http://people.brunel.ac.uk/~mastjjb/jeb/or/moreip.html

1 - Capital budgeting

There are four possible projects, which each run for 3 years and have the following characteristics.

		Capital requirements (£m)			
Project	Return (£m)	Year	1	2	3
1	0.2		0.5	0.3	0.2
2	0.3		1.0	0.8	0.2
3	0.5		1.5	1.5	0.3
4	0.1		0.1	0.4	0.1
Available capital (£m)			3.1	2.5	0.4

We have a decision problem here: Which projects would you choose in order to maximise the total return?

maximise $0.2x_1 + 0.3x_2 + 0.5x_3 + 0.1x_4$

subject to

$$0.5x_1 + 1.0x_2 + 1.5x_3 + 0.1x_4 <= 3.1$$
$$0.3x_1 + 0.8x_2 + 1.5x_3 + 0.4x_4 <= 2.5$$
$$0.2x_1 + 0.2x_2 + 0.3x_3 + 0.1x_4 <= 0.4$$
$$x_j \in \{0,1\} \quad j=1,\ldots,4$$

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$0.2x_1 + 0.2x_2 + 0.3x_3 + 0.1x_4 <= 0.4$

$x_j \in \{0,1\}$ $j=1,\ldots,4$

Suppose now that we have the additional condition that either project 1 or project 2 must be chosen (i.e. projects 1 and 2 are mutually exclusive).

$\mathbf{x_1 + x_2 = 1}$

2 - Blending

Consider the example of a manufacturer of animal feed who is producing feed mix for dairy cattle. In our simple example the feed mix contains two active ingredients **and a filler to provide bulk**. One kg of feed mix must contain a minimum quantity of each of four nutrients as below:

Nutrient	A	B	C	D
gram	90	50	20	2

The ingredients have the following nutrient values and cost

	A	B	C	D	Cost/kg
Ingredient 1 (gram/kg)	100	80	40	10	40
Ingredient 2 (gram/kg)	200	150	20	-	60

What should be the amounts of active ingredients and filler in one kg of feed mix?

minimise $W = 40x_1 + 60x_2$

$100x_1 + 200x_2 >= 90$ (nutrient A)

$80x_1 + 150x_2 >= 50$ (nutrient B)

$40x_1 + 20x_2 >= 20$ (nutrient C)

$10x_1 >= 2$ (nutrient D)

$x_1 + x_2 + x_3 = 1$ (balancing constraint)

$x_j >= 0$ j=1, 2

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$x_j >= 0$, j=1,2

Suppose now we have the additional conditions:

- **if we use any of ingredient 2 we incur a fixed cost of 15**
- **we need not satisfy all four nutrient constraints but need only satisfy three of them (i.e. whereas before the optimal solution required all four nutrient constraints to be satisfied now the optimal solution could (if it is worthwhile to do so) only have three (any three) of these nutrient constraints satisfied and the fourth violated.**

Variables
introducing a zero-one variable y defined by y = 1 if x_2>=0, 0 otherwise

and add the additional constraint

- $x_2 <= My$

introduce four zero-one variables z_i (i=1,2,3,4) where
z_i = 1 if nutrient constraint i (i=1,2,3,4) is satisfied, 0 otherwise

and add the constraint

- $z_1+z_2+z_3+z_4 >= 3$ (at least 3 constraints satisfied)

and alter the nutrient constraints to be

minimise $W = 40x_1 + 60x_2 + 15y$

$$x_1 + x_2 + x_3 = 1$$
$$100x_1 + 200x_2 >= 90z_1$$
$$80x_1 + 150x_2 >= 50z_2$$
$$40x_1 + 20x_2 >= 20z_3$$
$$10x_1 >= 2z_4$$
$$z_1 + z_2 + z_3 + z_4 >= 3$$
$$x_2 <= y$$
$$z_i \in \{0,1\} \quad i=1,2,3,4 \quad y \in \{0,1\}, \quad x_i >= 0 \quad i=1,2,3$$

3 - Integer programming example

In the planning of the monthly production for the next six months a company must, in each month, operate either a normal shift or an extended shift (if it produces at all). A normal shift costs £100,000 per month and can produce up to 5,000 units per month. An extended shift costs £180,000 per month and can produce up to 7,500 units per month. Note here that, for either type of shift, the cost incurred is fixed by a union guarantee agreement and so is independent of the amount produced.

