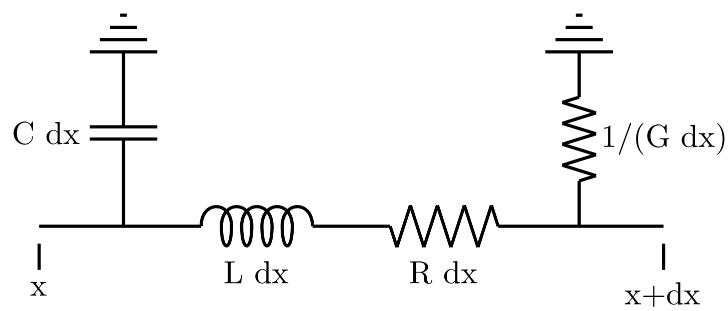
The Telegraph Equation

Model an infinitesmal piece of telegraph wire as an electrical circuit which consists of resistor of resistance Rdx and a coil of inductance Ldx. If $i(x,t)$ is the current through the wire, the voltage across the resistor is $iRdx$ while that across the coil is $i_t Ldx$. Denoting by $u(x,t)$ the voltage at position x and time t, we have that the change in voltage between the ends of the piece of wire is

$$du = -iRdx - i_t Ldx$$

Suppose further that current can escape from the wire to ground, either through a resistor of conductance Gdx or through a capacitor of capacitance Cdx. The amount that escapes



through the resistor is $uGdx$. Because the charge on the capacitor is $q = uCdx$, the amount that escapes from the capacitor is $u_t Cdx$. In total

$$di = -uGdx - u_t Cdx$$

Dividing by dx and taking the limit $dx \searrow 0$ we get the differential equations

$$u_x + Ri + Li_t = 0 \qquad (K1)$$

$$Cu_t + Gu + i_x = 0 \qquad (K2)$$

Solving $\frac{\partial}{\partial t}(K2)$ for

$$i_{xt} = -Cu_{tt} - Gu_t$$

and substituting the result into $\frac{\partial}{\partial x}(K1)$ gives

$$u_{xx} + Ri_x + L(-Cu_{tt} - Gu_t) = 0 \quad \implies \quad u_{xx} + R(-Cu_t - Gu) + L(-Cu_{tt} - Gu_t) = 0$$

Renaming some constants we get the *telegraph equation*

$$u_{tt} + (\alpha + \beta)u_t + \alpha\beta u = c^2 u_{xx}$$

where

$$c^2 = \tfrac{1}{LC} \qquad \alpha = \tfrac{G}{C} \qquad \beta = \tfrac{R}{L}$$

The Solution

We now solve the boundary value problem

(1) $$u_{tt} + (\alpha + \beta)u_t + \alpha\beta u = c^2 u_{xx}$$

(2) $$u(0, t) = 0$$

(3) $$u(\ell, t) = 0$$

(4) $$u(x, 0) = \delta(x - a)$$

(5) $$u_t(x, 0) = 0$$

for all $t > 0$, $0 < x < \ell$. This models a telegraph wire of length ℓ having the voltage at both ends $x = 0$ and $x = \ell$ clamped at zero. The initial conditions, (4) and (5), represent an idealized signal consisting of a spike at $x = a$ that is stationary at time zero.

The method of solution is separation of variables. So, we first try $u(x, t) = X(x)T(t)$.

$$\begin{aligned}(1) \quad &\implies \quad XT'' + (\alpha + \beta)XT' + \alpha\beta XT = c^2 X''T \\ &\implies \quad \tfrac{1}{c^2}\left\{\tfrac{T''}{T} + (\alpha + \beta)\tfrac{T'}{T} + \alpha\beta\right\} = \tfrac{X''}{X} = \sigma, \ \text{const}\end{aligned}$$

Imposing boundary conditions (2,3)

$$X(0) = X(\ell) = 0 \qquad X'' - \sigma X = 0 \quad \implies \quad X(x) = \text{const}\sin\left(\tfrac{n\pi x}{\ell}\right), \ \sigma = -\left(\tfrac{n\pi}{\ell}\right)^2$$

The equation for T is

$$T'' + (\alpha + \beta)T' + (\alpha\beta - \sigma c^2)T = 0$$

which has general solution

$$T = \text{const}e^{r_1 t} + \text{const}e^{r_2 t} \qquad \text{with} \qquad r_i = \tfrac{1}{2}\left(-\alpha - \beta \pm \sqrt{(\alpha + \beta)^2 - 4\alpha\beta + 4\sigma c^2}\right)$$

