UFPR
UNIVERSIDADE FEDERAL DO PARANÁ

PESQUISA OPERACIONAL

Algoritmo *Branch and Bound*

Prof. Volmir Wilhelm
Curitiba, Paraná, Brasil

Branch and Bound

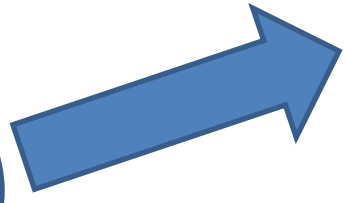
PLI

$$\begin{aligned} \max Z = \quad & 13x1 + 8x2 \\ \text{sa} \quad & x1 + 2x2 \leq 10 \\ & 5x1 + 2x2 \leq 20 \\ & x1, x2 \in Z_+ \end{aligned}$$

PLI Relaxado

PL 0

$$\begin{aligned} \max Z = \quad & 13x1 + 8x2 \\ \text{sa} \quad & x1 + 2x2 \leq 10 \\ & 5x1 + 2x2 \leq 20 \\ & x1, x2 \geq 0 \end{aligned}$$



Branch and Bound

PLI Relaxado -- PL^0

max Z =	$13x_1 + 8x_2$	Z = 62,50
sa	$x_1 + 2x_2 \leq 10$	$x_1 = 2{,}50, x_2 = 3{,}75$
	$5x_1 + 2x_2 \leq 20$	
	$x_1, x_2 \geq 0$	$x_1 \geq 3, \quad x_1 \leq 2$

$x_1 \geq 3$ -- PL^1

max Z =	$13x_1 + 8x_2$	Z = 59
sa	$x_1 + 2x_2 \leq 10$	$x_1 = 3 \quad x_2 = 2{,}50$
	$5x_1 + 2x_2 \leq 20$	
	$x_1 \geq 3$	$x_2 \geq 3, x_2 \leq 2$
	$x_1, x_2 \geq 0$	

$x_1 \leq 2$ -- PL^2

max Z =	$13x_1 + 8x_2$	Z = 58
sa	$x_1 + 2x_2 \leq 10$	$x_1 = 2 \quad x_2 = 4$
	$5x_1 + 2x_2 \leq 20$	
	$x_1 \leq 2$	*fathomed*
	$x_1, x_2 \geq 0$	

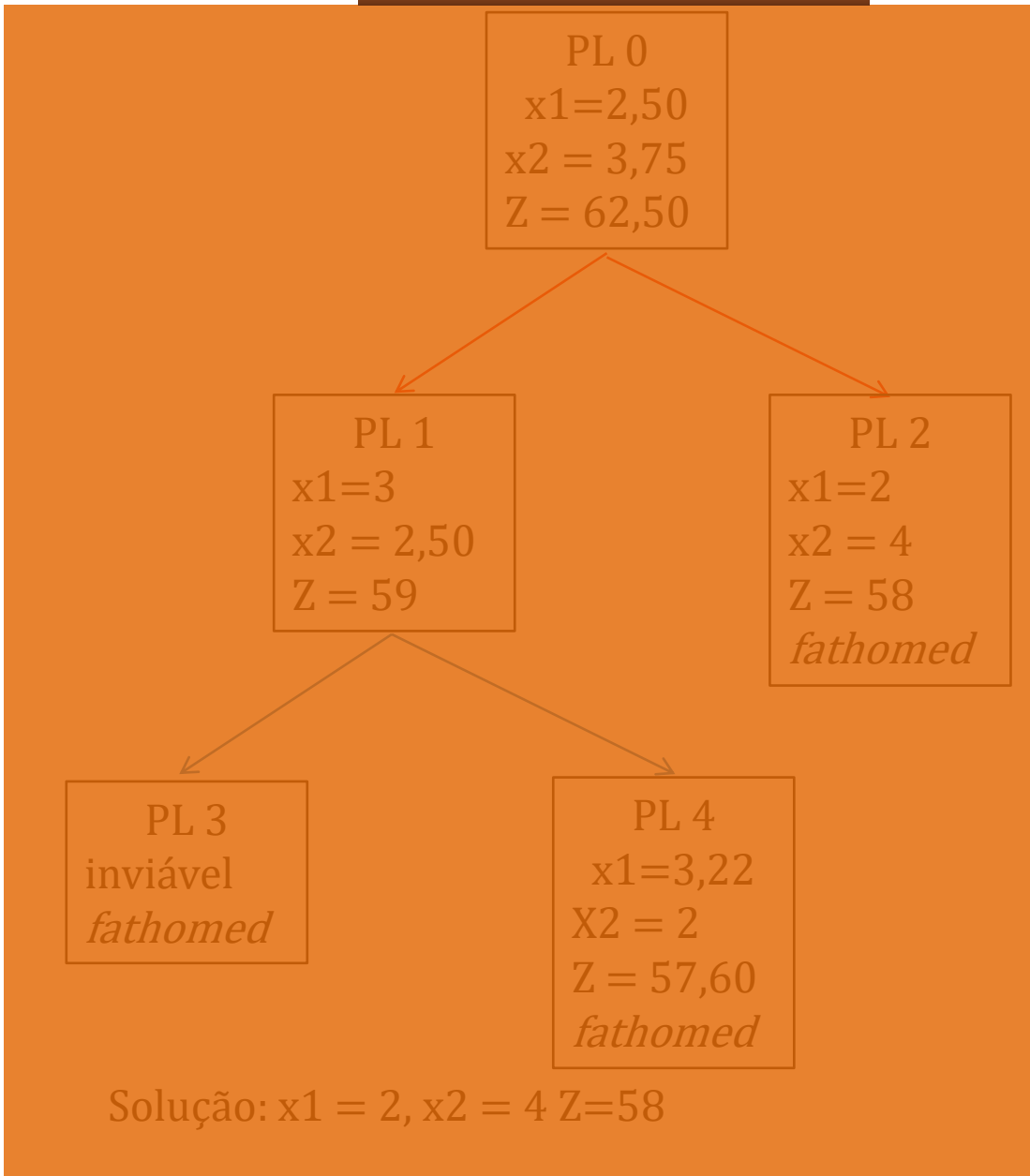
$x_2 \geq 3$ -- PL^3

max Z =	$13x_1 + 8x_2$	Inviável
sa	$x_1 + 2x_2 \leq 10$	
	$5x_1 + 2x_2 \leq 20$	*fathomed*
	$x_1 \geq 3$	
	$x_2 \geq 3$	
	$x_1, x_2 \geq 0$	