It is estimated that changing from a normal shift in one month to an extended shift in the next month costs an extra £15,000. No extra cost is incurred in changing from an extended shift in one month to a normal shift in the next month. The cost of holding stock is estimated to be £2 per unit per month (based on the stock held at the end of each month) and the initial stock is 3,000 units (produced by a normal shift). The amount in stock at the end of month 6 should be at least 2,000 units. The demand for the company's product in each of the next six months is estimated to be as shown below:

Month	1	2	3	4	5	6
Demand	6,000	6,500	7,500	7,000	6,000	6,000

Production constraints are such that if the company produces anything in a particular month it must produce at least 2,000 units. If the company wants a production plan for the next six months that avoids stockouts, formulate their problem as an integer program.

Hint: first formulate the problem allowing non-linear constraints and then attempt to make all the constraints linear.

Variables

- x_t = 1 if we operate a normal shift in month t (t=1,2,...,6), 0 otherwise
- y_t = 1 if we operate an extended shift in month t (t=1,2,...,6), 0 otherwise
- P_t (>= 0) be the amount produced in month t (t=1,2,...,6)
- z_t = 1 if we switch from a normal shift in month t-1 to an extended shift in month t (t=1,2,...,6), 0 otherwise
- I_t be the closing inventory (stock left) at the end of month t (t=1,2,...,6)
- w_t = 1 if we produce in month t, 0 otherwise (i.e. Pt = 0)

Restrictions

- only operate (at most) one shift each month: $x_t + y_t <= 1$ t=1,2,...,6

- production limits not exceeded: $P_t <= 5000x_t + 7500y_t$ t=1,2,...,6

- no stockouts: $I_t >= 0$ t=1,2,...,6

- we have an inventory continuity equation of the form

$I_t = I_{t-1} + P_t - D_t$ t=1,2,...,6

- the amount in stock at the end of month 6 should be at least 2000 units

$I_6 >= 2000$

- production constraints of the form "either $P_t = 0$ or $P_t >= 2,000$".

$P_t <= Mw_t$ t=1,2,...,6 $P_t >= 2000w_t$ t=1,2,...,6

- we also need to relate the shift change variable z_t to the shifts being operated: $z_t = x_{t-1}y_t$ t=1,2,...,6

Objective

SUM{t=1,...,6} W = $(100000x_t + 180000y_t + 15000z_t + 2I_t)$

Replace $z_t = x_{t-1}y_t$ t=1,2,...,6

$z_t <= (x_{t-1} + y_t)/2$ t=1,2,...,6 $z_t >= x_{t-1} + y_t - 1$ t=1,2,...,6

4 - Integer programming example 1996 MBA exam

A toy manufacturer is planning to produce new toys. The setup cost of the production facilities and the unit profit for each toy are given below:

Toy	Setup cost (£)	Profit (£)
1	45000	12
2	76000	16

The company has two factories that are capable of producing these toys. In order to avoid doubling the setup cost only *one* factory could be used.

The production rates of each toy are given below (**in units/hour**):

	Toy 1	Toy 2
Factory 1	52	38
Factory 2	42	23

Factories 1 and 2, respectively, **have 480 and 720 hours of production time available** for the production of these toys. The manufacturer wants to know *which* of the new toys to produce, *where* and *how many* of each (if any) should be produced so as to maximise the total profit.

Formulate the problem as an integer program. (Do not try to solve it).

Variables

- f_{ij} = 1 if factory i (i=1,2) is setup to produce toys of type j (j=1,2), 0 otherwise
- x_{ij} the number of toys of type j (j=1,2) produced in factory i (i=1,2), $x_{ij} \in Z^+$

Restrictions

- at each factory cannot exceed the production time available

$x_{11}/52 + x_{12}/38 <= 480$
$x_{21}/42 + x_{22}/23 <= 720$

- cannot produce a toy unless we are setup to do so

$x_{11} <= 52(480)f_{11}$
$x_{12} <= 38(480)f_{12}$
$x_{21} <= 42(720)f_{21}$
$x_{22} <= 23(720)f_{22}$

Objective

maximise $W = 12(x_{11}+ x_{21}) + 16(x_{12}+ x_{22}) - 45000(f_{11} + f_{21}) - 76000(f_{12} + f_{22})$

5 - Integer programming example 1995 MBA exam

A project manager in a company is considering a portfolio of 10 large project investments. These investments differ in the estimated long-run profit (net present value) they will generate as well as in the amount of capital required.