Call

$$4\omega_n^2 = 4\left(\frac{n\pi}{\ell}\right)^2 c^2 - (\alpha-\beta)^2 \qquad\qquad d = (\alpha+\beta)/2$$

Then $r_i = -d \pm i\omega_n$ and general solution to the T equation can be written

$$T(t) = A_n e^{-dt}\cos(\omega_n t - \phi_n)$$

with the amplitude A_n and phase ϕ_n arbitrary. So, for all A_n and ϕ_n,

$$u(x,t) = \sum_{n=1}^{\infty} A_n e^{-dt}\cos(\omega_n t - \phi_n)\sin\left(\frac{n\pi x}{\ell}\right)$$

satisfies the pde (1) and boundary conditions (2,3). It remains to choose the amplitudes and phases to satisfy the initial conditions (4,5).

$$\begin{aligned}
(5) \implies u_t(x,0) &= \sum_{n=1}^{\infty} A_n\left[-de^{-dt}\cos(\omega_n t - \phi_n) - \omega_n e^{-dt}\sin(\omega_n t - \phi_n)\right]\sin\left(\frac{n\pi x}{\ell}\right)\Big|_{t=0} \\
&= \sum_{n=1}^{\infty} A_n\left[-d\cos(\phi_n) + \omega_n\sin(\phi_n)\right]\sin\left(\frac{n\pi x}{\ell}\right) = 0 \\
&\implies \phi_n = \arctan\left(\frac{d}{\omega_n}\right) \\
(4) \implies u(x,0) &= \sum_{n=1}^{\infty} A_n\cos(\phi_n)\sin\left(\frac{n\pi x}{\ell}\right) = \delta(x-a) \\
&\implies A_n\cos\phi_n = \frac{2}{\ell}\int_0^{\ell}\delta(x-a)\sin\left(\frac{n\pi x}{\ell}\right) = \frac{2}{\ell}\sin\left(\frac{n\pi a}{\ell}\right)
\end{aligned}$$

The final solution is

$$u(x,t) = \sum_{n=1}^{\infty} A_n e^{-dt}\cos(\omega_n t - \phi_n)\sin\left(\frac{n\pi x}{\ell}\right)$$

$$\text{where } d = (\alpha+\beta)/2 \qquad \omega_n = \sqrt{\left(\frac{n\pi c}{\ell}\right)^2 - \frac{1}{4}(\alpha-\beta)^2}$$

$$\phi_n = \arctan\left(\frac{d}{\omega_n}\right) \qquad A_n = \frac{2}{\ell\cos\phi_n}\sin\left(\frac{n\pi a}{\ell}\right)$$

Interpretation of the Solution

To interpret this result, rewrite it as

$$u(x,t) = \sum_{n=1}^{\infty}\frac{1}{2}A_n e^{-dt}\left[\sin\left(\frac{n\pi x}{\ell} - \omega_n t + \phi_n\right) + \sin\left(\frac{n\pi x}{\ell} + \omega_n t - \phi_n\right)\right]$$

Suppose that we carefully tune the wire so that $\alpha = \beta$. Then

$$d = \alpha \qquad\qquad \omega_n = \tfrac{n\pi c}{\ell}$$

and

$$\begin{aligned} u(x,t) &= \sum_{n=1}^{\infty} \tfrac{1}{2} A_n e^{-\alpha t} \left[\sin(\tfrac{n\pi x}{\ell} - \tfrac{n\pi c}{\ell} t + \phi_n) + \sin(\tfrac{n\pi x}{\ell} + \tfrac{n\pi c}{\ell} t - \phi_n)\right] \\ &= e^{-\alpha t} f(x - ct) + e^{-\alpha t} g(x + ct) \end{aligned}$$

where

$$\begin{aligned} f(z) &= \sum_{n=1}^{\infty} \tfrac{1}{2} A_n \sin(\tfrac{n\pi}{\ell} z + \phi_n) \\ g(z) &= \sum_{n=1}^{\infty} \tfrac{1}{2} A_n \sin(\tfrac{n\pi}{\ell} z - \phi_n) \end{aligned}$$

Thus, assuming $\alpha = \beta > 0$, $u(x,t)$ is the sum of two signals, one moving to the right and the other moving to the left. They move without changing shape, but their amplitudes decrease with time, due to the factors $e^{-\alpha t}$. If $\alpha \neq \beta$, different frequency components of $u(x,t)$ move with different speeds, because ω_n depends on n, and the signals distort as they propagate. This phenomenon is called dispersion.