Séries de Potências

Lembremos do cálculo que uma **série de potências** (em potências de $x - x_0$) é uma série infinita da forma

(1) $$\sum_{m=0}^{\infty} a_m(x - x_0)^m = a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + \cdots.$$

Aqui, x é uma variável e a_0, a_1, a_2, $\cdots$ são constantes chamadas de **coeficientes** da série. x_0 é uma constante, chamada de **centro** da série. Em particular, se $x_0 = 0$, obtemos uma **série de potências expressa em potências de x**

(2) $$\sum_{m=0}^{\infty} a_m x^m = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots.$$

Suporemos que todas as variáveis e constantes sejam reais.

Exemplos familiares de séries de potências são as séries de Maclaurin

$$\frac{1}{1-x} = \sum_{m=0}^{\infty} x^m = 1 + x + x^2 + \cdots \qquad (|x| < 1, \textbf{ série geométrica})$$

$$e^x = \sum_{m=0}^{\infty} \frac{x^m}{m!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

$$\cos x = \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{(2m)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - + \cdots$$

$$\operatorname{sen} x = \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m+1}}{(2m+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - + \cdots.$$

Observamos que o termo "série de potências" usualmente se refere à forma (1) [ou (2)], porém ***não inclui*** séries de potências negativas ou fracionárias de x.

01) $y' - y = 0 \qquad y(t) = \sum_{n=0}^{\infty} a_n t^n, \qquad y'(t) = \sum_{n=0}^{\infty} a_n\, n t^{n-1}$

$$\sum_{n=0}^{\infty} a_n\, n t^{n-1} - \sum_{n=0}^{\infty} a_n t^n = 0, \quad \textit{Fazendo } k = n-1,\ n = k+1, \textit{no } 1^o \textit{ somatório. No } 2^o,\ n = k$$

$$\sum_{k=0}^{\infty} a_{k+1}(k+1)t^k - \sum_{k=0}^{\infty} a_k t^k = 0$$

$$\sum_{k=0}^{\infty} \left(a_{k+1}(k+1) - a_k\right)t^k = 0$$

$$a_{k+1}(k+1) - a_k = 0$$

$$k = 0, \quad a_1 = a_0/1$$

$$k = 1, \quad a_2(2) - a_1 = 0 ------> a_2 = a_1/2 = a_0/(1\times 2)$$

$$k = 2, \quad a_3(3) - a_2 = 0 ------> a_3 = a_2/2 = a_0/6 = a_0/(1\times 2\times 3)$$

$$k = 3, \quad a_4(4) - a_3 = 0 ------> a_4 = a_3/4 = a_0/(24) = a_0/(1\times 2\times 3\times 4)$$

$$y(t) = \sum_{n=0}^{\infty} a_0 \frac{t^n}{n!} = a_0 e^t$$

Fórmula Hadamard

$R = raio\ de\ \ convergência,\ I = \text{int}\, ervalo\, de\, convergência$

$$\frac{1}{R} = \lim_{n\to\infty}\left(\frac{a_{n+1}}{a_n}\right), \qquad a_n = \frac{1}{n!}$$

$$\frac{1}{R} = \lim_{n\to\infty}\left(\frac{\frac{1}{(n+1)!}}{\frac{1}{n!}}\right), \frac{1}{R} = \lim_{n\to\infty}\left(\frac{n!}{(n+1)!}\right), \frac{1}{R} = \lim_{n\to\infty}\left(\frac{n!}{(n+1)\times n!}\right), \frac{1}{R} = \lim_{n\to\infty}\left(\frac{1}{(n+1)}\right) \to 0$$

$R \to \infty$

$I = \text{int}\, ervalo\, de\, convergência$

$I = (c - R, c + R)$

$I = (-\infty, \infty)$

$$02)\, y''+y=0 \qquad y(t)=\sum_{n=0}^{\infty} a_n t^n, \qquad y'(t)=\sum_{n=0}^{\infty} a_n\, n\, t^{n-1}, \qquad y''(t)=\sum_{n=0}^{\infty} a_n\,(n)(n-1)\,t^{n-2}$$

$$\sum_{n=0}^{\infty} a_n\,(n)(n-1)\,t^{n-2}+\sum_{n=0}^{\infty} a_n t^n=0, \quad \textit{Fazendo } k=n-2,\ n=k+2, \textit{no } 1^{o} \textit{ somatório. No } 2^{o},\ n=k$$