Let P_j and C_j denote the estimated profit and capital required (both given in units of millions of £) for investment opportunity j (j=1,...,10) respectively. The total amount of capital available for these investments is Q (in units of millions of £)

Investment opportunities 3 and 4 are mutually exclusive and so are 5 and 6. Furthermore, neither 5 nor 6 can be undertaken unless either 3 or 4 is undertaken. (Além disso, nem 5 nem 6 podem ser executados a menos que 3 ou 4 seja executado.) At least two and at most four investment opportunities have to be undertaken from the set {1,2,7,8,9,10}.

The project manager wishes to select the combination of capital investments that will maximise the total estimated long-run profit subject to the restrictions described above.

Formulate this problem using an *integer programming*. (Do not actually solve it).

Variables
x_j = 1 if we use investment opportunity j (j=1,...,10), 0 otherwise

Restrictions
- total amount of capital available for these investments is Q

SUM{j=1,...,10}C_jx_j <= Q

- investment opportunities 3 and 4 are mutually exclusive and so are 5 and 6

$x_3 + x_4 <= 1$
$x_5 + x_6 <= 1$

- neither 5 nor 6 can be undertaken unless either 3 or 4 is undertaken

$x_5 <= x_3 + x_4$
$x_6 <= x_3 + x_4$

- at least two and at most four investment opportunities have to be undertaken from the set {1,2,7,8,9,10}

$x_1 + x_2 + x_7 + x_8 + x_9 + x_{10} >= 2$
$x_1 + x_2 + x_7 + x_8 + x_9 + x_{10} <= 4$

Objective
The objective is to maximise the total estimated long-run profit i.e.

maximise W = SUM{j=1,...,10}P_jx_j

6 - Integer programming example 1994 MBA exam

A food is manufactured by refining raw oils and blending them together. The raw oils come in two categories:

- Vegetable oil: VEG1, VEG2
- Non-vegetable oil: OIL1, OIL2, OIL3

The prices for buying each oil are given below (in £/tonne)

VEG1	VEG2	OIL1	OIL2	OIL3
115	128	132	109	114

The final product sells at £180 per tonne. Vegetable oils and non-vegetable oils require different production lines for refining. It is not possible to refine more than 210 tonnes of vegetable oils and more than 260 tonnes of non-vegetable oils. There is no loss of weight in the refining process and the cost of refining may be ignored.

There is a technical restriction relating to the hardness of the final product. In the units in which hardness is measured this must lie between 3.5 and 6.2. It is assumed that hardness blends linearly and that the hardness of the raw oils is:

VEG1	VEG2	OIL1	OIL2	OIL3
8.8	6.2	1.9	4.3	5.1

It is required to determine what to buy and how to blend the raw oils so that the company maximises its profit.

The following extra conditions are imposed on the food manufacture problem stated above as a result of the production process involved:

- the food may never be made up of more than 3 raw oils
- if an oil (vegetable or non-vegetable) is used, at least 30 tonnes of that oil must be used
- if either of VEG1 or VEG2 are used then OIL2 must also be used

Formulate the above problem as a *linear program*. (Do not actually solve it).

