = \int_{\partial\Omega} (\mathbf{n}\cdot\mathbf{v})\mathbf{w}\cdot\boldsymbol{\phi}\, ds.$$

In the remaining part of this article, we adopt the following notation: For any given function ϕ, we define

$$\hat{\phi} = e^{\alpha t}\phi.$$

III. A PRIORI ESTIMATES FOR THE EXACT SOLUTION

In this section, we derive some a priori bounds for the problem (1.2)–(1.4) which reflect exponential decay behavior in time. In the context of Oldroyd model, we refer to Pani et al. [14, 15] for similar analysis related to exponential decay property.

First of all, we state the main theorem of this section.

Theorem 3.1. *Let the assumptions (**A1**) and (**A2**) hold. Then, there exists a positive constant K depending on M, λ_1, α, κ, and ν such that for $0 \le \alpha < \frac{\nu\lambda_1}{2(1+\lambda_1\kappa)}$ the following estimate holds true:*

$$\|\mathbf{u}(t)\|_2^2 + \|\mathbf{u}_t(t)\|_2^2 + \|p(t)\|_{H^1/\mathbb{R}}^2 + \int_0^t e^{2\alpha s}\big(\|\mathbf{u}(s)\|_2^2 + \|\mathbf{u}_t(s)\|_2^2 + \|p(s)\|_{H^1/\mathbb{R}}^2\big)ds \le Ke^{-2\alpha t}, \quad t > 0.$$

The proof can be established using the following series of lemmas.

Lemma 3.1. *Let $0 \le \alpha < \frac{\nu\lambda_1}{2(1+\kappa\lambda_1)}$, and let the assumptions (**A1**)–(**A2**) hold. Then, the solution* **u** *of (2.6) satisfies*

$$\|\mathbf{u}(t)\|^2 + \kappa\|\nabla\mathbf{u}(t)\|^2 + \beta e^{-2\alpha t}\int_0^t e^{2\alpha s}\|\nabla\mathbf{u}(s)\|^2\,ds$$
$$\le e^{-2\alpha t}(\|\mathbf{u}_0\|^2 + \kappa\|\nabla\mathbf{u}_0\|^2) = M_0 e^{-2\alpha t}, \quad t > 0.$$

where $\beta = \nu - 2\alpha(\kappa + \lambda_1^{-1}) > 0$, and $M_0 = (1+\kappa)M^2$.

Proof. Setting $\hat{\mathbf{u}}(t) = e^{\alpha t}\mathbf{u}(t)$ for some $\alpha \ge 0$, we rewrite (2.6) as

$$(\hat{\mathbf{u}}_t, \boldsymbol{\phi}) - \alpha(\hat{\mathbf{u}}, \boldsymbol{\phi}) + \kappa(\nabla\hat{\mathbf{u}}_t, \nabla\boldsymbol{\phi}) - \kappa\alpha(\nabla\hat{\mathbf{u}}, \nabla\boldsymbol{\phi}) + \nu(\nabla\hat{\mathbf{u}}, \nabla\boldsymbol{\phi}) + e^{-\alpha t}(\hat{\mathbf{u}}\cdot\nabla\hat{\mathbf{u}}, \boldsymbol{\phi}) = 0 \quad \forall\boldsymbol{\phi} \in \mathbf{J}_1. \tag{3.1}$$

Choose $\boldsymbol{\phi} = \hat{\mathbf{u}}$ in (3.1). Using Lemma 2.1, $(\hat{\mathbf{u}}\cdot\nabla\hat{\mathbf{u}}, \hat{\mathbf{u}}) = 0$ and (2.3), we obtain

$$\frac{d}{dt}(\|\hat{\mathbf{u}}\|^2 + \kappa\|\nabla\hat{\mathbf{u}}\|^2) + 2\beta\|\nabla\hat{\mathbf{u}}\|^2 \le 0. \tag{3.2}$$

Integrate (3.2) from 0 to t with respect to time and use the assumption (**A2**) to complete the rest of the proof. ∎

Lemma 3.2. *Let $0 \le \alpha < \frac{\nu\lambda_1}{2(1+\lambda_1\kappa)}$ and let the assumptions (**A1**)–(**A2**) hold. Then, there exists a positive constant $K = K(\kappa, \nu, \lambda_1, \alpha, M)$ such that for all $t > 0$*

$$\|\nabla\mathbf{u}(t)\|^2 + \kappa\|\tilde{\Delta}\mathbf{u}(t)\|^2 + \beta e^{-2\alpha t}\int_0^t e^{2\alpha s}\|\tilde{\Delta}\mathbf{u}(s)\|^2\,ds \le K e^{-2\alpha t}$$

holds, where $\beta = \nu - 2\alpha(\kappa + \lambda_1^{-1}) > 0$.

Proof. Using the Stokes operator $\tilde{\Delta}$, we rewrite (3.1) as

$$(\hat{\mathbf{u}}_t, \boldsymbol{\phi}) - \alpha(\hat{\mathbf{u}}, \boldsymbol{\phi}) - \kappa(\tilde{\Delta}\hat{\mathbf{u}}_t, \boldsymbol{\phi}) + \kappa\alpha(\tilde{\Delta}\hat{\mathbf{u}}, \boldsymbol{\phi}) - \nu(\tilde{\Delta}\hat{\mathbf{u}}, \boldsymbol{\phi}) = -e^{-\alpha t}(\hat{\mathbf{u}}\cdot\nabla\hat{\mathbf{u}}, \boldsymbol{\phi}) \quad \forall\boldsymbol{\phi} \in \mathbf{J}_1. \tag{3.3}$$

With $\boldsymbol{\phi} = -\tilde{\Delta}\hat{\mathbf{u}}$ in (3.3), we note that

$$-(\hat{\mathbf{u}}_t, \tilde{\Delta}\hat{\mathbf{u}}) = \frac{1}{2}\frac{d}{dt}\|\nabla\hat{\mathbf{u}}\|^2,$$

and hence (3.3) becomes

$$\frac{d}{dt}(\|\nabla\hat{\mathbf{u}}\|^2 + \kappa\|\tilde{\Delta}\hat{\mathbf{u}}\|^2) + 2\nu\|\tilde{\Delta}\hat{\mathbf{u}}\|^2 - 2\alpha(\|\nabla\hat{\mathbf{u}}\|^2 + \kappa\|\tilde{\Delta}\hat{\mathbf{u}}\|^2) = 2e^{-\alpha t}(\hat{\mathbf{u}}\cdot\nabla\hat{\mathbf{u}}, \tilde{\Delta}\hat{\mathbf{u}}). \tag{3.4}$$

To estimate the term on the right hand side of (3.4), a use of Hölder's inequality yields

$$|I| = 2|e^{-\alpha t}(\hat{\mathbf{u}}\cdot\nabla\hat{\mathbf{u}}, \tilde{\Delta}\hat{\mathbf{u}})| \le e^{-\alpha t}\|\hat{\mathbf{u}}\|_{L^4}\|\nabla\hat{\mathbf{u}}\|_{L^4}\|\tilde{\Delta}\hat{\mathbf{u}}\|_{L^2}. \tag{3.5}$$

Using the Sobolev inequality for 3D, that is, when $d = 3$, (see [20], page no. 296) given by

$$\|\boldsymbol{\phi}\|_{L^4(\Omega)} \le C\|\boldsymbol{\phi}\|^{\frac{1}{4}}\|\nabla\boldsymbol{\phi}\|^{\frac{3}{4}}, \;\; \boldsymbol{\phi} \in \mathbf{H}_0^1(\Omega). \tag{3.6}$$

we arrive at

$$\begin{aligned}|I| &\le 2e^{-\alpha t}\|\hat{\mathbf{u}}\|_{L^4}\|\nabla\hat{\mathbf{u}}\|_{L^4}\|\tilde{\Delta}\hat{\mathbf{u}}\|,\\ &\le 2e^{-\alpha t}\left(\|\hat{\mathbf{u}}\|^{\frac{1}{4}}\|\nabla\hat{\mathbf{u}}\|^{\frac{3}{4}}\|\right)\left(\|\nabla\hat{\mathbf{u}}\|^{\frac{1}{4}}\|\tilde{\Delta}\hat{\mathbf{u}}\|^{\frac{3}{4}}\right)\|\|\tilde{\Delta}\hat{\mathbf{u}}\|,\\ &\le Ce^{-\alpha t}\|\hat{\mathbf{u}}\|^{\frac{1}{4}}\|\nabla\hat{\mathbf{u}}\|\|\tilde{\Delta}\hat{\mathbf{u}}\|^{\frac{7}{4}}.\end{aligned} \tag{3.7}$$

Applying Young's inequality $ab \le \frac{a^p}{p\epsilon^{p/q}} + \frac{\epsilon b^q}{q}, a, b \ge 0, \epsilon > 0$ with $p = 8$ and $q = \frac{8}{7}$, we obtain

$$|I| \le C\frac{\|\hat{\mathbf{u}}\|^2\|\nabla\hat{\mathbf{u}}\|^8}{8\epsilon^7} + \frac{7}{8}\epsilon\|\tilde{\Delta}\hat{\mathbf{u}}\|^2. \tag{3.8}$$

Choosing $\epsilon = \frac{4\nu}{7}$, we find that

$$|I| \le C\left(\frac{4\nu}{7}\right)^{-7}\frac{\|\hat{\mathbf{u}}\|^2\|\nabla\hat{\mathbf{u}}\|^8}{8} + \frac{\nu}{2}\|\tilde{\Delta}\hat{\mathbf{u}}\|^2. \tag{3.9}$$

Substitute (3.9) in (3.4) to arrive at

$$\frac{d}{dt}(\|\nabla\hat{\mathbf{u}}\|^2 + \kappa\|\tilde{\Delta}\hat{\mathbf{u}}\|^2) - 2\alpha(\|\nabla\hat{\mathbf{u}}\|^2 + \kappa\|\tilde{\Delta}\hat{\mathbf{u}}\|^2) + \nu\|\tilde{\Delta}\hat{\mathbf{u}}\|^2 \le C(\nu)e^{-4\alpha t}\|\hat{\mathbf{u}}\|^2\|\nabla\hat{\mathbf{u}}\|^8. \tag{3.10}$$

A use of (2.4) in (3.10) and integration with respect to time from 0 to t yield

$$\|\nabla\hat{\mathbf{u}}\|^2 + \kappa\|\tilde{\Delta}\hat{\mathbf{u}}\|^2 + \beta\int_0^t \|\tilde{\Delta}\hat{\mathbf{u}}\|^2 ds \le \|\nabla\mathbf{u}_0\|^2 + \kappa\|\tilde{\Delta}\mathbf{u}_0\|^2 + C(\nu)\int_0^t e^{-4\alpha s}\|\hat{\mathbf{u}}\|^2\|\nabla\hat{\mathbf{u}}\|^8 ds. \tag{3.11}$$

Using Lemma 3.1, we bound

$$\int_0^t e^{-4\alpha s}\|\hat{\mathbf{u}}\|^2\|\nabla\hat{\mathbf{u}}\|^8 ds \le K. \tag{3.12}$$

Substitute (3.12) in (3.11) to complete the rest of the proof. ∎

Lemma 3.3. *Let* $0 \le \alpha < \frac{\nu\lambda_1}{2(1+\lambda_1\kappa)}$ *and let the assumptions* (**A1**)*–*(**A2**) *hold. Then, there exists a positive constant* $K = K(\kappa, \nu, \lambda_1, \alpha, M)$ *such that for all* $t > 0$,

$$e^{-2\alpha t}\int_0^t e^{2\alpha s}(\|\mathbf{u}_t(s)\|^2 + \kappa\|\nabla\mathbf{u}_t(s)\|^2)ds + \|\nabla\mathbf{u}(t)\|^2 \le Ke^{-2\alpha t}.$$

Proof. Rewrite (2.6) as

$$(\mathbf{u}_t, \boldsymbol{\phi}) - \kappa(\tilde{\Delta}\mathbf{u}_t, \boldsymbol{\phi}) - \nu(\tilde{\Delta}\mathbf{u}, \boldsymbol{\phi}) + (\mathbf{u}.\nabla\mathbf{u}, \boldsymbol{\phi}) = 0 \quad \forall\boldsymbol{\phi} \in \mathbf{J}_1. \tag{3.13}$$

On multiplying (3.13) by $e^{\alpha t}$ and substituting $\boldsymbol{\phi} = e^{\alpha t}\mathbf{u}_t$, we arrive at

$$e^{2\alpha t}(\|\mathbf{u}_t\|^2 + \kappa\|\nabla\mathbf{u}_t\|^2) + \nu e^{2\alpha t}\frac{d}{dt}\|\nabla\mathbf{u}\|^2 = -e^{2\alpha t}(\mathbf{u}.\nabla\mathbf{u}, \mathbf{u}_t). \tag{3.14}$$

For the nonlinear term on the right hand side of (3.14), we use the generalized Hölder's inequality and Sobolev embedding theorem with (2.2) to obtain:

$$\begin{aligned}(\mathbf{u}.\nabla\mathbf{u}, \mathbf{w}) &\leq C\|\mathbf{u}\|_{\mathbf{L}^4}\ \|\nabla\mathbf{u}\|_{\mathbf{L}^4}\ \|\mathbf{w}\| \\ &\leq C\|\nabla\mathbf{u}\|\ \|\mathbf{u}\|_{\mathbf{H}^2}\ \|\mathbf{w}\| \\ &\leq C\|\nabla\mathbf{u}\|\ \|\tilde{\Delta}\mathbf{u}\|\ \|\mathbf{w}\|. \end{aligned} \tag{3.15}$$