$x_2 \leq 2$ -- PL^4

max Z =	$13x_1 + 8x_2$	Z = 57,6
sa	$x_1 + 2x_2 \leq 10$	$x_1 = 3{,}22 \quad x_2 = 2$
	$5x_1 + 2x_2 \leq 20$	
	$x_1 \geq 3$	*fathomed*
	$x_2 \leq 2$	
	$x_1, x_2 \geq 0$	

Árvore



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	$5x_1 + 2x_2 \leq 20$	*fathomed*
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	$x_2 \geq 3$	
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$x_2 \leq 2$ -- PL^4

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	$5x_1 + 2x_2 \leq 20$	
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	$x_2 \leq 2$	
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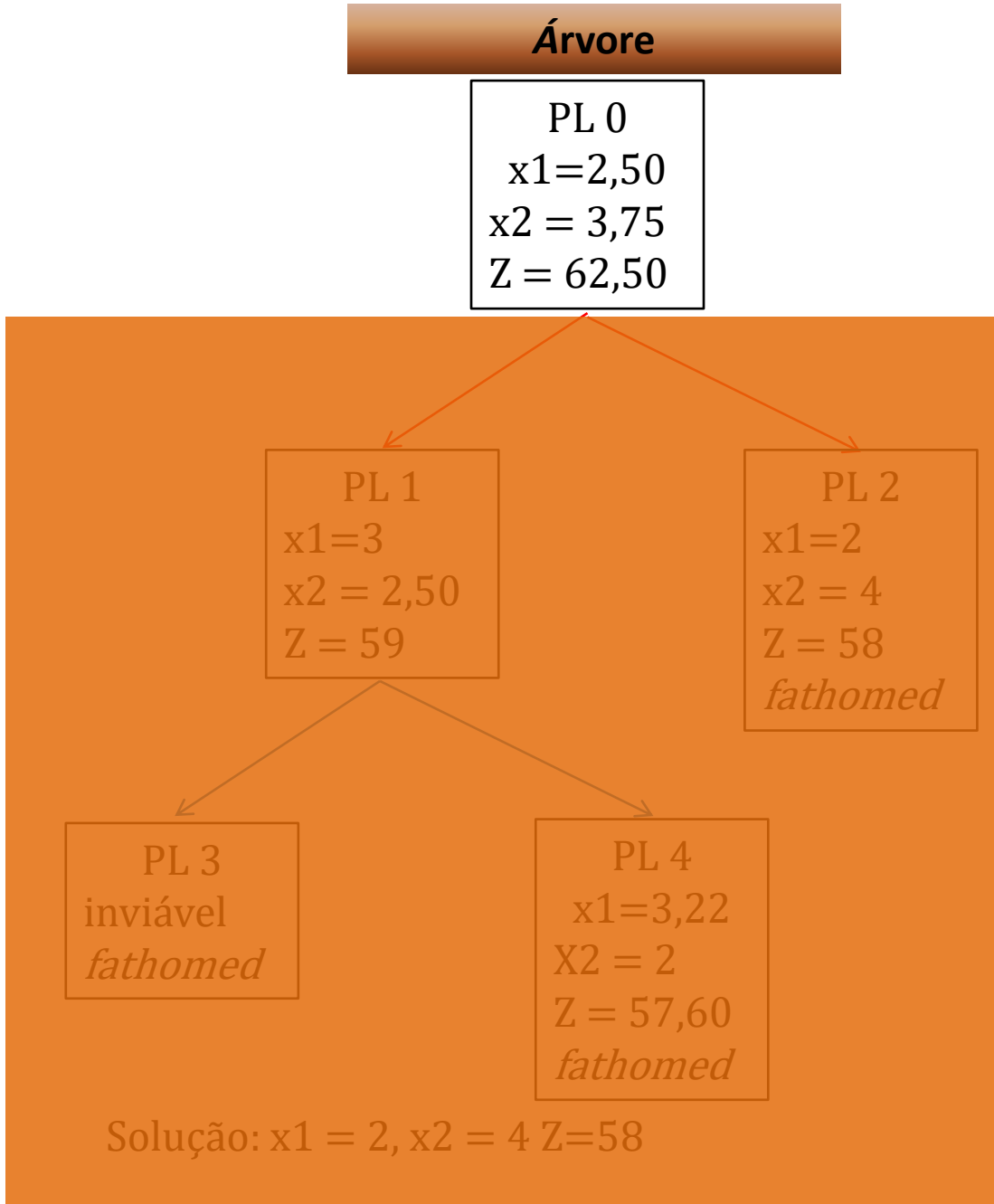
Árvore



Branch and Bound

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	$x_2 \geq 3$	
	$x_1, x_2 \geq 0$	