Variables

- x_i the number of tonnes of oil of type i used (i=1,...,5), where $x_i >= 0$ i=1,...,5
- $y_i = 1$ if we use any of oil i (i=1,...,5), 0 otherwise

Constraints

- cannot refine more than a certain amount of oil

  $x_1 + x_2 <= 210$ $\qquad$ $x_3 + x_4 + x_5 <= 260$

- hardness of the final product must lie between 3.5 and 6.2

  $(8.8x_1 + 6.2x_2 + 1.9x_3 + 4.3x_4 + 5.1x_5)/(x_1 + x_2 + x_3 + x_4 + x_5) >= 3.5$

  $(8.8x_1 + 6.2x_2 + 1.9x_3 + 4.3x_4 + 5.1x_5)/(x_1 + x_2 + x_3 + x_4 + x_5) <= 6.2$

- must relate the amount used (x variables) to the integer variables (y) that specify whether any is used or not

  $x_1 <= 210y_1$ $\quad$ $x_2 <= 210y_2$ $\quad$ $x_3 <= 260y_3$ $\quad$ $x_4 <= 260y_4$ $\quad$ $x_5 <= 260y_5$

- the food may never be made up of more than 3 raw oils

  $y_1 + y_2 + y_3 + y_4 + y_5 <= 3$

- if an oil (vegetable or non-vegetable) is used, at least 30 tonnes of that oil must be used

  $x_i >= 30y_i$, i=1,...,5

- if either of VEG1 or VEG2 are used then OIL2 must also be used

  $y_4 >= y_1$ $\qquad$ $y_4 >= y_2$

Objective

The objective is to maximise total profit, i.e.

maximise $W = 180(x_1 + x_2 + x_3 + x_4 + x_5) - 115x_1 - 128x_2 - 132x_3 - 109x_4 - 114x_5$

7 - Integer programming example 1987 UG exam

A company is attempting to decide the mix of products which it should produce next week. It has seven products, each with a profit (£) per unit and a production time (man-hours) per unit as shown below:

Product	**Profit** (£ per unit)	**Production time** (man-hours per unit)
1	10	1.0
2	22	2.0
3	35	3.7
4	19	2.4
5	55	4.5
6	10	0.7
7	115	9.5

The company has 720 man-hours available next week.

Incorporate the following additional restrictions into your integer program (retaining linear constraints and a linear objective):

- If any of product 7 are produced an additional fixed cost of £2000 is incurred.
- Each unit of product 2 that is produced over 100 units requires a production time of 3.0 man-hours instead of 2.0 man-hours (e.g. producing 101 units of product 2 requires 100(2.0) + 1(3.0) man-hours).
- If both product 3 and product 4 are produced 75 man-hours are needed for production line setup and hence the (effective) number of man-hours available falls to 720 - 75 = 645.

Formulate the problem of how many units (if any) of each product to produce next week as an integer program in which all the constraints are linear.

Variables

- x_i (integer >=0) be the number of units of product i produced then the integer program is
- $z_7 = 1$ if produce product 7 ($x_7 >= 1$), 0 otherwise
- y_2 = number of units of product 2 produced in excess of 100 units, $y_2 >= 0$
- $z_3 = 1$ if produce product 3, 0 otherwise
- $z_4 = 1$ if produce product 4, 0 otherwise
- $Z = 1$ if produce both product 3 and product 4, otherwise

Restrictions

$1.0x_1 + 2.0x_2 + 3.7x_3 + 2.4x_4 + 4.5x_5 + 0.7x_6 + 9.5x_7 <= 720$

- $x_7 <= Mz_7$
- $x_2 <= 100$ (This will work because x_2 and y_2 have the same objective function coefficient but y_2 requires longer to produce so will always get more flexibility by producing x_2 first (up to the 100 limit) before producing y_2.)
- $1.0x_1 + (2.0x_2 + 3.0y_2) + 3.7x_3 + 2.4x_4 + 4.5x_5 + 0.7x_6 + 9.5x_7 <= 720 - 75Z$
- $x_3 <= Mz_3$ and $x_4 <= Mz_4$
- $Z = z_3z_4$

which we linearise by replacing the non-linear constraint by the two linear constraints

