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A boundary element formulation for the heat equation with dissipative and heat generation terms

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ABSTRACT

This article presents a formulation of the Boundary Element Method (BEM) for the study of heat diffusion in isotropic and homogeneous media. The proposed formulation has a time independent fundamental solution obtained from the two-dimensional Laplace equation. Consequently, the formulation is called D-BEM since it has domain integrals in the basic integral equation. The first order time derivative that appears in the integral equations is approximated by a backward finite difference scheme. Internal dissipative and heat generation terms are considered in the analyses. The results from the numerical model are compared with the available analytical solutions. The correlation estimator R^2 is employed to validate the numerical model and to demonstrate the accuracy of the proposed formulation.

1. Introduction

This work is concerned with the development of a Boundary Element Method (BEM) formulation for the solution of the heat equation with the presence of dissipative and heat generation terms. Before presenting the proposed formulation, a brief discussion concerning the different formulations that can be used for the solution of time-dependent problems by the BEM must be carried out. Different BEM formulations arise according to the nature of the fundamental solution employed, that is, according to the use of time-dependent or time independent fundamental solutions. In the first case, the so-called TD-BEM formulations arise (TD means time-domain); see, for instance, Wrobel [1], Young et al. [2]. Here, only the boundary discretization is required for the solution of problems with null initial condition, whereas problems that present non-homogeneous initial condition are solved with the discretization of the part of the domain where it appears. On the other hand, the steady-state fundamental solution, much simpler than the time-dependent one, can be used for performing time-domain analyses. The counterpart of the simplicity of the fundamental solution is the presence of a domain integral, whose kernel is constituted by the product of the fundamental solution by the first order time derivative of the temperature, in the basic BEM integral equation. The transformation of this domain integral into boundary integrals, by means of suitable interpolation functions, generates the DR-BEM formulation, DR meaning dual reciprocity; see Tanaka et al. [3], Singh and Tanaka [4], Ochiai et al. [5,6]. If the domain integral is kept into the integral equation, one has the so-called D-BEM formulation, with D meaning domain. In the D-BEM formulation, the discretization of the entire domain is mandatory. The disadvantage of the domain discretization is counterbalanced by the simplicity of the formulation and reliable results it produces; see Taigbenu and Liggett [7] and Carrer et al. [8]. Both the DR-BEM and the D-BEM formulations require the adoption of a time-marching scheme, that is, an approximation for the first-order time derivative: the simpler choice usually falls on the backward finite difference scheme [9].

Although this scheme can always produce accurate results, other alternatives were sought recently: in Carrer et al. [8], the backward finite difference is combined with the Houbolt approximation [10] and also an approach based on the subdomain collocation method, Finlayson [11], is presented. The level of accuracy, however, was the same achieved by the backward difference scheme. It is important to mention that attention has also been devoted to meshless approaches; see for instance Boztosun and Charafi [12].

The context in which the present work is situated is based on a D-BEM formulation and employs the backward finite difference as an approximation for the first order time derivative of the temperature.

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The main contribution of the work is the incorporation of the heat generation and of the dissipative terms. The domain discretization employed triangular linear cells, in which it was assumed a constant variation for the time derivative of the temperature. The boundary discretization employed linear elements. It is important to note that extra domain integrals appear in the formulation due to the presence of the dissipative and heat generation terms. As these extra domain integrals contain the fundamental solution in their integrand, their evaluation does not present any additional effort, as the domain integral related to the time derivative of the temperature has been already computed. Three examples are included and the numerical results are compared with the analytical solutions.

2. Mathematical model

The heat equation in a two dimensional isotropic and homogeneous domain Ω with boundary Γ is written as follows

$$\nabla^2 u(X,t) = \frac{1}{\alpha}\frac{\partial u(X,t)}{\partial t} \quad X \in \Omega, \; X=(x,y) \tag{1}$$

where α represents the coefficient of thermal diffusivity measured in m^2/s, u is the temperature, X is the field point and t is the time variable.

The boundary conditions are:

Essential

$$u(X,t) = \hat{u}(X,t) \quad X \in \Gamma_u \tag{2}$$

Natural

$$q(X,t) = \frac{\partial u(X,t)}{\partial n(X)} = \hat{q}(X,t) \quad X \in \Gamma_q \tag{3}$$

The initial condition at $t=t_0$ is given by

$$u(X,t) = u_0(X,t_0) \quad X \in \Omega \tag{4}$$

3. D-BEM formulation

The integral equation of the D-BEM formulation for the heat equation can be written as follows

$$C(\xi)u(\xi,t) = \int_\Gamma u*(\xi,X)q(X,t)d\Gamma - \int_\Gamma q*(\xi,X)u(X,t)d\Gamma + -\frac{1}{\alpha}\int_\Omega \frac{\partial u(X,t)}{\partial t}u*(\xi,X)d\Omega \tag{5}$$

where $C(\xi)$ is a geometric coefficient at the collocation point ξ, q is the thermal flux and $u*$ and $q*$ are the fundamental solution and its normal derivative, respectively.

The expression of the fundamental solution $u*(\xi,X)$ is given by Greenberg [13],

$$u*(\xi,X) = \frac{1}{2\pi}\ln\left(\frac{1}{r}\right) \tag{6}$$

where $r=|X-\xi|$ is the distance between field, X, and collocation, ξ, points.

The derivative of the fundamental solution with respect to the normal direction to the boundary is given by

$$q*(\xi,X) = \frac{\partial u*}{\partial r}\frac{dr}{dn} = -\frac{1}{2\pi r}\frac{dr}{dn} \tag{7}$$

where n is the outward normal to the boundary.