Integration of (3.14) with respect to time from 0 to t along with use of (3.15) by replacing $\mathbf{w}$ by $\mathbf{u}_t$ and Young's inequality yields

$$\int_0^t e^{2\alpha s}(\|\mathbf{u}_t(s)\|^2 + \kappa\|\nabla\mathbf{u}_t(s)\|^2)ds + \nu e^{2\alpha t}\|\nabla\mathbf{u}\|^2 \leq C\left(\|\nabla\mathbf{u}(0)\|^2 + \int_0^t e^{2\alpha s}\|\nabla\mathbf{u}(s)\|^2 ds \right.$$
$$\left. + \int_0^t e^{2\alpha s}\|\nabla\mathbf{u}(s)\|^2\|\tilde{\Delta}\mathbf{u}(s)\|^2 ds\right).$$

Again, a use of a priori bounds for $\mathbf{u}$ obtained from the Lemmas 3.1 and 3.2, would provide us the desired result. ■

Lemma 3.4. *Let $0 \leq \alpha < \frac{\nu\lambda_1}{2(1+\lambda_1\kappa)}$ and let the assumptions (**A1**)–(**A2**) hold. Then, there exists a positive constant $K = K(\kappa, \nu, \lambda_1, \alpha, M)$ such that for all $t > 0$,*

$$\|\mathbf{u}_t(t)\|^2 + \kappa\|\nabla\mathbf{u}_t(t)\|^2 \leq Ke^{-2\alpha t}.$$

Proof. Substituting $\boldsymbol{\phi} = \mathbf{u}_t$ in (2.6), we obtain

$$\|\mathbf{u}_t\|^2 + \kappa\|\nabla\mathbf{u}_t\|^2 = -\nu(\nabla\mathbf{u}, \nabla\mathbf{u}_t) - (\mathbf{u}.\nabla\mathbf{u}, \mathbf{u}_t) = I_1 + I_2, \text{ say.} \tag{3.16}$$

To estimate $|I_1|$, we apply Cauchy-Schwarz's inequality and Young's inequality to arrive at

$$|I_1| \leq \frac{\nu}{2\epsilon}\|\nabla\mathbf{u}\|^2 + \frac{\epsilon}{2}\|\nabla\mathbf{u}_t\|^2.$$

Choose $\epsilon = \kappa$ to yield

$$|I_1| \leq \frac{\nu}{2\kappa}\|\nabla\mathbf{u}\|^2 + \frac{\kappa}{2}\|\nabla\mathbf{u}_t\|^2.$$

For I_2, apply (3.15) replacing $\mathbf{w}$ by $\mathbf{u}_t$ and use Young's inequality to obtain

$$|I_2| \leq C\|\nabla\mathbf{u}\|^2\|\tilde{\Delta}\mathbf{u}\|^2 + \frac{1}{2}\|\mathbf{u}_t\|^2.$$

Substitute bounds for $|I_1|$ and $|I_2|$ in (3.16) and use a priori estimates from Lemma 3.1 and 3.2 to complete the proof. ■

Lemma 3.5. *Let $0 \leq \alpha < \frac{\nu\lambda_1}{2(1+\lambda_1\kappa)}$ and let the assumptions (**A1**)–(**A2**) hold. Then, there exists a positive constant $K = K(\kappa, \nu, \lambda_1, \alpha, M)$ such that for all $t > 0$,*

$$\|\nabla\mathbf{u}_t(t)\|^2 + \frac{\kappa}{2}\|\tilde{\Delta}\mathbf{u}_t(t)\|^2 \leq Ke^{-2\alpha t}.$$

Proof. Setting $\boldsymbol{\phi} = -\tilde{\Delta}\mathbf{u}_t$ in (3.13), we obtain

$$\|\nabla \mathbf{u}_t\|^2 + \kappa\|\tilde{\Delta}\mathbf{u}_t\|^2 = -\nu(\tilde{\Delta}\mathbf{u}, \tilde{\Delta}\mathbf{u}_t) + (\mathbf{u}.\nabla\mathbf{u}, \tilde{\Delta}\mathbf{u}_t). \tag{3.17}$$

For the nonlinear term that is the last term on the right hand side of (3.17), we now use (3.15) replacing $\mathbf{w}$ by $\tilde{\Delta}\mathbf{u}_t$. Then, with the help of Cauchy–Schwarz's inequality and Young's inequality, we bound right hand side of (3.17) and use Lemmas 3.1 and 3.2 to complete the rest of the proof. ∎

Lemma 3.6. *Let* $0 \le \alpha < \frac{\nu\lambda_1}{2(1+\lambda_1\kappa)}$ *and let the assumptions* (**A1**)*-*(**A2**) *hold. Then, there exists a positive constant* $K = K(\kappa, \nu, \lambda_1, \alpha, M)$ *such that for all* $t > 0$*,*

$$e^{-2\alpha t}\int_0^t e^{2\alpha s}(\|\nabla \mathbf{u}_t(s)\|^2 + \kappa\|\tilde{\Delta}\mathbf{u}_t(s)\|^2)ds + \nu\|\tilde{\Delta}\mathbf{u}(t)\|^2 \le Ke^{-2\alpha t}.$$

Proof. Multiply (3.13) by $e^{\alpha t}$ and substitute $\boldsymbol{\phi} = -e^{\alpha t}\tilde{\Delta}\mathbf{u}_t$ to obtain

$$e^{2\alpha t}(\|\nabla \mathbf{u}_t\|^2 + \kappa\|\tilde{\Delta}\mathbf{u}_t\|^2) + \nu e^{2\alpha t}\frac{d}{dt}\|\tilde{\Delta}\mathbf{u}\|^2 = e^{2\alpha t}(\mathbf{u}.\nabla\mathbf{u}, \tilde{\Delta}\mathbf{u}_t). \tag{3.18}$$

Now for the nonlinear term, that is, the term on the right hand side of (3.18), we now use (3.15) replacing $\mathbf{w}$ by $\tilde{\Delta}\mathbf{u}_t$. Then, integrating with respect to time from 0 to t and using Young's inequality, we obtain

$$\int_0^t e^{2\alpha s}(\|\nabla \mathbf{u}_t\|^2 + \kappa\|\tilde{\Delta}\mathbf{u}_t\|^2)ds + \nu e^{2\alpha t}\|\tilde{\Delta}\mathbf{u}\|^2 \le C(\kappa)\left(\|\tilde{\Delta}\mathbf{u}(0)\|^2 + \int_0^t e^{2\alpha s}\|\tilde{\Delta}\mathbf{u}\|^2 ds \right.$$
$$\left. + \int_0^t e^{2\alpha s}\|\nabla\mathbf{u}\|^2\|\tilde{\Delta}\mathbf{u}\|^2 ds\right).$$

A use of Lemma 3.2 establishes the desired estimate and this completes the rest of the proof. ∎

Now, we derive the a priori bounds for the pressure p.

Lemma 3.7. *Let* $0 \le \alpha < \frac{\nu\lambda_1}{2(1+\lambda_1\kappa)}$ *and let the assumptions* (**A1**)*-*(**A2**) *hold. Then, there exists a positive constant* $K = K(\kappa, \nu, \lambda_1, \alpha, M)$ *such that for all* $t > 0$*, the following estimate holds true:*

$$\|p(t)\|^2_{L^2/\mathbb{R}} + \|p(t)\|^2_{H^1/\mathbb{R}} + e^{-2\alpha t}\int_0^t e^{2\alpha s}\|p(s)\|^2_{H^1/\mathbb{R}}ds \le Ke^{-2\alpha t}.$$

Proof. A use of Cauchy–Schwarz's inequality and Hölder's inequality in (2.5) yields

$$(p, \nabla.\boldsymbol{\phi}) \le C(\|\mathbf{u}_t\|\|\boldsymbol{\phi}\| + \kappa\|\nabla\mathbf{u}_t\|\|\nabla\boldsymbol{\phi}\| + \nu\|\nabla\mathbf{u}\|\|\nabla\boldsymbol{\phi}\| + \|\mathbf{u}\|_{L^4}\|\nabla\mathbf{u}\|\|\boldsymbol{\phi}\|_{L^4}). \tag{3.19}$$

Using Sobolev's embedding theorem (see, [20]), the Poincaré inequality, dividing by $\|\nabla\boldsymbol{\phi}\|$ and applying continuous inf-sup condition in (3.19), we obtain

$$\|p\|_{L^2/\mathbb{R}} \le C\frac{|(p, \nabla.\boldsymbol{\phi})|}{\|\nabla\boldsymbol{\phi}\|} \le C(\|\mathbf{u}_t\| + \kappa\|\nabla\mathbf{u}_t\| + \nu\|\nabla\mathbf{u}\| + \|\nabla\mathbf{u}\|^2). \tag{3.20}$$

An application of Lemmas 3.1 and 3.4 in (3.20) yields

$$\|p(t)\|_{L^2/\mathbb{R}} \le K(\kappa, \nu, \lambda_1, \alpha, M)e^{-\alpha t}. \tag{3.21}$$

Using the property of space $\mathbf{J}_1$ (see [20] page no 19, remark 1.9) in (2.6), we obtain

$$(\nabla p,\ \boldsymbol{\phi}) = (\mathbf{u}_t - \kappa\tilde{\Delta}\mathbf{u}_t - \nu\tilde{\Delta}\mathbf{u} + \mathbf{u}.\nabla\mathbf{u},\ \boldsymbol{\phi}) \quad \forall\boldsymbol{\phi} \in \mathbf{J}_1. \tag{3.22}$$

A use of Cauchy–Schwarz's inequality with the generalized Hölder's inequality in (3.22) yields

$$|(\nabla p,\ \boldsymbol{\phi})| \le C(\kappa, \nu)(\|\mathbf{u}_t\|\|\boldsymbol{\phi}\| + \|\tilde{\Delta}\mathbf{u}_t\|\|\boldsymbol{\phi}\| + \|\tilde{\Delta}\mathbf{u}\|\|\boldsymbol{\phi}\| + \|\mathbf{u}\|_{L^4}\|\nabla\mathbf{u}\|_{L^4}\|\boldsymbol{\phi}\|). \tag{3.23}$$

Applying the Sobolev embedding theorem (see [20]) in (3.23) with (2.2) and dividing by $\|\boldsymbol{\phi}\|$, we obtain

$$\|\nabla p\| \le C(\kappa, \nu)(\|\mathbf{u}_t\| + \kappa\|\tilde{\Delta}\mathbf{u}_t\| + \nu\|\tilde{\Delta}\mathbf{u}\| + \|\nabla\mathbf{u}\|\ \|\tilde{\Delta}\mathbf{u}\|). \tag{3.24}$$

A use of Lemmas 3.1, 3.2, 3.4 and 3.5 in (3.24) yields

$$\|p(t)\|_{H^1/\mathbb{R}} \le Ke^{-\alpha t}. \tag{3.25}$$

Taking square of both sides of (3.24), multiplying by $e^{2\alpha t}$ and integrating from 0 to t with respect to time, we obtain

$$\int_0^t e^{2\alpha s}\|\nabla p(s)\|^2 ds \le C(\kappa, \nu)\left(\int_0^t e^{2\alpha s}(\|\mathbf{u}_t(s)\|^2 + \|\tilde{\Delta}\mathbf{u}_t(s)\|^2)ds \right.$$
$$\left. + \int_0^t e^{2\alpha s}(\|\tilde{\Delta}\mathbf{u}(s)\|^2 + \|\nabla\mathbf{u}(s)\|^2\ \|\tilde{\Delta}\mathbf{u}\|^2)ds\right). \tag{3.26}$$

Using Lemmas 3.1, 3.2, 3.3, and 3.6, we arrive at

$$\int_0^t e^{2\alpha s}\|\nabla p(s)\|^2 ds \le K. \tag{3.27}$$

A use of (3.21), (3.25), and (3.27) would lead us to the desired result. ■

Proof of Theorem 3.1. Now the proof of Theorem 3.1 follows by combining Lemmas 3.1–3.7. ■

IV. THE SEMIDISCRETE PROBLEM

Let $h > 0$ be a discretization parameter. Further, let $\mathbf{H}_h$ and L_h, $0 < h < 1$ be finite dimensional subspaces of $\mathbf{H}_0^1$ and L^2, respectively, such that, there exist operators i_h and j_h satisfying the following approximation properties:

(B1). For each $\mathbf{w} \in \mathbf{J}_1 \cap \mathbf{H}^2$ and $q \in H^1/\mathbb{R}$, there exist approximations $i_h\mathbf{w} \in \mathbf{J}_h$ and $j_hq \in L_h$ such that

$$\|\mathbf{w} - i_h\mathbf{w}\| + h\|\nabla(\mathbf{w} - i_h\mathbf{w})\| \le K_0h^2\|\mathbf{w}\|_2, \quad \|q - j_hq\|_{L^2/\mathbb{R}} \le K_0h\|q\|_{H^1/\mathbb{R}}.$$

For defining the Galerkin approximations, for $\mathbf{v}, \mathbf{w}, \boldsymbol{\phi} \in \mathbf{H}_0^1$, set

$$a(\mathbf{v}, \boldsymbol{\phi}) = (\nabla \mathbf{v}, \nabla \boldsymbol{\phi})$$

and

$$b(\mathbf{v}, \mathbf{w}, \boldsymbol{\phi}) = \frac{1}{2}(\mathbf{v} \cdot \nabla \mathbf{w}, \boldsymbol{\phi}) - \frac{1}{2}(\mathbf{v} \cdot \nabla \boldsymbol{\phi}, \mathbf{w}).$$