- $Z >= z_3 + z_4 - 1$
- $Z <= (z_3 + z_4)/2$

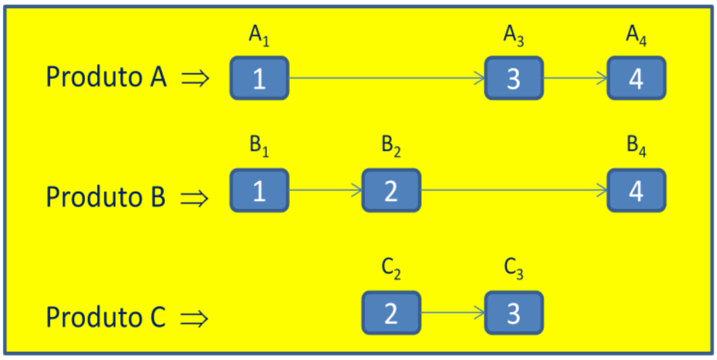
Objective

maximise $W = 10x_1 + 22(x_2 + y_2) + 35x_3 + 19x_4 + 55x_5 + 10x_6 + 115x_7$

8 – SCHEDULING

http://www.feg.unesp.br/~fmarins/po/slides/2o.s/PLinteira.pdf

Três produtos A, B e C serão produzidos usando quatro máquinas. A sequência tecnológica e os tempos de processamento (A_i, B_j, C_k) são mostrados abaixo

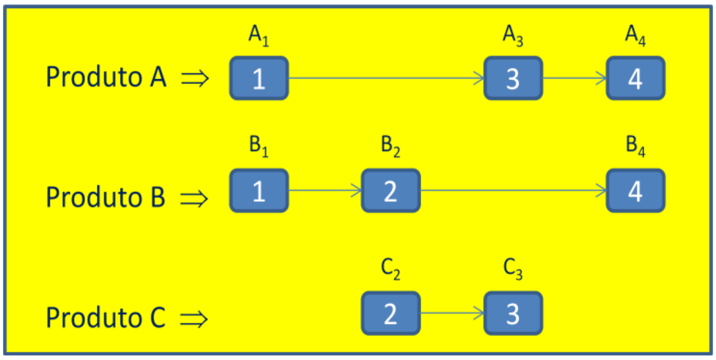


Cada máquina pode processar um produto de cada vez. Cada produto requer um conjunto diferente de ferramentas, de modo que cada máquina termina o processamento de um produto antes de iniciar o processamento de um outro produto.

Deseja-se que o tempo para concluir o produto B não seja maior que d horas após o início das atividades de processamento. O problema é determinar a sequência na qual os vários produtos devem ser processados nas máquinas de modo que se complete a fabricação de todos os produtos no menor tempo possível.

Variáveis

X_{Ai} o tempo (em horas), a partir do início dos trabalhos nas máquinas, quando o processamento do produto A é iniciado na máquina j, para j = 1, 3, 4. Analogamente, X_{Bj} para j = 1, 2, 4 e X_{Cj} para j = 2, 3.



Restrições

Restrições pela sequência tecnológica para produto A:

$X_{A1} + A_1 \leq X_{A3}$ $\qquad$ $X_{A3} + A_3 \leq X_{A4}$

Similarmente tem-se para os produtos B e C:

$X_{B1} + B_1 \leq X_{B2}$ $\qquad$ $X_{B2} + B_2 \leq X_{B4}$

$X_{C2} + C_2 \leq X_{C3}$

$X_{A1} + A_1 \leq X_{B1}$ ou $X_{B1} + B_1 \leq X_{A1}$.

$X_{A1} + A_1 \leq M\delta_1 + X_{B1}$ $\qquad$ $X_{B1} + B_1 \leq M(1 - \delta_1) + X_{A1}$

$X_{B2} + B_2 - X_{C2} \leq M\delta_2$ $\qquad$ $X_{C2} + C_2 - X_{B2} \leq M(1 - \delta_2)$

$X_{A3} + A_3 - X_{C3} \leq M\delta_3$ $\qquad$ $X_{C3} + C_3 - X_{A3} \leq M(1 - \delta_3)$

$X_{A4} + A_4 - X_{B4} \leq M\delta_4$ $\qquad$ $X_{B4} + B_4 - X_{A4} \leq M(1 - \delta_4)$

$X_{B4} + B_4 - X_{B1} \leq d$

Objetivo

min max $(X_{A4} + A_4, X_{B4} + B_4, X_{C3} + C_3)$.

$Y \geq X_{A4} + A_4$

$Y \geq X_{B4} + B_4$

$Y \geq X_{C3} + C_3$

Função objetivo linear: min W = Y.