$$\sum_{k=0}^{\infty}\ a_{k+2}\,(k+2)(k+1)\,t^k+\sum_{k=0} a_k t^k=0$$

$$\sum_{k=0}^{\infty}\ \left(a_{k+2}\,(k+2)(k+1)+a_k\right)t^k=0$$

$$a_{k+2}(k+2)(k+1)+a_k=0$$

$$a_{k+2}=-\frac{a_k}{(k+2)(k+1)}$$

$k=0,\ a_2=-a_0/2$

$k=1,\ a_3=-a_1/6$

$k=2,\ a_4=-a_2/12=a_0/(2.12)=a_0/(2.3.4)$

$k=3,\ a_5=-a_3/20=a_1/(6.20)=a_1/(2.3.4.5)$

$k=4,\ a_6=-a_4/30=-a_0/(2.12.30)=-a_0/(2.3.4.5.6)$

$k=5,\ a_7=-a_5/42=-a_1/(6.20.42)=-a_1/(2.3.4.5.6.7)$

$a0$	$a1$	$a2$	$a3$	$a4$	$a5$	$a6$	$a7$
a_0	a_1	$-\frac{a_0}{2}$	$-\frac{a_1}{2.3}$	$\frac{a_0}{2.3.4}$	$\frac{a_1}{2.3.4.5}$	$-\frac{a_0}{2.3.4.5.6}$	$-\frac{a_1}{2.3.4.5.6.7}$

$$a_0 \quad -\frac{a_0}{2} \quad + \quad \frac{a_0}{2.3.4} \quad - \quad \frac{a_0}{2.3.4.5.6}+....=\frac{a_0(-1)^n}{(2n)!}$$

$n=0 \qquad 1 \qquad 2.....$

$$a_1 \quad -\frac{a_1}{2.3} \quad + \quad \frac{a_1}{2.3.4.5} \quad -\frac{a_1}{2.3.4.5.6.7} \quad +...=\frac{a_1(-1)^n}{(2n+1)!}$$

$n=0 \qquad 1 \qquad 2.....$

$$y(t)=\sum_{n=0}^{\infty}\ \frac{a_0(-1)^n}{(2n)!}t^{2n}+\sum_{n=0}^{\infty}\ \frac{a_1(-1)^n}{(2n+1)!}t^{2n+1}=a_0\cos(t)+a_1 sen(t)$$

```
closeall

Ex=2
if Ex==1
%% 01)  Considere a EDO y'-y=0,  a0=k.
% A solução em série de potências é Somatório(a0/(n!)x^n).
% A solução analítica é y(x)=kexp(x)

%% SOLUÇÃO ANALÍTICA
k=1;
a0=k;
f = @(a0,x,n) a0/(factorial(n))*(x.^n);

%% PARÂMETROS
% Comprimento da barra ---> L
% Discretizar a barra em X + 1 elememtos
% Número de somatórios N
L=10;
X=100;
N=20;
s=[];
for x=0:L/X:L
soma=0;
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```
for n=0:N
soma=soma+f(a0,x,n);
end
        s=[s; soma];
end
x=0:L/X:L;
ye=a0*exp(x);  % Solução analítica
figure('Color',[1 1 1]);
plot(x,ye,'b')
holdon
plot(x,s,'or')
title(['Analytical Solution with N = ',num2str(N)])
xlabel('x')
ylabel('Value of y(x)')
boxoff
legend(['Analytical solution'],['Power series solution'])

else

%% 02)  Considere a EDO y''+ y = 0,  a0=k.
% A solução em série de potências é Somatórioa0*(-1).^n/factorial(2*n)*(x.^(2*n))+a1*(-1).^n/factorial(2*n+1)*(x.^(2*n+1)).
% A solução analítica é y(x)=a0cos(x)+a1sin(x)

%% SOLUÇÃO ANALÍTICA
a0=1;
a1=1;
f = @(a0,a1,x,n) a0*(-1).^n/factorial(2*n)*(x.^(2*n))+a1*(-1).^n/factorial(2*n+1)*(x.^(2*n+1));

%% PARÂMETROS
% Comprimento da barra ---> L
% Discretizar a barra em X + 1 elememtos
% Número de somatórios N
L=2*pi;
X=100;
N=100;
s=[];
for x=0:L/X:L
soma=0;
for n=0:N
soma=soma+f(a0,a1,x,n);
end
        s=[s; soma];
end
x=0:L/X:L;
ye=a0*cos(x)+a1*sin(x);  % Solução analítica
figure('Color',[1 1 1]);
plot(x,ye,'b')
holdon
plot(x,s,'or')
title(['Analytical Solution with N = ',num2str(N)])
xlabel('x')
ylabel('Value of y(x)')
boxoff
legend(['Analytical solution'],['Power series solution'])

end
```