When $\mathbf{v} \in \mathbf{J}_1$, $\mathbf{w}, \boldsymbol{\phi} \in \mathbf{H}_0^1$, using Lemma 2.1, we obtain

$$b(\mathbf{v}, \mathbf{w}, \boldsymbol{\phi}) = (\mathbf{v} \cdot \nabla \mathbf{w}, \boldsymbol{\phi}).$$

Note that the operator $b(\cdot, \cdot, \cdot)$ preserves the antisymmetric properties of the original nonlinear term, i.e.,

$$b(\mathbf{v}_h, \mathbf{w}_h, \mathbf{w}_h) = 0 \quad \forall \mathbf{v}_h, \mathbf{w}_h \in \mathbf{H}_h.$$

The discrete analogue of the weak formulation (2.5) is as follows:

Find $\mathbf{u}_h(t) \in \mathbf{H}_h$ and $p_h(t) \in L_h$ such that $\mathbf{u}_h(0) = \mathbf{u}_{0h}$ and for $t > 0$,

$$(\mathbf{u}_{ht}, \boldsymbol{\phi}_h) + \kappa a(\mathbf{u}_{ht}, \boldsymbol{\phi}_h) + \nu a(\mathbf{u}_h, \boldsymbol{\phi}_h) + b(\mathbf{u}_h, \mathbf{u}_h, \boldsymbol{\phi}_h) - (p_h, \nabla \cdot \boldsymbol{\phi}_h) = 0 \quad \forall \boldsymbol{\phi}_h \in \mathbf{H}_h,$$
$$(\nabla \cdot \mathbf{u}_h, \chi_h) = 0 \quad \forall \chi_h \in L_h, \tag{4.1}$$

where $\mathbf{u}_{0h} \in \mathbf{H}_h$ is a suitable approximation of $\mathbf{u}_0 \in \mathbf{J}_1$.

To consider a suitable approximation of $\mathbf{J}_1$, we introduce the discrete incompressibility condition in $\mathbf{H}_h$ and call the resulting subspace as $\mathbf{J}_h$. Thus, $\mathbf{J}_h$ is defined as

$$\mathbf{J}_h = \{\mathbf{v}_h \in \mathbf{H}_h : (\chi_h, \nabla \cdot \mathbf{v}_h) = 0 \quad \forall \chi_h \in L_h\}.$$

Note that, the space $\mathbf{J}_h$ is not a subspace of $\mathbf{J}_1$. We now define the finite dimensional problem:

Find $\mathbf{u}_h(t) \in \mathbf{J}_h$ such that $\mathbf{u}_h(0) = \mathbf{u}_{0h}$ and for $t > 0$,

$$(\mathbf{u}_{ht}, \boldsymbol{\phi}_h) + \kappa a(\mathbf{u}_{ht}, \boldsymbol{\phi}_h) + \nu a(\mathbf{u}_h, \boldsymbol{\phi}_h) = -b(\mathbf{u}_h, \mathbf{u}_h, \boldsymbol{\phi}_h) \quad \forall \boldsymbol{\phi}_h \in \mathbf{J}_h. \tag{4.2}$$

As $\mathbf{J}_h$ is finite dimensional, the problem (4.2) leads to a system of nonlinear partial differential equations. A use of Picard's theorem yields existence of a unique local solution in an interval $[0, t^*)$, for some $t^* > 0$. For continuation of solution beyond t^*, we need to establish an $L^\infty(\mathbf{L}^2)$ bound for the approximate solution $\mathbf{u}_h$. Setting $\boldsymbol{\phi}_h = \mathbf{u}_h$ in (4.2), we obtain

$$\frac{d}{dt}(\|\mathbf{u}_h\|^2 + \kappa\|\nabla \mathbf{u}_h\|^2) + 2\nu\|\nabla \mathbf{u}_h\|^2 = 0.$$

On integration with respect to the temporal variable t, we find that

$$\|\mathbf{u}_h(t)\|^2 + \kappa\|\nabla \mathbf{u}_h(t)\|^2 \leq (\|\mathbf{u}_{0h}\|^2 + \kappa\|\nabla \mathbf{u}_{0h}\|^2) \leq C \quad \forall t \geq 0,$$

provided $\|\nabla \mathbf{u}_{0h}\| \leq C\|\nabla \mathbf{u}_0\|$. This is indeed true, which we shall see later on. This shows the global existence of a unique Galerkin approximation $\mathbf{u}_h$ for all $t > 0$.

Once, we compute $\mathbf{u}_h(t) \in J_h$, the approximation $p_h(t) \in L_h$ to the pressure $p(t)$ can be found out by solving the following system

$$(p_h, \nabla \cdot \boldsymbol{\phi}_h) = (\mathbf{u}_{ht}, \boldsymbol{\phi}_h) + \kappa a(\mathbf{u}_{ht}, \boldsymbol{\phi}_h) + \nu a(\mathbf{u}_h, \boldsymbol{\phi}_h) + b(\mathbf{u}_h, \mathbf{u}_h, \boldsymbol{\phi}_h) \quad \forall \boldsymbol{\phi}_h \in \mathbf{H}_h. \tag{4.3}$$

For the solvability of the above system (4.3), we note that the right hand side defines a linear functional ℓ on $\mathbf{H}_h$, that is, $\boldsymbol{\phi}_h \mapsto \ell(\phi_h)$. By construction $\ell(\boldsymbol{\phi}_h) = 0$, for all $\boldsymbol{\phi}_h \in \mathbf{J}_h$. It is now easy to check that this condition implies existence of $p_h \in L_h$, see [21]. Uniqueness is obtained on the quotient space L_h/N_h, where

$$N_h = \{q_h \in L_h : (q_h, \nabla \cdot \boldsymbol{\phi}_h) = 0, \forall \boldsymbol{\phi}_h \in \mathbf{H}_h\}.$$

The norm on L_h/N_h is given by

$$\|q_h\|_{L^2/N_h} = \inf_{\chi_h \in N_h} \|q_h + \chi_h\|.$$

Furthermore, the pair $(\mathbf{H}_h, L_h/N_h)$ satisfies a uniform inf-sup condition:

(**B2**). For every $q_h \in L_h$, there exist a non-trivial function $\boldsymbol{\phi}_h \in \mathbf{H_h}$ and a positive constant K_1, independent of h, such that,

$$|(q_h, \nabla \cdot \boldsymbol{\phi}_h)| \geq K_1 \|\nabla \boldsymbol{\phi}_h\| \|q_h\|_{L^2/N_h}.$$

As a consequence of conditions (**B1**)–(**B2**), we have the following properties of the L^2 projection $P_h : \mathbf{L}^2 \to \mathbf{J}_h$. For $\boldsymbol{\phi} \in \mathbf{J_1}$, we note that, see [10,21],

$$\|\boldsymbol{\phi} - P_h\boldsymbol{\phi}\| + h\|\nabla P_h\boldsymbol{\phi}\| \leq Ch\|\nabla\boldsymbol{\phi}\|, \tag{4.4}$$

and for $\boldsymbol{\phi} \in \mathbf{J}_1 \cap \mathbf{H}^2$,

$$\|\boldsymbol{\phi} - P_h\boldsymbol{\phi}\| + h\|\nabla(\boldsymbol{\phi} - P_h\boldsymbol{\phi})\| \leq Ch^2\|\tilde{\Delta}\boldsymbol{\phi}\|. \tag{4.5}$$

We may define the discrete operator $\Delta_h : \mathbf{H}_h \to \mathbf{H}_h$ through the bilinear form $a(\cdot,\cdot)$ as

$$a(\mathbf{v}_h, \boldsymbol{\phi}_h) = (-\Delta_h \mathbf{v}_h, \boldsymbol{\phi}_h) \quad \forall \mathbf{v}_h, \boldsymbol{\phi}_h \in \mathbf{H}_h. \tag{4.6}$$

Set the discrete analogue of the Stokes operator $\tilde{\Delta} = P\Delta$ as $\tilde{\Delta}_h = P_h\Delta_h$. Examples of subspaces $\mathbf{H_h}$ and L_h satisfying assumptions (**B1**) and (**B2**) can be found in [10] and [22]. In the context of non conforming analysis, we would like to refer [10].

Next, we obtain some a priori bounds for the discrete solution $\mathbf{u}_h$ which will be helpful for our subsequent use. Using the definition of the discrete Stokes operator $\tilde{\Delta}_h$ in (4.6), we proceed along the lines of proof of Theorem 3.1 to derive the following bounds for $\mathbf{u}_h$.

Lemma 4.1. *For all $t > 0$, the semidiscrete Galerkin approximation $\mathbf{u}_h$ for the velocity satisfies*

$$\|\tilde{\Delta}_h\mathbf{u}_h(t)\|^2 + \|\mathbf{u}_{ht}(t)\|^2 + e^{-2\alpha t}\int_0^t e^{2\alpha s}\|\nabla\mathbf{u}_{ht}(s)\|^2\,ds \leq Ke^{-2\alpha t}.$$

Finally, we state below the main results of this article, which are related to the optimal error estimates of the velocity and the pressure, the proofs of which are established in Sections V and VI, respectively.

Theorem 4.1. *Let the assumptions (**A1**)–(**A2**) and (**B1**)–(**B2**) be satisfied. Further, let the discrete initial velocity $\mathbf{u}_{0h} = P_h\mathbf{u}_0$. Then, there exists a positive constant K which depends on κ, ν, λ_1, α, and M, such that, for all $t > 0$ and for $0 \le \alpha < \frac{\nu\lambda_1}{2(1+\lambda_1\kappa)}$, the following estimate holds true:*

$$\|(\mathbf{u}-\mathbf{u}_h)(t)\| + h\|\nabla(\mathbf{u}-\mathbf{u}_h)(t)\| \le Kh^2e^{-\alpha t}.$$

Theorem 4.2. *Under the hypotheses of Theorem 4.1, there exists a positive constant K depending on κ, ν, λ_1, α, and M, such that, for all $t > 0$, the following holds true:*

$$\|(p-p_h)(t)\|_{L^2/N_h} \le Khe^{-\alpha t}.$$

V. ERROR ESTIMATES FOR THE VELOCITY

In this section, we derive optimal error estimates of the velocity. As $\mathbf{J}_h$ is not a subspace of $\mathbf{J}_1$, the weak solution $\mathbf{u}$ satisfies

$$(\mathbf{u}_t,\boldsymbol{\phi}_h) + \kappa a(\mathbf{u}_t,\boldsymbol{\phi}_h) + \nu a(\mathbf{u},\boldsymbol{\phi}_h) = -b(\mathbf{u},\mathbf{u},\boldsymbol{\phi}_h) + (p,\nabla\cdot\boldsymbol{\phi}_h) \quad \forall\boldsymbol{\phi}_h \in \mathbf{J}_h. \tag{5.1}$$

Set $\mathbf{e} = \mathbf{u}-\mathbf{u}_h$. Then, from (5.1) and (4.2), we obtain

$$(\mathbf{e}_t,\boldsymbol{\phi}_h) + \kappa a(\mathbf{e}_t,\boldsymbol{\phi}_h) + \nu a(\mathbf{e},\boldsymbol{\phi}_h) = \Lambda(\boldsymbol{\phi}_h) + (p,\nabla\cdot\boldsymbol{\phi}_h), \tag{5.2}$$

where $\Lambda(\boldsymbol{\phi}_h) = -b(\mathbf{u},\mathbf{u},\boldsymbol{\phi}_h) + b(\mathbf{u}_h,\mathbf{u}_h,\boldsymbol{\phi}_h)$. Below, we derive an optimal error estimate of $||\nabla\mathbf{e}(t)||$, for $t > 0$.