$x_2 \leq 2$ -- PL^4

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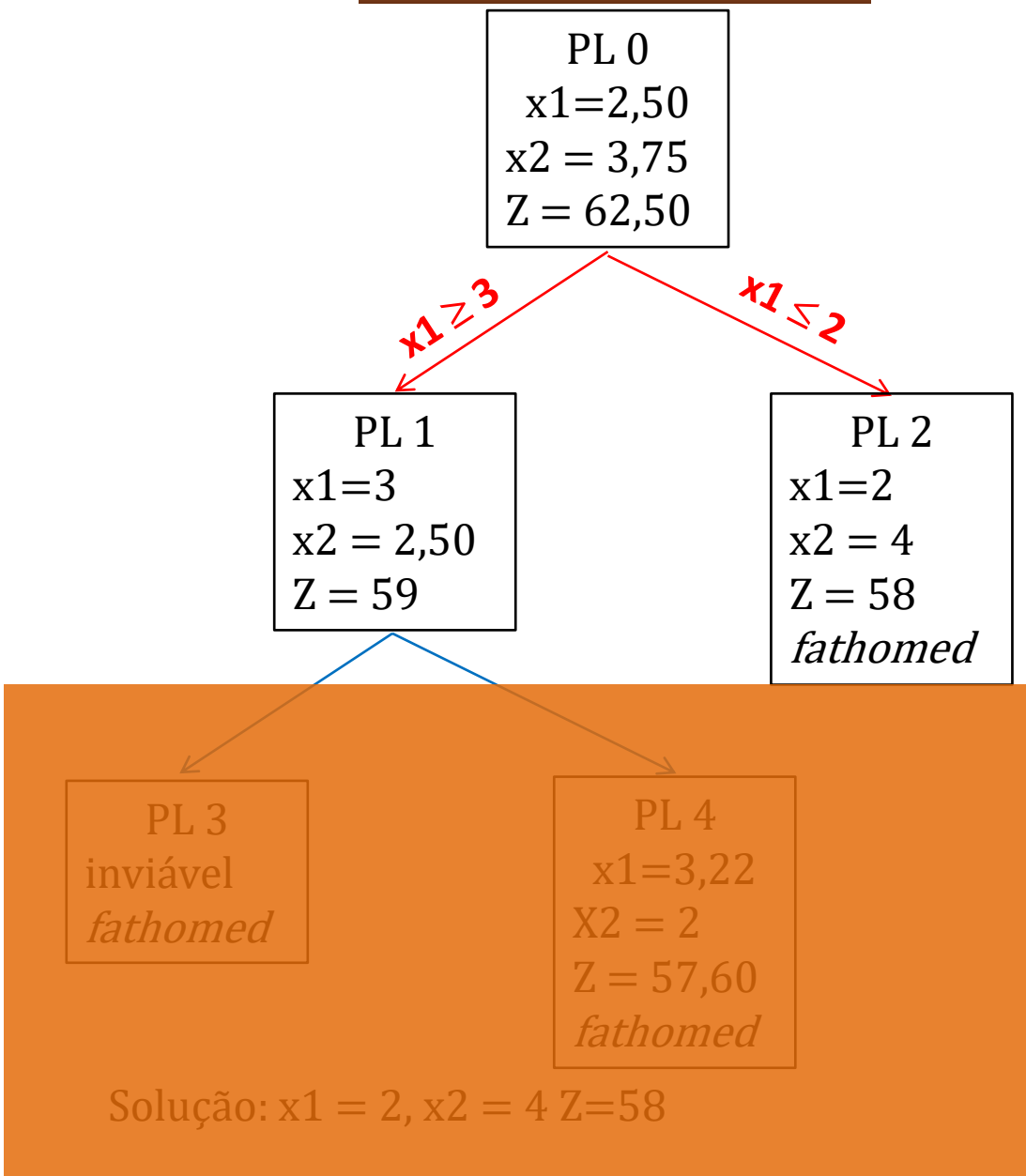
Árvore



Branch and Bound

PLI Relaxado -- PL^0

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$x_2 \geq 3$ -- PL^3

max Z =	$13x_1 + 8x_2$	Inviável
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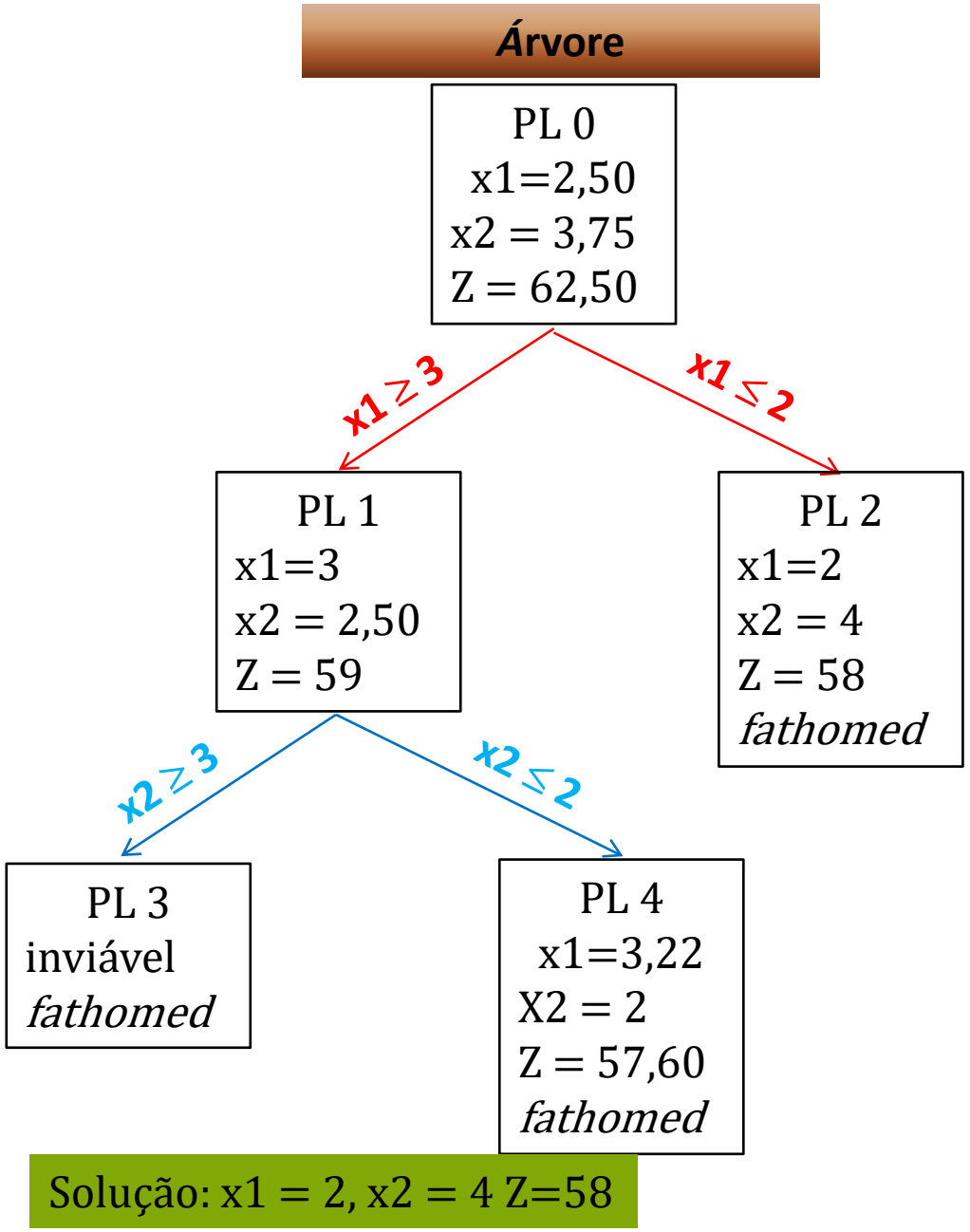
	$5x_1 + 2x_2 \leq 20$	*fathomed*
	$x_1 \geq 3$	
	$x_2 \geq 3$	
	$x_1, x_2 \geq 0$	