For simplicity, the time derivative presented in Eq. (5) is approximated by the backward finite difference formula [9]:

$$\frac{\partial u(X,t)}{\partial t} = \frac{u(X,t+\Delta t) - u(X,t)}{\Delta t} \tag{8}$$

Replacing (8) in (5) and grouping terms conveniently, one has

$$C(\xi)u(\xi,t+\Delta t) = \int_\Gamma u*(\xi,X)q(X,t+\Delta t)d\Gamma - \int_\Gamma q*(\xi,X)u(X,t+\Delta t)d\Gamma + -\frac{1}{\alpha\Delta t}\left(\int_\Omega u(X,t+\Delta t)u*(\xi,X)d\Omega - \int_\Omega u(X,t)u*(\xi,X)d\Omega\right) \tag{9}$$

Eq. (9) can be used recursively for the solution of the problem, starting at time t_m and determining the variables at time t_{m+1}. According to Wrobel [1], the critical time step, Δt_c, can be estimated as

$$\Delta t_c \le \frac{L_j^2}{2\alpha} \tag{10}$$

where L_j is the boundary element size.

For the solution of the problem, the boundary Γ is divided into boundary elements Γ_j, approximating the geometry of each element Γ_j with linear interpolation functions. Along each element the variables of the problem (potential and flux) are approximated by linear continuous approximation functions, see Brebbia [14].

In the domain discretization, it was assumed a constant behavior of the potential within each cell. A generic cell is defined by vertices $k_1(x_1, y_1)$, $k_2(x_2, y_2)$ and $k_3(x_3, y_3)$, as seen in Fig. 1.

The cell integrals are calculated numerically using a local coordinate transformation as illustrated in Fig. 2, and the global coordinates are defined by,

$$x = (1-U)x_1 + U[(1-V)x_2 + Vx_3]$$
$$y = (1-U)y_1 + U[(1-V)y_2 + Vy_3] \tag{11}$$

The Jacobian of the transformation (Eq. (12)) is equal to the double of the cell area A:

$$|J| = \begin{vmatrix} \frac{\partial x}{\partial U} & \frac{\partial x}{\partial V} \\ \frac{\partial y}{\partial U} & \frac{\partial y}{\partial V} \end{vmatrix} = 2A \tag{12}$$

The non-singular cell integrals are computed with Gaussian quadrature according to

$$\int_\Omega u*(\xi,X)d\Omega = \sum_{i=1}^{i=M}\sum_{j=1}^{j=M}\int_0^1\int_0^{1-U} u*(\xi_i,X_j)|J_j|dVdU \quad for \quad i \ne j \tag{13}$$

where M is the number of cells.

In the cases where the source point is located in the cell domain, a weak singularity is present and the integration is carried out using the third order coordinate transformation proposed by Telles [15]. For source points in the centroid of the cell, a cell subdivision is performed as illustrated in Fig. 3 and each triangular part is integrated with the same approach.

After applying Eq. (9) to the boundary nodes and internal points, one obtains the following system of equations:

$$\begin{bmatrix} \mathbf{H}^{bb} & \frac{1}{\alpha\Delta t}\mathbf{M}^{bd} \\ \mathbf{H}^{db} & \mathbf{I}+\frac{1}{\alpha\Delta t}\mathbf{M}^{dd} \end{bmatrix}\begin{bmatrix} \mathbf{u}^b \\ \mathbf{u}^d \end{bmatrix}_{m+1} = \begin{bmatrix} \mathbf{G}^{bb} \\ \mathbf{G}^{db} \end{bmatrix}\left[\mathbf{q}^b\right]_{m+1} + \frac{1}{\alpha\Delta t}\begin{bmatrix} \mathbf{M}^{bd} \\ \mathbf{M}^{dd} \end{bmatrix}\left[\mathbf{u}^d\right]_m \tag{14}$$

In Eq. (14), **H** and **G** are matrices which result from the boundary integrals related to $q*(\xi,X)u(x)$ and to $u*(\xi,X)q(x)$, respectively; the matrix **M** results from the domain integrals and **I** is the identity matrix. The first element of each superscript indicates the position of the source point and the second, the position of the field point, with b indicating boundary and d indicating domain. The subscript $m+1$ indicates the time $t_{m+1}=(m+1)$ and the subscript m, the time $t_m = m\Delta t$, where Δt