Lemma 5.1. *Let assumptions (**A1**)–(**A2**) and (**B1**)–(**B2**) be satisfied. With $\mathbf{u}_{0h} = P_h\mathbf{u}_0$, then, there exists a positive constant K depending on λ_1, κ, ν, α and M, such that, for all $t > 0$ and for $0 \le \alpha < \frac{\nu\lambda_1}{2(1+\lambda_1\kappa)}$, the following estimate holds true:*

$$\|(\mathbf{u}-\mathbf{u}_h)(t)\|^2 + \kappa\|\nabla(\mathbf{u}-\mathbf{u}_h)(t)\|^2 \le Kh^2e^{-2\alpha t}.$$

Proof. Choose $\boldsymbol{\phi}_h = e^{\alpha t}P_h\hat{\mathbf{e}} = \hat{\mathbf{e}} + (P_h\hat{\mathbf{u}} - \hat{\mathbf{u}})$ in (5.2) to rewrite it as:

$$\begin{aligned}(e^{\alpha t}\mathbf{e}_t,\hat{\mathbf{e}}) + \kappa a(e^{\alpha t}\mathbf{e}_t,\hat{\mathbf{e}}) + \nu a(\hat{\mathbf{e}},\hat{\mathbf{e}}) &= e^{\alpha t}\Lambda(P_h\hat{\mathbf{e}}) + (\hat{p},\nabla\cdot P_h\hat{\mathbf{e}})\\ &\quad + (e^{\alpha t}\mathbf{e}_t,\hat{\mathbf{u}}-P_h\hat{\mathbf{u}}) + \kappa\, a(e^{\alpha t}\mathbf{e}_t,\hat{\mathbf{u}}-P_h\hat{\mathbf{u}}) + \nu a(\hat{\mathbf{e}},\hat{\mathbf{u}}-P_h\hat{\mathbf{u}}).\end{aligned} \tag{5.3}$$

Note that

$$(e^{\alpha t}\mathbf{e}_t,\hat{\mathbf{e}}) + \kappa\, a(e^{\alpha t}\mathbf{e}_t,\hat{\mathbf{e}}) = \frac{1}{2}\frac{d}{dt}(\|\hat{\mathbf{e}}\|^2 + \kappa\|\nabla\hat{\mathbf{e}}\|^2) - \alpha(\|\hat{\mathbf{e}}\|^2 + \kappa\|\nabla\hat{\mathbf{e}}\|^2), \tag{5.4}$$

and

$$\begin{aligned}(e^{\alpha t}\mathbf{e}_t,\hat{\mathbf{u}}-P_h\hat{\mathbf{u}}) &= (\hat{\mathbf{e}}_t,\hat{\mathbf{u}}-P_h\hat{\mathbf{u}})) - \alpha(\hat{\mathbf{e}},\hat{\mathbf{u}}-P_h\hat{\mathbf{u}})\\ &= \frac{d}{dt}(\hat{\mathbf{e}},\hat{\mathbf{u}}-P_h\hat{\mathbf{u}}) - (\hat{\mathbf{e}},\hat{\mathbf{u}}_t - P_h\hat{\mathbf{u}}_t) - \alpha(\hat{\mathbf{e}},\hat{\mathbf{u}}-P_h\hat{\mathbf{u}}).\end{aligned} \tag{5.5}$$

Using (2.3), (5.4) and (5.5) in (5.3), we arrive at

$$\frac{d}{dt}(\|\hat{\mathbf{e}}\|^2+\kappa\|\nabla\hat{\mathbf{e}}\|^2)+(2\nu-2\alpha(\kappa+\lambda_1{}^{-1}))\|\nabla\hat{\mathbf{e}}\|^2\le 2e^{\alpha t}\Lambda(P_h\hat{\mathbf{e}})+2(\hat{p},\nabla\cdot P_h\hat{\mathbf{e}})$$
$$+2\frac{d}{dt}((\hat{\mathbf{e}},\hat{\mathbf{u}}-P_h\hat{\mathbf{u}})+\kappa\, a(\hat{\mathbf{e}},\hat{\mathbf{u}}-P_h\hat{\mathbf{u}}))-2((\hat{\mathbf{e}},\hat{\mathbf{u}}_t-P_h\hat{\mathbf{u}}_t)+\kappa\, a(\hat{\mathbf{e}},\hat{\mathbf{u}}_t-P_h\hat{\mathbf{u}}_t))$$
$$-2\alpha\big((\hat{\mathbf{e}},\hat{\mathbf{u}}-P_h\hat{\mathbf{u}})+\kappa\, a(\hat{\mathbf{e}},\hat{\mathbf{u}}-P_h\hat{\mathbf{u}})\big)+2\nu a(\hat{\mathbf{e}},\hat{\mathbf{u}}-P_h\hat{\mathbf{u}}). \tag{5.6}$$

Using Cauchy–Schwarz's inequality, Poincaré's inequality and Young's inequality, we estimate the last two terms on the right-hand side of (5.6) by

$$|2\alpha((\hat{\mathbf{e}},\hat{\mathbf{u}}-P_h\hat{\mathbf{u}})+\kappa\, a(\hat{\mathbf{e}},\hat{\mathbf{u}}-P_h\hat{\mathbf{u}})\big)+2\nu a(\hat{\mathbf{e}},\hat{\mathbf{u}}-P_h\hat{\mathbf{u}})|$$
$$\le C(\alpha,\kappa,\lambda_1,\nu,\epsilon)\|\nabla(\hat{\mathbf{u}}-P_h\hat{\mathbf{u}})\|^2+\frac{\epsilon}{2}\|\nabla\hat{\mathbf{e}}\|^2. \tag{5.7}$$

Similarly, using Cauchy–Schwarz's inequality, Poincaré's inequality and Young's inequality, we can bound

$$2|(\hat{\mathbf{e}},\hat{\mathbf{u}}_t-P_h\hat{\mathbf{u}}_t)+\kappa\, a(\hat{\mathbf{e}},\hat{\mathbf{u}}_t-P_h\hat{\mathbf{u}}_t)|\le C(\kappa,\epsilon)\|\nabla(\hat{\mathbf{u}}_t-P_h\hat{\mathbf{u}}_t)\|^2+\frac{\epsilon}{2}\|\nabla\hat{\mathbf{e}}\|^2. \tag{5.8}$$

For the second term on the right-hand side of (5.6), we use Cauchy–Schwarz's inequality, (4.4) and Young's inequality to obtain

$$2|(\hat{p},\nabla\cdot P_h\hat{\mathbf{e}})|\le 2\|\hat{p}-j_h\hat{p}\|\|\nabla\cdot P_h\hat{\mathbf{e}}\|\le C\|\hat{p}-j_h\hat{p}\|\|\nabla P_h\hat{\mathbf{e}}\|$$
$$\le C(\epsilon)\|\hat{p}-j_h\hat{p}\|^2+\frac{\epsilon}{2}\|\nabla\hat{\mathbf{e}}\|^2. \tag{5.9}$$

To estimate the first term on the right-hand side of (5.6), we rewrite it as

$$2e^{\alpha t}\Lambda(P_h\hat{\mathbf{e}})=2e^{-\alpha t}(b(\hat{\mathbf{e}},\hat{\mathbf{e}},P_h\hat{\mathbf{e}})-b(\hat{\mathbf{e}},\hat{\mathbf{u}},P_h\hat{\mathbf{e}})-b(\hat{\mathbf{u}},\hat{\mathbf{e}},P_h\hat{\mathbf{e}})).$$

Using the generalized Hölder's inequality, Agmon's inequality (see, [23] which is valid for 3D):

$$\|\mathbf{v}\|_{L^\infty}\le C\|\nabla\mathbf{v}\|\ \|\tilde{\Delta}\mathbf{v}\|,\quad \mathbf{v}\in\mathbf{H}^2\cap\mathbf{J}_1, \tag{5.10}$$

Young's inequality, the Sobolev embedding theorem, (2.2) and (4.4), we arrive at

$$2e^{-\alpha t}(|b(\hat{\mathbf{u}},\hat{\mathbf{e}},P_h\hat{\mathbf{e}})|+|b(\hat{\mathbf{e}},\hat{\mathbf{u}},P_h\hat{\mathbf{e}})|)\le 2e^{-\alpha t}(\|\hat{\mathbf{u}}\|_{L^\infty}\|\nabla\hat{\mathbf{e}}\|\|P_h\hat{\mathbf{e}}\|+\|\hat{\mathbf{e}}\|_{L^4}\|\nabla\hat{\mathbf{u}}\|_{L^4}\|P_h\hat{\mathbf{e}}\|\big)$$
$$\le 2e^{-\alpha t}\left(\|\nabla\hat{\mathbf{u}}\|^{\frac12}\|\tilde{\Delta}\hat{\mathbf{u}}\|^{\frac12}\|\nabla\hat{\mathbf{e}}\|\|P_h\hat{\mathbf{e}}\|+\|\nabla\hat{\mathbf{e}}\|\|\tilde{\Delta}\hat{\mathbf{u}}\|\|\hat{\mathbf{e}}\|\right)$$
$$\le 2e^{-\alpha t}\left(\|\nabla\hat{\mathbf{u}}\|^{\frac12}\|\tilde{\Delta}\hat{\mathbf{u}}\|^{\frac12}+\|\tilde{\Delta}\hat{\mathbf{u}}\|\right)\|\hat{\mathbf{e}}\|\|\nabla\hat{\mathbf{e}}\|$$
$$\le Ce^{-2\alpha t}(\|\nabla\hat{\mathbf{u}}\|\|\tilde{\Delta}\hat{\mathbf{u}}\|+\|\tilde{\Delta}\hat{\mathbf{u}}\|^2)\|\hat{\mathbf{e}}\|^2+\frac{\epsilon}{2}\|\nabla\hat{\mathbf{e}}\|^2. \tag{5.11}$$

Moreover, rewrite

$$b(\hat{\mathbf{e}},\hat{\mathbf{e}},P_h\hat{\mathbf{e}})=-b(\hat{\mathbf{e}},\hat{\mathbf{e}},\hat{\mathbf{u}}-P_h\hat{\mathbf{u}})+b(\hat{\mathbf{e}},\hat{\mathbf{e}},\hat{\mathbf{e}}). \tag{5.12}$$

As the last term on the right hand side of (5.12) vanishes because of the antisymmetric property of the trilinear form, we use Lemma 2.1, Hölder's inequality, the Sobolev embedding theorem, Young's inequality, Lemmas 3.1 and, 4.1 in (5.12) to obtain

$$\begin{aligned} |b(\hat{\mathbf{e}}, \hat{\mathbf{e}}, P_h\hat{\mathbf{e}})| &\leq Ce^{-\alpha t}\|\hat{\mathbf{e}}\|_{L^4}\|\nabla\hat{\mathbf{e}}\|\|\hat{\mathbf{u}} - P_h\hat{\mathbf{u}}\|_{L^4} \\ &\leq Ce^{-\alpha t}\|\nabla\hat{\mathbf{e}}\|\|\nabla\hat{\mathbf{e}}\|\|\nabla(\hat{\mathbf{u}} - P_h\hat{\mathbf{u}})\| \\ &\leq C(\|\nabla\hat{\mathbf{u}}\| + \|\nabla\hat{\mathbf{u}}_h\|)\|\nabla\hat{\mathbf{e}}\|\|\nabla(\hat{\mathbf{u}} - P_h\hat{\mathbf{u}})\| \\ &\leq C(\epsilon)\|\nabla(\hat{\mathbf{u}} - P_h\hat{\mathbf{u}})\|^2 + \frac{\epsilon}{2}\|\nabla\hat{\mathbf{e}}\|^2. \end{aligned} \tag{5.13}$$

Integrating (5.6) with respect to time from 0 to t, use bounds (5.7)–(5.13) with $\epsilon = \frac{2\nu}{5}$, to arrive at

$$\begin{aligned} &\|\hat{\mathbf{e}}(t)\|^2 + \kappa\|\nabla\hat{\mathbf{e}}(t)\|^2 + \beta\int_0^t \|\nabla\hat{\mathbf{e}}\|^2 ds \leq C(\|\mathbf{e}(0)\|^2 + \|\nabla\mathbf{e}(0)\|^2) \\ &\quad + C(\alpha,\kappa,\nu,\lambda_1,M)\left(\|\nabla(\hat{\mathbf{u}} - P_h\hat{\mathbf{u}})\|^2 + \int_0^t (\|\nabla(\hat{\mathbf{u}} - P_h\hat{\mathbf{u}})\|^2 + \|\nabla(\hat{\mathbf{u}}_t - P_h\hat{\mathbf{u}}_t)\|^2\right. \\ &\quad \left. + \|\hat{p} - j_h\hat{p}\|^2)ds\right) + C\int_0^t (\|\nabla\mathbf{u}\|\|\tilde{\Delta}\mathbf{u}\| + \|\tilde{\Delta}\mathbf{u}\|^2)\|\hat{\mathbf{e}}\|^2 ds. \end{aligned} \tag{5.14}$$

Using (4.5) and (**B1**) in (5.14), we find that

$$\begin{aligned} &\|\hat{\mathbf{e}}(t)\|^2 + \kappa||\nabla\hat{\mathbf{e}}(t)\|^2 + \beta\int_0^t \|\nabla\hat{\mathbf{e}}\|^2 ds \\ &\leq Ch^2\left(\|\mathbf{u}_0\|_2^2 + \|\hat{\mathbf{u}}\|_2^2 + \int_0^t \left(\|\hat{\mathbf{u}}\|_2^2 + \|\hat{\mathbf{u}}_t\|_2^2 + \|\hat{p}(t)\|^2_{H^1/\mathbb{R}}\right)ds\right) \\ &\quad + C\int_0^t (\|\nabla\mathbf{u}\|\|\tilde{\Delta}\mathbf{u}\| + \|\tilde{\Delta}\mathbf{u}\|^2)(\|\hat{\mathbf{e}}||^2 + \kappa\|\nabla\hat{\mathbf{e}}\|^2)ds. \end{aligned} \tag{5.15}$$

Use a priori bounds for $\mathbf{u}$, $\mathbf{u}_t$, and p (Theorem 3.1) to bound the first term on the right-hand side of (5.15) and then apply the Gronwall's lemma to obtain

$$\|\hat{\mathbf{e}}(t)\|^2 + \kappa\|\nabla\hat{\mathbf{e}}(t)\|^2 + \beta\int_0^t \|\nabla\hat{\mathbf{e}}\|^2 ds \leq C(\nu,\kappa,\alpha,\lambda_1,M)h^2\exp\left(\int_0^t (\|\tilde{\Delta}\mathbf{u}\|^2 + \|\nabla\mathbf{u}\|\|\tilde{\Delta}\mathbf{u}\|)\, ds\right).$$

A use of a priori bounds from Lemma 3.2 yields

$$\int_0^t (\|\nabla\mathbf{u}\|\ \|\tilde{\Delta}\mathbf{u}\| + \|\tilde{\Delta}\mathbf{u}\|^2)ds \leq C(M,\kappa,\lambda_1,\nu,\alpha)(1 - e^{-2\alpha t}) \leq C(M,\kappa,\lambda_1,\nu,\alpha) < \infty,$$

and hence, it completes the rest of the proof. ■

Note that, Theorem 5.1 provides a suboptimal error estimates for the velocity in $L^\infty(\mathbf{L}^2)$-norm. Therefore, in the remaining part of this section we derive an optimal error estimate for the velocity in $L^\infty(\mathbf{L}^2)$-norm. We shall achieve this by comparing our solutions with appropriate intermediate solutions and then making use of triangle inequality.