$x_2 \leq 2$ -- PL^4

max Z =	$13x_1 + 8x_2$	Z = 57,6
sa	$x_1 + 2x_2 \leq 10$	$x_1 = 3{,}22$ $x_2 = 2$
	$5x_1 + 2x_2 \leq 20$	
	$x_1 \geq 3$	*fathomed*
	$x_2 \leq 2$	
	$x_1, x_2 \geq 0$	

Árvore



Solução: x1 = 2, x2 = 4 Z=58

PLI Relaxado -- PL^0

$$\max Z = 13x_1 + 8x_2$$
$$\text{sa} \quad x_1 + 2x_2 \leq 10$$
$$\quad 5x_1 + 2x_2 \leq 20$$
$$\quad x_1, x_2 \geq 0$$

Base	X1	X2	F1	F1	b
zj	0	0	1,75	2,25	-62,5
X2	0	1	0,625	-0,125	3,75
X1	1	0	-0,25	0,25	2,5

$Z = 62{,}50 \quad x_1 = 2{,}50 \quad x_2 = 3{,}75$

$x_1 \geq 3, \quad x_1 \leq 2$

Pós otimização
Dual-Simplex

$x_1 \geq 3$ -- PL^1

$$\max Z = 13x_1 + 8x_2$$
$$\text{sa} \quad x_1 + 2x_2 \leq 10$$
$$\quad 5x_1 + 2x_2 \leq 20$$
$$\quad x_1 \geq 3$$
$$\quad x_1, x_2 \geq 0$$

$x_1 \geq 3 \Rightarrow -x_1 \leq -3 \Rightarrow -x_1 + S_1 = -3$

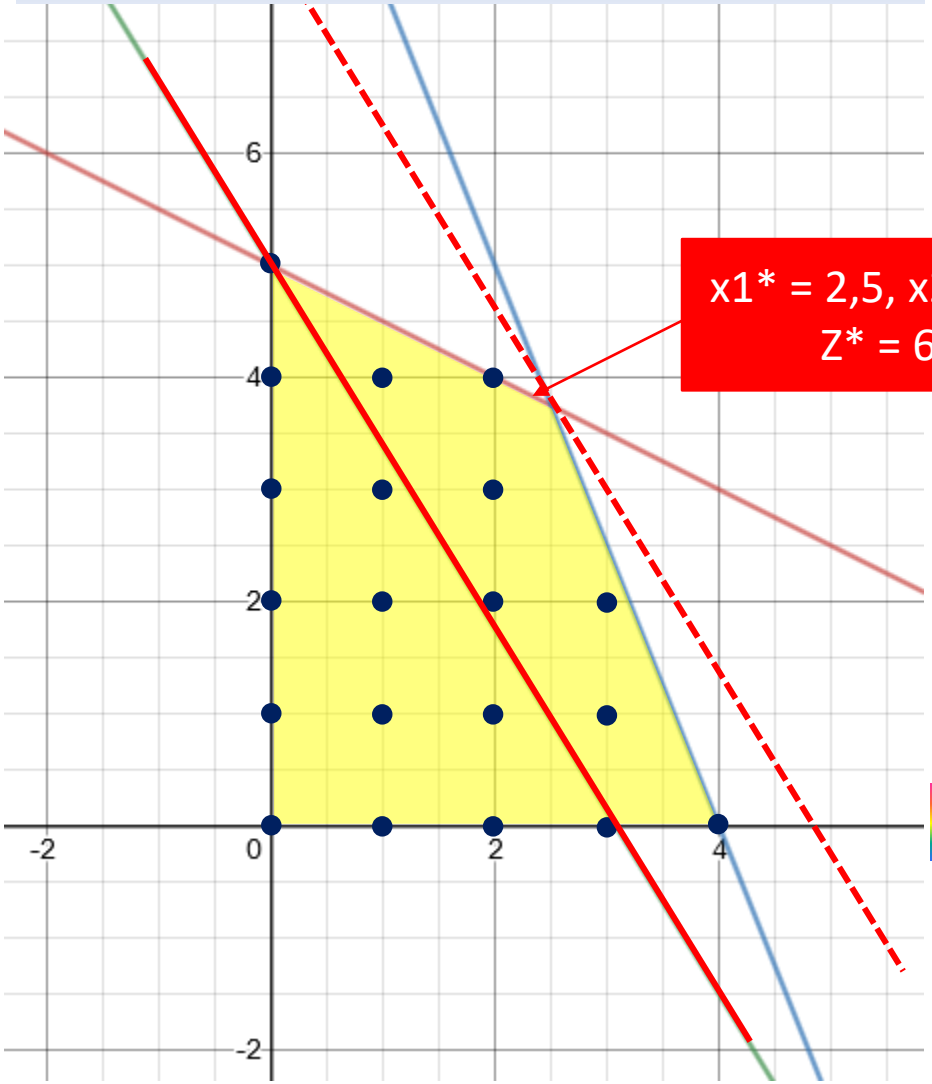
Base	X1	X2	F1	F1	S_1	b
zj	0	0	1,75	2,25		62,5
X2	0	1	0,625	-0,125		3,75
X1	1	0	-0,25	0,25		2,5
S_1	-1				1	-3