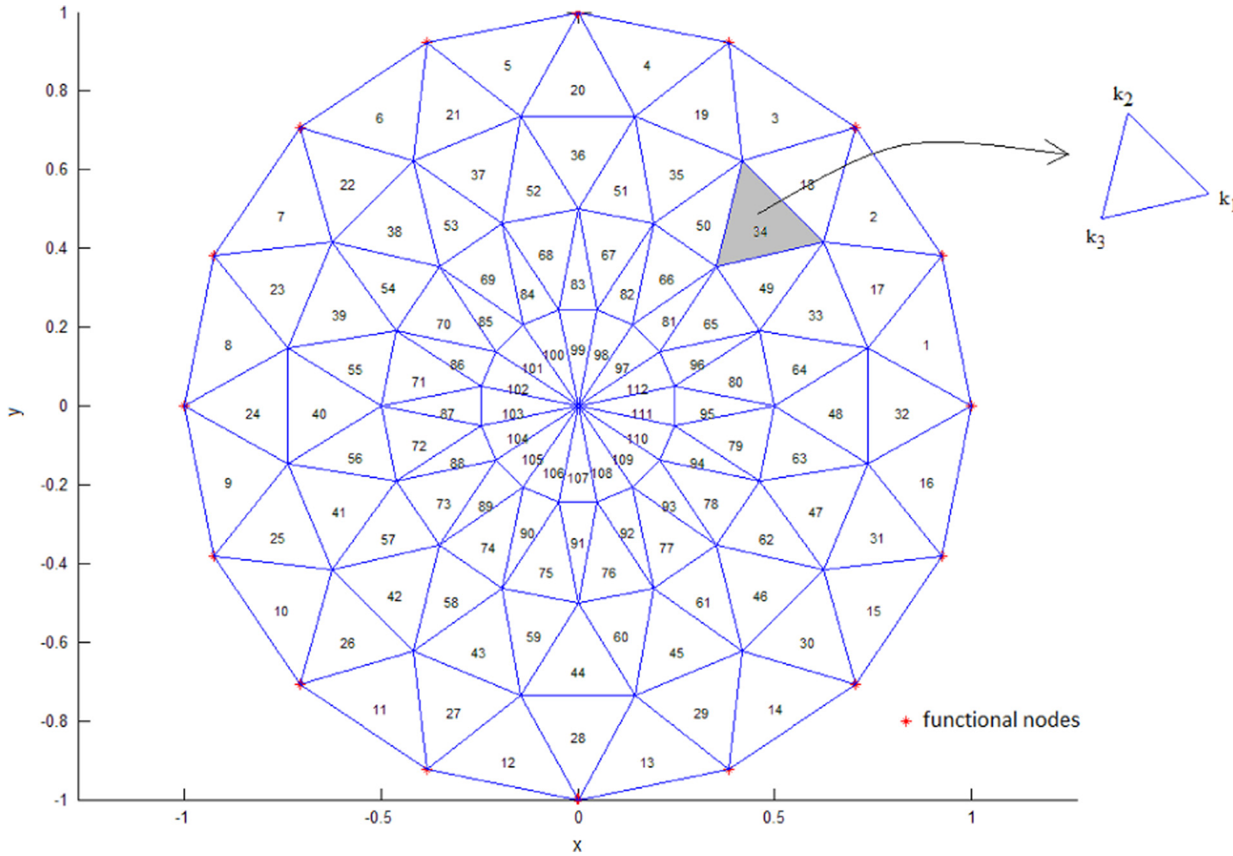

Fig. 1. Illustration of the discretization of the problem domain with triangular cells.

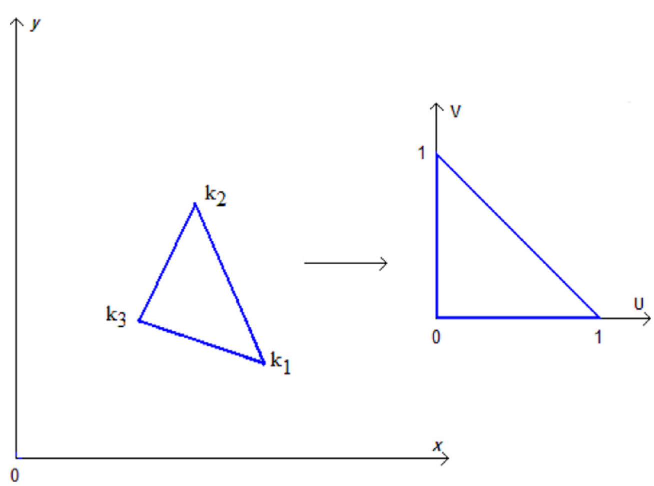

Fig. 2. Transformation of coordinates for the cell integration.

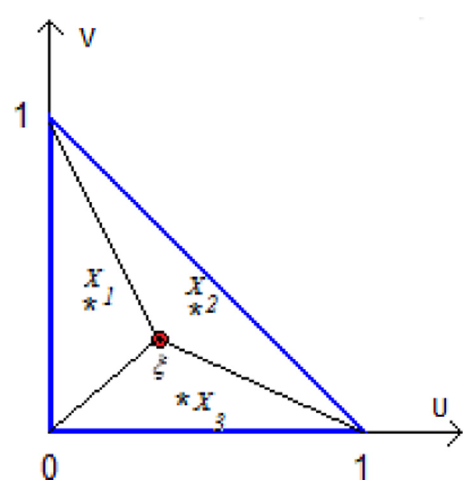

Fig. 3. Illustration of the cell subdivision in singular integrations.

is the time interval. In the formulation presented in this work was adopted a constant value for Δt, computed according to Eq. (10).

After imposing the initial conditions and the boundary conditions, Eq. (14) is solved and the unknowns (potential and/or flux at the boundary nodes and potential at the internal points) are determined for the time t_{m+1}. The potential values are updated and the procedure continues in a recursive way.

In general, after imposing the boundary conditions, one can write:

$$\mathbf{A}x^{m+1} = y^{m+1} + y^m \tag{15}$$

where $m \geq 0$ and:

- x^{m+1} is the vector of unknown nodal values at time t_{m+1};
- **A** is the coefficient matrix that contains terms relating to **H**, **G** and **M**;
- y^{m+1} is a vector that represents the contribution of the time t_{m+1}, and y^m, the contribution of the time t_m.

Thus:

$$x^{m+1} = \mathbf{A}^{-1}(y^{m+1} + y^m) \tag{16}$$

The boundary element formulation was implemented in Matlab software.

3.1. Numerical example: circular disk with prescribed temperature at the boundary

This problem consists of a flat disk with radius $R=1$ m (Fig. 4).

The problem to be solved presents the following boundary and initial conditions, respectively

$$u(X,t)=10 \quad X \in \Gamma \tag{17}$$

$$u_0(X,t_0)=0 \quad X \in \Omega \tag{18}$$

The problem was discretized with 16 linear boundary elements and 112 triangular cells, as shown in Fig. 1.

In order to assess the accuracy of the BEM results, a comparison is carried out with the analytical solution given by Greenberg [16]:

$$u(r,t)=\overline{u}-\frac{2\overline{u}}{R}\sum_{n=1}^{\infty}\frac{J_0(\lambda_n r)}{\lambda_n J_1(\lambda_n R)}e^{-\alpha\lambda_n^2 t} \tag{19}$$

where J_0 and J_1 are Bessel functions of the first kind of order zero and one, respectively. The parameters λ_n are the positive roots of the equation $J_0(\lambda_n)=0$. In this article, $n=100$.

To check the significance of the values obtained numerically, the statistical method of linear regression was applied to the numerical and the analytical results, both evaluated at the center of the disc, and the coefficient of determination R^2 was computed [17].