To dissociate the nonlinearity, we first introduce an intermediate solution $\mathbf{v}_h$, which is a finite element Galerkin approximation to a linearized Kelvin–Voigt equation, satisfying

$$(\mathbf{v}_{ht}, \boldsymbol{\phi}_h) + \kappa\, a(\mathbf{v}_{ht}, \boldsymbol{\phi}_h) + \nu a(\mathbf{v}_h, \boldsymbol{\phi}_h) = -b(\mathbf{u}, \mathbf{u}, \boldsymbol{\phi}_h) \quad \forall \boldsymbol{\phi}_h \in \mathbf{J}_h, \tag{5.16}$$

with $\mathbf{v}_h(0) = P_h \mathbf{u}_0$.

Now, we split $\mathbf{e}$ as

$$\mathbf{e} := \mathbf{u} - \mathbf{u}_h = (\mathbf{u} - \mathbf{v}_h) + (\mathbf{v}_h - \mathbf{u}_h) = \boldsymbol{\xi} + \boldsymbol{\eta}.$$

Here, $\boldsymbol{\xi}$ denotes the error due to the approximation using a linearized Kelvin–Voigt equation (5.16), whereas $\boldsymbol{\eta}$ represents the error due to the nonlinearity in the equation.

Subtracting (5.16) from (5.1), the equation in $\boldsymbol{\xi}$ can be written as

$$(\boldsymbol{\xi}_t, \boldsymbol{\phi}_h) + \kappa\, a(\boldsymbol{\xi}_t, \boldsymbol{\phi}_h) + \nu a(\boldsymbol{\xi}, \boldsymbol{\phi}_h) = (p, \nabla \cdot \boldsymbol{\phi}_h) \quad \forall \boldsymbol{\phi}_h \in \mathbf{J}_h. \tag{5.17}$$

For optimal error estimates of $\boldsymbol{\xi}$ in $L^\infty(L^2)$ and $L^\infty(H^1)$-norms, we again introduce the following auxiliary projection V_h such that $V_h\mathbf{u} : [0, \infty) \to J_h$ satisfying

$$\kappa a(\mathbf{u}_t - V_h\mathbf{u}_t, \boldsymbol{\phi}_h) + \nu a(\mathbf{u} - V_h\mathbf{u}, \boldsymbol{\phi}_h) = (p, \nabla \cdot \boldsymbol{\phi}_h) \quad \forall \boldsymbol{\phi}_h \in \mathbf{J}_h, \tag{5.18}$$

where $V_h\mathbf{u}(0) = P_h\mathbf{u}_0$.

With $V_h\mathbf{u}$ defined as above, we now split $\boldsymbol{\xi}$ as

$$\boldsymbol{\xi} := (\mathbf{u} - V_h\mathbf{u}) + (V_h\mathbf{u} - \mathbf{v}_h) = \boldsymbol{\zeta} + \boldsymbol{\rho}.$$

To obtain estimates for $\mathbf{e}$, first of all, we derive various estimates of $\boldsymbol{\zeta}$ in Lemmas 5.2, 5.3, 5.4, and 5.5. Then, we proceed to estimate $\|\boldsymbol{\rho}\|$ and $\|\nabla\boldsymbol{\rho}\|$ in Lemma 5.6. Combining these results, we obtain estimates for $\boldsymbol{\xi}$ in $L^\infty(\mathbf{L}^2)$ and $L^\infty(\mathbf{H}_0^1)$-norms in Lemma 5.7. Finally, we derive an estimate for $\boldsymbol{\eta}$ to complete the proof of Theorem 4.1.

Lemma 5.2. *Assume that (**A1**)–(**A2**) and (**B1**)–(**B2**) are satisfied. Then, there exists a positive constant $K = K(\nu, \lambda_1, \alpha, \kappa, M)$ such that for $0 \le \alpha < \frac{\nu\lambda_1}{2(1+\kappa\lambda_1)}$, the following estimate holds true:*

$$\|\nabla(\mathbf{u} - V_h\mathbf{u})(t)\|^2 + e^{-2\alpha t}\int_0^t e^{2\alpha s}\|\nabla(\mathbf{u} - V_h\mathbf{u})(s)\|^2 ds \le Kh^2 e^{-2\alpha t}.$$

Proof. On multiplying (5.18) by $e^{\alpha t}$ with $\boldsymbol{\zeta} = \mathbf{u} - V_h\mathbf{u}$, we find that

$$\kappa a(e^{\alpha t}\boldsymbol{\zeta}_t, \boldsymbol{\phi}_h) + \nu a(\hat{\boldsymbol{\zeta}}, \boldsymbol{\phi}_h) = (\hat{p}, \nabla \cdot \boldsymbol{\phi}_h) \forall \boldsymbol{\phi}_h \in \mathbf{J}_h. \tag{5.19}$$

Using $e^{\alpha t}\boldsymbol{\zeta}_t = \hat{\boldsymbol{\zeta}}_t - \alpha\hat{\boldsymbol{\zeta}}$ and choosing $\boldsymbol{\phi}_h = P_h\hat{\boldsymbol{\zeta}} = \hat{\boldsymbol{\zeta}} + (P_h\hat{\mathbf{u}} - \hat{\mathbf{u}})$ in (5.19), we arrive at

$$\kappa\frac{d}{dt}\|\nabla\hat{\boldsymbol{\zeta}}\|^2 + 2(\nu - \kappa\alpha)\|\nabla\hat{\boldsymbol{\zeta}}\|^2 = 2\kappa\frac{d}{dt}a(\hat{\boldsymbol{\zeta}}, \hat{\mathbf{u}} - P_h\hat{\mathbf{u}}) - 2\kappa a\left(\hat{\boldsymbol{\zeta}}, \frac{d}{dt}(\hat{\mathbf{u}} - P_h\hat{\mathbf{u}})\right)$$
$$+ 2(\nu - \kappa\alpha)a(\hat{\boldsymbol{\zeta}}, \hat{\mathbf{u}} - P_h\hat{\mathbf{u}}) + 2(\hat{p} - j_h\hat{p}, \nabla \cdot P_h\hat{\boldsymbol{\zeta}}). \tag{5.20}$$

Integrate (5.20) with respect to time from 0 to t and apply (4.4) along with Young's inequality to obtain

$$\kappa\|\nabla\hat{\boldsymbol{\zeta}}\|^2 + (\nu - \kappa\alpha)\int_0^t \|\nabla\hat{\boldsymbol{\zeta}}\|^2 ds \le C(\nu,\alpha,\kappa)\left(e^{2\alpha t}\|\nabla(\mathbf{u} - P_h\mathbf{u})\|^2 + \|\nabla(\mathbf{u}_0 - P_h\mathbf{u}_0)\|^2\right.$$
$$\left. + \int_0^t e^{2\alpha s}(\|\nabla(\mathbf{u}_t - P_h\mathbf{u}_t)\|^2 ds + \|\nabla(\mathbf{u} - P_h\mathbf{u})\|^2 + \|p - j_h p\|^2) ds\right). \quad (5.21)$$

A use of (4.5) with (**B1**) in (5.21) yields

$$\kappa\|\nabla\hat{\boldsymbol{\zeta}}\|^2 + (\nu - \kappa\alpha)\int_0^t \|\nabla\hat{\boldsymbol{\zeta}}\|^2 ds \le C(\nu,\alpha,\kappa)h^2\left(e^{2\alpha t}\|\tilde{\Delta}\mathbf{u}\|^2 + \|\tilde{\Delta}\mathbf{u}_0\|^2 + \int_0^t e^{2\alpha s}\|\nabla p\|^2 ds\right.$$
$$\left. + \int_0^t e^{2\alpha s}(\|\tilde{\Delta}\mathbf{u}_t\|^2 + \|\tilde{\Delta}\mathbf{u}\|^2) ds\right).$$

We now use a priori bounds for $\mathbf{u}$ and p derived in Lemmas 3.2, 3.6 and 3.7 to complete the proof. ■

For the estimation of time derivative, we have the following result.

Lemma 5.3. *Under the assumptions* (**A1**)*-*(**A2**) *and* (**B1**)*-*(**B2**)*, there exists a positive constant* $K = K(\nu,\lambda_1,\alpha,\kappa,M)$ *such that for* $0 \le \alpha < \frac{\nu\lambda_1}{2(1+\kappa\lambda_1)}$*, the following estimate holds true:*

$$\int_0^t e^{2\alpha s}\|\nabla(\mathbf{u}_t(s) - V_h\mathbf{u}_t(s))\|^2 ds \le Kh^2.$$

Proof. Recall (5.19) now with $\boldsymbol{\phi}_h = e^{\alpha t}P_h\boldsymbol{\zeta}_t = e^{\alpha t}\boldsymbol{\zeta}_t + e^{\alpha t}(P_h\mathbf{u}_t - \mathbf{u}_t)$ to find that

$$2\kappa\|e^{\alpha t}\nabla\boldsymbol{\zeta}_t\|^2 + \nu\frac{d}{dt}\|\nabla\hat{\boldsymbol{\zeta}}\|^2 = 2\nu\alpha\|\nabla\hat{\boldsymbol{\zeta}}\|^2 + 2(\hat{p}, e^{\alpha t}\nabla\cdot P_h\boldsymbol{\zeta}_t)$$
$$+ 2\kappa\, a(e^{\alpha t}\boldsymbol{\zeta}_t, e^{\alpha t}(\mathbf{u}_t - P_h\mathbf{u}_t)) + 2\nu a(\hat{\boldsymbol{\zeta}}, e^{\alpha t}(\mathbf{u}_t - P_h\mathbf{u}_t)). \quad (5.22)$$

An application of the Cauchy–Schwarz inequality, discrete incompressibility condition and (4.4) in (5.22) yields

$$2\kappa\|e^{\alpha t}\nabla\boldsymbol{\zeta}_t\|^2 + \nu\frac{d}{dt}\|\nabla\hat{\boldsymbol{\zeta}}\|^2 \le 2\nu\alpha\|\nabla\hat{\boldsymbol{\zeta}}\|^2 + 2\|\hat{p} - j_h\hat{p}\|\|e^{\alpha t}\nabla P_h\boldsymbol{\zeta}_t\|$$
$$+ 2\kappa\|e^{\alpha t}\nabla\boldsymbol{\zeta}_t\|\|e^{\alpha t}\nabla(\mathbf{u}_t - P_h\mathbf{u}_t)\| + 2\nu\|\nabla\hat{\boldsymbol{\zeta}}\|\|e^{\alpha t}\nabla(\mathbf{u}_t - P_h\mathbf{u}_t)\|. \quad (5.23)$$

Integrating (5.23) with respect to time from 0 to t, using Young's inequality, (**B1**) and (4.5), Lemmas 3.6, 3.7, and 5.2 and proceeding exactly as in the proof Lemma 5.2, we obtain the desired result. This completes the rest of the proof. ■

Below, we discuss the $\mathbf{L}^2$-estimate of $\boldsymbol{\zeta}(t)$.