Base	X1	X2	F1	F1	S_1	b
zj	0	0	1,75	2,25		62,5
X2	0	1	0,625	-0,125		3,75
X1	1	0	-0,25	0,25		2,5
S_1	0	0	-0,25	0,25	1	-0,5

Base	X1	X2	F1	F1	S_1	b
zj	0	0	0	4	7	59
X2	0	1	0	0,5	2,5	2,5
X1	1	0	0	0	-1	3
S_1	0	0	1	-1	-4	2

$Z = 59 \quad x_1 = 3 \quad x_2 = 2{,}50$

Branch and Bound - gráfico



$$\begin{aligned} \max Z = \quad & 13x1 + 8x2 \\ \text{sa} \quad & x1 + 2x2 \le 10 \\ & 5x1 + 2x2 \le 20 \\ & x1, x2 \in Z_+ \end{aligned}$$

PLI

$x1^* = 2{,}5,\ x2^* = 3{,}75$
$Z^* = 62{,}5$

$$\begin{aligned} \max Z = \quad & 13x1 + 8x2 \\ \text{sa} \quad & x1 + 2x2 \le 10 \\ & 5x1 + 2x2 \le 20 \\ & x1, x2 \ge 0 \end{aligned}$$

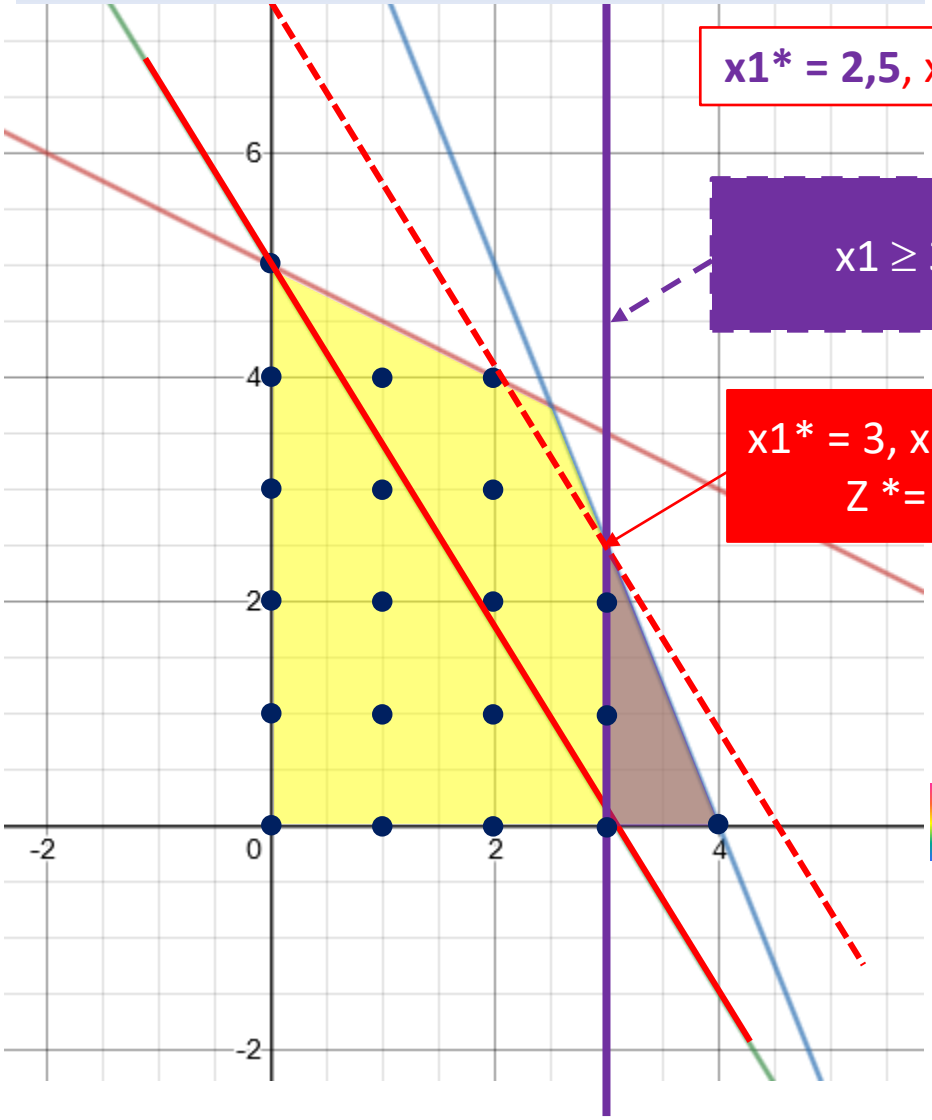
PL 0

$Z^{MAX} = --$

Não superado!!!
(pois $x1^*, x2^* \notin Z_+$)
Iremos gerar outros PL's a partir do PL 0.

https://www.desmos.com/calculator?lang=pt-BR

Branch and Bound - gráfico



x1* = 2,5, x2* = 3,75

$x1 \geq 3$

x1* = 3, x2* = 2,5
Z *= 59

Z^{MAX} = --

$$\begin{aligned} \max Z = \quad & 13x1 + 8x2 \\ \text{sa} \quad & x1 + 2x2 \leq 10 \\ & 5x1 + 2x2 \leq 20 \\ & x1, x2 \geq 0 \end{aligned}$$