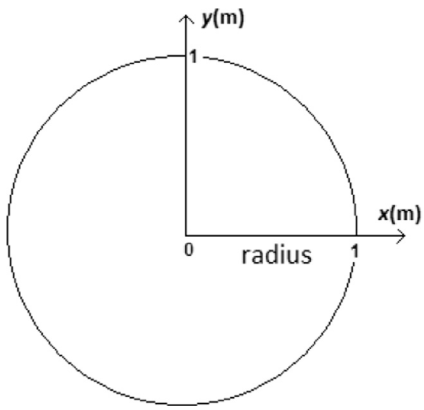

Fig. 4. Illustration of the problem.

Fig. 5 shows analytical and numerical values that represent the temperature evolution at the center of the disk for different thermal diffusivity values.

In the simulations, it was found $R^2=0.99992$, 0.99998, 0.99968 and 0.99686, for $\alpha=0.7$, 1.0, 1.5 and 3.0 m^2/s, respectively, indicating a high correlation ($|0.86| \le R^2 \le |1.0|$) between the numerical and analytical solutions. It is interesting to point out that these results were obtained by using a single integration point in each domain cell. Table 1 shows that the same level of accuracy was obtained for the results at the central point of the disk using 1, 4 or 9 integration points in each cell integration.

Fig. 6 illustrates the heat diffusion process at specific time instants for the case with $\alpha=1.0$ m^2/s.

From Fig. 6 it can be seen the gradual elevation of the temperature in the disk, where it is clearly seen the heat flowing from the boundary toward the center. Similar results were also observed for the cases $\alpha=0.7$, 1.5 and 3.0 m^2/s.

The boundary element formulation and results just presented are well known Wrobel [1]. This formulation is now extended to account for the presence of a dissipative term and a non-homogeneous heat generation term in the partial differential equation. The implementation of each term is performed separately.

4. D-BEM formulation with dissipative and non-homogeneous terms

4.1. Heat equation with dissipative term

The Heat Equation with dissipative term, according to Zill and Cullen [18], is given by

$$\nabla^2 u(X,t)-hu(X,t)=\frac{1}{\alpha}\frac{\partial u(X,t)}{\partial t}$$
$$X \in \Omega, \quad X=(x,y);\, h>0;\, t>0 \tag{20}$$

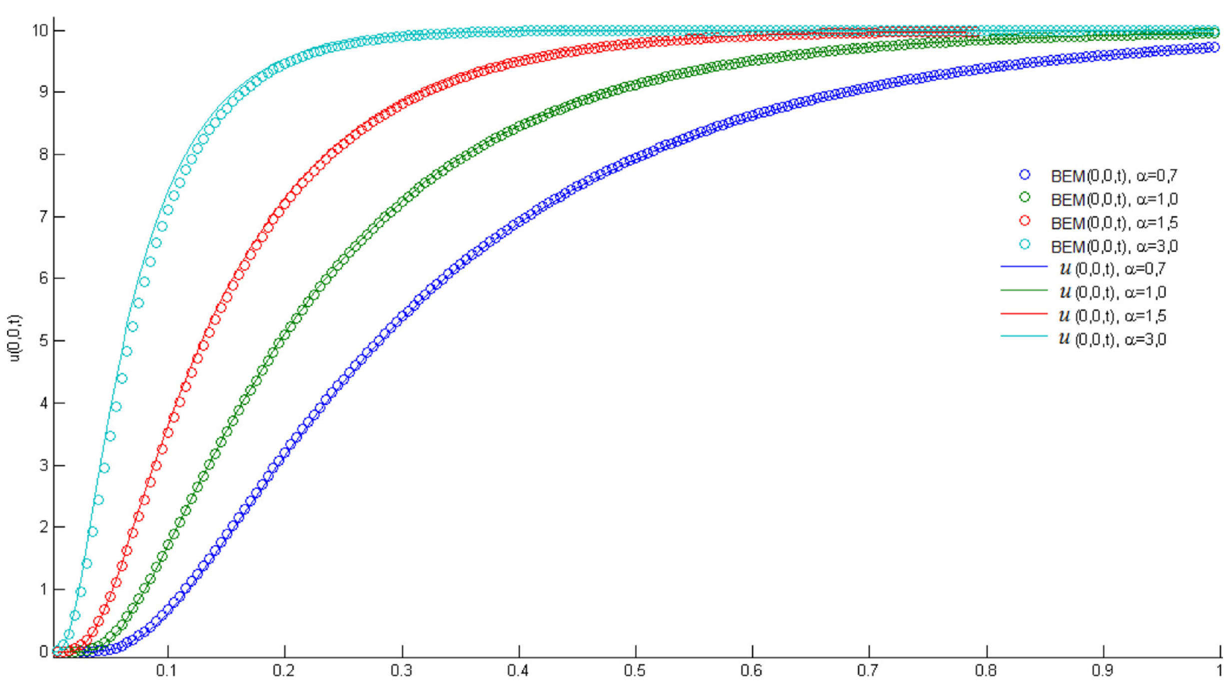

Fig. 5. BEM results at the central point of the circular disc.

The corresponding D-BEM integral equation is:

$$C(\xi)u(\xi,t)=\int_{\Gamma}u*(\xi,X)q(X,t)d\Gamma-\int_{\Gamma}q*(\xi,X)u(X,t)d\Gamma$$
$$+-\int_{\Omega}hu(X,t)u*(\xi,X)d\Omega-\frac{1}{\alpha}\int_{\Omega}\frac{\partial u(X,t)}{\partial t}u*(\xi,X)d\Omega \quad (21)$$

After approximating the first order time derivative by the backward finite difference, one has:

$$C(\xi)u(\xi,t+\Delta t)=\int_{\Gamma}u*(\xi,X)q(X,t+\Delta t)d\Gamma$$
$$+-\int_{\Gamma}q*(\xi,X)u(X,t+\Delta t)d\Gamma-\int_{\Omega}hu(X,t)u*(\xi,X)d\Omega$$
$$+-\frac{1}{\alpha\Delta t}\left(\int_{\Omega}u(X,t+\Delta t)u*(\xi,X)d\Omega-\int_{\Omega}u(X,t)u*(\xi,X)d\Omega\right) \quad (22)$$

Table 1
R^2 results at disk center point for different number of integration points at each domain cell.