Lemma 5.4. *Under the assumptions* (**A1**)*–*(**A2**) *and* (**B1**)*–*(**B2**)*, there exists a positive constant* $K = K(\nu, \lambda_1, \alpha, \kappa, M)$ *such that for* $0 \leq \alpha < \frac{\nu\lambda_1}{2(1+\kappa\lambda_1)}$*, the following estimate holds true for* $t > 0$*:*

$$\|\boldsymbol{\zeta}(t)\|^2 + e^{-2\alpha t}\int_0^t e^{2\alpha s}\|\boldsymbol{\zeta}(s)\|^2 ds \leq Kh^4 e^{-2\alpha t}.$$

Proof. For L^2 estimate, we recall the Aubin–Nitsche duality argument. Let $(\mathbf{w}, q)$ be the unique solution of the following steady state Stokes system:

$$-\nu\Delta\mathbf{w} + \nabla q = \hat{\boldsymbol{\zeta}} \ \text{ in } \ \Omega, \tag{5.24}$$

$$\nabla \cdot \mathbf{w} = 0 \quad \text{in } \ \Omega, \tag{5.25}$$

$$\mathbf{w}|_{\partial\Omega} = 0. \tag{5.26}$$

From assumption (**A1**), $(\mathbf{w}, q)$ satisfies the following regularity result:

$$\|\mathbf{w}\|_2 + \|q\|_{H^1/\mathbb{R}} \leq C\|\hat{\boldsymbol{\zeta}}\|. \tag{5.27}$$

Forming L^2-inner product between (5.24) and $\hat{\boldsymbol{\zeta}}$ and using discrete incompressibility condition, we obtain

$$\|\hat{\boldsymbol{\zeta}}\|^2 = \nu\, a(\mathbf{w} - P_h\mathbf{w}, \hat{\boldsymbol{\zeta}}) - (q - j_h q, \nabla \cdot \hat{\boldsymbol{\zeta}}) + \nu\, a(P_h\mathbf{w}, \hat{\boldsymbol{\zeta}}). \tag{5.28}$$

Now, using (5.19) with $\boldsymbol{\phi}_h$ replaced by $P_h\mathbf{w}$ and (5.25), the last term in (5.28) can be rewritten as

$$\nu\, a(P_h\mathbf{w}, \hat{\boldsymbol{\zeta}}) = (\hat{p} - j_h\hat{p}, \nabla \cdot (P_h\mathbf{w} - \mathbf{w})) - \kappa\, a(e^{\alpha t}\boldsymbol{\zeta}_t, P_h\mathbf{w} - \mathbf{w}) - \kappa\, a(e^{\alpha t}\boldsymbol{\zeta}_t, \mathbf{w}). \tag{5.29}$$

Once again, form L^2-inner product between (5.24) and $e^{\alpha t}\boldsymbol{\zeta}_t$, and use this in the last term of (5.29) to obtain

$$\kappa a(e^{\alpha t}\boldsymbol{\zeta}_t, \mathbf{w}) = \frac{\kappa}{\nu}(\hat{\boldsymbol{\zeta}}, \hat{\boldsymbol{\zeta}}_t) - \frac{\alpha\kappa}{\nu}\|\hat{\boldsymbol{\zeta}}\|^2 + \frac{\kappa}{\nu}(q - j_h q, \nabla \cdot e^{\alpha t}\boldsymbol{\zeta}_t). \tag{5.30}$$

Substituting (5.29) and (5.30) in (5.28), we obtain

$$\begin{aligned}\|\hat{\boldsymbol{\zeta}}\|^2 + \frac{\kappa}{\nu}\frac{d}{dt}\|\hat{\boldsymbol{\zeta}}\|^2 = \frac{\alpha\kappa}{\nu}\|\hat{\boldsymbol{\zeta}}\|^2 + \nu a(\mathbf{w} - P_h\mathbf{w}, \hat{\boldsymbol{\zeta}}) - (q - j_h q, \nabla \cdot \hat{\boldsymbol{\zeta}}) + (\hat{p} - j_h\hat{p}, \nabla \cdot (P_h\mathbf{w} - \mathbf{w}))\\ - \kappa\, a(e^{\alpha t}\boldsymbol{\zeta}_t, P_h\mathbf{w} - \mathbf{w}) - \frac{\kappa}{\nu}(q - j_h q, e^{\alpha t}\nabla \cdot \boldsymbol{\zeta}_t).\end{aligned} \tag{5.31}$$

Integrate (5.31) with respect to time from 0 to t, use (4.4) and then apply Cauchy Schwarz's inequality to yield

$$\begin{aligned}(\nu - \alpha\kappa)\int_0^t \|\hat{\boldsymbol{\zeta}}\|^2 ds + \kappa\|\hat{\boldsymbol{\zeta}}\|^2 \leq C(\kappa, \nu, \alpha)\Bigg(\|\boldsymbol{\zeta}(0)\|^2 + \int_0^t (\|\nabla(\mathbf{w} - P_h\mathbf{w})\|\|\nabla\hat{\boldsymbol{\zeta}}\|\\ + \|q - j_h q\|\|\nabla\hat{\boldsymbol{\zeta}}\| + \|\hat{p} - j_h\hat{p}\|\|\nabla(P_h\mathbf{w} - \mathbf{w})\|\\ + \|e^{\alpha t}\nabla\boldsymbol{\zeta}_t\|\|\nabla(P_h\mathbf{w} - \mathbf{w})\| + \|q - j_h q\|\|e^{\alpha t}\nabla\boldsymbol{\zeta}_t\|)ds\Bigg).\end{aligned}$$

Using (**B1**), (4.5) and (5.27), we arrive at

$$(\nu - \alpha\kappa)\int_0^t \|\hat{\boldsymbol{\zeta}}\|^2 ds + \kappa\|\hat{\boldsymbol{\zeta}}\|^2 \le C(\kappa,\nu,\alpha)\left(h^4\|\tilde{\Delta}\mathbf{u}_0\|^2 + h\int_0^t (\|\nabla\hat{\boldsymbol{\zeta}}\| + h\|\nabla\hat{p}\| + \|e^{\alpha t}\nabla\boldsymbol{\zeta}_t\|)\|\hat{\boldsymbol{\zeta}}\|ds\right). \quad (5.32)$$

As $0 \le \alpha < \frac{\nu\lambda_1}{2(1+\kappa\lambda_1)}$, $(\nu - \alpha\kappa) > 0$. Then, use Young's inequality appropriately and the estimates from Lemmas 3.7, 5.2, and 5.3 to complete the rest of the proof. ∎

Lemma 5.5. *Under the assumptions (*__A1__*)–(*__A2__*) and (*__B1__*)–(*__B2__*), there exists a positive constant* $K = K(\nu,\lambda_1,\alpha,\kappa,M)$ *such that for* $0 \le \alpha < \frac{\nu\lambda_1}{2(1+\kappa\lambda_1)}$, *the following holds true:*

$$\int_0^t e^{2\alpha s}\|\boldsymbol{\zeta}_t(s)\|^2 ds \le Kh^4.$$

The above lemma can be proved in an exactly similar fashion as the proof of Lemma 5.4 with the right hand side of (5.24) replaced by $e^{\alpha t}\boldsymbol{\zeta}_t$. but for completeness, we provide a short proof.

Proof. For obtaining the desired estimate of $\boldsymbol{\zeta}_t$, once again we appeal to the Aubin-Nitche's duality argument. Now recall the equation (5.19)

$$\kappa a(\boldsymbol{\zeta}_t, \boldsymbol{\phi}_h) + \nu a(\boldsymbol{\zeta}, \boldsymbol{\phi}_h) = (p, \nabla\cdot\boldsymbol{\phi}_h) \quad \forall\boldsymbol{\phi}_h \in \mathbf{J}_h.$$

In the dual problem (5.24), set $e^{\alpha t}\boldsymbol{\zeta}_t$ in stead of $\hat{\boldsymbol{\zeta}}$ on its right hand side and then form $\mathbf{L}^2$-inner product with $e^{\alpha t}\boldsymbol{\zeta}_t$ to obtain

$$\|e^{\alpha t}\boldsymbol{\zeta}_t\|^2 = \kappa a(e^{\alpha t}\boldsymbol{\zeta}_t, \mathbf{w} - P_h\mathbf{w}) - (e^{\alpha t}\nabla\cdot\boldsymbol{\zeta}_t, q) + \kappa a(e^{\alpha t}\boldsymbol{\zeta}_t, P_h\mathbf{w}).$$

From (5.19) with $\boldsymbol{\phi}_h = e^{\alpha t}P_h\mathbf{w}$, it now follows in a similar manner as in the $\mathbf{L}^2$-estimate of $\boldsymbol{\zeta}$ that

$$\begin{aligned}\|e^{\alpha t}\boldsymbol{\zeta}_t\|^2 &= \kappa a(e^{\alpha t}\boldsymbol{\zeta}_t, \mathbf{w} - P_h\mathbf{w}) - (q - j_hq, e^{\alpha t}\nabla\cdot\boldsymbol{\zeta}_t) - \nu a(e^{\alpha t}\boldsymbol{\zeta}, \mathbf{w}) \\ &\quad + e^{\alpha t}(p - j_hp, \nabla\cdot(P_h\mathbf{w} - \mathbf{w})) - \nu a(e^{\alpha t}\boldsymbol{\zeta}, P_h\mathbf{w} - \mathbf{w}). \quad (5.33)\end{aligned}$$

Using (5.24) with $\hat{\boldsymbol{\zeta}}$ replaced by $e^{\alpha t}\boldsymbol{\zeta}_t$ in the third term of (5.33) and the Cauchy–Schwarz inequality, we obtain

$$\begin{aligned}\|e^{\alpha t}\boldsymbol{\zeta}_t\|^2 &\le C(\nu,\lambda_1,\alpha,\kappa,M)[\|e^{\alpha t}\nabla\boldsymbol{\zeta}_t\|\|\nabla(\mathbf{w} - P_h\mathbf{w})\| + \|q - j_hq\|\|e^{\alpha t}\nabla\boldsymbol{\zeta}_t\| + \|\hat{\boldsymbol{\zeta}}\|\|e^{\alpha t}\boldsymbol{\zeta}_t\| \\ &\quad + \|\nabla\hat{\boldsymbol{\zeta}}\|\|\mathbf{w} - P_h\mathbf{w}\| + \|\hat{p} - j_h\hat{p}\|\|\nabla(\mathbf{w} - P_h\mathbf{w})\| + \|q - j_hq\|\|\nabla\hat{\boldsymbol{\zeta}}\|]. \quad (5.34)\end{aligned}$$

A use of (4.4) with (**B1**) yields

$$\begin{aligned}\|e^{\alpha t}\boldsymbol{\zeta}_t\|^2 &\le C(\nu,\lambda_1,\alpha,\kappa,M)(h\|e^{\alpha t}\nabla\boldsymbol{\zeta}_t\|\|\Delta\mathbf{w}\| + h\|e^{\alpha t}\nabla\boldsymbol{\zeta}_t\|\|\nabla q\| + \|\hat{\boldsymbol{\zeta}}\|\|e^{\alpha t}\boldsymbol{\zeta}_t\| \\ &\quad + h\|\nabla\hat{\boldsymbol{\zeta}}\|\|\Delta\mathbf{w}\| + h^2\|\nabla\hat{p}\|\|\Delta\mathbf{w}\| + h\|\nabla q\|\|\nabla\hat{\boldsymbol{\zeta}}\|). \quad (5.35)\end{aligned}$$

Using regularity result (5.27) now with right hand side $\|e^{\alpha t}\boldsymbol{\zeta}_t\|$, we arrive at

$$\|e^{\alpha t}\boldsymbol{\zeta}_t\|^2 \leq C(\nu,\lambda_1,\alpha,\kappa,M)((h\|e^{\alpha t}\nabla\boldsymbol{\zeta}_t\| + h\|\nabla\hat{\boldsymbol{\zeta}}\| + h^2\|\nabla\hat{p}\| + h\|\nabla\hat{\boldsymbol{\zeta}}\|)\|e^{\alpha t}\boldsymbol{\zeta}_t\|) + \|\hat{\boldsymbol{\zeta}}\|\|e^{\alpha t}\boldsymbol{\zeta}_t\|). \tag{5.36}$$

An application of Young's inequality yields

$$\|e^{\alpha t}\boldsymbol{\zeta}_t\|^2 \leq C(\nu,\lambda_1,\alpha,\kappa,M)(h^2\|e^{\alpha t}\nabla\boldsymbol{\zeta}_t\|^2 + h^2\|\nabla\hat{\boldsymbol{\zeta}}\|^2 + h^4\|\nabla\hat{p}\|^2 + \|\hat{\boldsymbol{\zeta}}\|^2). \tag{5.37}$$

Integrating (5.37) with respect to time from 0 to t, we obtain

$$\int_0^t \|e^{\alpha s}\boldsymbol{\zeta}_t(s)\|^2 \leq C(\nu,\lambda_1,\alpha,\kappa,M)\left(\int_0^t (h^2\|e^{\alpha t}\nabla\boldsymbol{\zeta}_t\|^2 + h^2\|\nabla\hat{\boldsymbol{\zeta}}\|^2 + h^4\|\nabla\hat{p}\|^2 + \|\hat{\boldsymbol{\zeta}}\|^2)ds\right). \tag{5.38}$$

A use of Lemmas 5.2, 5.3, 5.4, and 3.7 would lead us to

$$\int_0^t \|e^{\alpha s}\boldsymbol{\zeta}_t(s)\|^2 ds \leq C(\nu,\lambda_1,\alpha,\kappa,M)h^4, \tag{5.39}$$

and this completes the rest of the proof. ∎

As $\boldsymbol{\xi} = \boldsymbol{\zeta} + \boldsymbol{\rho}$ and the estimates of $\boldsymbol{\zeta}$ are already known, it suffices to derive estimate of $\boldsymbol{\rho}$ to obtain an estimate for $\boldsymbol{\xi}$.