$A)\ y''-ty=3, \qquad a_0=1,\ a_1=0$

$$\sum_{n=2}^{\infty} n(n-1)a_n t^{n-2} - t\sum_{n=0}^{\infty} a_n t^n = 3$$

$$\sum_{n=2}^{\infty} n(n-1)a_n t^{n-2} - \sum_{n=0}^{\infty} a_n t^{n+1} = 3 \ , \qquad n-2=k+1,\ \ n=k+3$$

$$\sum_{k+3=2}^{\infty} (k+3)(k+3-1)a_{k+3} t^{k+3-2} - \sum_{n=0}^{\infty} a_n t^{n+1} = 3$$

$$\sum_{k=-1}^{\infty} (k+3)(k+2)a_{k+3} t^{k+1} - \sum_{n=0}^{\infty} a_n t^{n+1} = 3$$

$$(-1+3)(-1+2)a_{-1+3}t^{-1+1} + \sum_{k=0}^{\infty} (k+3)(k+2)a_{k+3} t^{k+1} - \sum_{n=0}^{\infty} a_n t^{n+1} = 3$$

$$2a_2 + \sum_{k=0}^{\infty} (k+3)(k+2)a_{k+3} t^{k+1} - \sum_{n=0}^{\infty} a_n t^{n+1} = 3 + \sum_{n=0}^{\infty} (0) t^{n+1}$$

$2a_2 = 3------> a_2 = 3/2 \quad$ *e sabendo que* $a_0=1, a_1=0$

$$(k+3)(k+2)a_{k+3}t^{k+1} - a_k t^{k+1} = 0$$

$$t^{k+1}((k+3)(k+2)a_{k+3} - a_k) = 0 \ \ ----> (k+3)(k+2)a_{k+3} - a_k = 0$$

$$a_{k+3} = \frac{a_k}{(k+3)(k+2)}$$

			$k=0$	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$	$k=6$
a_0	a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8	a_9
1	0	$\frac{3}{2}$	$\frac{1}{2.3}$	0	$\frac{3}{2.4.5}$	$\frac{1}{2.3.5.6}$	0	$\frac{3}{2.4.5.7.8}$	$\frac{1}{2.3.5.6.8.9}$
1	0	$\frac{3}{2}$	$\frac{1}{3!}$	0	$\frac{3.3}{5!}$	$\frac{4}{6!}$	0	$\frac{2.3.3.3}{8!}$	$\frac{1.2.2.7}{9!}$

$$y(t) = 1 + 0t^1 + \frac{3}{2}t^2 + \frac{1}{3!}t^3 + 0t^4 + \frac{3.3}{5!}t^5 + \frac{4}{6!}t^6 + 0t^7 + \frac{2.3.3.3}{8!}t^8 + \frac{1.2.2.7}{9!}t^9 + ...$$

$$B)\, y''+y=1 \;\; ,a_0=1, a_1=1, \qquad y(t)=\sum_{n=0}^{\infty} a_n t^n, \quad y'(t)=\sum_{n=1}^{\infty} n a_n t^{n-1}, \quad y''(t)=\sum_{n=2}^{\infty} n(n-1) a_n t^{n-2}$$

$$\sum_{n=2}^{\infty} n(n-1) a_n t^{n-2} + \sum_{n=0}^{\infty} a_n t^n = 1$$

$Fazendo, \qquad n-2=k, \;\; n=k+2$

$$\sum_{k=0}^{\infty} (k+2)(k+1) a_{k+2} t^k + \sum_{k=0}^{\infty} a_k t^k = 1 + 0t^k$$

$$(k+2)(k+1)a_{k+2} + a_k = 0$$

$$k=0 \;---> \quad (2)(1)a_2 + a_0 = 1 -----> a_2 = 0$$

$$a_{k+2} = \frac{-a_k}{(k+1)(k+2)}$$

			$k=1$	$k=2$	$k=3$	$k=4$	$k=5$	$k=6$	$k=7$
a_0	a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8	a_9
1	1	0	$-\frac{1}{2.3}$	0	$\frac{1}{2.3.4.5}$	0	$-\frac{1}{2.3.4.5.6.7}$	0	$\frac{1}{2.3.4.5.6.7.8.9}$
1	1	0	$-\frac{1}{3!}$	0	$\frac{1}{5!}$	0	$-\frac{1}{7!}$	0	$\frac{1}{9!}$