Number of Gaussian points	R^2
1	0,99998
4	0,99992
9	0,99996

The matrix form of Eq. (22) is:

$$\begin{bmatrix}\mathbf{H}^{bb} & (\frac{1}{\alpha\Delta t}+h)\mathbf{M}^{bd}\\ \mathbf{H}^{db} & \mathbf{I}+(\frac{1}{\alpha\Delta t}+h)\mathbf{M}^{dd}\end{bmatrix}\begin{bmatrix}\mathbf{u}^{b}\\ \mathbf{u}^{d}\end{bmatrix}_{m+1}=\begin{bmatrix}\mathbf{G}^{bb}\\ \mathbf{G}^{db}\end{bmatrix}\begin{bmatrix}\mathbf{q}^{b}\end{bmatrix}_{m+1}+\frac{1}{\alpha\Delta t}\begin{bmatrix}\mathbf{M}^{bd}\\ \mathbf{M}^{dd}\end{bmatrix}\begin{bmatrix}\mathbf{u}^{d}\end{bmatrix}_{m} \quad (23)$$

4.2. Numerical example: circular disk with initial conditions

To validate the numerical implementation the same disk of the previous example is analyzed with the same boundary and domain discretizations (number of elements and cells) presented in the first problem. The applied boundary and initial conditions are:

$$u(X,t)=0 \quad X\in\Gamma \quad (24)$$

$$u_0(X,t_0)=1 \quad X\in\Omega \quad (25)$$

The analytical solution is given by (Zill and Cullen, 2001):

$$u(r,t)=2e^{-ht}\sum_{n=1}^{\infty}\frac{J_0(\lambda_n r)}{\lambda_n J_1(\lambda_n R)}e^{-\alpha\lambda_n^2 t} \quad (26)$$

Results are presented in Fig. 7 where the BEM and analytical temperature results are compared at the center of the disk for three different thermal diffusivity conditions. Again, a very good correlation was obtained in all cases: $R^2=0.99788$, 0.99988, 0.99949 and 0.99931, for $\alpha=0.7$, 1.0, 1.5 and 3.0 m^2/s, respectively. Fig. 8 illustrates the thermal status at the disk domain at different times for $\alpha=1.0$ m^2/s.

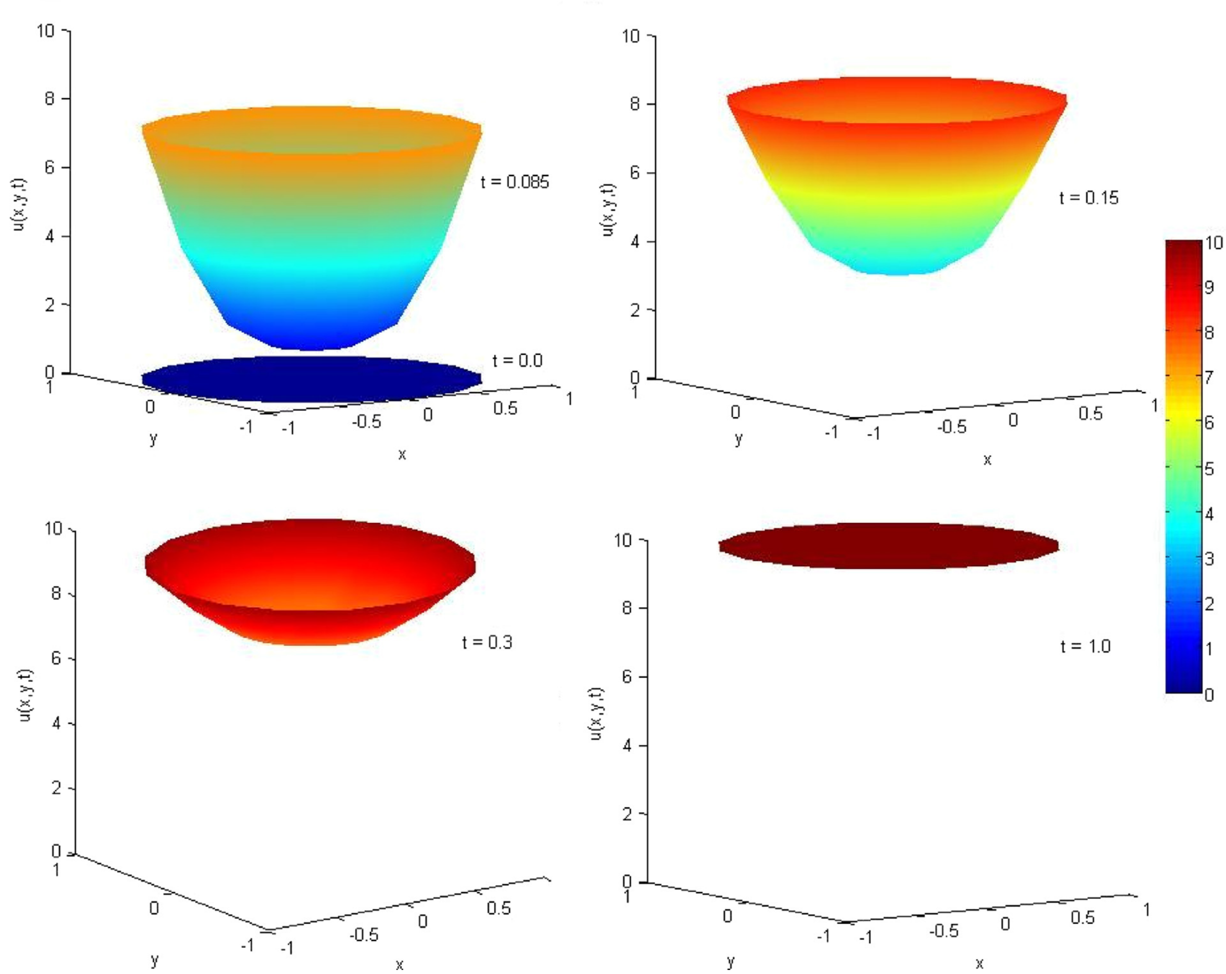

Fig. 6. Solution at the domain of the problem at different times.