Lemma 5.6. *Under the assumptions* (**A1**)*–*(**A2**) *and* (**B1**)*–*(**B2**)*, there exists a positive constant* $K = K(\nu,\lambda_1,\alpha,\kappa,M)$ *such that for* $0 \leq \alpha < \frac{\nu}{2(1+\kappa\lambda_1)}$*, the following estimate holds true:*

$$(\|\boldsymbol{\rho}\|^2 + \kappa\|\nabla\boldsymbol{\rho}\|^2) + 2\beta e^{-2\alpha t}\int_0^t e^{2\alpha s}\|\nabla\boldsymbol{\rho}(s)\|^2 ds \leq C(\nu,\lambda_1,\alpha,\kappa,M)h^4 e^{-2\alpha t}.$$

Proof. Subtracting (5.18) from (5.17), we find that

$$(\boldsymbol{\rho}_t,\boldsymbol{\phi}_h) + \kappa\, a(\boldsymbol{\rho}_t,\boldsymbol{\phi}_h) + \nu a(\boldsymbol{\rho},\boldsymbol{\phi}_h) = -(\boldsymbol{\zeta}_t,\boldsymbol{\phi}_h) \quad \forall\boldsymbol{\phi}_h \in \mathbf{J}_h. \tag{5.40}$$

Replace $\boldsymbol{\phi}_h$ by $e^{\alpha t}\hat{\boldsymbol{\rho}}$ in (5.40) to obtain

$$(e^{\alpha t}\boldsymbol{\rho}_t,\hat{\boldsymbol{\rho}}) + \kappa\, a(e^{\alpha t}\boldsymbol{\rho}_t,\hat{\boldsymbol{\rho}}) + \nu\|\nabla\hat{\boldsymbol{\rho}}\|^2 = -(e^{\alpha t}\boldsymbol{\zeta}_t,\hat{\boldsymbol{\rho}}) \quad \forall\boldsymbol{\phi}_h \in \mathbf{J}_h. \tag{5.41}$$

A use of Cauchy–Schwarz's inequality, (2.3) along with Young's inequality in (5.41) yields

$$\frac{d}{dt}(\|\hat{\boldsymbol{\rho}}\|^2 + \kappa\|\nabla\hat{\boldsymbol{\rho}}\|^2) + 2\beta\|\nabla\hat{\boldsymbol{\rho}}\|^2 \leq C(\kappa,\alpha,\lambda_1)\|e^{\alpha t}\boldsymbol{\zeta}_t\|^2. \tag{5.42}$$

Integrating (5.42) with respect to time from 0 to t, we arrive at

$$\|\hat{\boldsymbol{\rho}}\|^2 + \kappa\|\nabla\hat{\boldsymbol{\rho}}\|^2 + 2\beta\int_0^t \|\nabla\hat{\boldsymbol{\rho}}\|^2 ds \leq C(\kappa,\alpha,\lambda_1)\int_0^t \|e^{\alpha s}\boldsymbol{\zeta}_t(s)\|^2 ds. \tag{5.43}$$

The desired result follows after a use of Lemma 5.5 in (5.43). ∎

We now derive an estimate of $\boldsymbol{\xi}$ in $L^\infty(\mathbf{L}^2)$ and $L^\infty(\mathbf{H}_0^1)$-norms.

Lemma 5.7. *Let the assumptions (***A1***)–(***A2***) and (***B1***)–(***B2***) be satisfied. Then, there exists a positive constant* $K = K(\nu, \lambda_1, \alpha, \kappa, M)$ *such that for* $0 < \alpha < \frac{\nu\lambda_1}{2(1+\kappa\lambda_1)}$, *the following estimate holds true:*

$$\|\boldsymbol{\xi}\|^2 + e^{-2\alpha t}\int_0^t e^{2\alpha s}\|\nabla\boldsymbol{\xi}(s)\|^2 ds \le C(\nu, \lambda_1, \alpha, \kappa, M)h^4 e^{-2\alpha t}.$$

A use of the triangle inequality together with the Lemmas 5.4 and 5.6 would provide us the result. Now, we derive the proof of the main Theorem 4.1.

Proof of Theorem 4.1. As $e = \mathbf{u} - \mathbf{u}_h = (\mathbf{u} - \mathbf{v_h}) + (\mathbf{v_h} - \mathbf{u}_h) = \boldsymbol{\xi} + \boldsymbol{\eta}$ and the estimate of $\boldsymbol{\xi}$ is known from Lemma 5.7, we are left only with the estimate for $\boldsymbol{\eta}$. Subtracting (5.16) from (4.2), we obtain

$$(\boldsymbol{\eta}_t, \boldsymbol{\phi}_h) + \kappa\, a(\boldsymbol{\eta}_t, \boldsymbol{\phi}_h) + \nu a(\boldsymbol{\eta}, \boldsymbol{\phi}_h) = b(\mathbf{u}_h, \mathbf{u}_h, \boldsymbol{\phi}_h) - b(\mathbf{u}, \mathbf{u}, \boldsymbol{\phi}_h) \ \ \forall \boldsymbol{\phi}_h \in \mathbf{J}_h. \tag{5.44}$$

Choose $\boldsymbol{\phi}_h = e^{2\alpha t}\boldsymbol{\eta}$ and use (2.3) to find that

$$\frac{1}{2}\frac{d}{dt}(\|\hat{\boldsymbol{\eta}}\|^2 + \kappa\|\nabla\hat{\boldsymbol{\eta}}\|^2) + \left(\nu - \alpha\left(\kappa + \frac{1}{\lambda_1}\right)\right)\|\nabla\hat{\boldsymbol{\eta}}\|^2 = e^{\alpha t}\Lambda_h(\hat{\boldsymbol{\eta}}), \tag{5.45}$$

where

$$\Lambda_h(\boldsymbol{\phi}_h) = b(\mathbf{u}_h, \mathbf{u}_h, \boldsymbol{\phi}_h) - b(\mathbf{u}, \mathbf{u}, \boldsymbol{\phi}_h).$$

To estimate the right hand side term of (5.45), we note that

$$e^{\alpha t}\Lambda_h(\hat{\boldsymbol{\eta}}) = e^{-\alpha t}(-b(\hat{\mathbf{e}}, \hat{\mathbf{u}}_h, \hat{\boldsymbol{\eta}}) + b(\hat{\mathbf{u}}, \hat{\boldsymbol{\eta}}, \hat{\mathbf{e}})).$$

A use of Hölder's inequality with the Poincaré inequality, Agmon's inequality (5.10), and the discrete Sobolev inequality (see, Lemma 4.4 in [10]) yields

$$\begin{aligned} e^{\alpha t}|\Lambda_h(\hat{\boldsymbol{\eta}})| &\le Ce^{-\alpha t}(\|\hat{\mathbf{e}}\|\|\nabla\hat{\mathbf{u}}_h\|_{L^6}\|\hat{\boldsymbol{\eta}}\|_{L^3} + \|\hat{\mathbf{u}}\|_{L^\infty}\|\nabla\hat{\boldsymbol{\eta}}\|\|\hat{\mathbf{e}}\|) \\ &\le C\left(e^{-\alpha t}\|\tilde{\Delta}_h\mathbf{u}_h\|\|\nabla\hat{\boldsymbol{\eta}}\|\|\hat{\mathbf{e}}\| + \|\nabla\hat{\mathbf{u}}\|^{\frac{1}{2}}\|\tilde{\Delta}\hat{\mathbf{u}}\|^{\frac{1}{2}}\|\nabla\hat{\boldsymbol{\eta}}\|\|\,\hat{\mathbf{e}}\|\right) \\ &\le C(\epsilon)e^{-2\alpha t}(\|\tilde{\Delta}_h\hat{\mathbf{u}}_h\|^2 + \|\nabla\hat{\mathbf{u}}\|\|\tilde{\Delta}\hat{\mathbf{u}}\|\|)\|\hat{\mathbf{e}}\|^2 + \epsilon\|\nabla\hat{\boldsymbol{\eta}}\|^2. \end{aligned}$$

As $\mathbf{e} = \boldsymbol{\xi} + \boldsymbol{\eta}$, we obtain

$$e^{\alpha t}|\Lambda_h(\hat{\boldsymbol{\eta}})| \le C(\epsilon)e^{-2\alpha t}(\|\tilde{\Delta}_h\hat{\mathbf{u}}_h\|^2 + \|\nabla\hat{\mathbf{u}}\|\|\tilde{\Delta}\hat{\mathbf{u}}\|)(\|\hat{\boldsymbol{\xi}}\|^2 + \|\hat{\boldsymbol{\eta}}\|^2) + \epsilon\|\nabla\hat{\boldsymbol{\eta}}\|^2. \tag{5.46}$$

Using (5.46) in (5.45), we arrive at

$$\begin{aligned} \frac{d}{dt}(\|\hat{\boldsymbol{\eta}}\|^2 + \kappa\|\nabla\hat{\boldsymbol{\eta}}\|^2) + (\beta + \nu)\|\nabla\hat{\boldsymbol{\eta}}\|^2 &\le C(\epsilon)e^{-2\alpha t}((\|\hat{\boldsymbol{\xi}}\|^2 + \|\hat{\boldsymbol{\eta}}\|^2)\|\tilde{\Delta}_h\hat{\mathbf{u}}_h\|^2 \\ &\quad + (\|\hat{\boldsymbol{\xi}}\|^2 + \|\hat{\boldsymbol{\eta}}\|^2)\|\nabla\hat{\mathbf{u}}\|\|\tilde{\Delta}\hat{\mathbf{u}}\|) + 2\epsilon\|\nabla\hat{\boldsymbol{\eta}}\|^2. \end{aligned} \tag{5.47}$$

With choice of $\epsilon = \frac{\nu}{2}$, integration of (5.47) with respect to time from 0 to t yields

$$\|\hat{\boldsymbol{\eta}}\|^2 + \kappa\|\nabla\hat{\boldsymbol{\eta}}\|^2 + \beta\int_0^t \|\nabla\hat{\boldsymbol{\eta}}\|^2 ds \le C(\nu)\left(\int_0^t \|\hat{\boldsymbol{\xi}}\|^2(\|\nabla\mathbf{u}\|\|\tilde{\Delta}\mathbf{u}\| + \|\tilde{\Delta}_h\mathbf{u}_h\|^2)ds + \int_0^t \|\hat{\boldsymbol{\eta}}\|^2(\|\nabla\mathbf{u}\|\|\tilde{\Delta}\mathbf{u}\| + \|\tilde{\Delta}_h\mathbf{u}_h\|^2)ds\right). \quad (5.48)$$

Use Lemmas 3.2, 4.1, and 5.7 in the first term of the right side of (5.48) to obtain

$$\|\hat{\boldsymbol{\eta}}\|^2 + \kappa\|\nabla\hat{\boldsymbol{\eta}}\|^2 + \beta\int_0^t \|\nabla\hat{\boldsymbol{\eta}}\|^2 ds \le C(\nu,\lambda_1,\alpha,\kappa,M)h^4e^{-2\alpha t} + \int_0^t \|\hat{\boldsymbol{\eta}}\|^2(\|\nabla\hat{\mathbf{u}}\|\|\tilde{\Delta}\mathbf{u}\| + \|\tilde{\Delta}_h\mathbf{u}_h\|^2)ds. \quad (5.49)$$

An application of Gronwall's Lemma yields

$$\|\hat{\boldsymbol{\eta}}\|^2 + \kappa\|\nabla\hat{\boldsymbol{\eta}}\|^2 + \beta\int_0^t \|\nabla\hat{\boldsymbol{\eta}}(s)\|^2\,ds \le Kh^4\exp\left(\int_0^t (\|\nabla\mathbf{u}\|\|\tilde{\Delta}\mathbf{u}\| + \|\tilde{\Delta}_h\mathbf{u}_h\|^2)\,ds\right). \quad (5.50)$$

Once again, with the help of Lemmas 3.2 and 4.1, we obtain

$$\int_0^t (\|\nabla\mathbf{u}\|\|\tilde{\Delta}\mathbf{u}\| + \|\tilde{\Delta}_h\mathbf{u}_h\|^2)\,ds \le K(\kappa,\nu,\alpha,\lambda_1,M)(1-e^{-2\alpha t}) \le K. \quad (5.51)$$

Using (5.51) in (5.50), we derive estimate for $\boldsymbol{\eta}$ as

$$\|\eta\|^2 + \kappa\|\nabla\boldsymbol{\eta}\|^2 + 2\beta e^{-2\alpha t}\int_0^t e^{2\alpha s}\|\nabla\boldsymbol{\eta}(s)\|^2\,ds \le Kh^4e^{-2\alpha t}. \quad (5.52)$$

A use of triangle inequality along with (5.52) and Lemma 5.7 completes the optimal $L^\infty(\mathbf{L}^2)$-estimate of the velocity. For the rest part of proof of Theorem 4.1, we now appeal to Lemma 5.1 to complete the proof. ∎

VI. ERROR ESTIMATE FOR THE PRESSURE

In this section, we derive optimal error estimates for the Galerkin approximation p_h of the pressure p. The main result Theorem 4.2 follows from Lemmas 6.1, 6.2 and the approximation property for j_h. From (**B2**), we note that

$$\begin{aligned}\|(j_hp - p_h)(t)\|_{L^2/N_h} &\le C\sup_{\boldsymbol{\phi}_h\in\mathbf{H}_h/\{0\}}\left\{\frac{(j_hp - p_h,\nabla\cdot\boldsymbol{\phi}_h)}{\|\nabla\boldsymbol{\phi}_h\|}\right\},\\ &\le C\sup_{\boldsymbol{\phi}_h\in\mathbf{H}_h/\{0\}}\left\{\frac{(j_hp - p,\nabla\cdot\boldsymbol{\phi}_h)}{\|\nabla\boldsymbol{\phi}_h\|} + \frac{(p - p_h,\nabla\cdot\boldsymbol{\phi}_h)}{\|\nabla\boldsymbol{\phi}_h\|}\right\},\\ &\le C\left(\|j_hp - p\| + \sup_{\boldsymbol{\phi}_h\in\mathbf{H}_h/\{0\}}\left\{\frac{(p - p_h,\nabla\cdot\boldsymbol{\phi}_h)}{\|\nabla\boldsymbol{\phi}_h\|}\right\}\right). \end{aligned} \quad (6.1)$$