$$y(t) = 1 + \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m+1)!} t^{2m+1}$$

$$y(t) = 1 + sen(t)$$

$$C)\, y''+ty'+y=0, \quad a_0=1, \;\; a_1=0 \qquad y(t)=\sum_{n=0}^{\infty} a_n t^n, \quad y'(t)=\sum_{n=0}^{\infty} n a_n t^{n-1}, \quad y''(t)=\sum_{n=2}^{\infty} n(n-1) a_n t^{n-2}$$

$$\sum_{n=2}^{\infty} n(n-1) a_n t^{n-2} + t\sum_{n=0}^{\infty} n a_n t^{n-1} + \sum_{n=0}^{\infty} a_n t^n = 0$$

$Fazendo, \qquad n-2=k, \;\; n=k+2 \; no \; 1^o \; \sum$

$$\sum_{k=0}^{\infty} (k+2)(k+1) a_{k+2} t^k + \sum_{k=0}^{\infty} a_k (k) t^k + \sum_{k=0}^{\infty} a_k t^k = 0$$

$$a_{k+2} = \frac{-(k+1)a_k}{(k+1)(k+2)} ----> a_{k+2} = \frac{-a_k}{(k+2)}$$

		$k=0$	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$	$k=6$	$k=7$	
a_0	a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8	a_9	
1	0	$-\frac{1}{2}$	0	$\frac{1}{2.4}$	0	$-\frac{1}{2.4.6}$	0	$\frac{1}{2.4.6.8}$	0	$-\frac{1}{2.4.6.8.10}$
$1t^0$	$0t^1$	$-\frac{1}{2}t^2$	$0t^3$	$\frac{1}{2^2 2!}t^4$	0	$-\frac{1}{2^3.3!}$	0	$\frac{1}{2^4.4!}$	0	$-\frac{1}{2.4.6.8.10}$

$$y(t) = \sum_{m=0}^{\infty} \frac{(-1)^m}{2^m (m)!} t^{2m}$$

$D)\, y''+4y=0$

$$\sum_{n=2}^{\infty} n(n-1)a_n t^{n-2} + 4\sum_{n=0}^{\infty} a_n t^n = 0$$

$Fazendo, \quad n-2=k,\; n=k+2 \; no\; 1^{\text{o}} \; \sum$

$$\sum_{k=0}^{\infty} (k+2)(k+1)a_{k+2}t^k + 4\sum_{k=0}^{\infty} a_k t^k = 0$$

$$\sum_{k=0}^{\infty} \left[(k+2)(k+1)a_{k+2} + 4a_k\right]t^k = \sum_{k=0}^{\infty} [0]t^k$$

$$a_{k+2} = \frac{-4a_k}{(k+1)(k+2)}$$

		$k=0$	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$
a_0	a_1	a_2	a_3	a_4	a_5	a_6	a_7
a_0	a_1	$-\frac{4a_0}{1.2}$	$-\frac{4a_1}{1.2.3}$	$\frac{4^2 a_0}{1.2.3.4}$	$\frac{4^2 a_1}{1.2.3.4.5}$	$-\frac{4^3 a_0}{1.2.3.4.5.6}$	$-\frac{4^3 a_1}{1.2.3.4.5.6.7}$

$$a_0 t^0 \quad a_1 t^1 \quad -\frac{4a_0}{1.2}t^2 \quad -\frac{4a_1}{1.2.3}t^3 \quad \frac{4^2 a_0}{1.2.3.4}t^4 \quad \frac{4^2 a_1}{1.2.3.4.5}t^5 \quad .....$$

$$a_0\left(1-\frac{4}{1.2}t^2+\frac{4^2}{1.2.3.4}t^4-.......\right)+a_1\left(t^1-\frac{4}{1.2.3}t^3+\frac{4^2}{1.2.3.4.5}t^5-.....\right)$$

$$y(t) = a_0\sum_{m=0}^{\infty} \frac{(-4)^m}{(2m)!}t^{2m} + a_1\sum_{m=0}^{\infty} \frac{(-4)^m}{(2m+1)!}t^{2m+1}$$