4.3. Heat equation with non-homogeneous heat generation term

According to Wall [19], the heat equation containing a non-homogeneous internal heat generation term $F(X,t)$ is given by:

$$\nabla^2 u(X,t) + F(X,t) = \frac{1}{\alpha}\frac{\partial u(X,t)}{\partial t}$$
$$X \in \Omega, \quad X = (x,y) \tag{27}$$

The corresponding D-BEM integral equation is:

$$C(\xi)u(\xi,t) = \int_\Gamma u*(\xi,X)q(X,t)d\Gamma - \int_\Gamma q*(\xi,X)u(X,t)d\Gamma + -\frac{1}{\alpha}\int_\Omega \frac{\partial u(X,t)}{\partial t}u*(\xi,X)d\Omega + \int_\Omega F(X,t)u*(\xi,X)d\Omega \tag{28}$$

Again, approximating the first order time derivative in the first domain integral on the right-hand-side of Eq. (28), one has:

$$C(\xi)u(\xi,t+\Delta t) = \int_\Gamma u*(\xi,X)q(X,t+\Delta t)d\Gamma - \int_\Gamma q*(\xi,X)u(X,t+\Delta t)d\Gamma + -\frac{1}{\alpha\Delta t}\left(\int_\Omega u(X,t+\Delta t)u*(\xi,X)d\Omega - \int_\Omega u(X,t)u*(\xi,X)d\Omega\right) + \int_\Omega F(X,t)u*(\xi,X)d\Omega \tag{29}$$

Eq. (33), in matrix form, is written as:

$$\begin{bmatrix} \mathbf{H}^{bb} & \frac{1}{\alpha\Delta t}\mathbf{M}^{bd} \\ \mathbf{H}^{db} & \mathbf{I}+\frac{1}{\alpha\Delta t}\mathbf{M}^{dd} \end{bmatrix} \begin{bmatrix} \mathbf{u}^b \\ \mathbf{u}^d \end{bmatrix}_{m+1} = \begin{bmatrix} \mathbf{G}^{bb} \\ \mathbf{G}^{db} \end{bmatrix} \left[\mathbf{q}^b\right]_{m+1} + \frac{1}{\alpha\Delta t}\begin{bmatrix} \mathbf{M}^{bd} \\ \mathbf{M}^{dd} \end{bmatrix} \left[\mathbf{u}^d\right]_m + \begin{bmatrix} \mathbf{F}^{bb} \\ \mathbf{F}^{dd} \end{bmatrix}_{m+1} \tag{30}$$

where the vector **F** comes from the integral domain containing the non-homogeneous term.

4.4. Numerical example: circular disk with initial conditions

The numerical implementation also is validated with the same geometry and discretizations (number of elements and cells) presented in the first problem. The applied boundary and initial conditions are null and the term corresponding to the heat source is given by:

$$F(X,t) = \frac{1}{k} = 10 \quad X \in \Omega, \quad 0 < t < \infty \tag{31}$$

The analytical solution of this problem, in polar coordinates, is given by Wall [19]:

$$u(r,t) = \frac{R^2 - r^2}{4k} - \frac{2}{Rk}\sum_{n=1}^{\infty}\frac{J_0(\lambda_n r)}{\lambda_n^3 J_1(\lambda_n R)}e^{-\alpha\lambda_n^2 t} \tag{32}$$

Results are presented in Fig. 9, where the BEM and analytical results are compared for three different thermal diffusivity conditions. A very good correlation R^2, 0.99991, 0.99981, 0.99962 and 0.99888 was obtained, for α=0.7, 1.0, 1.5 and 3.0 m^2/s, respectively.

The thermal history is illustrated in Fig. 10 with domain results at different times for α=1.0 m^2/s.

5. Conclusions

A D-BEM formulation was implemented for the study of heat diffusion in isotropic and homogeneous media, including dissipative and heat generation terms. A time independent fundamental solution was employed and domain integrals were discretized with constant triangular cells. The first order time derivative in the formulation was approximated by a backward finite difference scheme. Cell domain numerical integrations were performed with a single Gaussian point.

The results from the numerical models were compared with analytical solutions showing very good correlation $R^2 > 0.99$ in all