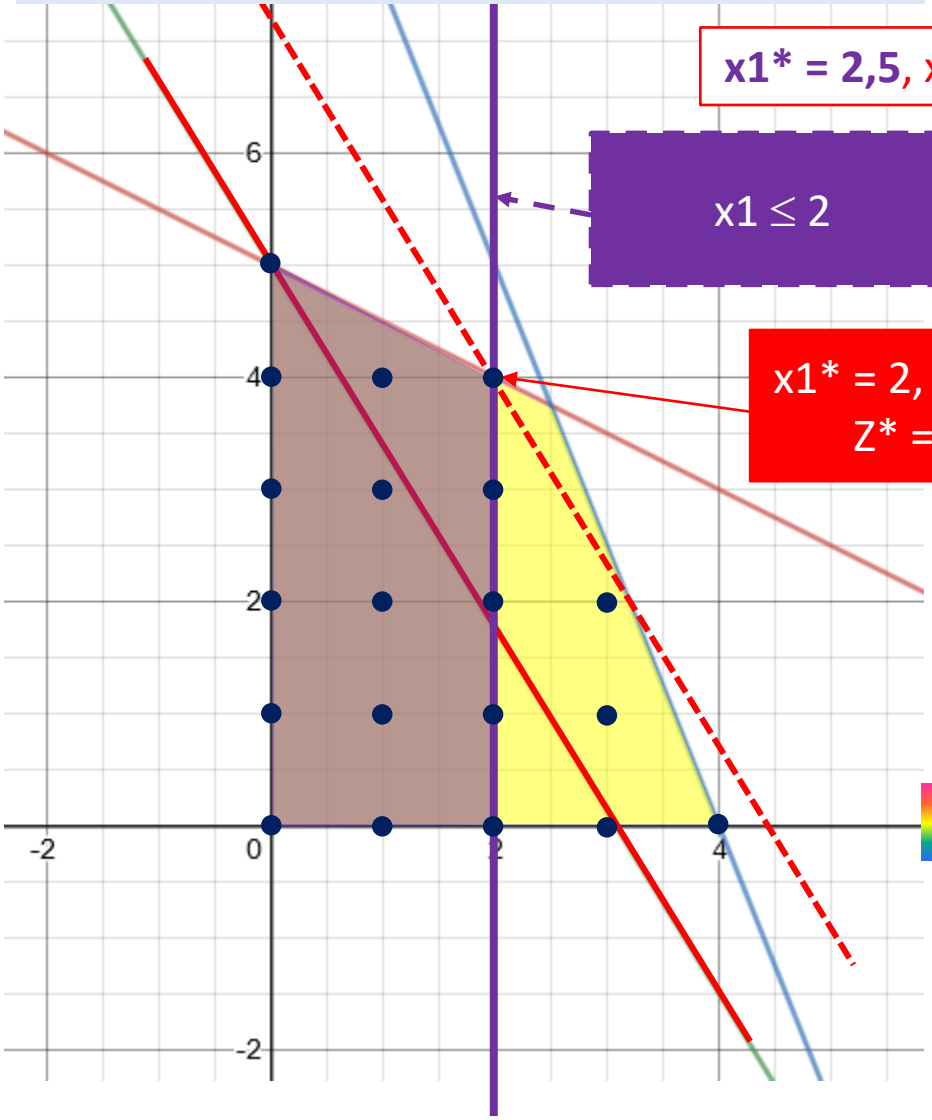
PL 0

$$\begin{aligned} \max Z = \quad & 13x1 + 8x2 \\ \text{sa} \quad & x1 + 2x2 \leq 10 \\ & 5x1 + 2x2 \leq 20 \\ & \mathbf{x1 \geq 3} \\ & x1, x2 \geq 0 \end{aligned}$$

PL 1

Não superado!!!
(pois x1*,x2* $\notin Z_+$)
Iremos gerar outros PL's a partir do PL 1.

Branch and Bound - gráfico



$$\begin{aligned} \max Z = \quad & 13x1 + 8x2 \\ \text{sa} \quad & x1 + 2x2 \le 10 \\ & 5x1 + 2x2 \le 20 \\ & x1, x2 \ge 0 \end{aligned}$$