As the estimate of the first term on the right hand side of (6.1) follows from (**B1**), it is sufficient to estimate the second term. Subtracting (4.3) from (5.1), we find that

$$(p - p_h, \nabla \cdot \boldsymbol{\phi}_h) = (\mathbf{e}_t, \boldsymbol{\phi}_h) + \kappa\, a(\mathbf{e}_t, \boldsymbol{\phi}_h) + \nu a(\mathbf{e}, \boldsymbol{\phi}_h) - \Lambda_h(\boldsymbol{\phi}_h) \;\; \forall \boldsymbol{\phi}_h \in \mathbf{H}_h,$$

where

$$-\Lambda_h(\boldsymbol{\phi}_h) = b(\mathbf{u}, \mathbf{u}, \boldsymbol{\phi}_h) - b(\mathbf{u}_h, \mathbf{u}_h, \boldsymbol{\phi}_h) = -b(\mathbf{e}, \mathbf{e}, \boldsymbol{\phi}_h) + b(\mathbf{u}, \mathbf{e}, \boldsymbol{\phi}_h) + b(\mathbf{e}, \mathbf{u}, \boldsymbol{\phi}_h).$$

Using Hölder's inequality, Sobolev's inequality and Lemma 5.1, we obtain

$$|\Lambda_h(\boldsymbol{\phi}_h)| \le C(\|\nabla \mathbf{u}\| + \|\mathbf{e}\|_{\mathbf{L}^4})\|\nabla \mathbf{e}\|\|\nabla \boldsymbol{\phi}_h\| \le C\|\nabla \mathbf{e}\|\|\nabla \boldsymbol{\phi}_h\|. \tag{6.2}$$

Thus,

$$(p - p_h, \nabla \cdot \boldsymbol{\phi}_h) \le C(\nu, \kappa)(\|\mathbf{e}_t\| + \|\nabla \mathbf{e}_t\| + \|\nabla \mathbf{e}\|)\|\nabla \boldsymbol{\phi}_h\|.$$

The results obtained can be summarized as

Lemma 6.1. *For all $t > 0$, the semidiscrete Galerkin approximation p_h of the pressure p satisfies*

$$\|(p - p_h)(t)\|_{L^2/N_h} \le C(\|\mathbf{e}_t\| + \|\nabla \mathbf{e}_t\| + \|\nabla \mathbf{e}\|). \tag{6.3}$$

From Theorem 5.1, the estimate $\|\nabla \mathbf{e}\|$ is known. We now derive bounds for $\|\mathbf{e}_t\|$ and $\|\nabla \mathbf{e}_t\|$.

Lemma 6.2. *For all $t > 0$, the error $\mathbf{e} = \mathbf{u} - \mathbf{u}_h$ in the velocity satisfies*

$$\|\mathbf{e}_t(t)\|^2 + \kappa\|\nabla \mathbf{e}_t(t)\|^2 \le Ch^2 e^{-2\alpha t}. \tag{6.4}$$

Proof. From (4.3) and (5.1), we obtain

$$(\mathbf{e}_t, \boldsymbol{\phi}_h) + \kappa\, a(\mathbf{e}_t, \boldsymbol{\phi}_h) + \nu a(\mathbf{e}, \boldsymbol{\phi}_h) = \Lambda_h(\boldsymbol{\phi}_h) + (p, \nabla \cdot \boldsymbol{\phi}_h), \;\; \boldsymbol{\phi}_h \in \mathbf{H}_h. \tag{6.5}$$

where

$$\Lambda_h(\boldsymbol{\phi}_h) = b(\mathbf{u}_h, \mathbf{u}_h, \boldsymbol{\phi}_h) - b(\mathbf{u}, \mathbf{u}, \boldsymbol{\phi}_h).$$

Choosing $\boldsymbol{\phi}_h = P_h \mathbf{e}_t = \mathbf{e}_t + (P_h \mathbf{u}_t - \mathbf{u}_t)$ in (6.5), we arrive at

$$\begin{aligned}(\mathbf{e}_t, \mathbf{e}_t) + \kappa\, a(\mathbf{e}_t, \mathbf{e}_t) &= -\nu a(\mathbf{e}, \mathbf{e}_t) + \Lambda_h(P_h \mathbf{e}_t) + (p, \nabla \cdot P_h \mathbf{e}_t) \\ &\quad + (\mathbf{e}_t, \mathbf{u}_t - P_h \mathbf{u}_t) + \kappa\, a(\mathbf{e}_t, \mathbf{u}_t - P_h \mathbf{u}_t) + \nu a(\mathbf{e}, \mathbf{u}_t - P_h \mathbf{u}_t).\end{aligned} \tag{6.6}$$

To estimate $(p, \nabla \cdot P_h \mathbf{e}_t)$, a use of the discrete incompressible condition with (4.4) yields

$$|(p, \nabla \cdot P_h \mathbf{e}_t)| = |(p - j_h p, \nabla \cdot P_h \mathbf{e}_t)| \le \|p - j_h p\|\|\nabla \mathbf{e}_t\|. \tag{6.7}$$

Now using Cauchy–Schwarz's inequality in (6.6), we arrive at

$$\begin{aligned}\|\mathbf{e}_t\|^2 + \kappa\|\nabla \mathbf{e}_t\|^2 &\le \nu\|\nabla \mathbf{e}\|\|\nabla \mathbf{e}_t\| + |\Lambda_h(P_h \mathbf{e}_t)| + \|p - j_h p\|\|\nabla.(P_h \mathbf{e}_t)\| \\ &\quad + \|\mathbf{e}_t\|\|\mathbf{u}_t - P_h \mathbf{u}_t\| + \kappa\|\nabla \mathbf{e}_t\|\|\nabla(\mathbf{u}_t - P_h \mathbf{u}_t)\| + \nu\|\nabla \mathbf{e}\|\|\nabla(\mathbf{u}_t - P_h \mathbf{u}_t)\|.\end{aligned} \tag{6.8}$$

Using (6.2) and (4.4), we obtain

$$|\Lambda_h(P_h\mathbf{e}_t)| \leq C\|\nabla\mathbf{e}\|\|\nabla\mathbf{e}_t\|. \tag{6.9}$$

Substitute (6.7), (6.9) in (6.8) and use Young's inequality to arrive at

$$\|\mathbf{e}_t\|^2 + \kappa\|\nabla\mathbf{e}_t\|^2 \leq C(\nu,\kappa)(\|\nabla\mathbf{e}\|^2 + (\|\nabla(\mathbf{u}_t - P_h\mathbf{u}_t)\|^2 + \|p - j_h p\|^2 + \|\mathbf{u}_t - P_h\mathbf{u}_t\|^2)).$$

Using (4.5) and (**B1**), we now obtain

$$\|\mathbf{e}_t\|^2 + \kappa\|\nabla\mathbf{e}_t\|^2 \leq C(\nu,\kappa)(\|\nabla\mathbf{e}\|^2 + h^2(\|\tilde{\Delta}\mathbf{u}_t\|^2 + \|\nabla p\|^2 + \|\nabla\mathbf{u}_t\|^2)).$$

An application of Lemmas 3.5, 3.7, and 5.1 would lead us to the result. This completes the rest of the proof. ∎

Proof of Theorem 4.2. The proof of Theorem 4.2 now follows from Lemma 6.2 and the approximation property (**B1**) of j_h. ∎

VII. NUMERICAL EXPERIMENTS

We use finite element method for spatial discretization and backward Euler method for temporal discretization. The approximating spaces $\mathbf{H}_h$ and L_h for velocity and pressure variables are respectively chosen as follows:

$$\mathbf{H}_h = \left\{\mathbf{v} \in \left(H_0^1(\Omega)\right)^2 \cap (C(\bar{\Omega}))^2 : \mathbf{v}|_K \in (P_2(K))^2, K \in \tau_h\right\}$$
$$L_h = \{q \in L^2(\Omega) : q|_K \in P_0(K), K \in \tau_h\},$$

where τ_h denotes an admissible triangulation of $\bar{\Omega}$ in to closed triangles. Let $0 = t_0 < t_1 < \cdots < t_N = T$, be a uniform subdivision of the time interval $(0,T]$ with $k = t_n - t_{n-1}$ and $\mathbf{U}^n$ be the approximation of $\mathbf{u}(t)$ in $\mathbf{H}_h$ at $t = t_n = nk$. Now, the completely discrete scheme based on backward Euler method can be stated as: given $\mathbf{U}^{n-1}$, find the pair $(\mathbf{U}^n, P^n)$ satisfying:

$$(\bar{\partial}_t\mathbf{U}^n, \mathbf{v}_h) + \kappa a(\bar{\partial}_t\mathbf{U}^n, \mathbf{v}_h) + \nu a(\mathbf{U}^n, \mathbf{v}_h) + b(\mathbf{U}^n, \mathbf{U}^n, \mathbf{v}_h) + (\mathbf{v}_h, \nabla P^n)$$
$$= (\mathbf{f}^n, \mathbf{v}_h), \quad \forall \mathbf{v}_h \in \mathbf{H}_h,$$
$$(\nabla \cdot \mathbf{U}^n, w_h) = 0, \qquad \forall w_h \in L_h, \tag{7.1}$$

where $\bar{\partial}_t\mathbf{U}^n = \frac{\mathbf{U}^n - \mathbf{U}^{n-1}}{k}$.

Example 1. In this example, we validate the theoretical error estimates obtained in Theorems 4.1-4.2. For verifying the convergence rates of the solution obtained numerically, we choose the right hand side function f in such a way that the exact solution $(\mathbf{u}, p) = ((u_1, u_2), p)$ of (1.2)–(1.4) is

$$u_1 = 2e^t x^2(x-1)^2 y(y-1)(2y-1), \quad u_2 = -2e^t y^2(y-1)^2 x(x-1)(2x-1), \quad p = e^t\, y.$$

TABLE I. Numerical errors and convergence rates with $k = h^2$.

S. No.	h	$\|\mathbf{u} - \mathbf{U}^n\|_{\mathbf{L}^2}$	Convergence rate	$\|\mathbf{u} - \mathbf{U}^n\|_{\mathbf{H}^1}$	Convergence rate	$\|p - P^n\|_{L^2}$	Convergence rate
1	1/2	0.0266		0.1039		1.0443	
2	1/4	0.0090	1.5653	0.0543	0.9357	0.5484	0.9291
3	1/8	0.0026	1.7790	0.0282	0.9428	0.2815	0.9618
4	1/16	0.0007	1.8938	0.0145	0.9601	0.1424	0.9827

TABLE II. Numerical errors and convergence rates with $k = h^2$.

S. No.	h	$\|\mathbf{u} - \mathbf{U}^n\|_{\mathbf{L}^2}$	Convergence rate	$\|\mathbf{u} - \mathbf{U}^n\|_{\mathbf{H}^1}$	Convergence rate
1	1/2	0.797874×10^{-3}		0.012952	
2	1/4	0.203886×10^{-3}	1.9683	0.006767	0.9366
3	1/8	0.051241×10^{-3}	1.9923	0.003418	0.9850
4	1/16	0.012817×10^{-3}	1.9992	0.001713	0.9964

We assume the viscosity of the fluid as $\nu = 1$ and retardation as $\kappa = 1$ with $\Omega = (0, 1) \times (0, 1)$ and time interval $(0, T]$ with $T = 1$. Here, Ω is subdivided into triangles with mesh size h. The theoretical analysis provides a convergence rate $\mathcal{O}(h^2)$ for the velocity in the L^2 norm and $\mathcal{O}(h)$ for the pressure. Table I gives the numerical errors and convergence rates obtained on successively refined meshes with time step size $k = h^2$. These results agree with the optimal theoretical convergence rates obtained in Theorems 4.1 and 4.2.

Example 2. In this example, we demonstrate the exponential decay property of the discrete solution. We choose $\nu = 1, \kappa = 1$ and $f = 0$ with $\mathbf{u}_0 = (2x^2(x-1)^2 y(y-1)(2y-1), -2y^2(y-1)^2 x(x-1)(2x-1),\ y)$ in (1.2)–(1.4). In this case, we replace exact solution $\mathbf{u}$ by finite element solution obtained in a refined mesh. The order of convergence is shown in Table II. In Fig. 1, for different values of time t, we plot $\|\mathbf{U}^n\|$ versus time and observe the exponential decay property for velocity.

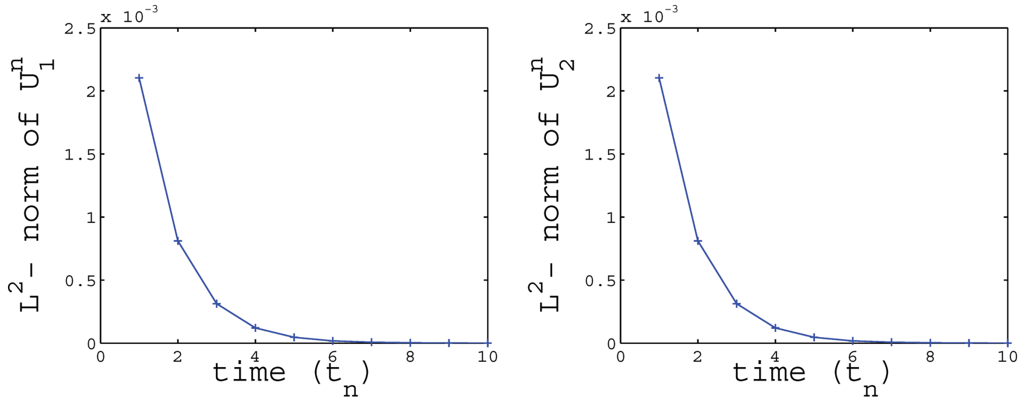

FIG. 1. Exponential decay property of the approximate solution $\|\mathbf{U}^n\|$. [Color figure can be viewed in the online issue, which is available at wileyonlinelibrary.com.]

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