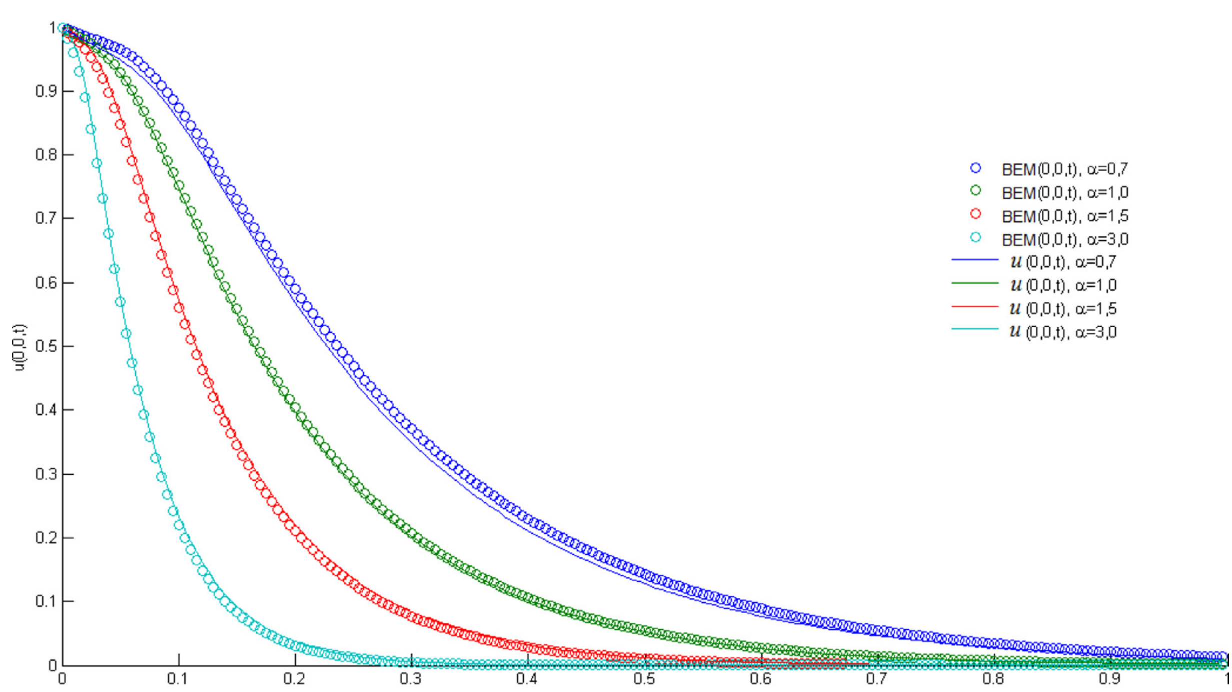

Fig. 7. BEM results at the central point of the circular disc for the heat equation with dissipative term.

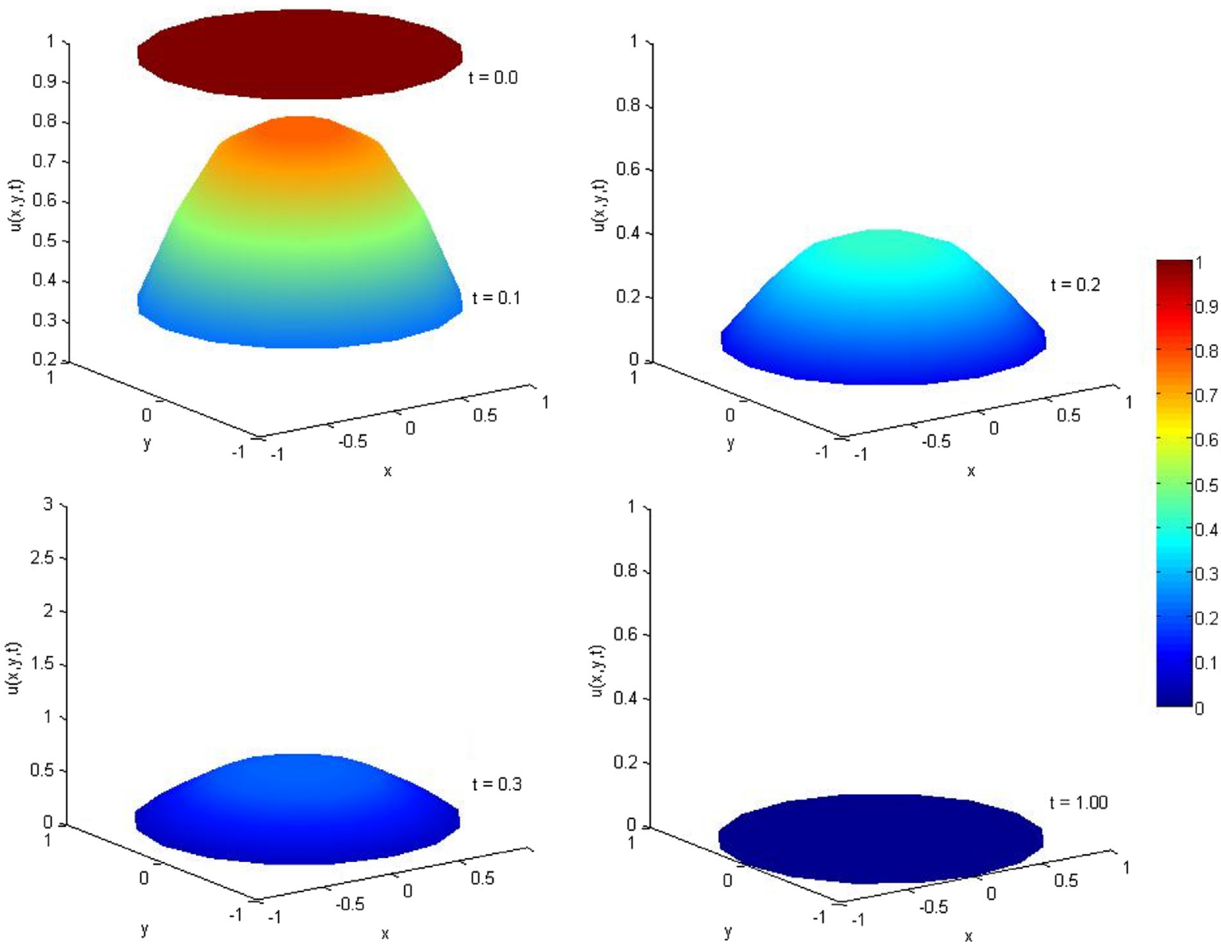

Fig. 8. Solution at the domain of the problem at different times for the heat equation with dissipative term and $\alpha = 1.0$ m^2/s.