PL 0

x1* = 2,5, x2* = 3,75

$x1 \le 2$

x1* = 2, x2* = 4
Z* = 58

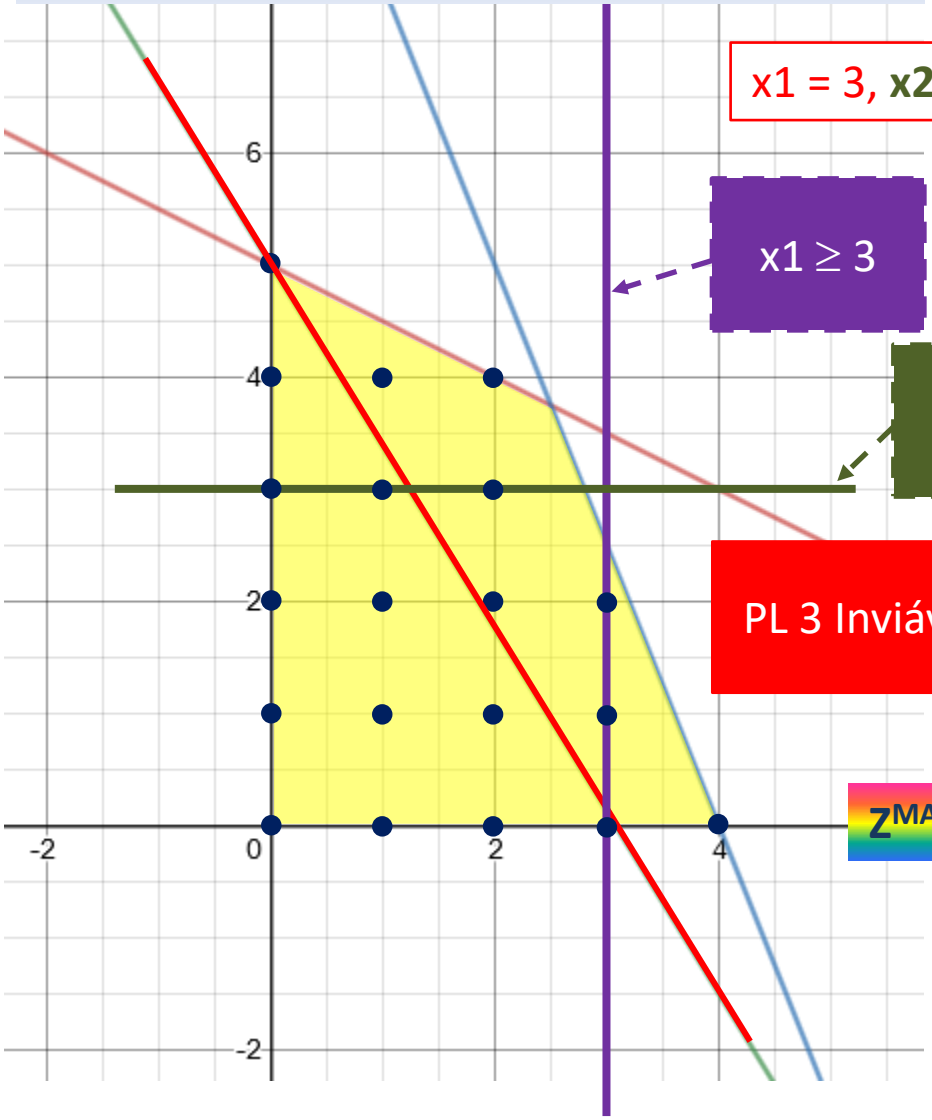
$$\begin{aligned} \max Z = \quad & 13x1 + 8x2 \\ \text{sa} \quad & x1 + 2x2 \le 10 \\ & 5x1 + 2x2 \le 20 \\ & \mathbf{x1 \le 2} \\ & x1, x2 \ge 0 \end{aligned}$$

PL 2

$Z^{MAX} = 58$

Superado/*fathomed*!!!
(pois $x1^*, x2^* \in Z_+$)
Não iremos gerar outros PL's a partir do PL 2.

Branch and Bound - gráfico



x1 = 3, x2 = 2,5

x1 ≥ 3

x2 ≥ 3

PL 3 Inviável!!!

$Z^{MAX} = 58$

$$\begin{aligned} \max Z = \quad & 13x1 + 8x2 \\ sa \quad & x1 + 2x2 \le 10 \\ & 5x1 + 2x2 \le 20 \\ & x1 \ge 3 \\ & x1, x2 \ge 0 \end{aligned}$$

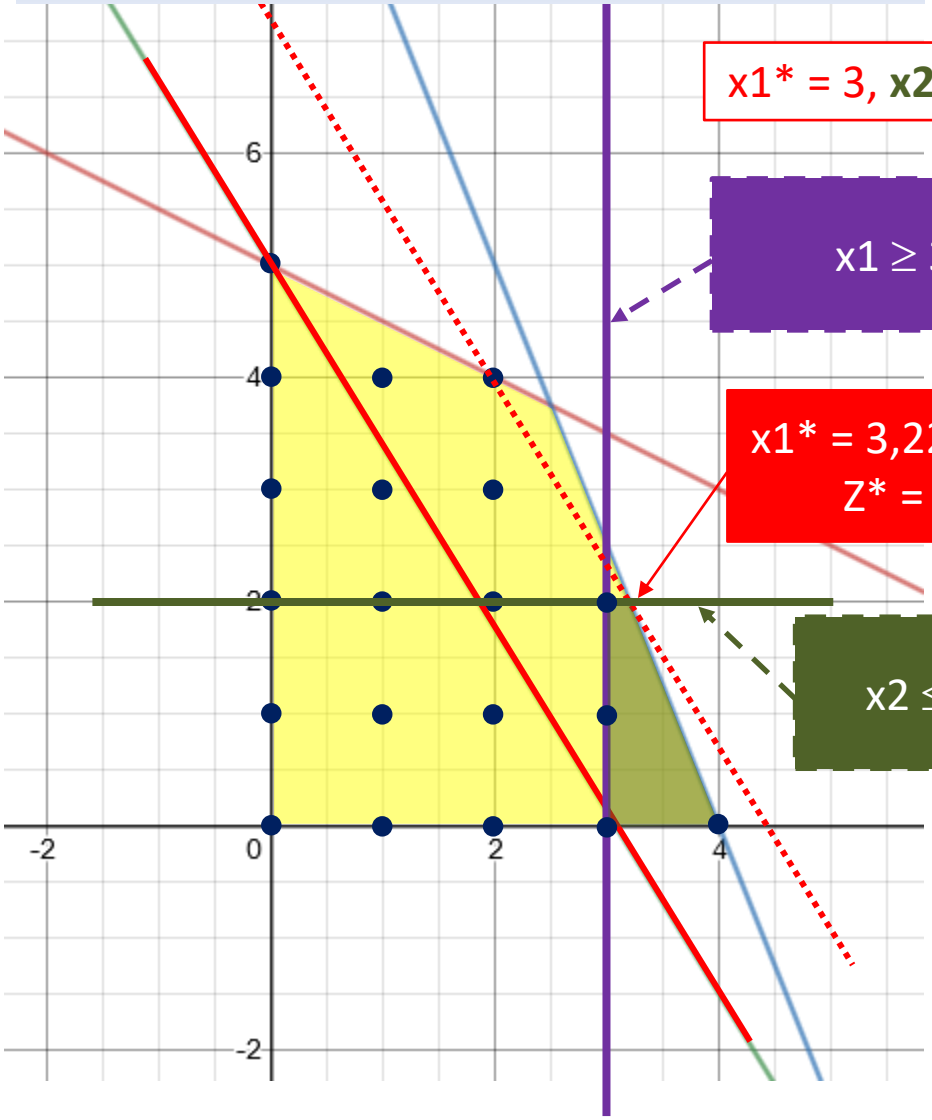
PL 1

$$\begin{aligned} \max Z = \quad & 13x1 + 8x2 \\ sa \quad & x1 + 2x2 \le 10 \\ & 5x1 + 2x2 \le 20 \\ & x1 \ge 3 \\ & x2 \ge 3 \\ & x1, x2 \ge 0 \end{aligned}$$

PL 3

Superado!!!
(pois PL 3 é inviável)
Iremos gerar outros PL's a partir do PL 3.