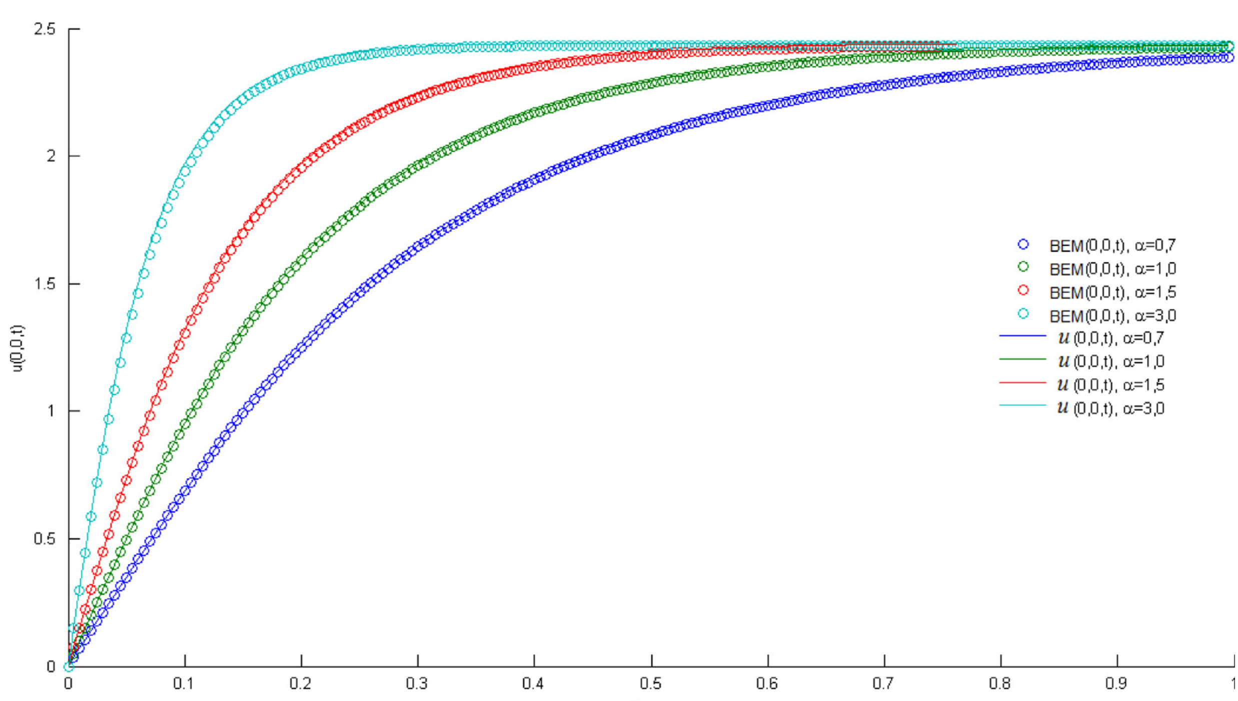

Fig. 9. BEM results for the heat equation with non-homogeneous term.

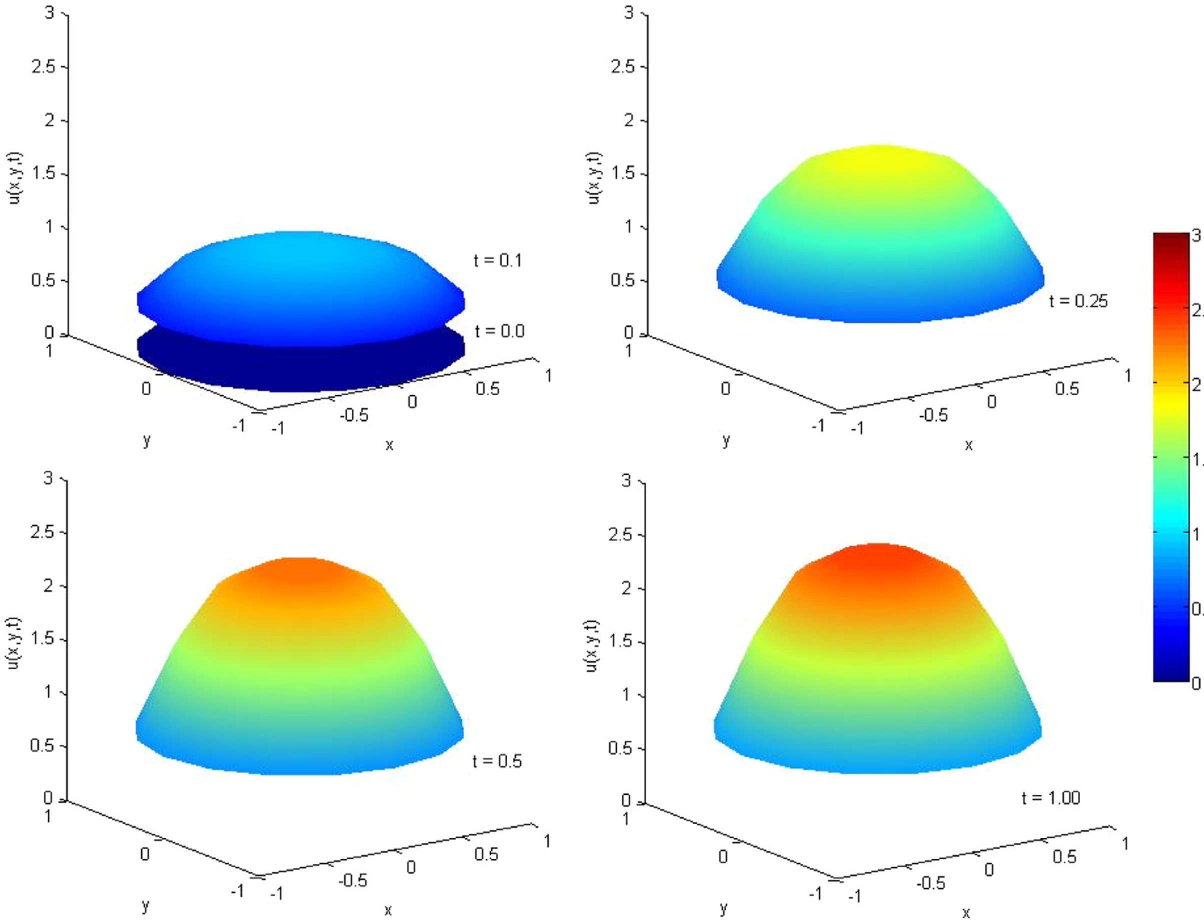

Fig. 10. Solution at the domain of the problem at different times for the heat equation with non-homogeneous term and $\alpha = 1.0\ m^2/s$.

cases: i) classical diffusion equation; ii) with dissipative term and; iii) with non-homogeneous heat generation term.

Despite the need for domain discretization, the results presented in the previous examples confirm the effectiveness of the D-BEM formulation for the evaluation of time-domain transient potential problems.

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