Branch and Bound - gráfico



$$\begin{aligned} \max Z = \quad & 13x1 + 8x2 \\ \text{sa} \quad & x1 + 2x2 \le 10 \\ & 5x1 + 2x2 \le 20 \\ & x1, x2 \ge 0 \end{aligned}$$

x1* = 3, **x2* = 2,5**

x1 ≥ 3

x1* = 3,22, x2* = 2
Z* = 57,6

x2 ≤ 2

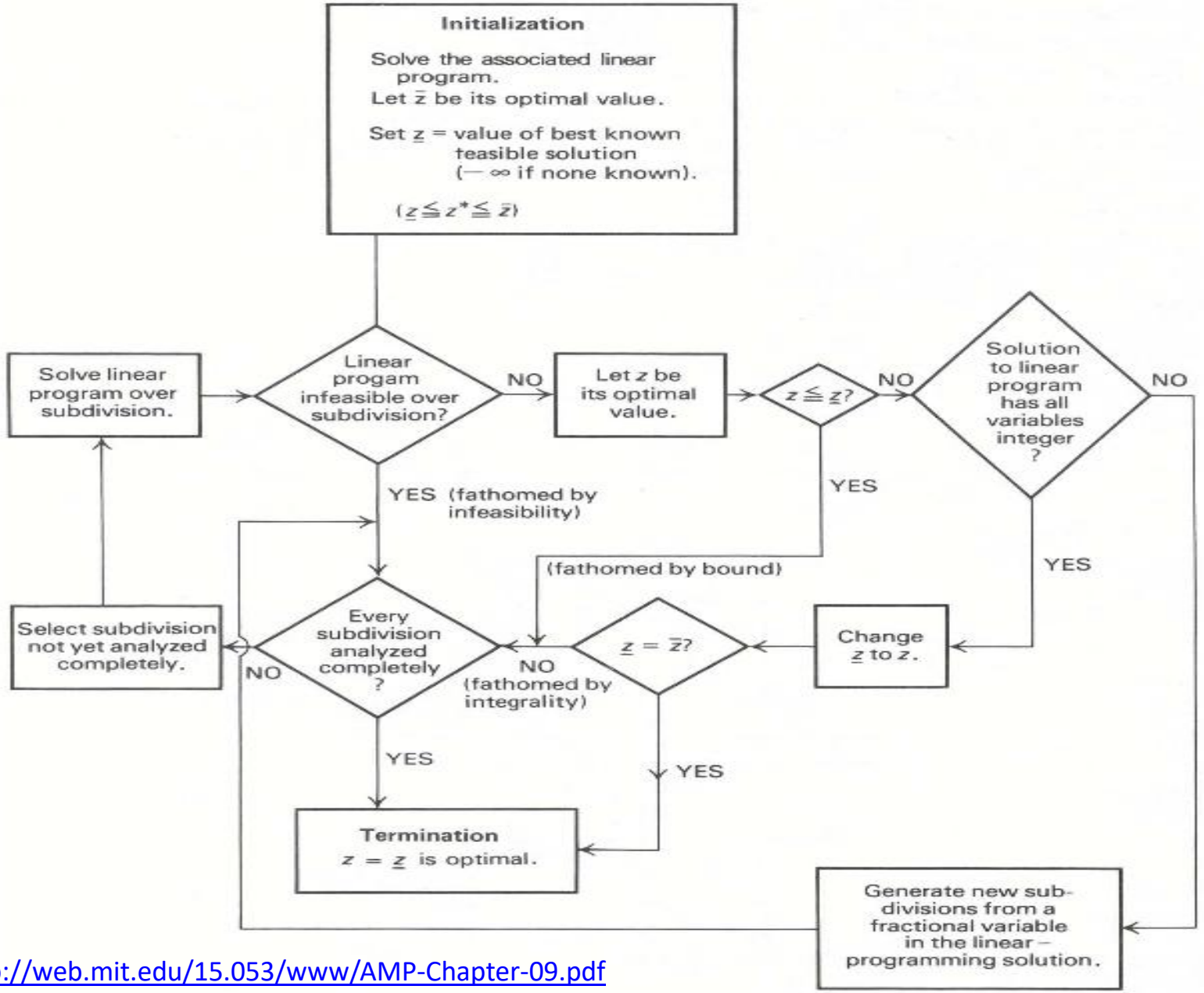
$Z^{MAX} = 58$

PL 0

$$\begin{aligned} \max Z = \quad & 13x1 + 8x2 \\ \text{sa} \quad & x1 + 2x2 \le 10 \\ & 5x1 + 2x2 \le 20 \\ & \mathbf{x1 \ge 3} \\ & \mathbf{x2 \le 2} \\ & x1, x2 \ge 0 \end{aligned}$$

PL 4

Superado!!!
(pois $Z^* < Z^{MAX}$)
Não iremos gerar outros PL's a partir do PL 4.



http://web.mit.edu/15.053/www/AMP-Chapter-09.pdf

Figure 9.17 Branch-and-bound for integer-programming maximization.

A ideia central do algoritmo *branch and bound*- B&B relaxar o problema de programação inteira e dividir o problema relaxado em vários problemas até encontrar soluções inteiras ou não viáveis, o ótimo é a melhor solução encontrada.

O algoritmo B&B é baseado na ideia de “dividir para conquistar”, ou seja, trabalhamos em problemas menores e mais fáceis de resolver em busca da solução ótima.

http://wiki.icmc.usp.br/images/3/32/Pi_aula_16_11_finalizacaoPI_mari.pdf

Busca em árvore

https://bit.ly/3r3y3In